IYGB

Mathematical Methods 1

Practice Paper D

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 17 questions in this question paper.

The total mark for this paper is 138.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained without attempting all the questions on the paper.

If the number of marks exceeds 100 then the paper will score maximum marks.

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Topics Examined Under Mathematical Methods 1

- Revision and extensions of Pre-University Algebra and Calculus, including intrinsic coordinates.
- Partial Differentiation
  [includes applications and the use of the gradient operator]
- Curve Sketching in 2 and 3 Dimensions
- Limits
- Ordinary Differential Equations
  [First and second order including substitutions, variation of parameters, D operator and simple Laplace transforms]
- Sequences and Series
- Recurrence Relations
- Product Operator
- Fourier Series
- Conics and Quadrics
- Differentiation and Integration under the Integral Sign

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Question 1

A curve C is defined parametrically

$$(x, y, z) = (3\cos t, 3\sin t, 4t), \quad 0 \le t \le 5\pi .$$

where t is a parameter.

Sketch the graph of C. **(2)**

Question 2

It is given that the following series converges.

$$\sum_{n=1}^{\infty} \frac{(5x)^n}{4n^2}, \; x \in \mathbb{R}, \; x > 0 .$$

Determine the range of possible values of x. **(4)**

Question 3

The functions F and G satisfy

$$G(x, y) \equiv F\left[u(x, y), \; v(x, y)\right],$$

where u and v satisfy the following transformation equations.

$$u = x\cos y, \quad v = x\sin y .$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial x}\right]^2 + \left[\frac{1}{x}\frac{\partial G}{\partial y}\right]^2 = \left[\frac{\partial F}{\partial u}\right]^2 + \left[\frac{\partial F}{\partial v}\right]^2 .$$

(6)

Question 4

Find each of the following Laplace transforms or inverse Laplace transforms, showing, where appropriate, the techniques used.

a) $\mathcal{L}\left(t^3 + 2\mathrm{e}^{-2t}\right)$ **(2)**

b) $\mathcal{L}\left(\mathrm{e}^{-2t}\cosh 3t\right)$ **(2)**

c) $\mathcal{L}\left(t^2 \sin t\right)$ **(3)**

d) $\mathcal{L}\left(\dfrac{\mathrm{e}^t - 1}{t}\right)$ **(3)**

e) $\mathcal{L}^{-1}\left(\dfrac{2}{2s-3}\right)$ **(1)**

f) $\mathcal{L}^{-1}\left(\dfrac{6s-17}{s^2-6s+9}\right)$ **(3)**

Question 5

Find the value of the following limit

$$\lim_{x\to\infty}\left[\left(1+\frac{1}{x^{\frac{3}{2}}}+\frac{1}{x^2}\right)^x\right].$$ **(7)**

Question 6

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{n=2}^{\infty}\left[1 - \frac{1}{2-2^n}\right].$$ **(7)**

Question 7

The function of three variables f is defined as

$$f(x,y,z) \equiv x^2 + y^2 + z^2 + xy - x + y, \quad x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}.$$

Find the stationary value of f, including the triple (x,y,z) which produces this value, further determining the nature of this stationary value. **(6)**

Question 8

$$\begin{aligned} 7x &\ +2y &\ -3z &\ = 30 \\ 3x &\ +4y &\ -5z &\ = 14 \\ 5x &\ -3y &\ +4z &\ = 18 \end{aligned}$$

Solve the above system of the simultaneous equations by using Cramer's rule. **(6)**

Question 9

A sequence of numbers u_1, u_2, u_3, u_4, $u_5 \ldots$, u_n, … is generated by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n + 4n\,, \quad u_1 = 1\,,\; u_2 = 3\,.$$

Determine an expression for the n^{th} term of this sequence. **(9)**

Question 10

Use the method of variation of parameters to find the particular integral of the following differential equation, and hence deduce its general solution.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\,\mathrm{e}^x\,, \quad x \neq 0\,.$$ **(10)**

Question 11

$$f(D) = 2D^2 - D + 1\,.$$

Show directly, by the definition of the D operator as $D \equiv \frac{d}{dx}(\;\;)$, that

$$f(D)\left\{\mathrm{e}^{kx}V(x)\right\} = \mathrm{e}^{kx} f(D+k)\left\{V(x)\right\},$$

where $V(x)$ is a smooth function of x and k is a constant. **(7)**

Question 12

A curve has Cartesian equation

$$y = \ln(\sin x), \ 0 < x < \pi .$$

Show that an intrinsic equation of the curve is

$$s = \ln\left|\frac{2}{\tan\psi + \sec\psi}\right| ,$$

where s is the arc length measured from the point where $\psi = \arctan\frac{3}{4}$, where ψ is the angle the tangent to the curve makes with the positive x axis **(9)**

Question 13

An integral I with variable limits is defined as

$$I(x) = \int_{x}^{x^2} e^{\sqrt{u}} \, du .$$

a) Use a suitable substitution followed by integration by parts to find a simplified expression for

$$\frac{d}{dx}\left[I(x)\right].$$ **(6)**

b) Verify the answer obtained in part **(a)** by carrying the differentiation over the integral sign. **(3)**

Question 14

Find the general solution of the following differential equation.

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 2x , \quad x > 0 .$$ **(10)**

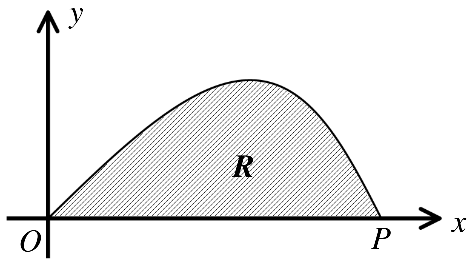
Question 15

Find, showing a detailed method, the area enclosed by the ellipse with the following Cartesian equation.

$$6x^2 + 4xy + 9y^2 - 12x - 4y = 4 .$$

(10)

Question 16



The figure above shows the finite region R, bounded by the coordinate axes and the curve with parametric equations

$$x = 3t + \sin t, \quad y = 2\sin t \ , \ 0 \le t \le \pi .$$

R is fully revolved about the y axis forming a solid of revolution.

Show that the volume of this solid is $39\pi^2$.

(10)

Question 17

A periodic function $f(t)$ is defined in the interval $(-L, L)$, $L > 0$, $f(t+2L) = f(t)$.

It is further given that $f(t)$ is continuous or piecewise continuous in $(-L, L)$ and has Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right],$$

where $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)\,dt$, $n = 0, 1, 2, 3, \ldots$

and $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)\,dt$, $n = 1, 2, 3, 4, \ldots$

Show that the complex Fourier series expansion of $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty}\left[c_n \mathrm{e}^{\frac{\mathrm{i}n\pi t}{L}}\right],$$

where $c_n = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(t)\,\mathrm{e}^{-\frac{\mathrm{i}n\pi t}{L}}\,dt$, $n \in \mathbb{Z}$ **(12)**