IYGB – FP2 PAPER K – QUESTION 1

PARALLEL TO THE INITIAL LINE IMPLIES $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = 0 \quad \Rightarrow \frac{dy}{d\theta} = 0$$

$$\Rightarrow \frac{d}{d\theta}(y) = 0$$

$$\Rightarrow \frac{d}{d\theta}(r\sin\theta) = 0$$

$$\Rightarrow \frac{d}{d\theta}\left((1+2\cos\theta)\sin\theta\right) = 0$$

$$\Rightarrow \frac{d}{d\theta}(\sin\theta + \sin 2\theta) = 0$$

$$\Rightarrow \cos\theta + 2\cos 2\theta = 0$$

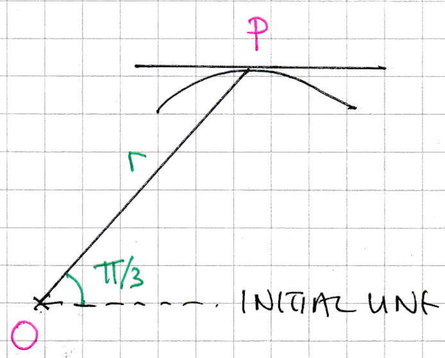
$$\Rightarrow \cos\theta + 2(2\cos^2\theta - 1) = 0$$

$$\Rightarrow 4\cos^2\theta + \cos\theta - 2 = 0$$

$$\therefore \cos\theta = \frac{-1 \pm \sqrt{33}}{8}$$

$$\cos\theta = \frac{-1+\sqrt{33}}{8} \quad \left(0 < \theta < \frac{\pi}{2}\right)$$

LOOKING AT THE DIAGRAM



$$\therefore |OP| = 1 + 2\cos\theta$$

$$= 1 + 2\left(\frac{-1+\sqrt{33}}{8}\right)$$

$$= 1 + \frac{-1+\sqrt{33}}{4}$$

$$= \frac{3+\sqrt{33}}{4}$$

IYGB - FP2 PAPER K - QUESTION 2

a)

$$f(x) = \sinh x \cos x + \sin x \cosh x$$

$$f'(x) = \cosh x \cos x + \sinh x(-\sin x) + \cos x \cosh x + \sin x \sinh x$$

$$f'(x) = 2\cosh x \cos x$$

b)

WORK WITH "SINES & COSINES"

$$\int \frac{2}{\tanh x + \tan x}\,dx = \int \frac{2}{\frac{\sinh x}{\cosh x} + \frac{\sin x}{\cos x}}\,dx$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION BY $\cos x \cosh x$

$$= \int \frac{2\cos x \cosh x}{\sinh x \cos x + \sin x \cosh x}\,dx$$

WHICH IS OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$

$$= \ln\left|\sinh x \cos x + \cosh x \sin x\right| + C$$

IYGB - FP2 PAPER K - QUESTION 3

a) USING THE SUBSTITUTION $u=\sqrt{x}$

$u=\sqrt{x}$

$x=u^2$

$dx=2u\,du$

& LIMITS

$x=1 \longmapsto u=1$

$x=4 \longmapsto u=2$

$$\int_1^4 \frac{3}{(x+9)\sqrt{x}}\,dx = \int_1^2 \frac{3}{(u^2+9)\cancel{u}}(2\cancel{u}\,du) = \int_1^2 \frac{6}{u^2+9}\,du$$

A STANDARD INTEGRAL

$$\dots = \frac{1}{3}\times 6\times\left[\arctan\left(\frac{u}{3}\right)\right]_1^2 = 2\left[\arctan\frac{2}{3}-\arctan\frac{1}{3}\right]$$

b) USING THE IDENTITY $\tan(A-B)\equiv\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$\tan\left(\arctan\frac{2}{3}-\arctan\frac{1}{3}\right) = \frac{\tan\left(\arctan\frac{2}{3}\right)-\tan\left(\arctan\frac{1}{3}\right)}{1+\tan\left(\arctan\frac{2}{3}\right)\tan\left(\arctan\frac{1}{3}\right)}$$

$$= \frac{\frac{2}{3}-\frac{1}{3}}{1+\frac{2}{3}\times\frac{1}{3}} = \frac{\frac{1}{3}}{1+\frac{2}{9}} = \frac{3}{11}$$

$\therefore\ \arctan\frac{2}{3}-\arctan\frac{1}{3}=\arctan\frac{3}{11}$

$\therefore\ I = 2\arctan\frac{3}{11}$

AS REQUIRED

IYGB - FP2 PAPER K - QUESTION 4

AUXILLIARY EQUATION FOR THE L.H.S OF THE O.D.E.

$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda-2)(\lambda-1) = 0$$
$$\lambda = \begin{cases} 1 \\ 2 \end{cases}$$

COMPLEMENTARY FUNCTION

$$y = Ae^{x} + Be^{2x}$$

PARTICULAR INTEGRAL, TRY $y = P\cos 2x + Q\sin 2x$

$$\frac{dy}{dx} = -2P\sin 2x + 2Q\cos 2x$$
$$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$$

SUB INTO THE O.D.E.

$$\frac{d^2y}{dx^2} = -4P\cos 2x - 4Q\sin 2x$$
$$-3\frac{dy}{dx} = -6Q\cos 2x + 6P\sin 2x$$
$$+2y = 2P\cos 2x + 2Q\sin 2x$$

$$(-2P-6Q)\cos 2x + (6P-2Q)\sin 2x \equiv 20\sin 2x$$

SOLVING SIMULTANEOUS EQUATIONS

- $-2P - 6Q = 0$
  $-6Q = 2P$
  $P = -3Q$

- $6P - 2Q = 20$
  $6(-3Q) - 2Q = 20$
  $-20Q = 20$
  $Q = -1$ & $P = 3$

IYGB - FP2 PAPER K - QUESTION 4

HENCE THE GENERAL SOLUTION IS

$$y = Ae^{x} + Be^{2x} + 3\cos 2x - \sin 2x$$

APPLY CONDITIONS

$(0,1) \implies \quad 1 = A + B + 3$

$A + B = -2$

$$\frac{dy}{dx} = Ae^{x} + 2Be^{2x} - 6\sin 2x - 2\cos 2x$$

$-5 = A + 2B - 2$

$-3 = A + 2B$

$$\therefore \left.\begin{aligned} A &= -2 - B \\ A &= -3 - 2B \end{aligned}\right\} \implies \begin{aligned} -2 - B &= -3 - 2B \\ B &= -1 \end{aligned}$$

$\& \quad A = -1$

$$\therefore \quad y = 3\cos 2x - \sin 2x - e^{x} - e^{2x}$$

IYGB - FP2 PAPER K - QUESTION 5

a) BY INSPECTION (COVER UP)

$$f(r) = \frac{2}{r(r+1)(r+2)} = \frac{\frac{2}{1\times 2}}{r} + \frac{\frac{2}{-1\times 1}}{r+1} \quad \frac{\frac{2}{(-2)(-1)}}{r+2}$$

$$f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

b) USING PART (a)

$$\boxed{f(r) \equiv \frac{2}{r(r+1)(r+2)} \equiv \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}}$$

- $r=1 \quad f(1) = \frac{2}{1\times2\times3} = \frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}$
- $r=2 \quad f(2) = \frac{2}{2\times3\times4} = \frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}$
- $r=3 \quad f(3) = \frac{2}{3\times4\times5} = \cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$
- $r=4 \quad f(4) = \frac{2}{4\times5\times6} = \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}$
- $r=5 \quad f(5) = \frac{2}{5\times6\times7} = \cancel{\frac{1}{5}} - \cancel{\frac{2}{6}} + \cancel{\frac{1}{7}}$

$\vdots \qquad \vdots \qquad \vdots$

- $r=n-1 \quad f(n-1) = \frac{2}{(n-1)n(n+1)} = \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \frac{1}{n+1}$
- $r=n \quad f(n) = \frac{2}{n(n+1)(n+2)} = \cancel{\frac{1}{n}} - \frac{2}{n+1} + \frac{1}{n+2}$

$$\Longrightarrow \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\Longrightarrow \sum_{r=1}^{n} f(r) = \frac{1}{2} + \frac{-1(n+2)+1(n+1)}{(n+1)(n+2)}$$

IYGB - FP2 PAPER K - QUESTION 5

$$\sum_{r=1}^{n} f(r) = \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

AS REQUIRED

c)

$$\frac{1}{5\times6\times7} + \frac{1}{6\times7\times8} + \frac{1}{7\times8\times9} + \dots = \sum_{r=5}^{\infty} \frac{1}{r(r+1)(r+2)}$$

$$\Longrightarrow \sum_{r=1}^{\infty} \frac{2}{r(r+1)(r+2)} = \frac{1}{2} \qquad \left(\text{AS } n \to \infty \quad \frac{1}{(n+1)(n+2)} \to 0\right)$$

$$\Longrightarrow \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Longrightarrow \underset{\uparrow\, r=1}{\frac{1}{1\times2\times3}} + \underset{\uparrow\, r=2}{\frac{1}{2\times3\times4}} + \underset{\uparrow\, r=3}{\frac{1}{3\times4\times5}} + \underset{\uparrow\, r=4}{\frac{1}{4\times5\times6}} + \sum_{r=5}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Longrightarrow \frac{1}{6} + \frac{1}{24} + \frac{1}{60} + \frac{1}{120} + \sum_{r=5}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

$$\Longrightarrow \sum_{r=5}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{60}$$

IYGB - FP2 PAPER K - QUESTION 6

$$y = 2x \arcsin 2x + (1-4x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2\arcsin 2x + 2x \times \frac{1}{\sqrt{1-4x^2}} \times 2 + \frac{1}{2}(1-4x^2)^{-\frac{1}{2}}(-8x)$$

$$\frac{dy}{dx} = 2\arcsin 2x + \cancel{\frac{4x}{\sqrt{1-4x^2}}} - \cancel{\frac{4x}{\sqrt{1-x^2}}}$$

DIFFERENTIATE AGAIN W.R.T x.

$$\Rightarrow \frac{d^2y}{dx^2} = 2 \times \frac{1}{\sqrt{1-4x^2}} \times 2 = \frac{4}{\sqrt{1-4x^2}}$$

$$\Rightarrow \sqrt{1-4x^2}\,\frac{d^2y}{dx^2} = 4$$

NOW REARRANGING THE ORIGINAL EQUATION

$$(1-4x)^{\frac{1}{2}} = y - 2x\arcsin 2x$$

$$\Rightarrow (y - 2x\arcsin x)\frac{d^2y}{dx^2} = 4$$

BUT

$$\frac{dy}{dx} = 2\arcsin 2x$$

$$\Rightarrow \left(y - x\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = 4$$

DIFFERENTIATE AGAIN W.R.T x

$$\left[\cancel{\frac{dy}{dx}} - 1 \times \cancel{\frac{dy}{dx}} - x\frac{d^2y}{dx^2}\right]\frac{d^2y}{dx^2} + \left[y - x\frac{dy}{dx}\right]\frac{d^3y}{dx^3} = 0$$

$$\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = x\left(\frac{d^2y}{dx^2}\right)^2$$

AS REQUIRED

IYGB – FP2 PAPER K – QUESTION 6

ALTERNATIVE / VARIATION

FROM PREVIOUS METHOD ....

$$\frac{dy}{dx} = 2\arcsin 2x \qquad \frac{d^2y}{dx^2} = 4(1-4x^2)^{-\frac{1}{2}}$$

DIFFERENTIATE ONCE MORE

$$\frac{d^3y}{dx^3} = -2(1-4x^2)^{-\frac{3}{2}}(-8x) = 16x(1-4x^2)^{-\frac{3}{2}}$$

NOW THE LHS GIVES

$$\left(y - x\frac{dy}{dx}\right)\frac{d^3y}{dx^3} = \left[\cancel{2x\arcsin 2x} + (1-4x^2)^{\frac{1}{2}} - \cancel{2x\arcsin 2x}\right]16x(1-4x^2)^{-\frac{3}{2}}$$

$$= 16x(1-4x^2)^{-1} = \frac{16x}{1-4x^2}$$

AND THE RHS GIVES

$$x\left(\frac{d^2y}{dx^2}\right)^2 = x\left(4(1-4x^2)^{-\frac{1}{2}}\right)^2 = x\left[16(1-4x^2)^{-1}\right]$$

$$= \frac{16x}{1-4x^2}$$

INDEED WE OBTAIN

$$\frac{d^3y}{dx^3}\left(y - x\frac{dy}{dx}\right) = x\left(\frac{d^2y}{dx^2}\right)^2 = \frac{16x}{1-4x^2}$$

IYGB - FP2 PAPER K - QUESTION 7

PROCEED AS FOLLOWS

$$\int_0^{\infty} \frac{2}{1+2x} - \frac{x}{1+x^2}\,dx$$

$$= \lim_{k\to\infty}\left[\int_0^{k} \frac{2}{1+2x} - \frac{x}{1+x^2}\,dx\right]$$

$$= \lim_{k\to\infty}\left[\Big[\ln(1+2x) - \tfrac{1}{2}\ln(1+x^2)\Big]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\Big[2\ln(1+2x) - \ln(1+x^2)\Big]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\left[\ln\left(\frac{(1+2x)^2}{1+x^2}\right)\right]_0^k\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\frac{(1+2k)^2}{1+k^2} - \cancel{\tfrac{1}{2}\ln 1}\right]$$

$$= \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\frac{4k^2+4k+1}{k^2+1}\right] = \lim_{k\to\infty}\left[\tfrac{1}{2}\ln\left(\frac{4+\frac{4}{k}+\frac{1}{k^2}}{1+\frac{1}{k^2}}\right)\right]$$

$$= \tfrac{1}{2}\ln 4$$

$$= \ln 2$$

IYGB - FP2 PAPER K - QUESTION 8

a) THIS IS A QUADRATIC IN z^3 (QUADRATIC FORMULA)

$$z^3 = \frac{-8 \pm \sqrt{8^2-4\times1\times64}}{2} = \frac{-8 \pm \sqrt{-3\times64}}{2}$$

$$= \frac{-8 \pm \sqrt{64}\sqrt{-3}}{2} = \frac{-8 \pm 8\sqrt{-3}}{2} = \frac{-8 \pm 8\sqrt{3}i}{2}$$

$$= -4 \pm 4\sqrt{3}i$$

b) USING EXPONENTIAL NOTATION FOR $w = -4 + 4\sqrt{3}i$

$$z^3 = 8e^{i\left(\frac{2\pi}{3}+2n\pi\right)}$$

$$z^3 = 8e^{i\frac{2\pi}{3}(1+3n)}$$

$$z = \left[8e^{i\frac{2\pi}{3}(1+3n)}\right]^{\frac{1}{3}}$$

$$z = 2e^{i\frac{2\pi}{9}(1+3n)}$$

- $|-4+4\sqrt{3}i| = 8$
- $\arg(-4+4\sqrt{3}i)$
  $= \pi + \arctan\left(\frac{4\sqrt{3}}{-4}\right)$
  $= \pi + \arctan(-\sqrt{3})$
  $= \frac{2\pi}{3}$

$$\therefore z = 2e^{i\frac{2\pi}{9}},\ 2e^{i\frac{8\pi}{9}},\ 2e^{-\frac{4\pi}{9}i}$$

& THEIR CONJUGATES FROM $-4-4\sqrt{3}i$

$$z = 2e^{-i\frac{2\pi}{9}},\ 2e^{-i\frac{8\pi}{9}},\ 2e^{\frac{4\pi}{9}i}$$

c)I) USING RELATIONSHIPS OF ROOTS

$$\text{SUM OF SIX ROOTS} = -\frac{\text{coeff of } z^5}{\text{coeff of } z^6} = 0$$

II) AS THE SUM OF ROOTS IS ZERO, REGROUP

$$2e^{i\frac{2\pi}{9}} + 2e^{-i\frac{2\pi}{9}} + 2e^{i\frac{8\pi}{9}} + 2e^{-i\frac{8\pi}{9}} + 2e^{\frac{4\pi}{9}i} + 2e^{-\frac{4\pi}{9}i} = 0$$

$$2\left[e^{\frac{2\pi}{9}i} + e^{-\frac{2\pi}{9}i}\right] + 2\left[e^{\frac{8\pi}{9}i} + e^{-\frac{8\pi}{9}i}\right] + 2\left[e^{\frac{4\pi}{9}i} + e^{-\frac{4\pi}{9}i}\right] = 0$$

IYGB – FP2 PAPER K – QUESTION 8

$$\Rightarrow 4\cosh\frac{2\pi i}{9} + 4\cosh\frac{8\pi i}{9} + 4\cosh\frac{4\pi i}{9} = 0$$

$$\Rightarrow 4\left[\cos\frac{2\pi}{9} + \cos\frac{8\pi}{9} + \cos\frac{4\pi}{9}\right] = 0$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = \cos\frac{6\pi}{9}$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = \cos\frac{2\pi}{3}$$

$$\Rightarrow \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}$$

AS REQUIRED

ALTERNATIVE INSTEAD OF USING HYPERBOLICS IN

$$2e^{i\frac{2\pi}{9}} + 2e^{-i\frac{2\pi}{9}} + 2e^{\frac{8\pi}{9}i} + 2e^{-i\frac{8\pi}{9}i} + 2e^{\frac{4\pi}{9}i} + 2e^{-i\frac{4\pi}{9}} = 0$$

IS TO WRITE IN TRIGONOMETRIC FORM & SET REAL PARTS EQUAL TO ZERO