IYGB - FP3 PAPER J - QUESTION 1

METHOD A

$$\frac{5x}{x^2+4} < x$$

AS $x^2+4>0$ WE MAY MULTIPLY ACROSS

$$5x < x^3+4x$$

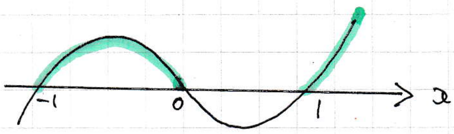
$$0 < x^3 - x$$

$$x^3 - x > 0$$

$$x(x^2-1) > 0$$

$$x(x+1)(x-1) > 0$$

$$\text{CV} = \begin{cases} -1 \\ 0 \\ 1 \end{cases}$$



$\therefore \; -1<x<0 \;\cup\; x>1$

METHOD B

$$\frac{5x}{x^2+4} < x$$

$$\frac{5x}{x^2+4} - x < 0$$

$$\frac{5x - x(x^2+4)}{x^2+4} < 0$$

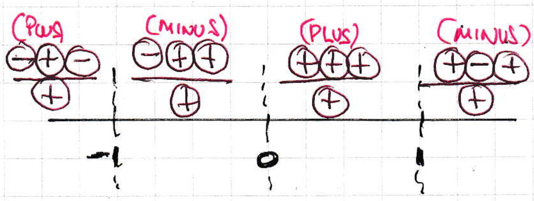
$$\frac{5x - x^3 - 4x}{x^2+4} < 0$$

$$\frac{x - x^3}{x^2+4} < 0$$

$$\frac{x(1-x^2)}{x^2+4} < 0$$

$$\frac{x(1-x)(1+x)}{x^2+4} < 0$$

THE CRITICAL VALUES ARE 0 & ±1 FROM THE NUMERATOR, AS THE DENOMINATOR IS IRREDUCIBLE



$\therefore \; -1<x<0 \;\cup\; x>1$

IYGB - FP3 PAPER J - QUESTION 2

a) TABULATE VALUES WITH A GAP OF 0.5

x	0	0.5	1	1.5	2
$\sqrt{4x-x^2}$	0	$\frac{1}{2}\sqrt{7}$	$\sqrt{3}$	$\frac{1}{2}\sqrt{15}$	2
	FIRST	ODD	EVEN	ODD	LAST

USING SIMPSON RULE

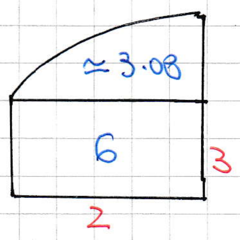
$$\text{AREA} \approx \frac{\text{"THICKNESS"}}{3}\left[\text{FIRST} + \text{LAST} + \text{"4} \times \text{ODDS"} + \text{"2} \times \text{EVENS"}\right]$$

$$\approx \frac{0.5}{3}\left[0 + 2 + 4\left(\tfrac{1}{2}\sqrt{7} + \tfrac{1}{2}\sqrt{15}\right) + 2 \times \sqrt{3}\right]$$

$$\approx 3.083595\ldots$$

$$\approx 3.08$$

b) EITHER BY GEOMETRY OR BY INTEGRATION



$\therefore$ APPROX 9.08

$$\int_0^2 3 + \sqrt{4x-x^2}\,dx$$

$$= \int_0^2 3\,dx + \int_0^2 \sqrt{4x-x^2}\,dx$$

$$= \left[3x\right]_0^2 + 3.08\ldots$$

$$= 6 + 3.08\ldots$$

$$\approx 9.08$$

IYGB - FP3 PAPER J - QUESTION 3.

AS THE SERIES EXPANSION OF $\arcsin x$ IS NOT USUALLY GIVEN IN EXAM FORMULA BOOK WE PROCEED BY L'HOSPITAL'S RULE

$$\lim_{x \to 0}\left[\frac{x\cos x}{x+\arcsin x}\right] = \dots \frac{\text{"ZERO"}}{\text{"ZERO"}} \dots$$

$$= \lim_{x \to 0}\left[\frac{\frac{d}{dx}(x\cos x)}{\frac{d}{dx}(x+\arcsin x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{\cos x - x\sin x}{1+\frac{1}{\sqrt{1-x^2}}}\right]$$

THIS LIMIT NOW EXISTS

$$= \frac{1-0}{1+1}$$

$$= \frac{1}{2}$$

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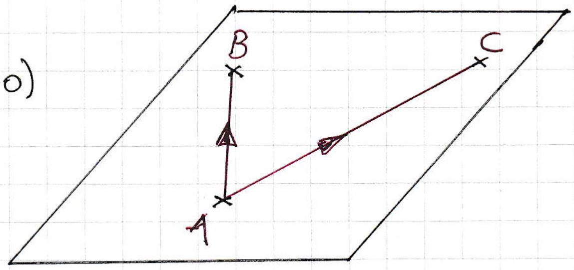
a) LOOKING AT THE DIAGRAM



- $\vec{AB} = \underline{b} - \underline{a} = (5,-2,1) - (1,1,1) = (4,-3,0)$
- $\vec{AC} = \underline{c} - \underline{a} = (3,2,6) - (1,1,1) = (2,1,5)$

"CROSSING" THE VECTORS $\vec{AB}$ & $\vec{AC}$ TO GET THE PLANE NORMAL

$$\vec{AB} \wedge \vec{AC} = \begin{vmatrix} i & j & k \\ 4 & -3 & 0 \\ 2 & 1 & 5 \end{vmatrix} = (-15-0,\ 0-20,\ 4+6) = (-15,-20,10)$$

SCALE THE NORMAL $\underline{n}$ TO $(3,4,-2)$

THE EQUATION OF THE PLANE MUST PASS THROUGH SAY $A(1,1,1)$

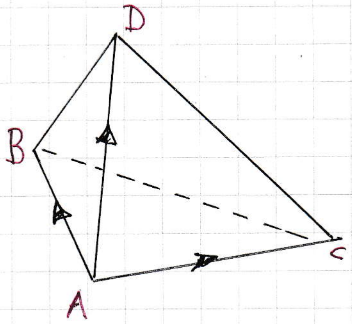
$$\Rightarrow 3x + 4y - 2z = \text{CONSTANT}$$

$$\Rightarrow 3 + 4 - 2 = \text{CONSTANT}$$

$$\therefore\ 3x + 4y - 2z = 5$$

b) START BY FINDING $\vec{AD}$

$$\vec{AD} = \underline{d} - \underline{a} = (1,5,6) - (1,1,1) = (0,4,5)$$



$$V = \frac{1}{6}\left| \vec{AB} \wedge \vec{AC} \cdot \vec{AD} \right|$$

$$= \frac{1}{6}\left| (-15,-20,10) \cdot (0,4,5) \right|$$

$$= \frac{1}{6}\left| 0 - 80 + 50 \right|$$

$$= 5$$

IYGB - FP3 PAPER J - QUESTION 5

$$\frac{d^2y}{dx^2} = \frac{x}{y^2} + \frac{1}{y} \qquad x = 0.5 \quad y = 1 \qquad x = 0.6 \quad y = 1.3$$

USING THE FORMULA $y_n'' \approx \dfrac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$

$$\Rightarrow y_{n+1} \approx y_n'' h^2 + 2y_n - y_{n-1}$$

$$\Rightarrow y_{n+1} \approx (0.1)^2 \left[\frac{x_n}{y_n^2} + \frac{1}{y_n} \right] + 2y_n - y_{n-1}$$

USING THE ABOVE WITH $x_0 = 0.5,\ y_0 = 1$ & $x_1 = 0.6,\ y_1 = 1.3$

$$\Rightarrow y_2 \approx 0.01 \left[\frac{x_1}{y_1^2} + \frac{1}{y_1} \right] + 2y_1 - y_0$$

$$\Rightarrow y_2 \approx 0.01 \left[\frac{0.6}{1.3^2} + \frac{1}{1.3} \right] + 2 \times 1.3 - 1$$

$$\Rightarrow y_2 \approx 1.611242604 \ldots \quad (\text{at } x = 0.7)$$

APPLY THE RECURSION ONCE MORE

$$\Rightarrow y_3 \approx 0.01 \left[\frac{x_2}{y_2^2} + \frac{1}{y_2} \right] + 2y_2 - y_1$$

$$\Rightarrow y_3 \approx 0.01 \left[\frac{0.7}{1.611242\ldots^2} + \frac{1}{1.6123\ldots} \right] + 2 \times 1.6112\ldots - 1.3$$

$$\Rightarrow y_3 \approx 1.93303607 \ldots \quad (\text{at } x = 0.8)$$

$\therefore$ THE APPROXIMATE VALUE OF y AT $x = 0.8$ IS 1.9330 //

IYGB - FP3 PAPER J - QUESTION 6

USING THE SUBSTITUTION GIVEN $y(x) = x\,V(x)$

$$\frac{dy}{dx} = \frac{d}{dx}\big(x\,V(x)\big) = 1 \times V(x) + x\,\frac{dV(x)}{dx}$$

$$\text{I.E } \frac{dy}{dx} = V + x\frac{dV}{dx}$$

SUBSTITUTE INTO THE O.D.E.

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{xy}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2+3(xV)^2}{x(xV)}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2+3x^2V^2}{x^2V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{x^2(1+3V^2)}{x^2V} - V$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2}{V} - V$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2-V^2}{V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+2V^2}{V}$$

SEPARATING VARIABLES

$$\Rightarrow \frac{V}{1+2V^2}\,dV = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{V}{1+2V^2}\,dV = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{4}\ln(1+2V^2) = \ln|x| + \ln A$$

$$\Rightarrow \ln(1+2V^2) = 4\ln(Ax)$$

$$\Rightarrow \ln(1+2V^2) = \ln(Bx^4) \quad (B=A^4)$$

$$\Rightarrow 1+2V^2 = Bx^4$$

$$\Rightarrow 1+2\left(\frac{y}{x}\right)^2 = Bx^4$$

$$\Rightarrow x^2+2y^2 = Bx^6$$

APPLY CONDITION $\left(1, \frac{1}{\sqrt{2}}\right)$

$$\Rightarrow 1+1 = B$$

$$\Rightarrow B = 2$$

$$\therefore\ x^2+2y^2 = 2x^6$$

$$2y^2 = 2x^6 - x^2$$

$$y^2 = x^6 - \frac{1}{2}x^2$$

AS REQUIRED

IYGB - FP3 PAPER J - QUESTION 7

a) NOTING THAT $1+\tan^2\theta \equiv \sec^2\theta$ WE HAVE

$$y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = 1 + \tan^2 x$$

$$\frac{dy}{dx} = 1 + y^2$$

DIFFERENTIATE AGAIN WITH RESPECT TO x

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(1+y^2\right)$$

$$\frac{d^2y}{dx^2} = 0 + 2y\frac{dy}{dx}$$

DIFFERENTIATE WITH RESPECT TO x ONCE MORE

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(2y\frac{dy}{dx}\right) \leftarrow \text{PRODUCT RULE}$$

$$\frac{d^3y}{dx^3} = 2y \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}(2y) \times \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx} \times \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$$

AS REQUIRED

IYGB - FP3 PAPER J - QUESTION 7

b) EVALUATE AT $x=\frac{\pi}{4}$

$y=\tan\frac{\pi}{4}=1$

$\frac{dy}{dx}=1+y^2=1+1=2$

$\frac{d^2y}{dx^2}=2y\frac{dy}{dx}=2\times1\times2=4$

$\frac{d^3y}{dx^3}=2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=2\times1\times4+2\times2^2=8+8=16$

HENCE WE NOW HAVE

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\dots$$

$$\tan x=1+\left(x-\frac{\pi}{4}\right)\times2+\frac{\left(x-\frac{\pi}{4}\right)^2}{2}\times4+\frac{\left(x-\frac{\pi}{4}\right)^3}{6}\times16+\dots$$

$$\tan x=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+\frac{8}{3}\left(x-\frac{\pi}{4}\right)^3+\dots$$

c) LET $x=\frac{5}{18}\pi$ IN THE ABOVE EXPANSION

FIRST $\frac{5\pi}{18}-\frac{\pi}{4}=\frac{\pi}{36}$

$$\therefore\ \tan\frac{5\pi}{18}\approx1+2\times\frac{\pi}{36}+2\times\left(\frac{\pi}{36}\right)^2+\frac{8}{3}\left(\frac{\pi}{36}\right)^3$$

$$\tan\frac{5\pi}{18}\approx1+\frac{\pi}{18}+\frac{\pi^2}{648}+\frac{\pi^3}{17496}$$

AS REQUIRED

IYGB - FP3 PAPER J - QUESTION 8

a) DIFFERENTIATE IMPLICITLY WITH RESPECT TO x TO FIND GRADIENT AT $(4p^2, 8p)$

$$y^2 = 16x$$

$$2y\frac{dy}{dx} = 16$$

$$\frac{dy}{dx} = \frac{8}{y}$$

$$\left.\frac{dy}{dx}\right|_{y=8p} = \frac{8}{8p} = \frac{1}{p}$$

HENCE THE EQUATION OF THE TANGENT SATISFIES

$$y - 8p = \frac{1}{p}(x - 4p^2)$$

$$py - 8p^2 = x - 4p^2$$

$$py = x + 4p^2$$

AS REQUIRED

b) OBTAIN THE "PARTICULARS" OF THE PARABOLA

$$y^2 = 4(4x) \quad \Longrightarrow \quad \left\{\begin{array}{ll} \text{DIRECTRIX IS} & x = -a \\ \text{FOCUS IS AT} & F(a,0) \end{array}\right\}$$

("a" = 4)

$$\Longrightarrow \left\{\begin{array}{l} \therefore x = -4 \\ F(4,0) \end{array}\right\}$$

THE TANGENT MUST PASS THROUGH $A\left(-4, \frac{42}{5}\right)$

$$\Longrightarrow \frac{42}{5}p = -4 + 4p^2$$

$$\Longrightarrow 42p = -20 + 20p^2$$

$$\Longrightarrow 0 = 20p^2 - 42p - 20$$

$$\Longrightarrow 10p^2 - 21p - 10 = 0$$

$$\Longrightarrow (5p+2)(2p-5) = 0$$

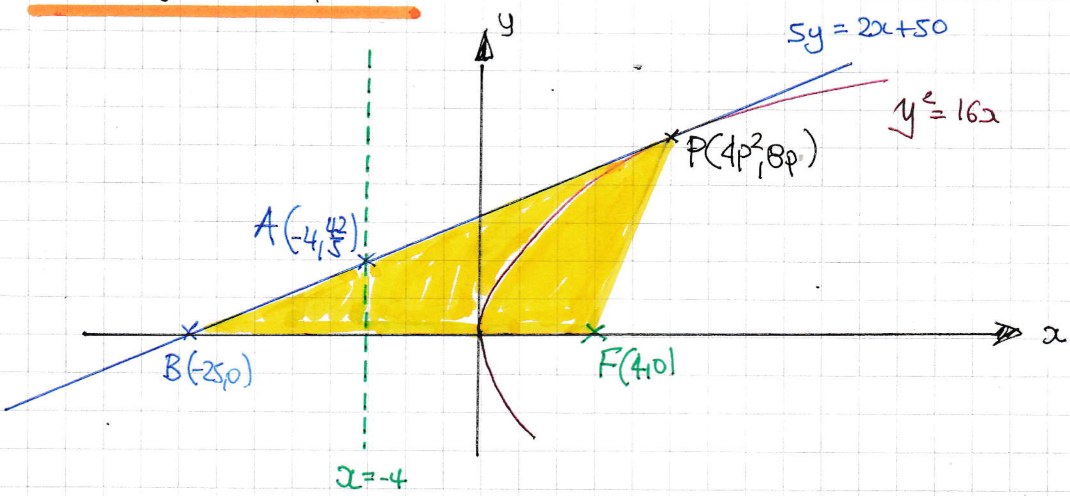
$$p = \begin{cases} 5/2 \\ \cancel{-2/5} \end{cases} \qquad p > 0$$

IYGB - FP3 PAPER J - QUESTION 8

HENCE THE EQUATION OF THE REQUIRED TANGENT IS

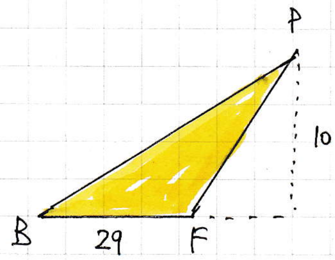
$$yp = x + 4p^2 \implies \frac{5}{2}y = x + 4\left(\frac{5}{2}\right)^2$$
$$\implies \frac{5}{2}y = x + 25$$
$$\implies \underline{\underline{5y = 2x + 50}}$$

DRAWING A DIAGRAM



THE x INTERCEPT OF THE TANGENT IS -25 (BY INSPECTION)

$$\text{AREA OF } \overset{\triangle}{FBP} = \frac{1}{2} \times \text{BASE} \times \text{HEIGHT}$$
$$= \frac{1}{2} \times |BF| \times 8p$$
$$= \frac{1}{2} \times 29 \times \left(8 \times \frac{5}{2}\right)$$
$$= \frac{1}{2} \times 29 \times 20$$
$$= \underline{\underline{290}}$$



IYGB - FP3 PAPER J - QUESTION 9

USING THE SUBSTITUTION GIVEN

$$t = \tan\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$$

$$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$$

$$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$$

$$dx = \frac{dt}{\frac{1}{2}(1+t^2)}$$

$$dx = \frac{2}{1+t^2}\,dt$$

CHANGING THE LIMITS

$$x=0 \longmapsto t=0$$

$$x=\frac{\pi}{2} \longmapsto t=1$$

OBTAIN AN EXPRESSION FOR $\sin x$ IN TERMS OF t BY ANY SUITABLE METHOD/MANIPULATION

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\sin x = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}} \times \cos^2\frac{x}{2}$$

$$\sin x = 2\tan\frac{x}{2} \times \frac{1}{\sec^2\frac{x}{2}}$$

$$\sin x = 2\tan\frac{x}{2} \times \frac{1}{1+\tan^2\frac{x}{2}}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\sin x = \frac{2t}{1+t^2}$$

TRANSFORMING THE INTEGRAL USING ALL THE RESULTS FROM ABOVE

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx = \ldots \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt = \int_0^1 \frac{2}{1+t^2+2t}\,dt$$

$$= \int_0^1 \frac{2}{t^2+2t+1}\,dt = \int_0^1 \frac{2}{(t+1)^2}\,dt$$

$$= \left[-\frac{2}{t+1}\right]_0^1 = \left[\frac{2}{t+1}\right]_1^0 = 2-1$$

$$= 1$$