IYGB – FP3 PAPER O – QUESTION 1

FROM THE DEFINITION OF THE "DOT PRODUCT"

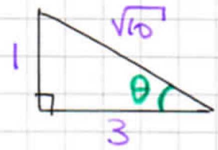
$$\Rightarrow \underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$$

$$\Rightarrow 30 = \sqrt{10} \times 10 \times \cos\theta$$

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \theta \simeq 18.43^\circ \ldots$$

OR



$\sin\theta = \frac{1}{\sqrt{10}}$

HENCE WE OBTAIN BY THE DEFINITION OF THE "CROSS" PRODUCT

$$\Rightarrow \underline{a} \wedge \underline{b} = |\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}}$$

$$\Rightarrow |\underline{a} \wedge \underline{b}| = \left|\,|\underline{a}||\underline{b}|\sin\theta\,\hat{\underline{n}}\,\right|$$

$$\Rightarrow |\underline{a} \wedge \underline{b}| = |\underline{a}||\underline{b}||\sin\theta||\hat{\underline{n}}|$$

$|\hat{\underline{n}}| = 1$

$$\Rightarrow |\underline{a} \wedge \underline{b}| = \sqrt{10} \times 10 \times \frac{1}{\sqrt{10}} \times 1$$

(or $\sin 18.43^\circ$)

$$\therefore |\underline{a} \wedge \underline{b}| = 10$$

IYGB - FP3 PAPER O - QUESTION 2

METHOD 1

$$\Rightarrow \frac{4x-3}{2-x} < 1$$

$$\Rightarrow \frac{4x-3}{2-x} - 1 < 0$$

$$\Rightarrow \frac{4x-3-(2-x)}{2-x} < 0$$

$$\Rightarrow \frac{4x-3-2+x}{2-x} < 0$$

$$\Rightarrow \frac{5x-5}{2-x} < 0$$

$$\Rightarrow \frac{5(x-1)}{x-2} > 0$$

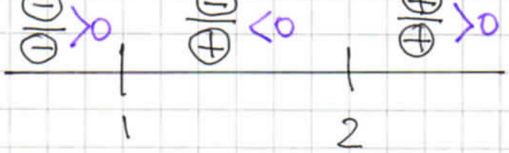
THE CRITICAL VALUES ARE

$x=2$ (VERTICAL ASYMPTOTE)

&

$x=1$ (x INTERCEPT)

USING A NUMBER LINE



$\therefore\ x<1$ OR $x>2$

METHOD 2

$$\Rightarrow \frac{4x-3}{2-x} < 1$$

$$\Rightarrow \frac{(4x-3)(2-x)}{(2-x)(2-x)} < 1$$

$$\Rightarrow \frac{(4x-3)(2-x)}{(2-x)^2} < 1$$

$$\Rightarrow (4x-3)(2-x) < (2-x)^2$$

$$\Rightarrow (4x-3)(2-x) - (2-x)^2 < 0$$

$$\Rightarrow (2-x)\left[(4x-3)-(2-x)\right] < 0$$

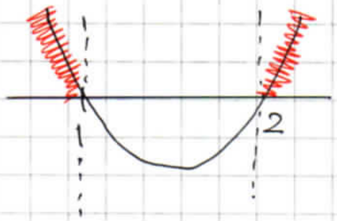
$$\Rightarrow (2-x)(5x-5) < 0$$

$$\Rightarrow -5(x-2)(x-1) < 0$$

$$\Rightarrow (x-2)(x-1) > 0$$

LOOKING AT THE QUADRATIC

CRITICAL VALUES $<$ 1, 2



$\therefore\ x<1$ OR $x>2$

IYGB - FP3 PAPER O - QUESTION 2

METHOD 3

$\Longrightarrow \dfrac{4x-3}{2-x} < 1$

SPLIT INTO 2 CASES

- IF $x>2$ (IE $2-x<0$)

$4x-3 > 2-x$

$5x > 5$

$x > 1$

IE $x>2 \cap x>1$

IE $x>2$

- IF $x<2$ (IE $2-x>0$)

$4x-3 < 2-x$

$5x < 5$

$x < 1$

IE $x<2 \cap x<1$

IE $x<1$

$x<1$ OR $x>2$ //

IYGB – FP3 PAPER O – QUESTION 3

$$\boxed{\frac{dy}{dx} = \frac{1}{1+\sqrt{x}} \quad P(9,6)}$$

USING EULER'S METHOD BASED ON THE DERIVATIVE (TAYLOR SERIES)

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \qquad \text{(FOR SMALL } h\text{)}$$

$$\boxed{f(x+h) \approx hf'(x) + f(x)}$$

WRITE THE ABOVE AS A RECURRENCE RELATION

$$y_{n+1} \approx hy'_n + y_n$$

$$y_{n+1} \approx \frac{h}{1+\sqrt{x_n}} + y_n$$

$$\boxed{y_{n+1} \approx y_n + \frac{0.25}{1+\sqrt{x_n}}} \qquad (h = 0.25)$$

USING THE ABOVE FORMULA TWICE

- $y_1 \approx y_0 + \dfrac{0.25}{1+\sqrt{x_0}}$ $\qquad (x_0 = 9, \ y_0 = 6)$

  $y_1 \approx 6 + \dfrac{0.25}{1+\sqrt{9}}$

  $y_1 \approx 6.0625$

- $y_2 \approx y_1 + \dfrac{0.25}{1+\sqrt{x_1}}$

  $y_2 \approx 6.0625 + \dfrac{0.25}{1+\sqrt{9.25}}$

  $y_2 \approx 6.124360038\ldots$

HENCE $y(9.5) \approx 6.1244$

IYGB - FP3 PAPER O - QUESTION 4

a) DIFFERENTIATE & EVALUATE DERIVATIVES AT $x=\frac{\pi}{4}$

$f(x) = \cos 2x$ $\qquad f(\pi/4) = 0$

$f'(x) = -2\sin 2x$ $\qquad f'(\frac{\pi}{4}) = -2$

$f''(x) = -4\cos 2x$ $\qquad f''(\frac{\pi}{4}) = 0$

$f'''(x) = 8\sin 2x$ $\qquad f'''(\frac{\pi}{4}) = 8$

$f''''(x) = 16\cos 2x$ $\qquad f''''(\frac{\pi}{4}) = 0$

$f'''''(x) = -32\sin 2x$ $\qquad f'''''(x) = -32$

USING TAYLOR THEOREM

$$f(x) = f(\tfrac{\pi}{4}) + \frac{(x-\pi/4)}{1!}f'(\tfrac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^2}{2!}f''(\tfrac{\pi}{4}) + \frac{(x-\frac{\pi}{4})^3}{3!}f'''(\tfrac{\pi}{4}) + \dots$$

$$\cos 2x = -2(x-\tfrac{\pi}{4}) + \frac{8}{3!}(x-\tfrac{\pi}{4})^3 - \frac{32}{5!}(x-\pi/4)^5 + O\left[(x-\tfrac{\pi}{4})^7\right]$$

$$\cos 2x = -2(x-\tfrac{\pi}{4}) + \tfrac{4}{3}(x-\tfrac{\pi}{4})^3 - \tfrac{4}{15}(x-\tfrac{\pi}{4})^5 + O\left[(x-\tfrac{\pi}{4})^7\right]$$

b) LETTING $x=1$ IN THE ABOVE EXPANSION WE OBTAIN

$\Rightarrow \cos 2 \simeq -2(1-\frac{\pi}{4}) + \frac{4}{3}(1-\frac{\pi}{4})^3 - \frac{4}{15}(1-\frac{\pi}{4})^5$

$\Rightarrow \cos 2 \simeq -0.416147367\dots$

$\Rightarrow \cos 2 \simeq -0.416$

AS REQUIRED

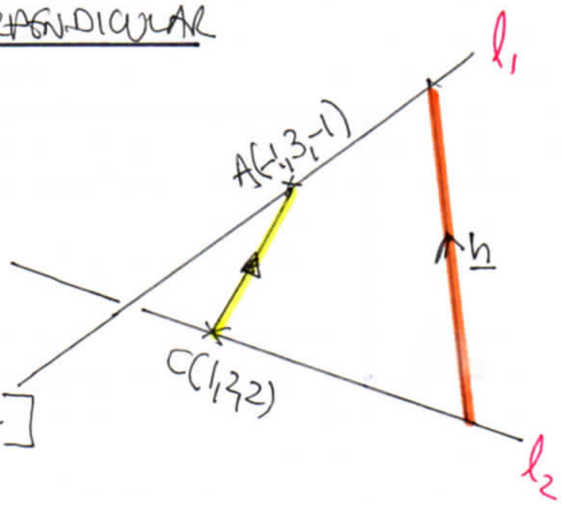
IYGB - FP3 PAPER O - QUESTION 5

a) START BY OBTAINING THE DIRECTION VECTORS OF THE TWO LINES

$$\vec{AB} = \underline{b} - \underline{a} = (1,2,-2) - (-1,3,-1) = (2,-1,-1)$$

$$\vec{CD} = \underline{d} - \underline{c} = (k,k,k) - (1,2,2) = (k-1, k-2, k-2)$$

FIND THE DIRECTION OF THE COMMON PERPENDICULAR



$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ k-1 & k-2 & k-2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$\underline{n} = [-k+2+k-2,\ 2k-4+k-1,\ -k+1-2k+4]$$

$$\underline{n} = (0,\ 3k-5,\ -3k+5)$$

$$\underline{n} = (3k-5)[0,1,-1]$$

SCALE n AND MAKING IT UNIT YIELDS $\frac{1}{\sqrt{2}}(0,1,-1)$

FINALLY OBTAIN THE VECTOR $\vec{CA}$ & PROJECT IT ONTO THE UNIT PERPENDICULAR BETWEEN THE TWO LINES

$$\vec{CA} = \underline{a} - \underline{c} = (-1,3,-1) - (1,2,2) = (-2,1,-3)$$

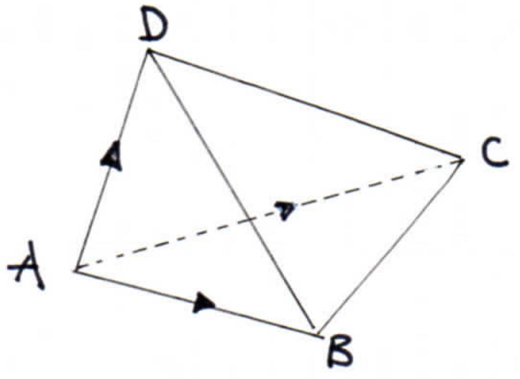
$$d = |\vec{CA} \cdot \hat{\underline{n}}| = \left|(-2,1,-3) \cdot \frac{1}{\sqrt{2}}(0,1,-1)\right| = \frac{1}{\sqrt{2}}|0+1+3|$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

IYGB - FP3 PAPER O - QUESTION 5

b)



$\overrightarrow{AB} = (2,-1,-1)$ (FROM EARLIER)

$\overrightarrow{AC} = \underline{c} - \underline{a} = (1,2,2) - (-1,3,-1) = (2,-1,3)$

$\overrightarrow{AD} = \underline{d} - \underline{a} = (k,k,k) - (-1,3,-1) = (k+1, k-3, k+1)$

- THE REQUIRED VOLUME IS GIVEN BY

$$V = \frac{1}{6}\left|\overrightarrow{AB} \wedge \overrightarrow{AC} \cdot \overrightarrow{AD}\right| = \frac{1}{6}\left\| \begin{matrix} k+1 & k-3 & k+1 \\ 2 & -1 & -1 \\ 2 & -1 & 3 \end{matrix} \right\|$$

$r_{23}(-1)$

$$= \frac{1}{6}\left\| \begin{matrix} k+1 & k-3 & k+1 \\ 2 & -1 & -1 \\ 0 & 0 & 4 \end{matrix} \right\|$$

$$= \frac{1}{6} \times \left| 4(-k-1-2k+6) \right|$$

$$= \frac{2}{3}\left|5-3k\right|$$

OR $\frac{2}{3}\left|3k-5\right|$

IYGB – FP3 PAPER O – QUESTION 6

AS THE LIMIT CURRENTLY YIELDS $\frac{0}{0}$ APPLY L'HOSPITAL'S RULE

$$\lim_{x \to 0}\left[\frac{\tan x - x}{\sin 2x - \sin x - x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(\sin 2x - \sin x - x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{\sec^2 x - 1}{2\cos 2x - \cos x - 1}\right]$$

THE ABOVE LIMIT YIELDS $\frac{0}{0}$ AGAIN, SO APPLY L'HOSPITAL'S RULE ONCE MORE

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}(2\cos 2x - \cos x - 1)}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sec^2 x \tan x}{-4\sin 2x + \sin x}\right]$$

THIS GIVES $\frac{0}{0}$ AGAIN, SO PROCEED BY L'HOSPITAL'S RULE FOR A THIRD TIME OR REMOVE THE SINGULARITY BY IDENTITIES

$$= \lim_{x \to 0}\left[\frac{2\sec^2 x \tan x}{-8\sin x \cos x + \sin x}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sec^2 x \times \frac{1}{\cos x}}{-8\cos x + 1} \times \cancel{\frac{\sin x}{\sin x}}\right]$$

$$= \lim_{x \to 0}\left[\frac{2\sec^3 x}{1 - 8\cos x}\right]$$

$$= -\frac{2}{7}$$

IYGB - FP3 PAPER O - QUESTION 6

ALTERNATIVE - USING L HOSPITAL'S RULE FOR A THIRD TIME

$$\ldots = \lim_{x \to 0}\left[\frac{2\sec^2 x \tan x}{-4\sin 2x - \sin x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}\left(2\sec^2 x \tan x\right)}{\frac{d}{dx}\left(-4\sin 2x + \sin x\right)}\right]$$

$$= \lim_{x \to 0}\left[\frac{4\sec^2 x \tan^2 x + 2\sec^4 x}{-8\cos 2x + \cos x}\right]$$

$$= \frac{0+2}{-8+1}$$

$$= -\frac{2}{7}$$

AS BEFORE

IYGB - FP3 PAPER O - QUESTION 7

USING THE SUBSTITUTION GIVEN

$$\Rightarrow u = \frac{dy}{dx} - 2x$$

$$\Rightarrow \frac{dy}{dx} = u + 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{du}{dx} + 2$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow (x^3+1)\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} = 2 - 4x^3$$

$$\Rightarrow (x^3+1)\left(\frac{du}{dx}+2\right) - 3x^2(u+2x) = 2 - 4x^3$$

$$\Rightarrow (x^3+1)\frac{du}{dx} + 2(x^3+1) - 3x^2u - 6x^3 = 2 - 4x^3$$

$$\Rightarrow (x^3+1)\frac{du}{dx} + 2x^3 \cancel{+2} - 3ux^2 - 6x^3 = \cancel{2} - 4x^3$$

$$\Rightarrow (x^3+1)\frac{du}{dx} - 3ux^2 \cancel{-4x^3} = \cancel{-4x^3}$$

$$\Rightarrow (x^3+1)\frac{du}{dx} = 3ux^2$$

SEPARATE VARIABLES

$$\Rightarrow \frac{1}{u}\,du = \frac{3x^2}{x^3+1}\,dx$$

$$\Rightarrow \int \frac{1}{u}\,du = \int \frac{3x^2}{x^3+1}\,dx$$

$$\Rightarrow \ln|u| = \ln|x^3+1| + \ln A$$

$$\Rightarrow \ln|u| = \ln|A(x^3+1)|$$

$$\Rightarrow \boxed{u = A(x^3+1)}$$

IYGB - FP3 PAPER O - QUESTION 7

REVERSING THE TRANSFORMATION

$$\Rightarrow \frac{dy}{dx} - 2x = A(x^3+1)$$

$$\Rightarrow \frac{dy}{dx} = A(x^3+1) + 2x$$

INTEGRATING W.R.T x

$$\Rightarrow y = A\left(\tfrac{1}{4}x^4 + x\right) + x^2 + B$$

USING THE CONDITION GIVEN

$$x=0,\ y=0 \Rightarrow 0 = B$$

$$x=0,\ \frac{dy}{dx}=4 \Rightarrow 4 = A$$

$$\therefore\ y = 4\left(\tfrac{1}{4}x^4+x\right) + x^2$$

$$y = x^4 + 4x + x^2$$

$$y = x^4 + x^2 + 4x$$

IYGB – FP3 PAPER O – QUESTION 8

USING LEIBNIZ RULE FOR PRODUCTS

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{d^1 v}{dx^1} + \frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + \frac{n(n-1)(n-2)}{3!}\frac{d^{n-3}u}{dx^{n-3}}\frac{d^3 v}{dx^3} + \cdots$$

HERE $y = x^4 \cos x$

x^4: VANISHES AFTER A FEW DIFFERENTIATIONS

$\cos x$: u (ITS DERIVATIVE HAS A PATTERN)

$$\boxed{\frac{d^k}{dx^k}\left[\cos(ax)\right] = a^k \cos\left(ax + \frac{k\pi}{2}\right)}$$

USING THE RULE WE OBTAIN

$$\frac{d^6}{dx^6}\left(x^4 \cos x\right) = 1\frac{d^6}{dx^6}(\cos x)\,x^4 + 6\frac{d^5}{dx^5}(\cos x)\frac{d^1}{dx^1}(x^4) + \frac{6\times5}{2!}\frac{d^4}{dx^4}(\cos x)\frac{d^2}{dx^2}(x^4) + \frac{6\times5\times4}{3!}\frac{d^3}{dx^3}(\cos x)\frac{d^3}{dx^3}(x^4)$$

$$+ \frac{6\times5\times4\times3}{4!}\frac{d^2}{dx^2}(\cos x)\frac{d^4}{dx^4}(x^4) + \cancel{\frac{6\times5\times4\times3\times2}{5!}\frac{d^1}{dx^1}(\cos x)\frac{d^5}{dx^5}(x^4)} + \cancel{1\frac{d^0}{dx^0}(\cos x)\frac{d^6}{dx^6}(x^4)}$$

$$= x^4\cos(x+3\pi) + 6\cos\left(x+\tfrac{5}{2}\pi\right)\cdot 4x^3 + 15\cos(x+2\pi)\cdot 12x^2 + 20\cos\left(x+\tfrac{3}{2}\pi\right)\cdot 24x + 15\cos(x+\pi)\cdot 24$$

$$= x^4\cos(x+\pi) + 24x^3\cos\left(x+\tfrac{\pi}{2}\right) + 180x^2\cos x + 480x\cos\left(x-\tfrac{\pi}{2}\right) + 360\cos(x+\pi)$$

$$= x^4(-\cos x) + 24x^3(-\sin x) + 180x^2\cos x + 480x\sin x + 360(-\cos x)$$

$$= 24x(20-x^2)\sin x - (x^4 - 180x^2 + 360)\cos x$$

IYGB - FP3 PAPER O - QUESTION 9

a) REARRANGE & DIFFERENTIATE

$$\Rightarrow xy = 4$$
$$\Rightarrow y = \frac{4}{x}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{4}{x^2}$$
$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=2t} = -\frac{4}{(2t)^2} = -\frac{4}{4t^2} = -\frac{1}{t^2}$$

TANGENT GRADIENT

FINALLY WE OBTAIN THE NORMAL

$$y - \frac{2}{t} = t^2(x - 2t)$$
$$ty - 2 = t^3(x - 2t)$$
$$ty - 2 = t^3x - 2t^4$$
$$ty - t^3x = 2 - 2t^4$$

b) PROCEED BY SOLVING SIMULTANEOUSLY THE CURVE & THE NORMAL — NOTE THAT THE POINT OF NORMALITY MUST BE A SOLUTION

$$\Rightarrow ty - t^3x = 2 - 2t^4 \qquad \times x$$
$$\Rightarrow txy - t^3x^2 = (2 - 2t^4)x$$
$$\Rightarrow 4t - t^3x^2 = (2 - 2t^4)x$$
$$\Rightarrow 0 = t^3x^2 + (2 - 2t^4)x - 4t$$
$$\Rightarrow (t^3x + 2)(x - 2t) = 0$$

POINT OF NORMALITY $P\left(2t, \frac{2}{t}\right)$

POINT Q (REINTERSECTION)

IYGB - FP3 PAPER O - QUESTION 9

FINDING THE CO-ORDINATES OF Q & M

WHEN $x = -\frac{2}{t^3}$ $\quad y = \frac{4}{-\frac{2}{t^3}} = -2t^3$ $\quad Q\left(-\frac{2}{t^3}, -2t^3\right)$

$$M\left(\frac{2t-\frac{2}{t^3}}{2}, \frac{\frac{2}{t}-2t^3}{2}\right) = M\left(t-\frac{1}{t^3}, \frac{1}{t}-t^3\right)$$

FINALLY ELIMINATE t, TO OBTAIN A CARTESIAN EXPRESSION

$$\left.\begin{array}{l} X = t - \frac{1}{t^3} \\ Y = \frac{1}{t} - t^3 \end{array}\right\} \Longrightarrow \begin{array}{l} X = \frac{t^4-1}{t^3} \\ Y = \frac{1-t^4}{t} = \frac{t^4-1}{-t} \end{array}$$

DIVIDING THE EQUATIONS ABOVE

$$\frac{Y}{X} = Y\left(\frac{1}{X}\right) = \frac{\cancel{t^4-1}}{-t}\left(\frac{t^3}{\cancel{t^4-1}}\right)$$

$$\frac{Y}{X} = -t^2$$

SUB INTO EITHER PARAMETRIC

$$\Longrightarrow Y = \frac{t^4-1}{-t}$$

$$\Longrightarrow Y^2 = \frac{(t^4-1)^2}{t^2}$$

$$\Longrightarrow Y^2t^2 = (t^4-1)^2$$

IYGB - FP3 PAPER O - QUESTION 9

$$\Rightarrow y^2\left(-\frac{y}{x}\right) = \left[\left(-\frac{y}{x}\right)^2 - 1\right]^2$$

$$\Rightarrow -\frac{y^3}{x} = \left[\frac{y^2}{x^2} - 1\right]^2$$

$$\Rightarrow -\frac{y^3}{x} = \left(\frac{y^2 - x^2}{x^2}\right)^2$$

$$\Rightarrow -\frac{y^3}{x} = \frac{(y^2 - x^2)^2}{x^4}$$

$$\Rightarrow -y^3x^3 = (y^2 - x^2)^2$$

$$\Rightarrow (y^2 - x^2)^2 + x^3y^3 = 0$$