IYGB GCE

Mathematics M456

Advanced Level

Practice Paper A

Difficulty Rating: 3.44

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

This practice paper follows closely the Advanced Level Further Mechanics Syllabi, assessed between 1993 and 2005 and with minor omissions between 2006 and 2017.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 14 questions in this question paper.
The total mark for this paper is 175.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1

A particle P is moving on the Cartesian plane so that its position vector $\mathbf{r}$ m at time t s satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} = 6(\mathbf{r} + t\,\mathbf{i} - 2\mathbf{j}).$$

When $t = 0$, P has position vector $(\mathbf{i} + 2\mathbf{j})$ m and moving with velocity $(3\mathbf{i} - \mathbf{j})$ ms^{-1}.

Express $\mathbf{r}$ in terms of t. **(12)**

Question 2

The standard unit vectors $\mathbf{i}$ and $\mathbf{j}$ are oriented in the positive x direction and positive y direction, respectively.

Three forces

$$\mathbf{F}_1 = (4\mathbf{i} + b\mathbf{j})\,\text{N}, \quad \mathbf{F}_2 = (3a\mathbf{i} + 2b\mathbf{j})\,\text{N} \quad \text{and} \quad \mathbf{F}_3 = (10b\mathbf{i} + 3\mathbf{j})\,\text{N},$$

where a and b are scalar constants, are acting at the points $A(1,2)$, $B(4,-2)$ and $C(-3,-5)$, respectively.

a) Given that the resultant of the three forces is zero, determine the magnitude and direction of the total moment of these three forces about O. **(4)**

b) Find, by direct calculation, the magnitude and direction of the total moment of these three forces about C. **(5)**

Question 3

The point A lies on a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{5}{12}$.

A particle is projected from A, up a line of greatest slope of the plane, with a speed of 24.5 ms^{-1} at an angle of elevation $\alpha+\theta$, where $\tan\theta = \frac{3}{4}$.

The particle is moving freely under gravity and first hits the plane at B.

Given that the coefficient of restitution between the plane and the particle is $\frac{1}{2}\sqrt{3}$, show that the particle first rebounds from B with a speed of 14.7 ms^{-1}. **(12)**

Question 4

A rocket, of initial mass M, propels itself forward by ejecting burned fuel.

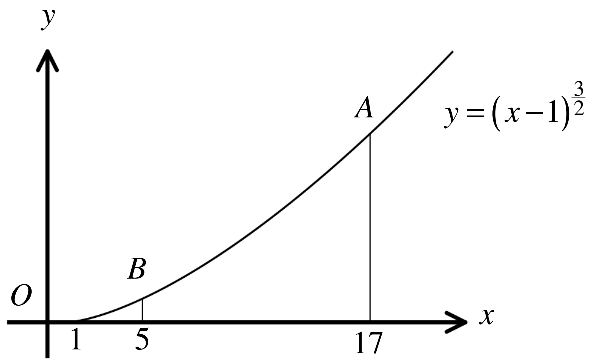
The initial speed of the rocket is U.

The burned fuel is ejected with constant speed u, relative to the rocket, in an opposite direction to that of the rocket's motion.

When all the fuel has been consumed, the mass of the rocket is $\frac{1}{4}M$.

By modelling the rocket as a particle and further assuming that there are no external forces acting on the rocket, determine, in terms of u and U, the speed of the rocket when all its fuel has been consumed. **(10)**

Question 5



A rollercoaster car, of mass 200 kg, is constrained to move along a rail path with Cartesian equation

$$y=(x-1)^{\frac{3}{2}}.$$

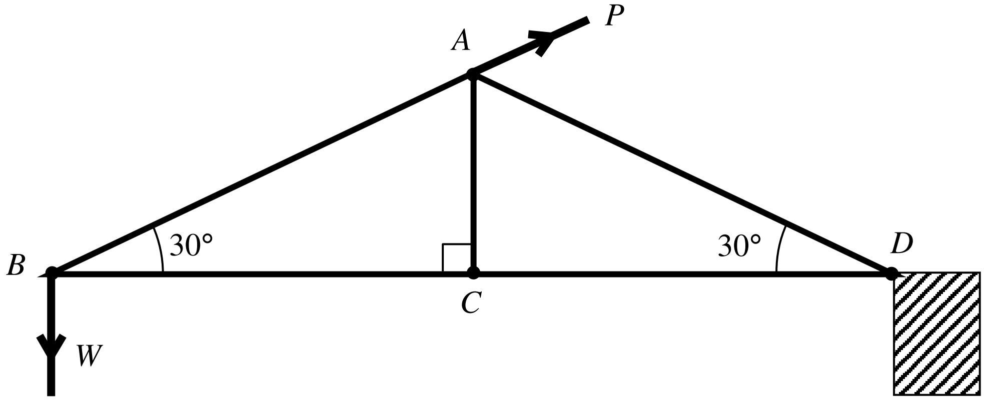
The car comes to instantaneous rest at the point A, where $x=17$, and immediately begins to freely slide downwards towards the origin O, as shown in the figure above. The point B, lies on the same rail path, where $x=5$. The car is modelled as a particle moving along a smooth rail path without any air resistance.

As the car passes through B, calculate ...

a) ... the magnitude of the acceleration of the car as it passes through B. **(10)**

b) ... the magnitude of the normal reaction exerted by the rails onto the car **(5)**

Question 6



A light rigid framework consists of 5 light pin jointed rods, AB, AC, AD, BC and BD, where $|AB| = |AD|$, $|BC| = |BD|$, $\measuredangle ABC = \measuredangle ADC = 30°$ and $\measuredangle ACD = 90°$, as shown in the figure above.

The framework is freely hinged at D and a weight W is supported at B.

The framework is supported in equilibrium, with BCD horizontal, by a force P which acts at B in the direction of BA

a) Find the magnitude, in terms of W, and the direction of the reaction force acting on the framework at D. **(6)**

b) Determine, in terms of W, the magnitude of the internal force acting on each of the rods, classifying them where applicable as tension or thrust. **(9)**

Question 7

A pulley is modelled as a uniform circular disc of mass $16m$ and radius a. The pulley is free to rotate about a fixed horizontal axis through its centre and perpendicular to its plane. A light inextensible string passes over the pulley and two particles A and B, of respective mass $2m$ and $5m$ are attached to the two ends of the string.

The particles are released from rest with the string taut.

A constant couple of magnitude of mga resists the rotation of the pulley about its axis.

In the consequent motion there is no slipping between the string and the pulley.

Determine, in terms of mg, the tension in each of the two sections of the string to which the two particles are attached. **(12)**

Question 8

A uniform rod AB, of mass m and length $8a$, is free to rotate about an axis L which passes through the point C, where $|AC| = 2a$.

a) Given that the moment of inertia of the rod about L is λma^2, use integration to find the value of λ.

A different rod AB, also of mass m and length $8a$ is free to rotate about a smooth fixed axis L', which passes through the point C, where $|AC| = 2a$. The mass density of the section AC is twice as large as the mass density of the section CB. **(5)**

b) Given that the moment of inertia of this rod about L' is μma^2, determine the value of μ. **(5)**

Question 9

An object is released from rest from a great height, and allowed to fall down through still air all the way to the ground.

Let v ms^{-1} be the velocity of the object t seconds after it was released.

The velocity of the object is increasing at the constant rate of $10\ \text{ms}^{-1}$ every second, but at the same time due to the air resistance its velocity is decreasing at a rate proportional its velocity at that time.

The maximum velocity that the particle can achieve is $100\ \text{ms}^{-1}$.

By forming and solving a differential equation, show that

$$v = 100\left(1 - e^{-0.1t}\right).$$

(10)

Question 10

At noon a frigate is $18\ \text{km}$ away from a ship and at that time the bearing of the frigate relative to the ship is $120°$.

The ship is sailing east at a constant speed of $20\ \text{km}\,\text{h}^{-1}$.

a) Determine the minimum speed with which the frigate can intercept the ship. **(3)**

The frigate sets off to intercept the ship by sailing at a constant speed of $15\ \text{km}\,\text{h}^{-1}$.

b) Calculate, to the nearest degree, the two possible bearings which the frigate can follow, and hence find the shorter of the two possible interception times, correct to the nearest minute. **(9)**

Question 11

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r, θ), where O is the pole.

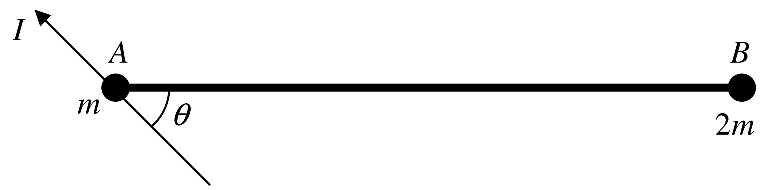
The radius vector OP rotates with constant angular speed ω.

The radial component of the acceleration of P has magnitude $2r\omega^2$, and is directed towards O.

Initially, P is at the point with coordinates $(a, 0)$, where a is a positive constant, and has radial velocity $\sqrt{3}\,a\omega$.

Determine, in terms of a, a polar equation for the path of P. **(12)**

Question 12



Two particles, A and B, of respective masses m and $2m$ are connected by a light rigid rod of length $2a$. The system is lies at rest on a smooth horizontal surface when it receives an impulse of magnitude I at A. The direction of the impulse is at an acute angle θ to AB, as shown in the figure above.

a) Determine the speed of each of the particles immediately after the impulse is received, in terms of m, I and θ. **(12)**

b) Find the gain in the kinetic energy of the system, as a result of this impulse, in terms of m, I and θ. **(3)**

Question 13

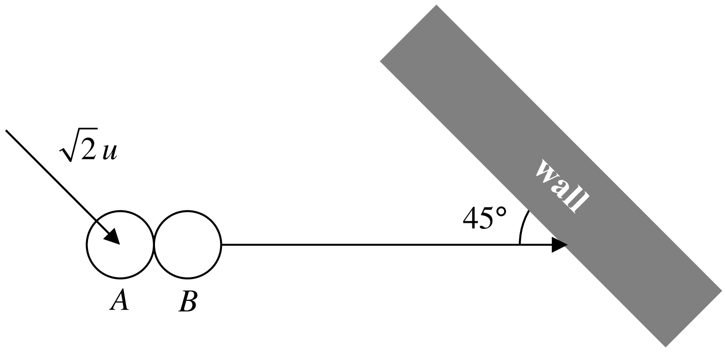
A smooth wire is bent into the shape of a circle of radius a, centre at O. The wire is fixed in a vertical plane and the straight line AOB is a horizontal diameter of the circle. A small ring P, of mass m, is threaded on the wire and a light elastic string is also threaded through the ring. The two ends of the string are attached at A and B.

The natural length of the string is $2a$ and its modulus of elasticity is kmg, $k > 0$.

The angle between the radius OP and the downward vertical through O is denoted by 2θ, where $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$.

If $\theta = \frac{1}{6}\pi$ is a position of equilibrium for the above described system determine the exact value of k. **(14)**

Question 14



A smooth sphere B is at rest on a smooth horizontal surface at a fixed distance from a smooth, long vertical wall. An identical sphere A is moving with speed $\sqrt{2}u$ in a straight line, on the same surface, in a direction parallel to the wall.

There is a collision between the two spheres and L is the straight line joining the centres of the two spheres at the moment of impact, which is at 45° to the wall as shown in the figure above.

The coefficient of restitution between the two spheres is e.

After B collides with the wall, its direction of motion is parallel to the direction of motion of A after the collision.

Show that the coefficient of restitution between B and the wall is

$$\frac{1+e}{3-e}.$$ **(17)**