IYGB

Mathematical Methods 1

Practice Paper B
Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to MOST questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 18 questions in this question paper.
The total mark for this paper is 150.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained without attempting all the questions on the paper.

If the number of marks exceeds 100 then the paper will score maximum marks.

Topics Examined Under Mathematical Methods 1

- Revision and extensions of Pre-University Algebra and Calculus, including intrinsic coordinates.
- Partial Differentiation
  [includes applications and the use of the gradient operator]
- Curve Sketching in 2 and 3 Dimensions
- Limits
- Ordinary Differential Equations
  [First and second order including substitutions, variation of parameters, D operator and simple Laplace transforms]
- Sequences and Series
- Recurrence Relations
- Product Operator
- Fourier Series
- Conics and Quadrics
- Differentiation and Integration under the Integral Sign

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Question 1

A surface S is given by the Cartesian equation

$$x^2 + y^2 = 25.$$

Draw a sketch of S, and describe it geometrically. **(3)**

Question 2

Simplify each of the following expressions.

a) $\dfrac{1}{D^2+4D+3}\left\{30\mathrm{e}^{-2x}\right\}$. **(1)**

b) $\dfrac{1}{D^2+4D+3}\{30\sin 2x\}$. **(3)**

c) $\dfrac{1}{D^2+4D+4}\left\{30x^2\mathrm{e}^{-2x}\right\}$. **(3)**

d) $\dfrac{1}{D^2+4D+4}\{30\}$. **(2)**

e) $\dfrac{1}{D^2+2D}\{30\}$. **(3)**

Question 3

$$\begin{aligned} x+2y+3z &= 5 \\ 3x+y+2z &= 18 \\ 4x-y+z &= 27 \end{aligned}$$

Solve the above system of the simultaneous equations ...

a) ... by manipulating their augmented matrix into reduced row echelon form. **(4)**

b) ... by using Cramer's rule. **(4)**

Question 4

Investigate the convergence or divergence of each of the following series justifying every step in the workings.

a) $$\sum_{k=1}^{\infty}\left[\frac{\sqrt{k}}{k^2+4k+1}\right].$$ **(3)**

b) $$\sum_{n=1}^{\infty}\left[\frac{3^n+2}{2^n+3}\right].$$ **(3)**

Question 5

It is given that the following integral converges

$$\int_0^1 x^m [\ln x]^n \, dx,$$

where n is a positive integer and m is a positive constant.

By carrying out a suitable differentiation under the integral sign, show that

$$\int_0^1 x^m [\ln x]^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}.$$ **(7)**

You may not use standard integration techniques in this question.

Question 6

Find the value of the following limit

$$\lim_{x \to 0+}\left[x^{-\sin x}\right].$$ **(5)**

Question 7

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method, the following expression.

$$\prod_{r=1}^{n}\left[\frac{2r}{2r+1}\right].$$

Give the final answer as a single simplified fraction. **(6)**

Question 8

Use Laplace transforms to solve the differential equation

$$\frac{dx}{dt} - 2x = 4, \ t \geq 0,$$

subject to the initial condition $x = 1$ at $t = 0$. **(6)**

Question 9

$$f(x) = x, \ x \in \mathbb{R}, \ -\pi \leq x \leq \pi .$$

$$f(x) = f(x + 2\pi).$$

Determine a simplified expression for the Fourier series expansion of $f(x)$. **(7)**

Question 10

The function z depends on x and y so that

$$z = r^2 \tan\theta, \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

a) Express r and θ in terms of x and y and hence determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, in terms of x and y. **(5)**

Give each of the answers as a single simplified fraction.

b) Verify the answer to part **(a)** by implicit differentiation using Jacobians. **(9)**

Question 11

A solid's base is bounded by the right angled triangle in the x-y plane whose vertices have coordinates $(0,0)$, $(1,0)$ and $(0,4)$.

Every vertical cross-section of the solid, perpendicular to the y axis, is a semicircle with its diameter lying on the base of the solid.

Use calculus to determine the volume of the above described solid. **(6)**

Question 12

A sequence of numbers is generated by the recurrence relation

$$u_{n+2} = 2u_{n+1} - 2u_n, \quad n \in \mathbb{N},$$

with $u_1 = 1$, $u_2 = 6$.

Determine a simplified expression for the n^{th} term of this sequence. **(10)**

The final answer may not contain complex numbers

Question 13

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}, \ x > 0.$$ **(8)**

Question 14

$$5x^2 - 4xy + 8y^2 = 36.$$

The conic with the above Cartesian equation is rotated out of its standard position.

By suitably removing the xy term, give a Cartesian equation for the conic in the rotated system. **(8)**

You are expected to sketch and name the conic.

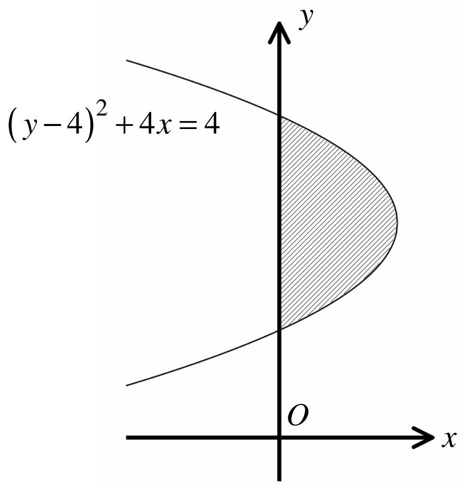
Question 15

$$f(x,y) = x^2 + y^2, \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

The region R in the x-y plane is a circle centred at $(-1,1)$ and of radius 1.

Use partial differentiation to determine the maximum and the minimum value of f, whose projection onto the x-y plane is the region R. **(10)**

Question 16



The figure above shows the curve with equation

$$(y-4)^2+4x=4.$$

The finite region bounded the curve and the y axis, shown shaded in the figure, is rotated by a full turn about the x axis to form a solid of revolution.

Find, in exact form, the volume of the solid. **(8)**

Question 17

a) Use the suitable substitution to solve the differential equation

$$x^2\frac{dy}{dx}+xy=y^2, \qquad y\left(\tfrac{1}{2}\right)=2.$$

Give the answer in the form $y=f(x)$. **(8)**

b) Verify the answer of part **(a)** by solving the above differential equation with an alternative method. **(8)**

Question 18

A curve C has intrinsic equation

$$s = 4\sin\psi, \quad 0 \le \psi \le \pi,$$

where s is the arc length is measured from the Cartesian origin O, and ψ is the angle the tangent to C makes with the positive x axis.

It is given that the tangent to C at O has zero gradient.

Show that the parametric equations of C are

$$x = t + \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi.$$ **(10)**

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