IYGB – FP3 PAPER K – QUESTION 1

TABULATING VALUES, NOTING THAT x MUST BE IN RADIANS

x	0	0.25	0.5	0.75	1
$y=\sqrt{1+\sin x}$	1	1.11687	1.21632	1.29678	1.3570
	FIRST	ODD	EVEN	ODD	LAST

USING SIMPSON'S RULE

$$\int_0^1 \sqrt{1+\sin x}\,dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + \text{"}4\times\text{ODD"} + \text{"}2\times\text{EVEN"}\right]$$

$$\simeq \frac{0.25}{3}\left[1 + 1.3570 + 4(1.11687 + 1.29678) + 2(1.21632)\right]$$

$$\simeq 1.2036\ldots$$

$$\simeq 1.204$$

IYGB – FP3 PAPER K – QUESTION 2

a) LINEARLY DEPENDENT $\Longrightarrow$ "THEY DO NOT SPAN 3D SPACE"

$\Longrightarrow$ "VOLUME OF THE PARALLELEPIPED THEY DEFINE MUST BE ZERO

HENCE WE FORM $\vec{OA} \wedge \vec{OB} \cdot \vec{OC}$

$$\begin{vmatrix} -1 & 2 & 2 \\ 3 & 4 & -1 \\ 1 & 4 & 1 \end{vmatrix} = -1\begin{vmatrix} 4 & -1 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix}$$

$$= -1(4+4) - 2(3+1) + 2(12-4)$$

$$= -8 - 8 + 16$$

$$= 0$$

INDEED LINEARLY DEPENDENT

b) WORK OUT ANY TWO SIDES OF ABC

- $\vec{AB} = \underline{b} - \underline{a} = (3,4,-1) - (-1,2,2) = (4,2,-3)$
- $\vec{AC} = \underline{c} - \underline{a} = (1,4,1) - (-1,2,2) = (2,2,-1)$

- $\text{AREA} = \frac{1}{2}\left|\vec{AB} \wedge \vec{AC}\right| = \frac{1}{2}\left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ 4 & 2 & -3 \\ 2 & 2 & -1 \end{matrix}\right\| = \frac{1}{2}\left|-2+6, -6+4, 8-4\right|$

$$= \frac{1}{2}\left|4, -2, 4\right| = \frac{1}{2}\sqrt{16+4+16}$$

$$= 3$$

IYGB - FP3 PAPER K - QUESTION 3

BY SERIES EXPANSIONS

$$\lim_{x\to 0}\left[\frac{\cos^2 3x - 1}{x^2}\right] = \lim_{x\to 0}\left[\frac{1-\cos^2 3x}{-x^2}\right] = \lim_{x\to 0}\left[\frac{\sin^2 3x}{-x^2}\right]$$

NOW $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + O(x^5)$$

$$\sin 3x = 3x - \frac{9}{2}x^3 + O(x^5)$$

$$\dots = \lim_{x\to 0}\left[\frac{\left[3x - \frac{9}{2}x^3 + O(x^5)\right]^2}{-x^2}\right] = \lim_{x\to 0}\left[\frac{9x^2 - 27x^4 + O(x^6)}{-x^2}\right]$$

$$= \lim_{x\to 0}\left[-9 + 27x^2 + O(x^4)\right] = -9$$

OR BY L'HOSPITAL RULE AS THE LIMIT IS ZERO OVER ZERO

$$\lim_{x\to 0}\left[\frac{\cos^2 3x - 1}{x^2}\right] = \dots \text{"}\frac{0}{0}\text{"} = \lim_{x\to 0}\left[\frac{\frac{d}{dx}\left[\cos^2 3x - 1\right]}{\frac{d}{dx}(x^2)}\right]$$

$$= \lim_{x\to 0}\left[\frac{-6\cos 3x \sin 3x}{2x}\right] = \lim_{x\to 0}\left[\frac{-3(2\cos 3x \sin 3x)}{2x}\right] = \lim_{x\to 0}\left[\frac{-3\sin 6x}{2x}\right]$$

THIS AGAIN IS OF THE FORM ZERO OVER ZERO

$$= \lim_{x\to 0}\left[\frac{\frac{d}{dx}\left[-3\sin 6x\right]}{\frac{d}{dx}(2x)}\right] = \lim_{x\to 0}\left[\frac{-18\cos 6x}{2}\right] = \lim_{x\to}\left[-9\cos 6x\right]$$

$$= -9$$

IYGB - FP3 PAPER K - QUESTION 4

$$\boxed{\frac{dy}{dx} = \sin(x^2+y^2) \qquad x=1 \quad y=2 \qquad h=0.01}$$

USING THE STANDARD APPROXIMATION $f'(x) \approx \frac{f(x+h)-f(x)}{h}$

$$\Rightarrow y_n' \approx \frac{y_{n+1}-y_n}{h}$$

$$\Rightarrow y_{n+1} \approx h y_n' + y_n$$

$$\Rightarrow \boxed{y_{n+1} \approx h\sin(x_n^2+y_n^2) + y_n}$$

APPLYING THE ABOVE WITH $h=0.01$

$$\Rightarrow y_1 \approx 0.01\sin(x_0^2+y_0^2)+y_0 \qquad (x_0=1, y_0=2)$$

$$\Rightarrow y_1 \approx 0.01\sin 5 + 2$$

$$\Rightarrow y_1 \approx 1.99041\ldots$$

$$\Rightarrow y_2 \approx 0.01\sin(x_1^2+y_1^2)+y_1 \qquad (x_1=1.01, y_1=1.99041\ldots)$$

$$\Rightarrow y_2 \approx 0.01\sin(1.01^2+1.99041\ldots^2)+1.99041\ldots$$

$$\Rightarrow y_2 \approx 1.98077\ldots$$

$$\Rightarrow y_3 \approx 0.01\sin(x_2^2+y_2^2)+y_2 \qquad (x_2=1.02, y_2=1.98077\ldots)$$

$$\Rightarrow y_3 \approx 0.01\sin(1.02^2+1.98077^2)+1.98077\ldots$$

$$\Rightarrow y_3 \approx 1.97109\ldots$$

$\therefore$ THE VALUE OF y AT $x=1.03$ IS APPROXIMATELY 1.9711

IYGB - FP3 PAPER K - QUESTION 5

a) WRITE THE ELLIPSE IN "STANDARD" FORM

$$x^2 - 8x + 4y^2 + 12 = 0$$
$$(x-4)^2 - 16 + 4y^2 + 12 = 0$$
$$(x-4)^2 + 4y^2 = 4$$
$$\frac{(x-4)^2}{4} + y^2 = 1$$

$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1 \quad a > b$ HAS FOCI AT $(\pm ae, 0)$
DIRECTRICES AT $x = \pm \frac{a}{e}$

HERE $a=2 \quad b=1$

$$b^2 = a^2(1-e^2) \implies 1 = 4(1-e^2)$$
$$\implies \frac{1}{4} = 1 - e^2$$
$$e^2 = \frac{3}{4}$$
$$e = +\sqrt{3}/2$$

HENCE THE ELLIPSE $\frac{x^2}{4} + \frac{y^2}{1} = 1$ HAS

FOCI AT $\left(\pm 2\frac{\sqrt{3}}{2}, 0\right) = (\pm\sqrt{3}, 0)$

DIRECTRICES AT $x = \pm \frac{2}{\sqrt{3}/2} = \pm \frac{4}{\sqrt{3}} = \pm \frac{4}{3}\sqrt{3}$

AS "OUR ELLIPSE" IS A HORIZONTAL TRANSLATION BY $+4$

- FOCI AT $(4 \pm \sqrt{3})$
- DIRECTRICES $x = \begin{cases} 4 + \frac{4}{3}\sqrt{3} \\ 4 - \frac{4}{3}\sqrt{3} \end{cases}$

IYGB – FP3 PAPER K – QUESTION 5

b) LET THE STRAIGHT LINE (SOON TO BE A TANGENT) HAVE EQUATION $y = mx$, $m > 0$

$$\left.\begin{array}{l} x^2 + 4y^2 - 8x + 12 = 0 \\ y = mx \end{array}\right\} \Rightarrow x^2 + 4m^2x^2 - 8x + 12 = 0$$

$$\Rightarrow (1 + 4m^2)x^2 - 8x + 12 = 0$$

IF TANGENT $b^2 - 4ac = 0$

$$\Rightarrow (-8)^2 - 4(1 + 4m^2) \times 12 = 0$$
$$\Rightarrow 64 - 48(1 + 4m^2) = 0$$
$$\Rightarrow 64 = 48(1 + 4m^2)$$
$$\Rightarrow \frac{4}{3} = 1 + 4m^2$$
$$\Rightarrow \left(m^2 = \frac{1}{12}\right)$$
$$\Rightarrow \left(m = +\frac{1}{\sqrt{12}}\right)$$

SUBSTITUTE INTO THE QUADRATIC $4m^2 + 1 = \frac{4}{3}$

$$\Rightarrow \frac{4}{3}x^2 - 8x + 12 = 0 \quad \leftarrow \text{"EXPECT A PERFECT SQUARE"}$$
$$\Rightarrow 4x^2 - 24x + 36 = 0$$
$$\Rightarrow x^2 - 6x + 9 = 0$$
$$\Rightarrow (x - 3)^2 = 0$$
$$\Rightarrow \underline{x = 3}$$

FINALLY $y = mx$ WITH $x = 3$ & $m = \frac{1}{\sqrt{12}}$

$$\Rightarrow y = \frac{1}{\sqrt{12}} \times 3 = \frac{1}{2\sqrt{3}} \times 3 = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{1}{2}\sqrt{3}$$

$$\therefore \underline{A\left(3, \tfrac{1}{2}\sqrt{3}\right)}$$

IYGB - FP3 PAPER K - QUESTION 6

USING LEIBNIZ RULE FOR $y = e^{2x}\sin x$ WITH $u = e^{2x}$ & $v = \sin x$

$$\frac{d^6}{dx^6}\left(e^{2x}\sin x\right) = \binom{6}{0}\frac{d^6}{dx^6}\left(e^{2x}\right)\sin x + \binom{6}{1}\frac{d^5}{dx^5}\left(e^{2x}\right)\frac{d}{dx}(\sin x) + \binom{6}{2}\frac{d^4}{dx^4}\left(e^{2x}\right)\frac{d^2}{dx^2}(\sin x)$$

$$+ \binom{6}{3}\frac{d^3}{dx^3}\left(e^{2x}\right)\frac{d^3}{dx^3}(\sin x) + \binom{6}{4}\frac{d^2}{dx^2}\left(e^{2x}\right)\frac{d^4}{dx^4}(\sin x)$$

$$+ \binom{6}{5}\frac{d}{dx}\left(e^{2x}\right)\frac{d^5}{dx^5}(\sin x) + \binom{6}{6}e^{2x}\frac{d^6}{dx^6}(\sin x)$$

NOTE THAT $\frac{d^n}{dx^n}\left(e^{ax}\right) = a^n e^{ax}$.

ALSO THE DERIVATIVES OF $\sin x$ HAVE A PATTERN

DERIVATIVES:	0	1	2	3	4	5	6
	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$	$-\sin x$

HENCE WE HAVE

$$\frac{d^6}{dx^6}\left(e^{2x}\sin x\right) = \Big[\left(1\times 2^6\times \sin x\right) + \left(6\times 2^5\times\cos x\right) + \left[15\times 2^4\times(-\sin x)\right] + \left[20\times 2^3\times(-\cos x)\right]$$

$$+ \left(15\times 2^2\times \sin x\right) + \left(6\times 2\times \cos x\right) + \left[1\times 2^0\times(-\sin x)\right]\Big]e^{2x}$$

$$= \left[64\sin x + 192\cos x - 240\sin x - 160\cos x + 60\sin x + 12\cos x - \sin x\right]e^{2x}$$

$$= \left(44\cos x - 117\sin x\right)e^{2x}$$

IYGB - FP3 PAPER K - QUESTION 7

USING THE SUBSTITUTION GIVEN $u(x) = x\,y(x)$

$$\frac{d}{dx}\left(u(x)\right) = \frac{d}{dx}\left(x\,y(x)\right)$$

$$\frac{du}{dx} = x \times \frac{dy}{dx} + 1 \times y$$

$$\frac{du}{dx} = x\frac{dy}{dx} + y$$

$$\boxed{x\frac{dy}{dx} = \frac{du}{dx} - y}$$

DIFFERENTIATE THE ABOVE AGAIN WITH RESPECT TO x

$$\frac{d}{dx}\left[x\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{du}{dx} - y\right]$$

$$1 \times \frac{dy}{dx} + x\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} - \frac{dy}{dx}$$

$$\boxed{x\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2} - 2\frac{dy}{dx}}$$

TRANSFORM THE O.D.E.

$$\Rightarrow x\frac{d^2y}{dx^2} + (6x+2)\frac{dy}{dx} + 9xy = 27x - 6y$$

$$\Rightarrow \frac{d^2u}{dx^2} \cancel{- 2\frac{dy}{dx}} + 6x\frac{dy}{dx} + \cancel{2\frac{dy}{dx}} + 9u = 27x - 6y$$

$$\Rightarrow \frac{d^2u}{dx^2} + 6x\frac{dy}{dx} + 9u = 27x - 6y$$

$$\Rightarrow \frac{d^2u}{dx^2} + 6\left(\frac{du}{dx} - y\right) + 9u = 27x - 6y$$

$$\Rightarrow \frac{d^2u}{dx^2} + 6\frac{du}{dx} \cancel{- 6y} + 9u = 27x \cancel{- 6y}$$

$$\Rightarrow \boxed{\frac{d^2u}{dx^2} + 6\frac{du}{dx} + 9u = 27x}$$

IYGB - FP3 PAPER K - QUESTION 7

THE AUXILIARY EQUATION FOR THE LHS IS

$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda+3)^2 = 0$$
$$\lambda = -3$$

COMPLEMENTARY FUNCTION

$$u = Ae^{-3x} + Bxe^{-3x}$$

PARTICULAR INTEGRAL BY INSPECTION

$$u = Px + Q$$
$$u' = P$$
$$u'' = 0$$

$$\therefore \ 0 + 6P + 9(Px+Q) \equiv 27x$$
$$(6P + 9Q) + 9Px \equiv 27x$$

$P = 3$ & $6P + 9Q = 0$

$18 + 9Q = 0$

$Q = -2$

THUS WE HAVE

$$u(x) = (A + Bx)e^{-3x} + 3x - 2$$

REVERSING THE TRANSFORMATION

$$xy = (A + Bx)e^{-3x} + 3x - 2$$
$$y = \left(\frac{A}{x} + B\right)e^{-3x} + 3 - \frac{2}{x}$$

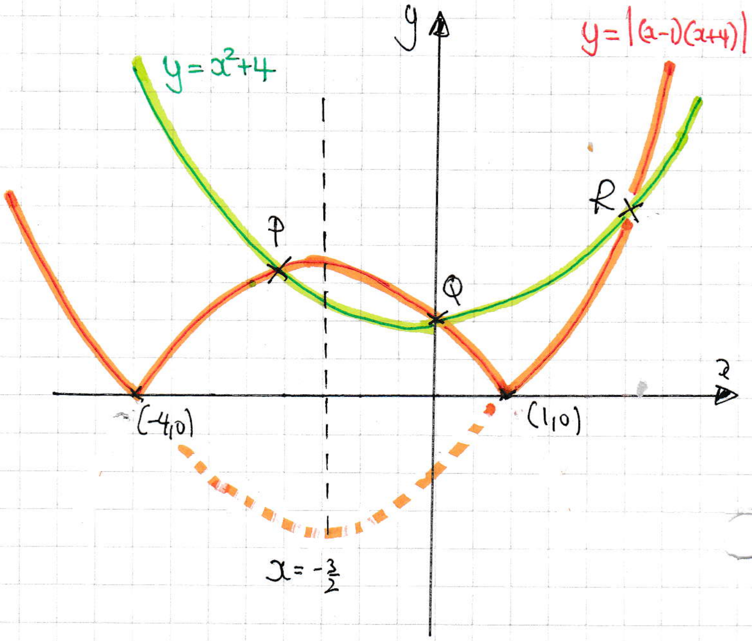
IYGB – FP3 PAPER K – QUESTION 8

GRAPHICAL APPROACH

$$\left|\frac{(x-1)(x+4)}{x^2+4}\right| < 1$$

$$|(x-1)(x+4)| < x^2+4$$

(AS $x^2+4>0$)



"TO FIND P & Q"

$x^2+4 = -(x-1)(x+4)$

$x^2+4 = -(x^2+3x-4)$

$x^2+4 = -x^2-3x+4$

$2x^2+3x=0$

$x(2x+3)=0$

$x = \begin{cases} 0 \\ -\frac{3}{2} \end{cases}$

"TO FIND R"

$x^2+4 = (x+4)(x-1)$

$\cancel{x^2}+4 = \cancel{x^2}+3x-4$

$8 = 3x$

$x = \frac{8}{3}$

WE REQUIRE THE "ORANGE CURVE" TO BE LOWER THAN THE "GREEN CURVE"

$\Longrightarrow$ THIS HAPPENS TO THE "LEFT OF P" & "BETWEEN Q & R"

$x=-\frac{3}{2}$ $\quad$ $x=0$ $\quad$ $x=\frac{8}{3}$

$\therefore \quad x < -\frac{3}{2}$ OR $0 < x < \frac{8}{3}$

IYGB - FP3 PAPER K - QUESTION 8

ALGEBRAIC APPROACH

$$\left|\frac{(x-1)(x+4)}{x^2+4}\right| < 1$$

$$\frac{|(x-1)(x+4)|}{x^2+4} < 1$$

$$|(x-1)(x+4)| < x^2+4$$

(NOTE THAT $x^2+4>0$)

THE "CRITICAL VALUES" FOR THIS INEQUALITY ARE 1 & -4

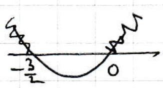
- IF $x \leqslant -4$

$$(x-1)(x+4) < x^2+4$$
$$\cancel{x^2}+3x-4 < \cancel{x^2}+4$$
$$3x < 8$$
$$x < \frac{8}{3}$$

$\therefore$ $x \leqslant -4$

- IF $-4 \leqslant x \leqslant 1$

$$-(x-1)(x+4) < x^2+4$$
$$-x^2-3x+\cancel{4} < x^2+\cancel{4}$$
$$-2x^2-3x < 0$$
$$2x^2+3x > 0$$
$$x(2x+3) > 0$$



$x < -\frac{3}{2}$ OR $x > 0$

$\therefore$ $-4 \leqslant x < -\frac{3}{2}$

OR

$0 < x \leqslant 1$

- IF $x \geqslant 1$

$$(x-1)(x+4) < x^2+4$$
$$\vdots$$
$$x < \frac{8}{3}$$

$\therefore$ $1 \leqslant x < \frac{8}{3}$

COMBINING RESULTS WE HAVE

$$x < -\frac{3}{2} \quad \text{OR} \quad 0 < x < \frac{8}{3}$$

IYGB - FP3 PAPER K - QUESTION 9

USING THE SUBSTITUTION GIVEN

$\Rightarrow t = \tan\frac{x}{2}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\Rightarrow dx = \frac{1}{\frac{1}{2}(1+t^2)}\,dt$

$\Rightarrow \boxed{dx = \frac{2}{1+t^2}\,dt}$

CHANGE THE LIMITS

$x=0 \longmapsto t=0$

$x=\frac{\pi}{2} \longmapsto t=1$

GETTING AN EXPRESSION FOR COSx IN TERMS OF t

$\cos x = 2\cos^2\frac{x}{2} - 1$

$\cos x = \frac{2}{\sec^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+\tan^2\frac{x}{2}} - 1$

$\cos x = \frac{2}{1+t^2} - 1$

$\cos x = \frac{2-(1+t^2)}{1+t^2}$

$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$

TRANSFORMING THE GIVEN INTEGRAL USING THE ABOVE RESULTS

$$\int_0^{\frac{\pi}{2}} \frac{3\sqrt{3}}{2-\cos x}\,dx = \ldots\ldots = \int_0^1 \frac{3\sqrt{3}}{2-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{6\sqrt{3}}{2(1+t^2)-(1-t^2)}\,dt = \int_0^1 \frac{6\sqrt{3}}{1+3t^2}\,dt = \int_0^1 \frac{2\sqrt{3}}{\underbrace{\frac{1}{3}}_{\left(\frac{1}{\sqrt{3}}\right)^2}+t^2}\,dt$$

THIS IS A "STANDARD ARCTAN" INTEGRAL

$$= \left[\frac{2\sqrt{3}}{1/\sqrt{3}}\arctan\left(\frac{t}{1/\sqrt{3}}\right)\right]_0^1 = \left[6\arctan(\sqrt{3}t)\right]_0^1$$

$$= 6\arctan\sqrt{3} - \cancel{6\arctan 0} = 6\times\frac{\pi}{3} = 2\pi$$