IYGB – FP2 PAPER V – QUESTION 1

AUXILIARY EQUATION FOR THE O.D.E IS

$$\lambda^2 + 4\lambda + 13 = 0$$
$$(\lambda+2)^2 - 4 + 13 = 0$$
$$(\lambda+2)^2 = -9$$
$$\lambda + 2 = \pm 3i$$
$$\lambda = -2 \pm 3i$$

$\therefore$ GENERAL SOLUTION IS

$$y = e^{-2x}(A\cos 3x + B\sin 3x)$$

IYGB - FP2 PAPER V - QUESTION 2

a) $z^5 = i$

$|i| = 1$

$\arg i = \frac{\pi}{2}$

WORKING IN EXPONENTIALS

$\Rightarrow z^5 = 1 \times e^{\left(\frac{\pi}{2} + 2k\pi\right)i} \qquad k \in \mathbb{Z}$

$\Rightarrow z^5 = e^{\frac{\pi}{2}i(1+4k)}$

$\Rightarrow (z^5)^{\frac{1}{5}} = \left[e^{\frac{\pi}{2}i(1+4k)}\right]^{\frac{1}{5}}$

$\Rightarrow z = e^{i\frac{\pi}{10}(4k+1)}$

$k = 0, 1, 2, 3, 4$ OR WE MAY HAVE TO GO NEGATIVE

$k=0 \qquad z_0 = e^{i\frac{\pi}{10}}$

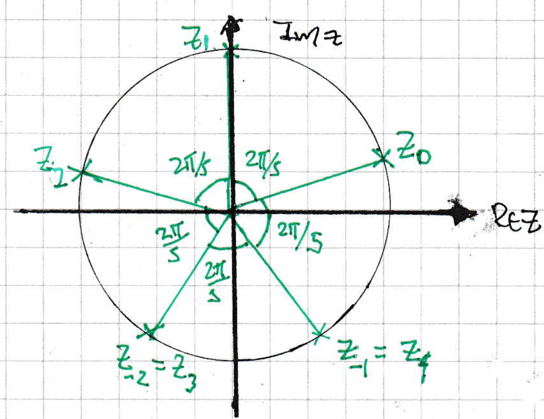
$k=1 \qquad z_1 = e^{i\frac{\pi}{2}}$

$k=2 \qquad z_2 = e^{i\frac{9\pi}{10}}$

$k=3 \qquad z_3 = e^{i\frac{13}{10}\pi} \qquad$ (OR $k=-2 \qquad z_{-2} = e^{-\frac{7}{10}\pi i}$)

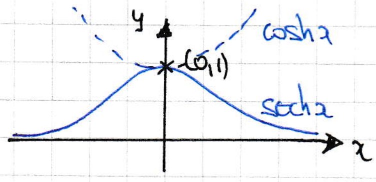
$k=4 \qquad z_4 = e^{i\frac{17\pi}{10}} \qquad$ (OR $k=-1 \qquad z_{-1} = e^{-\frac{3\pi}{10}i}$)

b) THE ROOTS ARE EQUALLY SPACED AND OF RADIUS 1



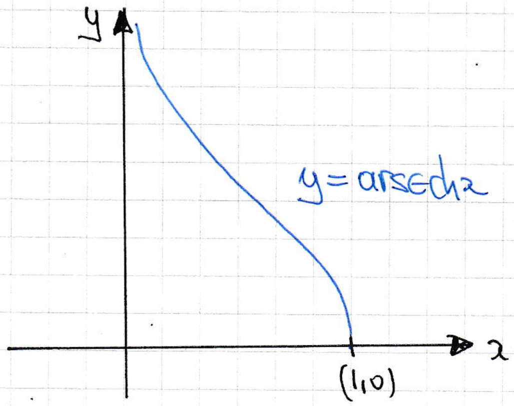
IYGB - FP2 PAPER V - QUESTION 3

a) STARTING WITH THE GRAPH OF $y = \text{sech}\,x$, REFLECTED IN $y = x$



$$y = \text{sech}\,x = \frac{1}{\cosh x}$$

PICKING FOR ONE TO ONE PURPOSES THE POSITIVE BRANCH



b) USING THE INVERSE RULE

$$y = \text{arsech}\,x$$
$$\text{sech}\,y = x$$
$$x = \text{sech}\,y$$
$$\frac{dx}{dy} = -\text{sech}\,y \tanh y$$
$$\frac{dy}{dx} = -\frac{1}{\text{sech}\,y \tanh y}$$
$$\frac{dy}{dx} = -\frac{1}{\text{sech}\,y \times \left(+\sqrt{1 - \text{sech}^2 x}\right)}$$

(PLUS AS THE GRADIENT MUST REMAIN NEGATIVE - SEE GRAPH)

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}}$$

IYGB - FP2 PAPER V - QUESTION 3

c) $$\int_{\frac{1}{2}}^{1} \text{arsech}\,x \, dx = \int_{\frac{1}{2}}^{1} 1 \times \text{arsech}\,x \, dx$$

BY PARTS

$\text{arsech}\,x$	$\frac{-1}{x\sqrt{1-x^2}}$
x	1

$$= \left[x\,\text{arsech}\,x\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} \frac{-x}{x\sqrt{1-x^2}} \, dx$$

$$= \left[x\,\text{arsech}\,x\right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= \left[x\,\text{arsech}\,x + \arcsin x\right]_{\frac{1}{2}}^{1}$$

$$\left(\cancel{\text{arsech}\,1} + \arcsin 1\right) - \left(\tfrac{1}{2}\text{arsech}\tfrac{1}{2} + \arcsin\tfrac{1}{2}\right)$$

FROM GRAPH

$$= \frac{\pi}{2} - \frac{1}{2}\text{arsech}\frac{1}{2} - \frac{\pi}{6}$$

$$= \frac{\pi}{3} - \frac{1}{2}\text{arsech}\,\frac{1}{2}$$

FINALLY WE HAVE

$$k = \text{arsech}\,\frac{1}{2}$$
$$\text{sech}\,k = \frac{1}{2}$$
$$\cosh k = 2$$
$$k = \text{arcosh}\,2$$
$$k = \ln(2+\sqrt{3})$$

$$\therefore \int_{\frac{1}{2}}^{1} \text{arsech}\,x \, dx = \frac{\pi}{3} - \frac{1}{2}\ln(2+\sqrt{3})$$

$$= \frac{1}{6}\left[2\pi - 3\ln(2+\sqrt{3})\right]$$

IE $\lambda = \frac{1}{6}$

IYGB - FP2 PAPER V - QUESTION 4

START BY REPLACING "0" AS THE BOTTOM LIMIT WITH k, $k>0$

$$\int_{k}^{\frac{\pi}{4}} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x}\,dx = \Big[\ln x - \tfrac{1}{2}\ln(1-\cos 2x)\Big]_{k}^{\pi/4}$$

$$= \frac{1}{2}\Big[2\ln x - \ln(1-\cos 2x)\Big]_{k}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\Big[\ln x^2 - \ln(1-\cos 2x)\Big]_{k}^{\pi/4}$$

$$= \frac{1}{2}\Big[\ln\Big(\frac{x^2}{1-\cos 2x}\Big)\Big]_{k}^{\pi/4}$$

EVALUATING AND TAKE LIMITS

$$= \frac{1}{2}\ln\Big(\frac{\pi^2/16}{1-\cancel{\cos\pi/2}}\Big) - \frac{1}{2}\ln\Big(\frac{k^2}{1-\cos 2k}\Big)$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\Bigg[\frac{k^2}{1-\big(1-\frac{4k^2}{2!}+\frac{16k^4}{4!}-\dots\big)}\Bigg]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\Bigg[\frac{k^2}{2k^2-\frac{2}{3}k^4+O(k^6)}\Bigg]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\Bigg(\frac{1}{2-\frac{2}{3}k^2+O(k^4)}\Bigg)$$

$$\therefore \int_{0}^{\pi/4} \frac{1}{x} - \frac{\sin 2x}{1-\cos 2x}\,dx = \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\lim_{k\to 0}\Bigg[\ln\Bigg(\frac{1}{2-\frac{2}{3}k^2+O(k^4)}\Bigg)\Bigg]$$

$$= \frac{1}{2}\ln\frac{\pi^2}{16} - \frac{1}{2}\ln\Big(\frac{1}{2}\Big)$$

$$= \ln\Big(\frac{\pi}{4}\Big) - \ln\Big(\frac{1}{2}\Big)^{\frac{1}{2}}$$

$$= \ln\frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}$$

$$= \ln\frac{\pi}{4} + \ln\frac{2}{\sqrt{2}}$$

$$= \ln\Big(\frac{\pi}{2\sqrt{2}}\Big)$$

$$= \ln\Big(\frac{\pi\sqrt{2}}{4}\Big)$$

ie $n=4$

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a) START BY OBTAINING THE GENERAL TERM IN SIGMA NOTATION

$$\frac{3}{1\times2} - \frac{5}{2\times3} + \frac{7}{3\times4} - \frac{9}{4\times5} + \frac{11}{5\times6} - \ldots \quad \sum_{r=1}^{\infty}\left[(-1)^{r+1}\frac{2r+1}{r(r+1)}\right]$$

IGNORING $(-1)^{r+1}$ EXPRESS THE REST INTO PARTIAL FRACTIONS BY COVER UP.

$$\frac{2r+1}{r(r+1)} = \frac{1}{r} + \frac{1}{r+1}$$

NOW WE HAVE

$$r=1 \qquad \frac{3}{1\times2} = \frac{1}{1} + \cancel{\frac{1}{2}}$$

$$r=2 \qquad -\frac{5}{2\times3} = -\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$

$$r=3 \qquad \frac{7}{3\times4} = \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}}$$

$$r=4 \qquad -\frac{9}{4\times5} = -\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$r=n \qquad (-1)^{n+1}\frac{2n+1}{n(n+1)} = \cancel{(-1)^{n+1}\frac{1}{n}} + (-1)^{n+1}\frac{1}{n+1}$$

$$\sum_{r=1}^{n}\left[(-1)^{r+1}\frac{2r+1}{r(r+1)}\right] = 1 + (-1)^{n+1}\frac{1}{n+1}$$

$\therefore$ AS $n \to \infty$ THE SUM TO INFINITY IS 1

b) LOOKING AT THE EXPANSION OF $\ln(1+x)$, VALID FOR $-1 < x \le 1$

- $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots = \sum_{r=1}^{\infty}\left[(-1)^{r+1}\frac{x^r}{r}\right]$

LET $x=1$

- $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}$

IYGB - FP2 PAPER V - QUESTION 5

USING THE PARTIAL FRACTIONS FROM PART (a)

$$\sum_{r=1}^{\infty}\left[\frac{(-1)^{r+1}(2r+1)}{r(r+1)}\right] = \sum_{r=1}^{\infty}\left[(-1)^{r+1}\left[\frac{1}{r}+\frac{1}{r+1}\right]\right]$$

$$= \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r} + \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1}$$

$$= \ln 2 + \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1}$$

RE-INDEXING AND MANIPULATING

$$= \ln 2 + \left[-\sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$= \ln 2 + \left[1 - 1 - \sum_{r=2}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$= \ln 2 + \left[1 - \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}\right]$$

$$= \ln 2 + (1 - \ln 2)$$

$$= 1$$

ALTERNATIVE TO RE-INDEXING & MANIPULATING

$$S = \sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r+1} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots$$

$$-S = -\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} \dots$$

$$1 - S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$$1 - S = \ln 2$$

$$S = 1 - \ln 2$$

AS ABOVE

IYGB – FP2 PAPER V – QUESTION 6

IT IS ACTUALLY BETTER TO WORK IN CARTESIAN, AT LEAST FOR THE SKETCH

$$r = \frac{1}{\cos\theta - \sin\theta}$$

$$r\cos\theta - r\sin\theta = 1$$

$$x - y = 1$$

$$y = x - 1$$

$$r = \cos\theta$$

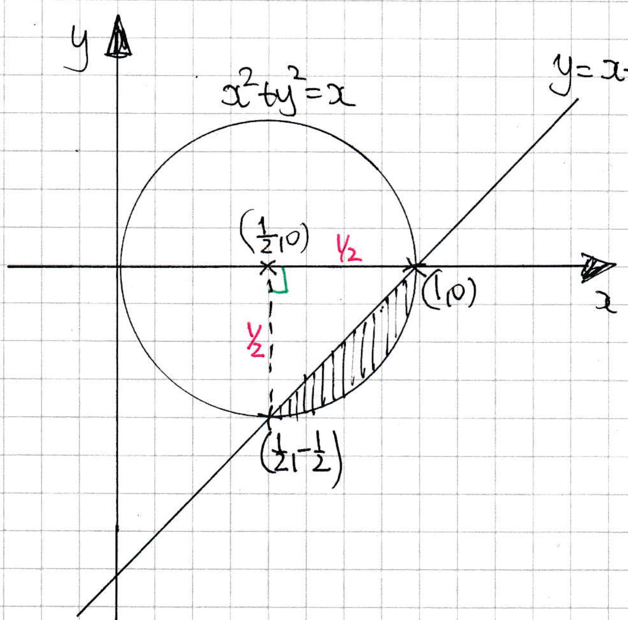
$$r^2 = r\cos\theta$$

$$x^2 + y^2 = x$$

$$x^2 - x + y^2 = 0$$

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}$$

DRAWING A SKETCH AND IDENTIFY THE REGION



$$\left.\begin{array}{l} y^2 + x^2 = x \\ y = x - 1 \end{array}\right\} \Rightarrow$$

$$(x-1)^2 + x^2 = x$$

$$x^2 - 2x + 1 + x^2 = x$$

$$2x^2 - 3x + 1 = 0$$

$$(2x-1)(x-1) = 0$$

$$x = \left\langle \begin{array}{l} 1 \\ \frac{1}{2} \end{array} \right. \quad y = \left\langle \begin{array}{l} 0 \\ -\frac{1}{2} \end{array} \right.$$

BY SIMPLE GEOMETRY THE AREA CAN BE FOUND

REQUIRED AREA = AREA OF QUARTER CIRCLE – AREA OF ISOSCELES, RIGHT ANGLE TRIANGLE

$$= \frac{1}{4} \times \pi \times \left(\frac{1}{2}\right)^2 - \frac{1}{2} \times \left(\frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{\pi}{16} - \frac{1}{8}$$

$$= \frac{\pi - 2}{16}$$

IYGB – FP2 PAPER V – QUESTION 7

- START BY A SUBSTITUTION

$$\int_0^{\frac{\pi}{4}} \frac{10}{2-\tan x}\,dx = \int_0^1 \frac{10}{2-u}\left(\frac{1}{1+u^2}\,du\right)$$

$$= \int_0^1 \frac{10}{(2-u)(1+u^2)}\,du$$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$dx = \frac{du}{\sec^2 x}$

$dx = \frac{du}{1+\tan^2 x}$

$dx = \frac{du}{1+u^2}$

$x = \frac{\pi}{4} \mapsto u = 1$

$x = 0 \mapsto u = 0$

- BY PARTIAL FRACTIONS

$$\frac{10}{(2-u)(1+u^2)} \equiv \frac{Au+B}{u^2+1} + \frac{C}{2-u}$$

$$\boxed{10 \equiv (2-u)(Au+B) + C(u^2+1)}$$

- If $u=2$

$10 = 5C$

$C = 2$

- If $u=0$

$10 = 2B + C$

$10 = 2B + 2$

$8 = 2B$

$B = 4$

- If $u=1$

$10 = A + B + 2C$

$10 = A + 4 + 4$

$A = 2$

- RETURNING TO THE INTEGRAL

$$\dots = \int_0^1 \frac{2u+4}{u^2+1} + \frac{2}{2-u}\,du = \int_0^1 \frac{4}{u^2+1} + \frac{2u}{u^2+1} + \frac{2}{2-u}\,du$$

$$= \Big[4\arctan u + \ln(u^2+1) - 2\ln|2-u|\Big]_0^1$$

$$= \left(4\arctan 1 + \ln 2 - \cancel{2\ln 1}\right) - \left(0 + \cancel{\ln 1} - 2\ln 2\right)$$

$$= 4 \times \frac{\pi}{4} + 3\ln 2$$

$$= \pi + 3\ln 2$$

IYGB – FP2 PAPER V – QUESTION 8

$$f(x) = 2\arcsin\sqrt{x} - \arcsin(2x-1) \qquad 0 \le 0 \le 1$$

DIFFERENTIATE W.R.T x USING $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$

$$f'(x) = 2 \times \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x}) - \frac{1}{\sqrt{1-(2x-1)^2}} \times \frac{d}{dx}(2x-1)$$

$$= \frac{2}{\sqrt{1-x}} \times \frac{1}{2}x^{-\frac{1}{2}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$

$$= \frac{2}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} - \frac{2}{\sqrt{1-4x^2+4x-1}}$$

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{4x-4x^2}}$$

$$= \frac{1}{\sqrt{x(1-x)}} - \frac{\cancel{2}}{2\sqrt{x-x^2}}$$

$$= \frac{1}{\sqrt{x-x^2}} - \frac{1}{\sqrt{x-x^2}}$$

$$= 0$$

AS THE DERIVATIVE IS 0, THE FUNCTION IS CONSTANT IN ITS DOMAIN

EVALUATING SAY AT $x=0$

$$f(0) = 2\arcsin 0 - \arcsin(-1)$$
$$= 0 - \left(-\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2}$$