IYGB - FP2 PAPER N - QUESTION 1

a) OBTAIN A COMMON DENOMINATOR AND ADD

$$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} = \frac{(r+2)-r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$

b) USING THE IDENTITY PROVEN IN PART (a)

$$\boxed{\frac{2}{r(r+1)(r+2)} \equiv \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}}$$

- IF $r=1$: $\frac{2}{1\times2\times3} = \frac{1}{1\times2} - \frac{1}{2\times3}$
- IF $r=2$: $\frac{2}{2\times3\times4} = \frac{1}{2\times3} - \frac{1}{3\times4}$
- IF $r=3$: $\frac{2}{3\times4\times5} = \frac{1}{3\times4} - \frac{1}{4\times5}$

$\vdots$

- IF $r=20$ $\frac{2}{20\times21\times22} = \frac{1}{20\times21} - \frac{1}{21\times22}$

$$\Rightarrow \sum_{r=1}^{20}\left[\frac{2}{r(r+1)(r+2)}\right] = \frac{1}{1\times2} - \frac{1}{21\times22}$$

ADDING

$$\Rightarrow 2\sum_{r=1}^{20}\left[\frac{1}{r(r+1)(r+2)}\right] = \frac{1}{2} - \frac{1}{462}$$

$$\Rightarrow \sum_{r=1}^{20}\left[\frac{1}{r(r+1)(r+2)}\right] = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{462}\right) = \frac{115}{462}$$

AS REQUIRED

IYGB – FP2 PAPER N – QUESTION 2

USING STANDARD EXPANSIONS

- $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O(x^5)$

$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + O(x^5)$

$e^{-2x} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)$

- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)$

$\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} + O(x^6)$

$\cos 4x = 1 - 8x^2 + \frac{32}{3}x^4 + O(x^6)$

COMBINING THESE RESULTS

$f(x) = e^{-2x}\cos 4x = (\cos 4x)(e^{-2x})$

$f(x) = \left[1 - 8x^2 + \frac{32}{3}x^4 + O(x^6)\right]\left[1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + O(x^5)\right]$

$$
\begin{aligned}
f(x) = 1 - 2x &+ 2x^2 - \tfrac{4}{3}x^3 + \tfrac{2}{3}x^4 + O(x^5)\\
&- 8x^2 + 16x^3 - 16x^4 + O(x^5)\\
&+ \tfrac{32}{3}x^4 + O(x^5)
\end{aligned}
$$

$f(x) = 1 - 2x - 6x^2 + \frac{44}{3}x^3 - \frac{14}{3}x^4 + O(x^5)$

IYGB - FP2 PAPER N - QUESTION 3

a) FACTORIZE THE DENOMINATOR FULLY

$$f(y) = \frac{4y}{y^4-1} = \frac{4y}{(y^2-1)(y^2+1)} = \frac{4y}{(y-1)(y+1)(y^2+1)}$$

NOW WE HAVE

$$\frac{4y}{(y-1)(y+1)(y^2+1)} \equiv \frac{A}{y-1} + \frac{B}{y+1} + \frac{Cx+D}{y^2+1}$$

$$\Longrightarrow \boxed{4y \equiv A(y+1)(y^2+1) + B(y-1)(y^2+1) + (y-1)(y+1)(Cx+D)}$$

- IF $y=1$: $4 = 4A$, $\underline{A=1}$
- IF $y=-1$: $-4 = -4B$, $\underline{B=1}$
- IF $y=0$: $0 = A - B - D$, $\underline{D=0}$
- IF $y=2$: $8 = 15A + 5B + 6C$, $8 = 15 + 5 + 6C$, $-12 = 6C$, $\underline{C=-2}$

$$\therefore f(y) = \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}$$

b)
$$\int_2^{\infty} f(y)\,dy = \int_2^{\infty} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy$$

$$= \lim_{a \to \infty}\left[\int_2^{a} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy\right]$$

$$= \lim_{a \to \infty}\left[\left[\ln(y-1) + \ln(y+1) - \ln(y^2+1)\right]_2^a\right]$$

$$= \lim_{a \to \infty}\left[\left[\ln\left(\frac{y^2-1}{y^2+1}\right)\right]_2^a\right]$$

IYGB – FP2 PAPER N – QUESTION 3

$$= \lim_{a \to \infty}\left[\ln\left(\frac{a^2-1}{a^2+1}\right) - \ln\left(\frac{3}{5}\right)\right]$$

$$= \ln 1 - \ln \frac{3}{5}$$

$$= \ln \frac{5}{3}$$

c) USING THE INTEGRATION OF PART (b) WE HAVE

MEAN OF $f(y)$, OVER $[2,4]$ $= \frac{1}{4-2}\int_2^4 f(y)\,dy$

$$= \frac{1}{2}\left[\ln\left(\frac{a^2-1}{a^2+1}\right) - \ln\left(\frac{3}{5}\right)\right]$$

WHERE $a=4$

$$= \frac{1}{2}\left[\ln \frac{15}{17} - \ln \frac{3}{5}\right]$$

$$= \frac{1}{2}\left[\ln \frac{15}{17} + \ln \frac{5}{3}\right]$$

$$= \frac{1}{2}\ln \frac{25}{17}$$

OR $\ln\left(\frac{5}{\sqrt{17}}\right)$

IYGB - FP2 PAPER N - QUESTION 4

a) WORKING IN EXPONENTIALS

$$z = e^{i\theta} \implies z^n = (e^{i\theta})^n$$
$$\implies z^n = e^{in\theta}$$
$$\implies z^{-n} = e^{-in\theta}$$

HENCE WE HAVE

I) $z^n + \frac{1}{z^n} = z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = 2\cosh(in\theta)$

$= 2\cos n\theta$

II) $z^n - \frac{1}{z^n} = z^n - z^{-n} = e^{in\theta} - e^{-in\theta} = 2\sinh(in\theta)$

$= 2i\sin n\theta$

OR USING TRIGONOMETRIC FUNCTIONS VIA EULER'S FORMULA

$$z^n + \frac{1}{z^n} = e^{in\theta} + e^{-in\theta} = (\cos n\theta + \cancel{i\sin n\theta}) + (\cos n\theta - \cancel{i\sin n\theta})$$
$$= 2\cos n\theta$$

& SIMILARLY THE OTHER

b) START BY NOTING THAT IF $n=1$

$$z + \frac{1}{z} = 2\cos\theta \quad \& \quad z - \frac{1}{z} = 2i\sin\theta$$

SUBSTITUTE & EXPAND BINOMIALLY

$$\implies \left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = (2\cos\theta)^4(2i\sin\theta)^2$$

IYGB - FP2 PAPER N - QUESTION 4

$$\Longrightarrow (16\cos^4\theta)(-4\sin^2\theta) = \left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^4$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = \left(z-\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^2\left(z+\frac{1}{z}\right)^2$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = \left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = \left(z^4-2+\frac{1}{z^4}\right)\left(z^2+2+\frac{1}{z^2}\right)$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = \begin{aligned} z^6 + 2z^4 &+ z^2 \\ -2z^2 &- 4 - \frac{2}{z^2} \\ &+ \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6} \end{aligned}$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = z^6 + 2z^4 - z^2 - 4 - \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = \left(z^6+\frac{1}{z^6}\right) + 2\left(z^4+\frac{1}{z^4}\right) - \left(z^2+\frac{1}{z^2}\right) - 4$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = (2\cos 6\theta) + 2(2\cos 4\theta) - (2\cos 2\theta) - 4$$

$$\Longrightarrow -64\cos^4\theta\sin^2\theta = -4 - 2\cos 2\theta + 4\cos 4\theta + 2\cos 6\theta$$

$$\Longrightarrow \cos^4\theta\sin^2\theta = \frac{1}{16} + \frac{1}{32}\cos 2\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 6\theta$$

AS REQUIRED

IYGB - FP2 PAPER N - QUESTION 5

$$f(x) = e^{2x+2}\left(e^{2x}-4\right), \; x \in \mathbb{R}$$

IF $x = \ln\left(2\cosh\frac{1}{2}\right)$

$$e^{2x} = e^{2\ln\left(2\cosh\frac{1}{2}\right)} = e^{\ln\left(2\cosh\frac{1}{2}\right)^2} = e^{\ln\left(4\cosh^2\frac{1}{2}\right)} = 4\cosh^2\frac{1}{2}$$

$$e^{2x+2} = e^2\left(4\cosh^2\frac{1}{2}\right) = 4e^2\cosh^2\frac{1}{2}$$

HENCE WE HAVE

$$f\left(\ln\left(2\cosh\frac{1}{2}\right)\right) = 4e^2\cosh^2\frac{1}{2}\left(4\cosh^2\frac{1}{2}-4\right)$$

$$= 16e^2\cosh^2\frac{1}{2}\left(\cosh^2\frac{1}{2}-1\right)$$

$\cosh^2 A - \sinh^2 A \equiv 1$

$$= 16e^2\cosh^2\frac{1}{2}\left(\sinh^2\frac{1}{2}\right)$$

$$= 4e^2\left(4\sinh^2\frac{1}{2}\cosh^2\frac{1}{2}\right)$$

$$= 4e^2\left(2\sinh\frac{1}{2}\cosh\frac{1}{2}\right)^2$$

$\sinh 2A \equiv 2\sinh A\cosh A$

$$= 4e^2\left(\sinh\left(2\times\frac{1}{2}\right)\right)^2$$

$$= 4e^2\sinh^2 1$$

$$= \left(2e\sinh 1\right)^2$$

$$= \left[\cancel{2}e \times \frac{1}{\cancel{2}}\left(e^1 - e^{-1}\right)\right]^2$$

$$= \left(e^2-1\right)^2$$

IYGB - FP2 PAPER N - QUESTION 6

a) CONDITION FOR A "HORIZONTAL TANGENT"

$$\frac{dy}{dx}=0 \Rightarrow \frac{dy/d\theta}{dx/d\theta}=0$$

$$\Rightarrow dy/d\theta = 0$$

$$\Rightarrow \frac{d}{d\theta}(y)=0$$

$$\Rightarrow \frac{d}{d\theta}(r\sin\theta)$$

$$\Rightarrow \frac{d}{d\theta}\left[4(1-\sin\theta)\sin\theta\right]=0$$

$$\Rightarrow \cancel{4}\frac{d}{d\theta}\left[\sin\theta-\sin^2\theta\right]=0$$

$$\Rightarrow \cos\theta - 2\sin\theta\cos\theta = 0$$

$$\Rightarrow \cos\theta(1-2\sin\theta)=0$$

HENCE THE SOLUTIONS FOR $0 \leq \theta \leq \pi$

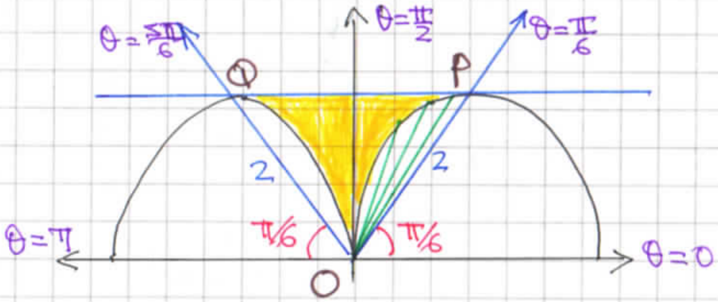
$$\cos\theta = 0 \Rightarrow \cancel{\theta = \pi/2}$$

$$\sin\theta = \frac{1}{2} \Rightarrow \theta = \begin{cases} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{cases}$$

$$\Rightarrow r = \begin{cases} 4(1-\frac{1}{2}) = 2 \\ 4(1-\frac{1}{2}) = 2 \end{cases}$$

$\therefore P(2, \frac{\pi}{6})$ & $Q(2, \frac{5\pi}{6})$ //

b) LOOKING AT THE DIAGRAM BELOW



$$\text{AREA OF } \overset{\triangle}{OPQ} = \frac{1}{2}|OP||OQ|\sin\left(\pi - 2\times\frac{\pi}{6}\right)$$

$$= \frac{1}{2}\times 2\times 2\times \sin\left(\frac{2\pi}{3}\right)$$

$$= \sqrt{3}$$

IYGB - FP2 PAPER N - QUESTION 6

AREA OF THE "GREEN" POLAR SECTORS FROM $\theta = \frac{\pi}{6}$ TO $\theta = \frac{\pi}{2}$

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta = \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}} \frac{1}{2}\left[4(1-\sin\theta)\right]^2 d\theta = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{2}} 8(1-\sin\theta)^2 \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8(1 - 2\sin\theta + \sin^2\theta)\, d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8 - 16\sin\theta + 8\sin^2\theta \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8 - 16\sin\theta + 8\left(\frac{1}{2} - \frac{1}{2}\cos2\theta\right) d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12 - 16\sin\theta - 4\cos2\theta \, d\theta = \left[12\theta + 16\cos\theta - 2\sin2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= (6\pi + 0 - 0) - (2\pi + 8\sqrt{3} - \sqrt{3}) = 4\pi - 7\sqrt{3}$$

THUS THE REQUIRED AREA IS GIVEN BY $\sqrt{3} - 2(4\pi - 7\sqrt{3})$

$$= \sqrt{3} - 8\pi + 14\sqrt{3}$$

$$= 15\sqrt{3} - 8\pi$$

AS REQUIRED

IYGB - FP2 PAPER N - QUESTION 7

START BY WRITING THE R.H.S OF THE EQUATION IN EXPONENTIAL FORM

$$|1+i\sqrt{3}| = \sqrt{1^2+\sqrt{3}^2} = \sqrt{1+3} = 2$$

$$\arg(1+i\sqrt{3}) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\boxed{1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}}$$

$$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\arg(1-i) = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

$$\boxed{1-i = \sqrt{2}e^{-i\frac{\pi}{4}}}$$

REWRITING THE EQUATION

$$z^3 = (1+i\sqrt{3})^8(1-i)^5$$

$$z^3 = \left[2e^{i\frac{\pi}{3}}\right]^8\left[\sqrt{2}e^{-i\frac{\pi}{4}}\right]^5$$ (IGNORE MULTIPLES OF 2π AT THIS STAGE)

$$z^3 = 2^8e^{i\frac{8\pi}{3}} \times 4\sqrt{2}e^{-i\frac{5\pi}{4}}$$

$$z^3 = 2^8 \times 2^2 \times 2^{\frac{1}{2}} \times e^{i\frac{17\pi}{12}}$$

$$z^3 = 2^{\frac{21}{2}}e^{i\frac{17\pi}{12}+2k\pi}, \quad k \in \mathbb{Z}$$ (INTRODUCE MULTIPLES OF 2π)

$$(z^3)^{\frac{1}{3}} = \left[2^{\frac{21}{2}}e^{i\frac{\pi}{12}(17+24k)}\right]^{\frac{1}{3}}$$

$$z = 2^{\frac{7}{2}}e^{i\frac{\pi}{36}(24k+17)}$$

COLLECTING THE SOLUTIONS FOR $-\pi < \theta \le \pi$

$z_0 = 8\sqrt{2}e^{i\frac{17}{36}\pi}$ $\quad$ $z_{-1} = 8\sqrt{2}e^{-i\frac{7\pi}{36}}$ $\quad$ $z_{-2} = 8\sqrt{2}e^{-i\frac{31\pi}{36}}$

IYGB - FP2 PAPER N - QUESTION 8

$$\frac{dx}{dt} = x + \frac{2}{3}y \qquad \frac{dy}{dt} = 3y - \frac{3}{2}x \qquad \begin{array}{l} t=0 \\ x=1 \\ y=3 \end{array}$$

DIFFERENTIATE THE FIRST O.D.E WITH RESPECT TO t

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3}\frac{dy}{dt}} \text{ — I}$$

SUBSTITUTE THE SECOND O.D.E. INTO THE ABOVE EXPRESSION

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3}\left(3y - \frac{3}{2}x\right)$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2y - x} \text{ — II}$$

REARRANGE THE FIRST O.D.E

$$\Rightarrow \frac{dx}{dt} = x + \frac{2}{3}y$$

$$\Rightarrow 3\frac{dx}{dt} = 3x + 2y$$

$$\Rightarrow \boxed{2y = 3\frac{dx}{dt} - 3x} \text{ — III}$$

COMBINING (II) & (III) WE OBTAIN

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + \left(3\frac{dx}{dt} - 3x\right) + x$$

IYGB - FP2 PAPER N - QUESTION 8

$$\Rightarrow \frac{d^2x}{dt^2} = 4\frac{dx}{dt} - 4x$$

$$\Rightarrow \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$$

AUXILIARY EQUATION FOR THE ABOVE O.D.E

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda = 2 \quad \text{(REPEATED)}$$

GENERAL SOLUTION FOR $x = f(t)$

$$\Rightarrow x = f(t) = Ae^{2t} + Bte^{2t}$$

$$\Rightarrow x = f(t) = e^{2t}(A + Bt)$$

APPLY CONDITION $t=0$, $x=1$ YIELDS $A=1$

$$\Rightarrow \boxed{x = f(t) = e^{2t}(1 + Bt)}$$

NOW DIFFERENTIATE x & SUB INTO THE FIRST O.D.E

$$\Rightarrow \frac{dx}{dt} = 2e^{2t}(1+Bt) + Be^{2t}$$

$$= e^{2t}(2 + B + 2Bt)$$

$$\Rightarrow e^{2t}(2+B+2Bt) = e^{2t}(1+Bt) + \frac{2}{3}y$$

$$\frac{dx}{dt} \quad = \quad x \quad + \quad \frac{2}{3}y$$

IYGB - FP2 PAPER N - QUESTION 8

$$\Rightarrow \frac{2}{3}y = e^{2t}(2+B+2Bt) - e^{2t}(1+Bt)$$

$$\Rightarrow \frac{2}{3}y = e^{2t}(1+B+Bt)$$

$$\Rightarrow \boxed{y = \frac{3}{2}e^{2t}(1+B+Bt)}$$

FINALLY APPLY THE CONDITION, $t=0$ $y=3$

$$\Rightarrow 3 = \frac{3}{2}(1+B)$$

$$\Rightarrow 2 = B+1$$

$$\Rightarrow B = 1$$

$$x = f(t) = e^{2t}(1+t)$$

$$y = g(t) = \frac{3}{2}e^{2t}(2+t)$$