IYGB GCE

Mathematics M456

Advanced Level

Practice Paper C

Difficulty Rating: 3.52

Time: 3 hours 30 minutes

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This practice paper follows closely the Advanced Level Further Mechanics Syllabi, assessed between 1993 and 2005 and with minor omissions between 2006 and 2017.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 16 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

The vectors $\mathbf{i}$ *and* $\mathbf{j}$ *are horizontal unit vectors perpendicular to each other.*

A bead of mass 0.2 kg is threaded on a smooth straight horizontal wire.

The bead is at rest at the point A with position vector $(2\mathbf{i}+5\mathbf{j})$ m.

A single force $(2.6\mathbf{i}-0.1\mathbf{j})$ N acts on the bead and moves it to the point B with position vector $(17\mathbf{i}-5\mathbf{j})$ m.

Find the speed of the bead at B. **(5)**

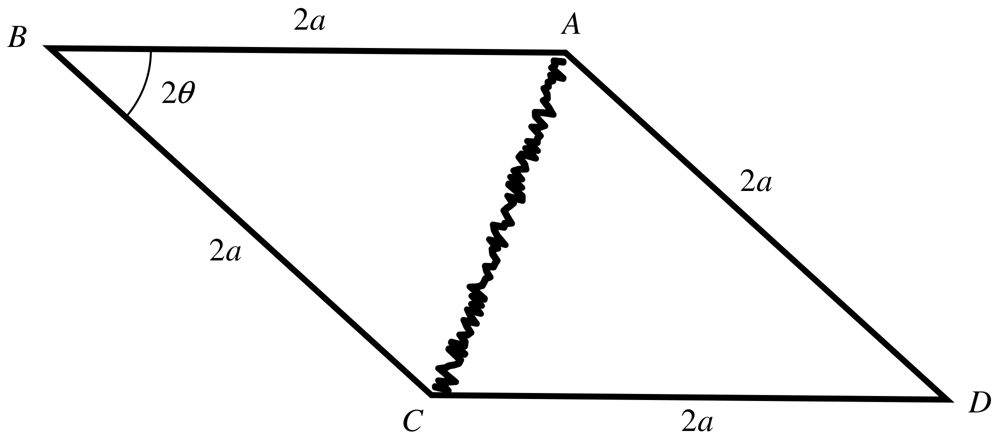
Question 2

Use integration to show that the moment of inertia I of a thin uniform rod AB, of length $2a$ and mass m, about an axis through A and perpendicular to the length of the rod is given by

$$I = \frac{4}{3}ma^2 .$$

(6)

Question 3



Four identical uniform rods each of mass m and length $2a$ are smoothly joined together to form a rhombus $ABCD$.

The rod AB is fixed in a horizontal position and a light elastic string connects the joints A and C.

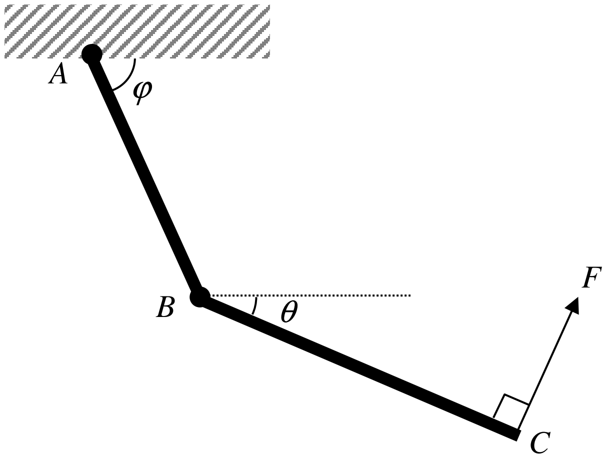
The string has natural length a and modulus of elasticity mg.

The rhombus hangs in equilibrium with C and D at the same horizontal level, vertically below AB.

Given that $\measuredangle ABC = 2\theta$, $0° < \theta < 45°$, show that the positions of equilibrium of the system are solutions of the equation

$$2\sin 2\theta - 2\cos 2\theta - \cos\theta = 0 .$$ **(11)**

Question 4



Two identical uniform rigid rods AB and BC, each of weight 100 N, are freely joined at B and lie in the same vertical plane.

The rod AB is freely joined at A, a fixed point on a horizontal ceiling.

The system is held in equilibrium by a force F N acting at C, in a perpendicular direction to BC, as shown in the figure above.

AB and BC form angles φ and θ to the horizontal, respectively, where $\tan\theta = \frac{3}{4}$.

a) Find the value of F. **(4)**

b) Calculate the magnitude of the horizontal reaction and vertical reaction, acting on BC at B. **(4)**

c) Determine, in degrees, the size of the angle φ. **(4)**

Question 5

A rocket has initial mass M, which includes the fuel for its flight.

The rocket is initially at rest on the surface of the earth pointing vertically upwards. At time $t = 0$ the rocket begins to propel itself by ejecting mass backwards at constant rate λ, and with speed u relative to the rocket.

At time t the speed of the rocket is v.

The rocket is modelled as a particle moving vertically upwards without air resistance.

The motion takes place close to the surface of the earth and it is assumed that g is the constant gravitational acceleration throughout the motion.

a) Determine an expression, in terms of u, g, λ, M and t, for the acceleration of the rocket and hence deduce that if the rocket lifts off immediately $\lambda > \frac{Mg}{u}$. **(10)**

It is now given that $\lambda = \frac{3Mg}{u}$.

b) Find, in terms of u, the speed of the rocket when its mass is $\frac{3}{4}M$. **(8)**

Question 6

A small smooth sphere B of mass m is at rest on a smooth horizontal surface.

Another smooth sphere A of mass $2m$ and of equal radius as that of B is moving with speed u in a straight line, on the same surface.

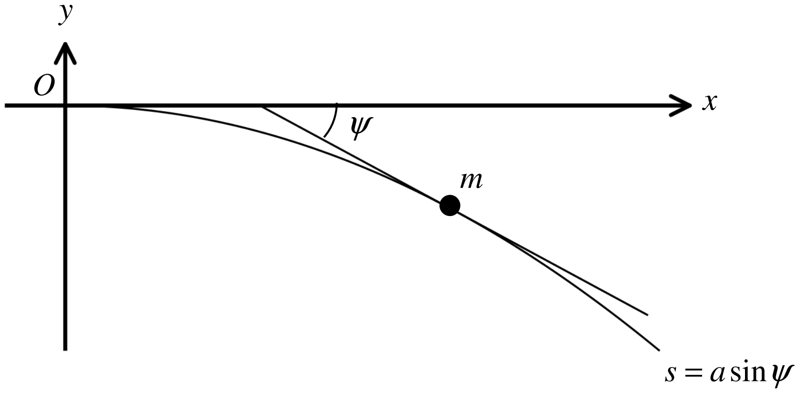
There is a collision between the two spheres and L is the straight line joining the centres of the two spheres at the moment of impact.

The path of A immediately before the collision makes an angle of $30°$ with L.

a) Determine simplified expressions for the speeds of A and B, in terms of u and e, where e is the coefficient of restitution between the two spheres. **(10)**

b) Given further that 20% of the kinetic energy of the system is lost due to the impact, show that $e^2 = \frac{1}{5}$. **(7)**

Question 7



The figure above shows a particle of mass m, which is free to slide along a smooth surface, whose vertical cross section is the curve C with intrinsic equation

$$s = a\sin\psi \, , \; 0 \le \psi < \frac{\pi}{2}$$

where a is a positive constant.

The arclength s is measured from the origin O, and the angle ψ is the angle the tangent to C makes with the positive x axis, as shown in the figure above.

The particle is projected from O with speed $\sqrt{\frac{1}{2}ag}$ and leaves the surface at the point P.

a) Find the value of s at P. **(10)**

b) Determine the magnitude of the acceleration at P. **(5)**

Question 8

The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the positive x direction and positive y direction, respectively.

Three forces

$$\left[(3a+1)\mathbf{i}+3\mathbf{j}\right]\text{N}, \quad \left[(a-10)\mathbf{i}-2\mathbf{j}\right]\text{N} \quad \text{and} \quad \left[\mathbf{i}+(1-a)\mathbf{j}\right]\text{N},$$

where a is a constant, act at the points $A(1,2)$, $B(2,0)$ and $C(4,-1)$, respectively.

Distances are measured in m, relative to a fixed origin O.

a) Given that the system of the three forces reduces to a couple about O, find the magnitude and direction of this couple. **(5)**

b) Given instead that the system of the three forces reduces to single force $\mathbf{F}$, determine the equation of the line of action of $\mathbf{F}$. **(7)**

Question 9

The unit vectors $\mathbf{i}$ *and* $\mathbf{j}$ *are oriented due east and due north, respectively.*

Two boats, A and B, are moving in the open sea with velocities $(7\mathbf{i}+3\mathbf{j})\ \text{km h}^{-1}$ and $(-3\mathbf{i}+9\mathbf{j})\ \text{km h}^{-1}$, respectively.

At noon, B is on a bearing of $120°$ from A, $12\ \text{km}$ away.

Calculate, correct to the nearest m, the closest distance between the two boats and the time when they are at that closest distance. **(10)**

Question 10

A particle, of mass 0.5 kg, is moving on a straight line, under the action of a single force of magnitude

$$\left[\frac{25}{x^2}-\frac{50}{x^3}\right] \text{N}, \ x>0,$$

where x is the distance of the particle from a fixed origin O.

The particle is released from the point where $x=1$, with speed 13 ms^{-1}, in the direction of x increasing.

It is further given that in moving the particle from $x=1$ to a point where $x=k$, $k>1$, the force does work of -4 J.

a) Determine the possible values of k. **(5)**

b) Find the least speed of the particle in its consequent motion. **(7)**

Question 11

Relative to a fixed origin O, a particle P is moving with constant angular velocity ω on the curve with polar equation

$$r = k\,\mathrm{e}^{\theta \cot \alpha},$$

where k and α are positive constants with $0<\alpha<\frac{1}{4}\pi$.

Show that the magnitude of the acceleration of the particle is $\frac{v^2}{r}$, where v is the speed of the particle and r is the distance OP. **(13)**

Question 12

A uniform rod AB, of mass m, is free to rotate about a smooth fixed horizontal axis L, which passes through A.

The rotation of the rod takes place in a vertical plane.

The rod is held so that AB makes an angle of 60° with the upward vertical and released from rest.

a) Given that the moment of inertia of the rod about L is $12ma^2$, show that in the subsequent motion

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{8a}(1-2\cos\theta),$$

where θ is the angle that AB makes with the upward vertical. **(8)**

b) Determine, in terms of m, g and θ, the magnitude and direction of the radial force exerted on L by the rod, **(6)**

Question 13

A small raindrop of mass m kg, is released from rest from a rain cloud and is falling through still air under the action of its own weight.

The raindrop is subject to air resistance of magnitude kmv N, where v ms^{-1} is the speed of the raindrop x m below the point of release, and k is a positive constant.

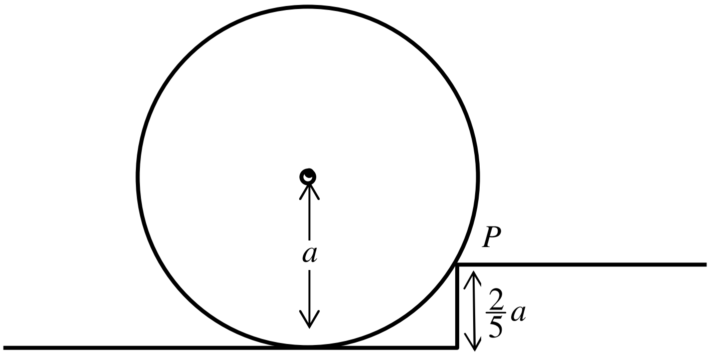
a) Show, by forming and solving a differential equation, that

$$x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}. \qquad \textbf{(9)}$$

The raindrop has a limiting speed V.

b) Show further that the raindrop reaches a speed of $\frac{1}{2}V$, after a falling through a distance of $\frac{V^2}{2g}(-1+\ln 4)$ metres. **(4)**

Question 14



A uniform solid sphere, of radius a, is rolling without slipping on a rough horizontal plane, with constant speed V.

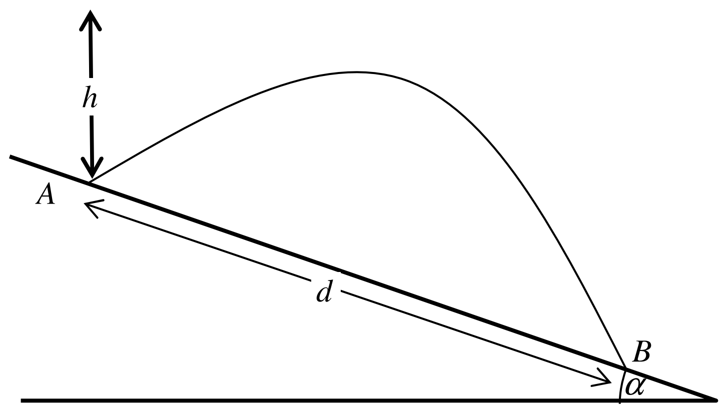
The sphere reaches a vertical step of height $\frac{2}{5}a$, which is at right angles to its direction of motion, as shown in the figure above.

When the sphere touches the step at the point P, it begins to rotate about P, without slipping or loss of contact.

Show that

$$V < \frac{147}{125}ag \, .$$ **(12)**

Question 15



The figure above shows the path of a particle, released from rest, from a height h above a smooth plane, inclined at an angle α to the horizontal.

The particle strikes the plane at the point A, and rebounds striking the plane for the second time at the point B.

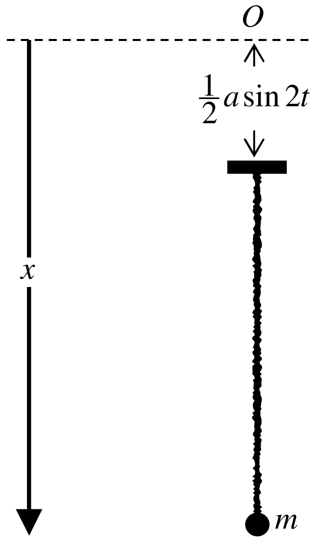
The gravitational acceleration g is assumed constant and air resistance is ignored.

The coefficient of restitution between the plane and the particle is e.

Given that $|AB| = d$, show that

$$d = 4eh(e+1)\sin\alpha .$$ **(12)**

Question 16



A particle, of mass m, is attached to one end of a light elastic spring of natural length a and modulus of elasticity $\frac{1}{2}mg$. The other end of the spring is initially stationary at the point O so that the particle is hanging in equilibrium vertically below O.

At time $t=0$, the end of the spring which is at O begins to oscillate so that its **positive** displacement from O is given by $\frac{1}{2}a\sin 2t$.

If x denotes the distance of the particle from O at time t, show that

$$x = 3a + \frac{a\omega}{2(\omega^2-4)}[\omega\sin 2t - 2\sin\omega t],$$

where $\omega^2 = \frac{g}{2a}$. **(18)**

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