IYGB

Mathematical Methods 2

Practice Paper D

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 9 questions in this question paper.

The total mark for this paper is 100.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained by attempting all the questions on the paper.

Topics Examined Under Mathematical Methods 2

- Vector Operators: Gradient, Divergence and Curl
- Index Summation Notation
- Double Integrals in Cartesian and Polar Coordinates
- Triple Integrals in Cartesian, Cylindrical and Spherical Coordinates
- Jacobians
- Volume Integrals
- Surface Integrals
- Line Integrals
- Multiple and Vector Integration in Parametric Form
- Applications of Multiple Integration and Vector Integration
  [Mass, Work, Flux, Pressure, Centre of Mass, Moment of Inertia etc.]
- Green's Theorem and Applications
- Divergence Theorem and Applications
- Stokes Theorem and Applications

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Question 1

Use index summation notation to prove the validity of the following vector identity

$$\nabla \wedge (\varphi \mathbf{A}) \equiv \nabla \varphi \wedge \mathbf{A} + \varphi (\nabla \wedge \mathbf{A}),$$

where $\varphi = \varphi(x, y, z)$ is a smooth scalar function and $\mathbf{A} = \mathbf{A}(x, y, z)$ is a smooth vector function. **(6)**

Question 2

Evaluate the line integral

$$\oint_C \left[y^3 dx + (xy) dy \right],$$

where C is a circle of radius 1, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question. **(9)**

Question 3

$$I = \int_0^2 \int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} \frac{2y}{x^2 + y^2} \, dx \, dy .$$

Use polar coordinates to find an exact simplified answer for I . **(11)**

Question 4

The surface S is the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = 1.$$

By using Spherical Polar coordinates (r, θ, φ), or otherwise, evaluate

$$\oiint_S \left(x^2 + y + z\right) dS .$$ **(11)**

Question 5

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{m}$ is a constant vector, show that

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{|\mathbf{r}|^3} \right) = 0$$ **(8)**

Question 6

The functions $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives.

a) State formally Green's theorem in the plane, with reference to the functions, P and Q. **(2)**

b) Evaluate the integral

$$\int_{-1}^{1} \int_{x^2}^{1} \left(x^2 - 7y^2\right) dy\, dx .$$ **(6)**

c) By considering a line integral over a suitable contour C, use Green's theorem in the plane to independently verify the answer to part **(b)**. **(7)**

Question 7

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta,\varphi)=(R+r\cos\theta)\cos\varphi, \quad y(\theta,\varphi)=(R+r\cos\theta)\sin\varphi, \quad z(\theta,\varphi)=r\sin\theta,$$

where $0\le\theta\le 2\pi$ and $0\le\varphi\le 2\pi$.

a) Find a general Cartesian equation for the surface of a torus. **(3)**

A torus T has Cartesian equation

$$\left(4-\sqrt{x^2+y^2}\right)^2=1-z^2.$$

b) Use a suitable parameterization of T to find its surface area. **(12)**

Question 8

The finite region R is bounded by the straight lines with equations

$$y=x, \quad x=1 \quad \text{and} \quad y=0.$$

Use the transformation equations

$$u=x+y \quad \text{and} \quad v=\frac{y}{x},$$

to find an exact value for

$$\iint_R \left(\frac{x+y}{x^2}\right)e^{x+y}\,dx\,dy.$$ **(12)**

Question 9

The smooth vector field $\mathbf{F}$ exists around the open, two sided, surface S, with closed boundary C.

a) State Stokes' Integral Theorem for open surfaces, fully defining all the quantities involved. **(2)**

b) Hence show, that if φ a smooth vector field defined everywhere, and C is any path between two fixed points, then

$$\int_C \nabla\varphi \cdot \mathbf{dr},$$

is independent of the path of C. **(3)**

c) Given further that $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ evaluate

$$\int_C \left[\frac{\mathbf{r}}{|\mathbf{r}|^3} + x\mathbf{i} \right] \cdot \mathbf{dr},$$

where C is the straight line segment from $(2,1,2)$ to $(6,3,2)$. **(8)**

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