IYGB

Mathematical Methods 1

Practice Paper A

Time: 3 hours

Candidates may use any non programmable, non graphical calculator which does not have the capability of storing data or manipulating algebraic expressions.

Information for Candidates

This practice paper follows the most common syllabi of Mathematical Methods used the United Kingdom Universities for Mathematics, Physics and Engineering degrees.

Booklets of *Mathematical formulae and statistical tables* may NOT be used.

Full marks may be obtained for answers to MOST questions.

The marks for the parts of questions are shown in round brackets, e.g. **(2)**.

There are 18 questions in this question paper.

The total mark for this paper is 131.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.

You must show sufficient working to make your methods clear to the Examiner.

Answers without working may not gain full credit.

Non exact answers should be given to an appropriate degree of accuracy.

The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

The maximum number of marks on this paper is 100, which can be obtained without attempting all the questions on the paper.

If the number of marks exceeds 100 then the paper will score maximum marks.

Topics Examined Under Mathematical Methods 1

- Revision and extensions of Pre-University Algebra and Calculus, including intrinsic coordinates.
- Partial Differentiation
  [includes applications and the use of the gradient operator]
- Curve Sketching in 2 and 3 Dimensions
- Limits
- Ordinary Differential Equations
  [First and second order including substitutions, variation of parameters, D operator and simple Laplace transforms]
- Sequences and Series
- Recurrence Relations
- Product Operator
- Fourier Series
- Conics and Quadrics
- Differentiation and Integration under the Integral Sign

Question 1

$$\begin{aligned} x + 3y + 5z &= -2 \\ 3x - 4y + 2z &= 7 \\ 3x + 11y + 13z &= 2 \end{aligned}$$

Solve the above system of simultaneous equations by manipulating their augmented matrix into reduced row echelon form.

(5)

Question 2

It is given that the following integral converges.

$$\int_0^1 x^{\frac{4}{3}} \ln x \; dx .$$

a) Evaluate the above integral by introducing a parameter and carrying out a suitable differentiation under the integral sign. **(6)**

b) Verify the answer obtained in part **(a)** by evaluating the integral by standard integration by parts. **(2)**

Question 3

Use D operator techniques, to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{4x} .$$

(5)

Question 4

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Use a clear method to show that

$$\prod_{m=1}^{3}\prod_{n=1}^{4}\left[\sqrt{mn}\right] = k^3\sqrt{6},$$

where k is a positive integer to be found. **(5)**

Question 5

Use standard expansions of functions to find the value of the following limit.

$$\lim_{x \to 0}\left[\frac{\cos 7x - 1}{x \sin x}\right].$$ **(4)**

Question 6

The function of two variables f is defined as

$$f(x, y) \equiv xy(x+2) - y(y+3), \quad x \in \mathbb{R}, \ y \in \mathbb{R}.$$

Find the coordinates of each of the stationary points of f, where $z = f(x, y)$, and further determine their nature. **(8)**

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Question 7

Investigate the convergence or divergence of the following series justifying every step in the workings.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{(n+1)^2}{n^3}$. **(2)**

b) $\displaystyle\sum_{r=1}^{\infty} \frac{1}{2r+2^r}$. **(2)**

Question 8

A quadratic curve C has equation

$$y=(4-x)(2+x), \quad x \in \mathbb{R}.$$

The finite region bounded by C and the x axis is fully revolved about the y axis, forming a solid of revolution S.

Determine in exact form the volume of S. **(6)**

Question 9

A sequence of numbers $u_1, u_2, u_3, u_4, u_5 \ldots, u_n, \ldots$ is generated by the following recurrence relation

$$u_{n+2}=u_{n+1}+6u_n, \quad u_1=2, \quad u_2=1.$$

Determine an expression for the n^{th} term of this sequence. **(7)**

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Question 10

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 2x\mathrm{e}^x .$$ **(8)**

Question 11

$$x^2 + 2xy + y^2 + 8x + y = 0 .$$

The conic with the above Cartesian equation has been rotated and translated out of its standard position.

Show that a Cartesian equation of the conic in a suitable coordinate system is

$$4X^2 + 9\sqrt{2}\,X + 7\sqrt{2}\,Y = 0 .$$

You are not expected to sketch the conic. **(8)**

Question 12

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 9x^8 .$$

Determine the solution of the above differential equation subject to the boundary conditions

$$y = \frac{3}{2},\ \frac{dy}{dx} = 2 \text{ at } x = 1 .$$ **(8)**

Question 13

A curve C is defined in the largest real domain by the equation

$$y = -\sqrt{\frac{x(1-x)}{4-x^2}}.$$

Sketch the graph of C.

The sketch must include

- ... the equations of any asymptotes of C.
- ... the coordinates of any point where C meets the coordinate axes.
- ... the coordinates of the stationary points of C, giving the answer in the form $\left[2k + k\sqrt{3}, -\frac{1}{2k}\left(\sqrt{3k}+\sqrt{k}\right)\right]$, where k is a positive integer.

(10)

Question 14

It is given that a curve with equation $f(x, y) = 0$ passes through the point $(0,1)$ and satisfies the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}.$$

By solving the differential equation, show that an equation for the curve is

$$y = \exp\left[\frac{x^2}{2y^2}\right].$$

(8)

Question 15

$$\frac{dx}{dt} + y = e^{-t} \quad \text{and} \quad \frac{dy}{dt} - x = e^{t} .$$

Use Laplace transformations to solve the above simultaneous differential equations, subject to the initial conditions $x = 0$, $y = 0$ at $t = 0$. **(10)**

Question 16

It is given that for $x \in \mathbb{R}$, $-\pi \le x \le \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos\left[(2n-1)x\right]}{(2n-1)^2}, \qquad |x| = |x + 2\pi| .$$

a) Use the above Fourier series expansion to deduce the Fourier series expansion of $\operatorname{sgn}(x)$. **(2)**

b) Verify the answer of part **(a)** by obtaining directly the Fourier series expansion of $\operatorname{sgn}(x)$. **(6)**

a) Hence determine the exact value of

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2r-1} .$$ **(2)**

Question 17

The function φ depends on u and v so that

$$x = 2u + e^{2v} \quad \text{and} \quad y = 2v + e^{-2v} .$$

Without using standard results involving Jacobians, determine simplified expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, in terms of u and v. **(10)**

Question 18

An ellipse E has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are positive constants.

Use integral calculus to show that the volume of a right elliptical cone with base E and height h, is given by

$$\frac{1}{3}\pi abh.$$

You may assume the standard result for the area of an ellipse without proof. **(7)**