IYGB

Mathematical Methods 1

Practice Paper C

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 17 questions in this question paper.

The total mark for this paper is 153.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained without attempting all the questions on the paper.

If the number of marks exceeds 100 then the paper will score maximum marks.

Topics Examined Under Mathematical Methods 1

- Revision and extensions of Pre-University Algebra and Calculus, including intrinsic coordinates.
- Partial Differentiation
  [includes applications and the use of the gradient operator]
- Curve Sketching in 2 and 3 Dimensions
- Limits
- Ordinary Differential Equations
  [First and second order including substitutions, variation of parameters, D operator and simple Laplace transforms]
- Sequences and Series
- Recurrence Relations
- Product Operator
- Fourier Series
- Conics and Quadrics
- Differentiation and Integration under the Integral Sign

I.Y.G.B. www.madasmaths.com

Question 1

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the value of

$$\prod_{3}^{16}\left[1+\frac{4}{r-2}\right].$$ (3)

Question 2

The 3×3 matrix $\mathbf{C}$ is given below.

$$\mathbf{C}=\begin{pmatrix}1 & 2 & 1\\ 2 & 1 & 1\\ 1 & 4 & 2\end{pmatrix}.$$

a) Use the standard method for finding the inverse of a 3×3 matrix, to determine the elements of $\mathbf{C}^{-1}$. (4)

b) Verify the answer of part **(a)** by obtaining the elements of $\mathbf{C}^{-1}$, by using a method involving elementary row operations. (4)

Question 3

$$z(x,y)=x^4+y^4-4xy .$$

Investigate the critical points of z. (8)

Question 4

Use integration to find the Laplace Transform of

$$f(t) = \cosh(at), \ t \geq 0$$

where a is non zero constant. **(6)**

Question 5

It is given that the following integral converges.

$$I = \int_0^1 \frac{x-1}{\ln x} \, dx .$$

Evaluate I by carrying out a suitable differentiation under the integral sign.

You may not use standard integration techniques in this question. **(8)**

Question 6

A sequence of numbers, u_1, u_2, u_3, u_4, ... , is defined by

$$u_{n+1} = 3u_n - 1, \quad u_1 = 2 .$$

Determine, in terms of n, a simplified expression for

$$\sum_{r=1}^{n} u_r .$$ **(9)**

Question 7

A function $f(x)$ is defined in an interval $(-L, L)$, $L > 0$.

a) State the general formula for the Fourier series of $f(x)$ in $(-L, L)$, giving general expressions for the coefficients of the series. **(2)**

b) Find the Fourier series of

$$f(x) = \frac{1}{2} + \frac{1}{2}\text{sign}(x), \quad -\pi \le x \le \pi .$$ **(5)**

c) Prove the validity of Parseval's identity for the Fourier series of $f(x)$ in the interval $(-\pi, \pi)$. **(3)**

d) Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} .$$ **(4)**

Question 8

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = \frac{(5x-2)e^{4x}}{x^3} .$$

You may assume that $\frac{d}{dx}\left(\frac{e^{5x}}{x^2}\right) = \frac{(5x-2)e^{5x}}{x^3}$. **(8)**

Question 9

The function f depends on u and v where

$$u = 2xy \qquad \text{and} \qquad v = x^2 - y^2 .$$

Assuming $x \neq y$, $x \neq 0$ and $y \neq 0$, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left[u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right].$$

(7)

Question 10

A solid's base is bounded by the circle with equation

$$x^2 + y^2 = 9 .$$

Determine the volume of the solid, given that every vertical cross-section of the solid, which is perpendicular to the x axis, is a square. (7)

Question 11

$$x^2\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + 9y = 0, \; x > 0 .$$

Use the fact that $y = Ax^{\frac{3}{2}}$ satisfies the above differential equation, to find the full solution subject to $y = 4$ and $\frac{dy}{dx} = 10$ at $x = 1$. (11)

Question 12

A curve C has intrinsic equation

$$s = 8\left(\sec^3\psi - 1\right),\ 0 \le \psi < \frac{\pi}{2},$$

where s is the arc length is measured from a Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis. It is further given that $\psi = 0$ at O.

Show that a Cartesian equation of C is

$$y^2 = \frac{x^3}{27}.$$ **(9)**

Question 13

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 10\mathrm{e}^{-2x},$$

a) ... by using D-operator techniques only. **(7)**

b) ... by using the substitution $Y = (D+2)y$, in a method involving D-operator techniques only. **(7)**

Question 14

$$x^2 + y^2 + z^2 + xy + yz + xz = 2.$$

The quadric surface, with the above Cartesian equation, is rotated out of its standard position.

By suitably removing the cross terms, give the Cartesian equation of the quadric surface in the rotated system.

You are expected to name the conic and give the direction of any axes of symmetry in the original coordinate system. **(12)**

Question 15

Use two distinct methods to evaluate the following limit

$$\lim_{n \to \infty} \left[\sqrt{n^2 + 3n} - n \right].$$

You may not use the L' Hospital's rule in this question. **(10)**

Question 16

A curve C is defined in the largest real domain by the equation

$$y = \frac{2 - \sqrt{x}}{x - 9\sqrt{x} + 18}.$$

Sketch the graph of C.

The sketch must include

- ... the equations of any asymptotes of C.
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates any stationary points of C. **(12)**

Question 17

Use the ratio test to show that the following series converges

$$\sum_{n=1}^{\infty} \left[\frac{5^n + 1}{n^n + 8} \right].$$

You may assume without proof that $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{-n} = \frac{1}{e}$. **(7)**
