IYGB - FP3 PAPER W - QUESTION 1

$$\frac{dy}{dx} = x + y + y^2 \qquad \begin{array}{ll} x_1 = 0.8 & y_1 = k \\ x_2 = 0.9 & y_2 = 3.75 \\ x_3 = 1 & y_3 = 4 \end{array}$$

USING THE APPROXIMATION GIVEN

$$\Rightarrow \left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$$

$$\Rightarrow y'_r \approx \frac{y_{r+1} - y_{r-1}}{2h}$$

$$\Rightarrow 2hy'_r \approx y_{r+1} - y_{r-1}$$

$$\Rightarrow y_{r-1} \approx y_{r+1} - 2hy'_r$$

LET $r=2$

$$\Rightarrow y_1 \approx y_3 - 2hy'_2$$

$$\Rightarrow y_1 \approx y_3 - 2h\left(x_2 + y_2 + y_2^2\right)$$

$$\Rightarrow k \approx 4 - 2 \times 0.1\left(0.9 + 3.75 + 3.75^2\right)$$

$$\Rightarrow k \approx 0.2575$$

IYGB - FP3 PAPER W - QUESTION 2

a) FILLING A STANDARD TABLE

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$
$y = e^{\sec^2 x}$	2.718	2.921	3.794	7.389	54.598
	FIRST	ODD	EVEN	ODD	LAST

BY SIMPSON'S RULE

$$\int_0^{\pi/3} e^{\sec^2 x}\,dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$\simeq \frac{\pi/12}{3}\left[2.718 + 54.598 + 4(2.921 + 7.389) + 2\times 3.794\right]$$

$$\simeq 9.26$$

b) USING $1 + \tan^2 x = \sec^2 x$

$$\int_0^{\pi/3} e^{\tan^2 x}\,dx = \int_0^{\pi/3} e^{\sec^2 x - 1}\,dx = \int_0^{\pi/3} e^{\sec^2 x} \times e^{-1}\,dx = \frac{1}{e}\int_0^{\pi/3} e^{\sec^2 x}\,dx$$

$$= \frac{1}{e} \times 9.26\ldots \simeq 3.41$$

c) THE GRAPH OF $y = e^{\sec^2 x}$ IS STRICTLY INCREASING AND AS WE GET CLOSE TO $\frac{\pi}{3}$ VERY RAPIDLY

THEREFORE THE ESTIMATES ARE LIKELY TO BE INACCURATE

IYGB - FP3 PAPER W - QUESTION 3

USING LEIBNIZ RULE FOR DIFFERENTIATION

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{d^1 v}{dx^1} + \frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \frac{n(n-1)(n-2)}{3!}\frac{d^{n-3}u}{dx^{n-3}}\frac{d^3 v}{dx^3}$$

$\frac{d^0 v}{dx^0}$

TAKE $u = e^{3x}$ & $v = x^4$ (THIS WILL VANISH AFTER A FEW DIFFERENTIATIONS)

$$\frac{d^k}{dx^k}(x^4 e^{3x}) = 3^k e^{3x}(x^4) + k\cdot 3^{k-1}e^{3x}\cdot 4x^3 + \frac{1}{2}k(k-1)\cdot 3^{k-2}e^{3x}\cdot 12x^2$$
$$+ \frac{1}{6}k(k-1)(k-2)\cdot 3^{k-3}e^{3x}\cdot 24x + \frac{1}{24}k(k-1)(k-2)(k-3)\cdot 3^{k-4}e^{3x}\cdot 24 + \ldots$$

REST IS ZERO HERE

TIDY UP BY FACTORIZING

$$\frac{d^k}{dx^k}(x^4 e^{3x}) = 3^{k-4}e^{3x}\Big[3^4 x^4 + k\cdot 3^3\cdot 4x^3 + \frac{1}{2}k(k-1)3^2\cdot 12x^2 + \frac{1}{6}k(k-1)(k-2)\cdot 3^1\cdot 24x$$
$$+ \frac{1}{24}k(k-1)(k-2)(k-3)\cdot 24\Big]$$

$$\frac{d^k}{dx^k}(x^4 e^{3x}) = 3^{k-4}e^{3x}\underbrace{\left[81x^4 + 108kx^3 + 54k(k-1)x^2 + 12k(k-1)(k-2)x + k(k-1)(k-2)(k-3)\right]}_{f(x,k)}$$

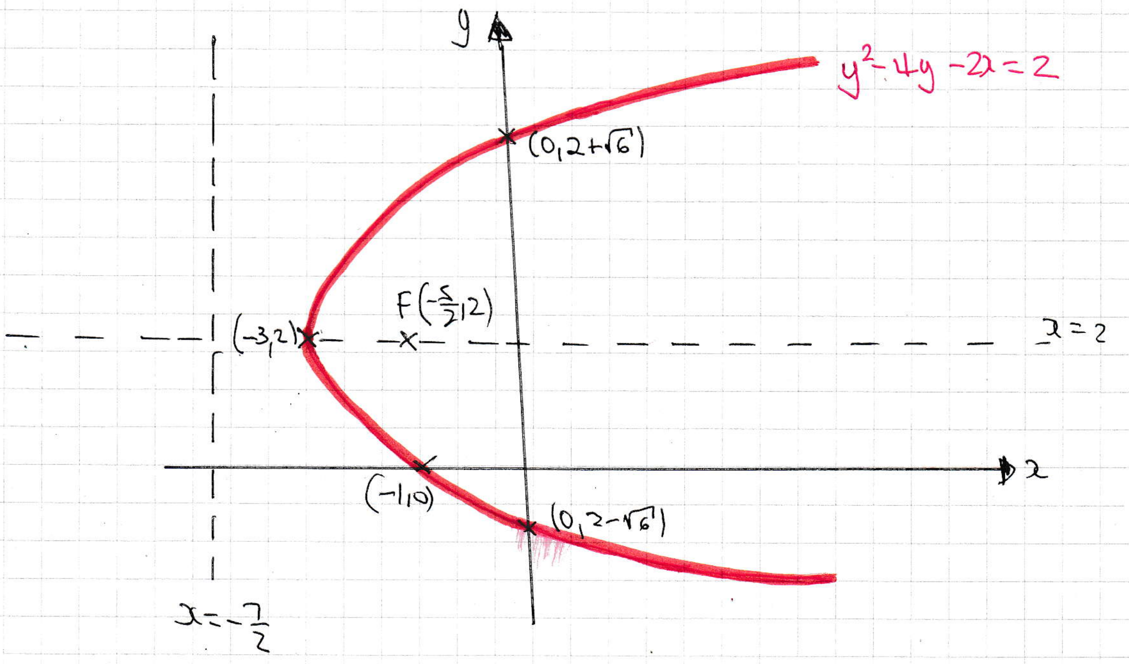
IYGB - FP3 PAPER W - QUESTION 4

REWRITE IN THE "STANDARD" FORM

$$\Rightarrow y^2 - 4y - 2x = 2$$
$$\Rightarrow (y-2)^2 - 4 = 2x + 2$$
$$\Rightarrow (y-2)^2 = 2x + 6$$
$$\Rightarrow (y-2)^2 = 2(x+3)$$
$$Y^2 = 2X$$

PARABOLA WITH $a = \frac{1}{2}$ WHEN COMPARED WITH $Y^2 = 4aX$, TRANSLATED BY THE VECTOR $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

- VERTEX $(0,0) \longmapsto (-3,2)$
- FOCUS $(\frac{1}{2},0) \longmapsto (-\frac{5}{2},2)$
- DIRECTRIX $x = -\frac{1}{2} \longmapsto x = -\frac{7}{2}$
- $x=0$ $\quad (y-2)^2 = 6$
  $y - 2 = \pm\sqrt{6}$
  $y = 2 \pm \sqrt{6}$
- $y=0$ $\quad x = -1$



IYGB - FP3 PAPER W - QUESTION 5

START WITH THE SUBSTITUTION GIVEN

- $x = z^{\frac{1}{2}}$

$$\frac{dx}{dy} = \frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dy}$$

$$\frac{dx}{dy} = \frac{1}{2z^{\frac{1}{2}}}\frac{dz}{dy}$$

$$\frac{dy}{dx} = 2z^{\frac{1}{2}}\frac{dy}{dz}$$

OR

$$\frac{dy}{dz} = \frac{1}{2z^{\frac{1}{2}}}\frac{dy}{dx}$$

- $x = z^{\frac{1}{2}}$

$$\frac{dx}{dz} = \frac{1}{2}z^{-\frac{1}{2}}$$

$$\frac{dx}{dz} = \frac{1}{2z^{\frac{1}{2}}}$$

OR

$$\frac{dz}{dx} = 2z^{\frac{1}{2}}$$

NOW THE SECOND DERIVATIVES

$$\frac{dy}{dx} = 2z^{\frac{1}{2}}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = z^{-\frac{1}{2}}\frac{dz}{dx}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}\left[\frac{1}{z^{\frac{1}{2}}}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\right]$$

$$\frac{d^2y}{dx^2} = 2z^{\frac{1}{2}}\left[\frac{1}{z^{\frac{1}{2}}}\frac{dy}{dz} + 2z^{\frac{1}{2}}\frac{d^2y}{dz^2}\right]$$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}$$

NOW SUBSTITUTE INTO THE O.D.E

$$\Rightarrow x\frac{d^2y}{dx^2} - \frac{dy}{dx} - x^3y + x^5 = 0$$

$$\Rightarrow z^{\frac{1}{2}}\left[2\frac{dy}{dz} + 4z\frac{d^2y}{dz^2}\right] - 2z^{\frac{1}{2}}\frac{dy}{dz} - z^{\frac{3}{2}}y + z^{\frac{5}{2}} = 0$$

$$\Rightarrow \cancel{2z^{\frac{1}{2}}\frac{dy}{dz}} + 4z^{\frac{3}{2}}\frac{d^2y}{dz^2} - \cancel{2z^{\frac{1}{2}}\frac{dy}{dz}} - z^{\frac{3}{2}}y + z^{\frac{5}{2}} = 0$$

IYGB - FP3 PAPER W - QUESTION 5

$$\Rightarrow 4z^{\frac{3}{2}}\frac{d^2y}{dz^2} - yz^{\frac{3}{2}} + z^{\frac{5}{2}} = 0$$

$\div z^{\frac{3}{2}}$

$$\Rightarrow 4\frac{d^2y}{dz^2} - y + z = 0$$

AUXILIARY EQUATION FOR $4\frac{d^2y}{dz^2} - y = -z$

$$4\lambda^2 - 1 = 0$$
$$\lambda^2 = \frac{1}{4}$$
$$\lambda = \pm\frac{1}{2}$$

PARTICULAR INTEGRAL (BY INSPECTION)

$$y = z$$

GENERAL SOLUTION IS

$$y = Ae^{\frac{1}{2}z} + Be^{-\frac{1}{2}z} + z$$

$$y = Ae^{\frac{1}{2}x^2} + Be^{-\frac{1}{2}x^2} + x^2$$

$x = z^{\frac{1}{2}}$
$x^2 = z$

IYGB - FP3 PAPER W - QUESTION 6

a) START WITH DIFFERENTIATIONS

- $y = \tan^2 x = \sec^2 x - 1$

- $\frac{dy}{dx} = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

- $\frac{d^2y}{dx^2} = 4\sec x(\sec x \tan x)\tan x + 2\sec^2 x \sec^2 x$

$= 4\sec^2 x \tan^2 x + 2\sec^4 x$

$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$

$= 4\sec^4 x - 4\sec^2 x + 2\sec^4 x$

$= 6\sec^4 x - 4\sec^2 x$

- $\frac{d^3y}{dx^3} = 24\sec^3 x(\sec x \tan x) - 8\sec x(\sec x \tan x)$

$= 24\sec^4 x \tan x - 8\sec^2 x \tan x$

- $\frac{d^4y}{dx^4} = 96\sec^3 x(\sec x \tan x)\tan x + 24\sec^4 x \sec^2 x - 16\sec x(\sec x \tan x)\tan x - 8\sec^2 x \sec^2 x$

$= 96\sec^4 x \tan^2 x + 24\sec^6 x - 16\sec^2 x \tan^2 x - 8\sec^4 x$

$= 96\sec^4 x(\sec^2 x - 1) + 24\sec^6 x - 16\sec^2 x(\sec^2 x - 1) - 8\sec^4 x$

$= 96\sec^6 x - 96\sec^4 x + 24\sec^6 x - 16\sec^4 x + 16\sec^2 x - 8\sec^4 x$

$= 120\sec^6 x - 112\sec^4 x + 16\sec^2 x$ // AS REQUIRED

b) EVALUATE THESE AT $\pi/3$ SO $\tan\frac{\pi}{3} = \sqrt{3}$ & $\sec\frac{\pi}{3} = 2$

- $y = 3$

- $\frac{dy}{dx} = 2 \times 4 \times \sqrt{3} = 8\sqrt{3}$

- $\frac{d^2y}{dx^2} = 6 \times 16 - 4 \times 4 = 80$

- $\frac{d^3y}{dx^3} = 24 \times 16\sqrt{3} - 8 \times 4\sqrt{3} = 384\sqrt{3} - 32\sqrt{3} = 352\sqrt{3}$

IYGB - FP3 PAPER W - QUESTION 6

- $\frac{d^4y}{dx^4} = 120 \times 2^6 - 120 \times 2^4 + 16 \times 2^2$

$= 7680 - 1920 + 64$

$= 5824$

APPLYING TAYLOR'S THEOREM

$$f(x) = f\left(\tfrac{\pi}{3}\right) + \frac{f'(\pi/3)}{1!}\left(x - \tfrac{\pi}{3}\right) + \frac{f''(\pi/3)}{2!}\left(x - \tfrac{\pi}{3}\right)^2 + \frac{f'''(\pi/3)}{3!}\left(x - \tfrac{\pi}{3}\right)^3 + \dots$$

$$\tan^2 x = 3 + 8\sqrt{3}\left(x - \tfrac{\pi}{3}\right) + 40\left(x - \tfrac{\pi}{3}\right)^2 + \frac{176}{3}\sqrt{3}\left(x - \tfrac{\pi}{3}\right)^3 + \frac{728}{3}\left(x - \tfrac{\pi}{3}\right)^4 + \dots$$

IYGB – FP3 PAPER W – QUESTION 7

AS WITH ALL EXPONENTIAL LIMITS, TAKE NATURAL LOGS

$$\Rightarrow \lim_{x \to \infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = L$$

$$\Rightarrow \ln\left[\lim_{x \to \infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right]\right] = \ln L$$

$$\Rightarrow \lim_{x \to \infty}\left[\ln\left(1+\frac{a}{x}\right)^{bx}\right] = \ln L$$

$$\Rightarrow \lim_{x \to \infty}\left[bx\ln\left(1+\frac{a}{x}\right)\right] = \ln L$$

NOW THE LIMIT IS INDETERMINATE OF THE FORM "∞" $\times$ "0" SO REWRITE IT

$$\Rightarrow \lim_{x \to \infty}\left[\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}\right] = \ln L$$

NOW IT IS OF THE FORM ZERO OVER ZERO, SO BY L'HOSPITAL RULE

$$\Rightarrow \lim_{x \to \infty}\left[\frac{\frac{1}{1+\frac{a}{x}} \times \frac{-a}{x^2}}{-\frac{1}{bx^2}}\right] = \ln L$$

$$\Rightarrow \lim_{x \to \infty}\left[\frac{\frac{-a}{x^2+ax}}{-\frac{1}{bx^2}}\right] = \ln L$$

$$\Rightarrow \lim_{x \to \infty}\left[\frac{-a}{x^2+ax} \times \frac{-bx^2}{1}\right] = \ln L$$

$$\Rightarrow \lim_{x \to \infty}\left[\frac{abx^2}{x^2+ax}\right] = \ln L$$

IYGB - FP3 PAPER W - QUESTION 7

$$\Rightarrow \lim_{x \to \infty}\left[\frac{ab}{1+\frac{a}{x}}\right] = \ln L$$

$$\Rightarrow ab = \ln L$$

FINALLY INVERTING THE LOGARITHM

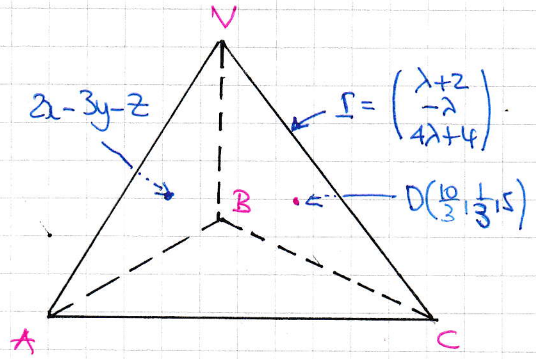
$$\Rightarrow L = e^{ab}$$

$$\therefore \lim_{x \to \infty}\left[\left(1+\frac{a}{x}\right)^{bx}\right] = e^{ab}$$

IYGB - FP3 PAPER W - QUESTION 8

STARTING WITH A DIAGRAM

THE LINE VB IS THE INTERSECTION OF THE PLANES VBA (GIVEN) AND THE PLANE VBC (TO BE FOUND)



TAKE 3 POINTS ON VBC

$\lambda = 0 \quad P(2,0,4)$

$\lambda = -1 \quad Q(1,1,0)$

$D\left(\frac{10}{3}, \frac{1}{3}, 5\right)$

$\overrightarrow{PQ} = \underline{q} - \underline{p} = (1,1,0) - (2,0,4) = (-1,1,-4)$ SCALE IT TO $(1,-1,4)$

$\overrightarrow{PD} = \underline{d} - \underline{p} = \left(\frac{10}{3}, \frac{1}{3}, 5\right) - (2,0,4) = \left(\frac{4}{3}, \frac{1}{3}, 1\right)$ SCALE IT TO $(4,1,3)$

CROSSING THESE DIRECTIONS TO GET THE NORMAL OF VBC

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 4 \\ 4 & 1 & 3 \end{vmatrix} = (-7, 13, 5) \leftarrow \text{NORMAL OF VBC}$$

NEXT CROSSING THE NORMALS OF ABV & VBC TO GET THE DIRECTION OF VB

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -7 & 13 & 5 \\ 2 & -3 & -1 \end{vmatrix} = (2, 3, -5) \leftarrow \text{DIRECTION VECTOR OF VB}$$

VBC

ABV

IYGB — FP3 PAPER W — QUESTION 8

NOW INTERSECTING THE PLANE ABV & VC TO FIND V

$2x - 3y - z = 1$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda+2 \\ -\lambda \\ 4\lambda+4 \end{pmatrix}$

$\Rightarrow \quad 2(\lambda+2) - 3(-\lambda) - (4\lambda+4) = 1$

$\Rightarrow \quad 2\lambda + \cancel{4} + 3\lambda - 4\lambda - \cancel{4} = 1$

$\Rightarrow \quad \lambda = 1$

$\therefore \; V(3,-1,8)$

FINALLY THE LINE VB, USING $V(3,-1,8)$ & DIRECTION $(2,3,-5)$

$\underline{r} = (3,-1,8) + \mu(2,3,-5)$

AS REQUIRED