IYGB – FP1 PAPER L – QUESTION 1

AS THE QUADRATIC IS SYMMETRICAL CONSIDER THE REVOLUTION BY 2π OF HALF THE AREA (OR THE ENTIRE AREA BY π)

$$y = 4 - x^2 = (2-x)(x+2)$$

$$\Rightarrow V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx$$

$$\Rightarrow V = \pi \int_0^2 (4-x^2)^2 \, dx$$

$$\Rightarrow V = \pi \int_0^2 16 - 8x^2 + x^4 \, dx$$

$$\Rightarrow V = \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^2$$

$$\Rightarrow V = \pi \left[\left(32 - \frac{64}{3} + \frac{32}{5} \right) - 0 \right]$$

$$\therefore V = \frac{256}{15}\pi$$

IYGB - FP1 PAPER L - QUESTION 2

START BY FINDING AN EXPRESSION FOR THE SUM OF THE FIRST n TERM (NOTE THAT k=1 YIELDS ZERO)

$$\sum_{k=2}^{n}\left[(k-1)(k+2)\right] = \sum_{k=1}^{n}\left[k^2+k-2\right]$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k - 2\sum_{k=1}^{n} 1$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) - 2\times n$$

$$= \frac{1}{6}n\left[(n+1)(2n+1) + 3(n+1) - 12\right]$$

$$= \frac{1}{6}n\left[2n^2+3n+1+3n+3-12\right]$$

$$= \frac{1}{6}n(2n^2+6n-8)$$

$$= \frac{1}{3}n(n^2+3n-4)$$

$$= \frac{1}{3}n(n-1)(n+4)$$

NOW LETTING n=16

$$\sum_{k=2}^{16}\left[(k-1)(k+2)\right] = \frac{1}{3}\times16\times15\times20 = 1600$$

IYGB - FP1 PAPER L - QUESTION 3

a)

- $$\underline{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- $\det \underline{A} = 1$

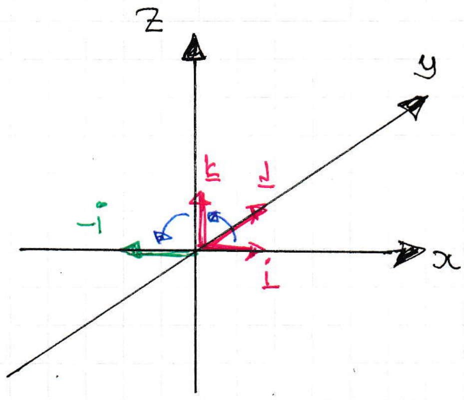
(NO REFLECTION) IS INVOLVED

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$$

$$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underline{j}$$

$$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\underline{i}$$

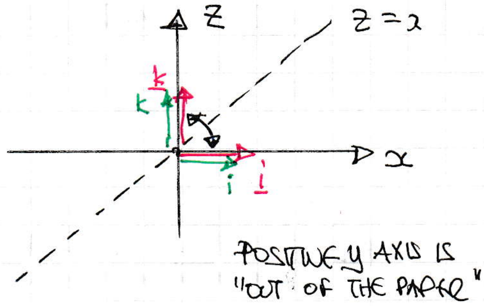
- LOOKING AT THE AXES



$\therefore$ ROTATION BY $90°$, CLOCKWISE, ABOUT THE y AXIS

b)

- LOOKING AT THE AXES, FROM THE POSITIVE y-AXIS



IE

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$$

$$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\underline{j}$$

$$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underline{i}$$

$$\therefore \underline{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

NOTE THAT THE ABOVE SET OF AXES IS IN 2-D, WHERE y IS "OUT" OF THE PAPER - ON ROTATION BY $180°$, THE y AXIS WILL BE "INTO THE PLANE OF THE PAPER"

IYGB - FP1 PAPER L - QUESTION 3

c) • FIND THE MATRIX FOR THE COMPOSITION

$$\underline{C} = \underline{A}\,\underline{B} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

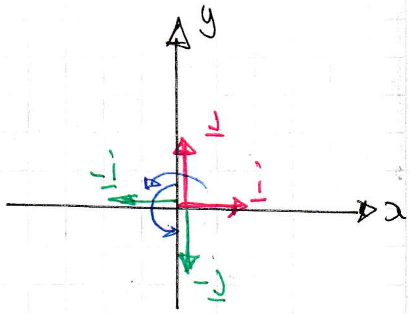
• $\det \underline{C} = +1$ (NO REFLECTION)

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} = -\underline{i}$$

$$\underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\underline{j}$$

$$\underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \underline{k}$$

• LOOKING AT A SET OF AXES FROM THE POSITIVE Z AXIS, "STICKING OUT OF THE PAPER"



ROTATION ABOUT THE Z AXIS, BY 180°

IYGB – FP1 PAPER L – QUESTION 4

START BY WRITING THE EQUATIONS AS A "3 TERM QUADRATIC" IN z

$$\Rightarrow z^2 - z + 8 + 2(z+1)i = 0$$

$$\Rightarrow z^2 - z + 8 + 2zi + 2i = 0$$

$$\Rightarrow z^2 + (-1+2i)z + (8+2i) = 0$$

USING THE QUADRATIC FORMULA

$$\Rightarrow z = \frac{-(-1+2i) \pm \sqrt{(-1+2i)^2 - 4 \times 1 \times (8+2i)}}{2 \times 1}$$

$$\Rightarrow z = \frac{1-2i \pm \sqrt{1-4i-4-32-8i}}{2}$$

$$\Rightarrow z = \frac{1-2i \pm \sqrt{-35-12i}}{2}$$

NOW WE NEED TO EVALUATE THE SQUARE ROOT

$$\Rightarrow (a+bi)^2 \equiv -35-12i$$

$$\Rightarrow a^2 + 2abi - b^2 \equiv -35 - 12i$$

$$\Rightarrow \left(\begin{array}{l} a^2 - b^2 = -35 \\ ab = -6 \end{array} \right) \Rightarrow b = -\frac{6}{a}$$

$$\Rightarrow a^2 - \left(\frac{-6}{a}\right)^2 = -35$$

$$\Rightarrow a^2 - \frac{36}{a^2} = -35$$

$$\Rightarrow a^4 - 36 = -35a$$

$$\Rightarrow a^4 + 35a - 36 = 0$$

IYGB - FP1 PAPER L - QUESTION 4

$$\Longrightarrow \left(a^2+36\right)\left(a^2-1\right)=0$$

$$\Longrightarrow a^2 = \begin{cases} \cancel{-36} \\ 1 \end{cases}$$

$$\Longrightarrow a = \begin{cases} 1 \\ -1 \end{cases} \qquad b = \frac{-6}{a} = \begin{cases} -6 \\ 6 \end{cases}$$

RETURNING TO THE QUADRATIC FORMULA

$$z = \frac{1-2i \pm (1-6i)}{2}$$

$$z = \begin{cases} \dfrac{1-2i+1-6i}{2} = \dfrac{2-8i}{2} = 1-4i \\[2ex] \dfrac{1-2i-1+6i}{2} = \dfrac{4i}{2} = 2i \end{cases}$$

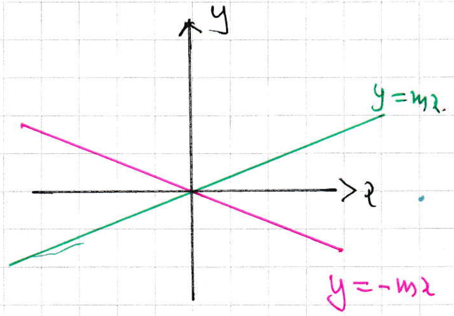
THE REQUIRED SOLUTIONS ARE

$$z_1 = 1-4i \quad \text{OR} \quad z_2 = 2i$$

IYGB – FP1 PAPER L – QUESTION 5

LET THE REQUIRED LINE HAVE EQUATION $y = mx$

LOOKING AT THE DIAGRAM THE REFLECTED LINE WILL HAVE EQUATION $y = -mx$



$$\Longrightarrow \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2x + mx \\ -6x + 3mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$$

$$\Longrightarrow \left.\begin{array}{l} 2x + mx = X \\ -6x + 3mx = -mX \end{array}\right\}$$ DIVIDING

$$\frac{2+m}{-6+3m} = \frac{1}{-m}$$

$$-6 + 3m = -2m - m^2$$

$$m^2 + 5m - 6 = 0$$

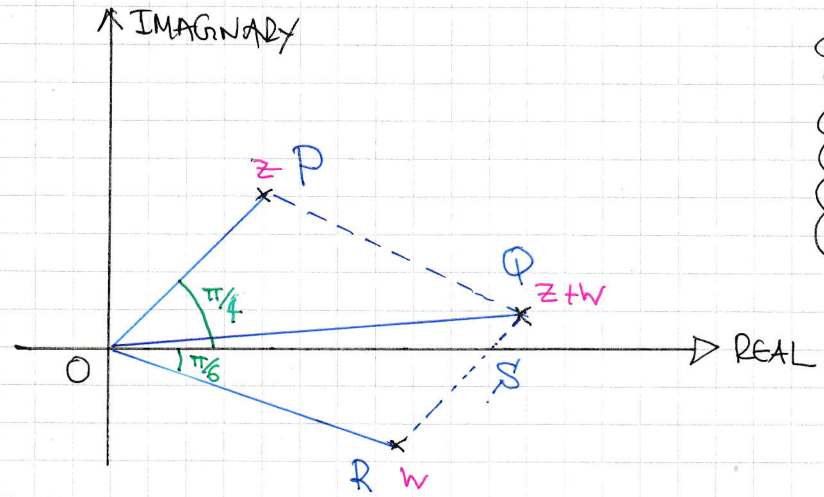
$$(m-1)(m+6) = 0$$

$$\therefore m = \begin{cases} 1 \\ -6 \end{cases}$$

$$\therefore \quad y = x \quad \text{OR} \quad y = -6x$$

IYGB – FP1 PAPER L – QUESTION 6

SKETCHING THE REQUIRED POINTS



$z = \sqrt{2} + \sqrt{2}\,i$

$w = \sqrt{3} - i$

$z + w = (\sqrt{2} + \sqrt{3}) + (\sqrt{2} - 1)i$

- $\arg z = P\hat{O}S$

  $\arctan\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = P\hat{O}S$

  $P\hat{O}S = \frac{\pi}{4}$

- $|\arg w| = S\hat{O}R$

  $\left|\arctan\left(\frac{\sqrt{3}}{-1}\right)\right| = S\hat{O}R$

  $S\hat{O}R = \pi/6$

NOW WE HAVE BY ORDINARY GEOMETRY

$P\hat{O}R = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$

$Q\hat{O}R = \frac{1}{2} \times \frac{5\pi}{12} = \frac{5\pi}{24}$ (RHOMBUS GEOMETRY)

$Q\hat{O}S = \frac{5\pi}{24} - \frac{\pi}{6} = \frac{\pi}{24}$

THUS THE ANGLE $Q\hat{O}S$ IS THE ARGUMENT OF $z+w$

$\Rightarrow \arg(z+w) = \frac{\pi}{24}$

$\Rightarrow \arctan\left(\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}\right) = \frac{\pi}{24}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}$

$\Rightarrow \tan\frac{\pi}{24} = \frac{\sqrt{6} - 2 - \sqrt{3} + \sqrt{2}}{3-2}$

$\therefore \tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2$

IYGB - FP1 PAPER L - QUESTION 7

IF $ax^3+bx^2+cx+d=0$

- $\alpha+\beta+\gamma = -\frac{b}{a}$ — I
- $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$ — II
- $\alpha\beta\gamma = -\frac{d}{a}$ — III

BUT TWO ROOTS, WITHOUT LOSS OF GENERALITY α & β MULTIPLY TO 1

(III) $\alpha\beta\gamma = -\frac{d}{a}$

$\gamma = -\frac{d}{a}$

SUBSTITUTE INTO II & I

- $\alpha+\beta-\frac{d}{a} = -\frac{b}{a}$

  $\alpha+\beta = \frac{d-b}{a}$

- $1+\gamma(\alpha+\beta) = \frac{c}{a}$

  $1+\left(-\frac{d}{a}\right)(\alpha+\beta) = \frac{c}{a}$

COMBINING WE OBTAIN

$$1-\frac{d}{a}\left(\frac{d-b}{a}\right) = \frac{c}{a}$$

$$1-\frac{d(d-b)}{a^2} = \frac{c}{a}$$

$$a^2-d(d-b) = ca$$

$$a^2-d^2+bd = ac$$

$$a^2-d^2 = ac-bd$$

AS REQUIRED

IYGB - FP1 PAPER L - QUESTION 8

ESTABLISH A BASE CASE

$$\text{LHS} = \sum_{r=1}^{1} \frac{3r+2}{r(r+1)(r+2)} = \frac{3\times1+2}{1\times2\times3} = \frac{5}{6}$$

$$\text{RHS} = \frac{1\times(2\times1+3)}{2\times3} = \frac{1\times5}{6} = \frac{5}{6}$$

THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n=k\in\mathbb{N}$

$$\Longrightarrow \sum_{r=1}^{k} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)}$$

$(k+1)^{\text{TH}}$ TERM

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)} + \frac{3(k+1)+2}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)}{(k+1)(k+2)} + \frac{3k+5}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{k(2k+3)(k+3)+(3k+5)}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3+9k^2+9k+3k+5}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^3+9k^2+12k+5}{(k+1)(k+2)(k+3)}$$

NOW WE "EXPECT" THAT $(k+1)$ IS A FACTOR FOR THE INDUCTION TO WORK

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^2(k+1)+7k(k+1)+5(k+1)}{(k+1)(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{\cancel{(k+1)}(2k^2+7k+5)}{\cancel{(k+1)}(k+2)(k+3)}$$

IYGB – FP1 PAPER L – QUESTION 8

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{2k^2+7k+5}{(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(2k+5)(k+1)}{(k+2)(k+3)}$$

$$\Longrightarrow \sum_{r=1}^{k+1} \frac{3r+2}{r(r+1)(r+2)} = \frac{(k+1)\left[2(k+1)+3\right]}{\left[(k+1)+1\right]\left[(k+1)+2\right]}$$

IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL n

IYGB - FP1 PAPER L - QUESTION 9

a) WRITING THE EQUATIONS OF THE LINES IN PARAMETRIC

$$\Gamma_1 = (5,3,1) + \lambda(1,1,1) = (\lambda+5, \lambda+3, \lambda+1)$$
$$\Gamma_2 = (-3,4,8) + \mu(2,-1,-3) = (2\mu-3, 4-\mu, 8-3\mu)$$

EQUATE $\underline{i}$ & $\underline{j}$ COMPONENTS

$\underline{i}$: $\lambda+5 = 2\mu-3$

$\underline{j}$: $\lambda+3 = 4-\mu$

SUBTRACTING $\Rightarrow 2 = 3\mu - 7$

$\Rightarrow 9 = 3\mu$

$\Rightarrow \underline{\mu = 3}$

$\Rightarrow \lambda + 3 = 4 - 3$

$\Rightarrow \underline{\lambda = -2}$

CHECKING $\underline{k}$

- $\lambda+1 = -2+1 = -1$
- $8-3\mu = 8-3\times3 = -1$

AS ALL 3 COMPONENTS AGREE WHEN $\lambda=-2$ & $\mu=3$, THE LINES INTERSECT AT SOME POINT

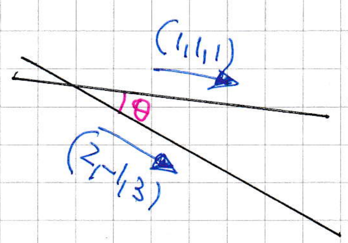
USING $\lambda=-2$ OR $\mu=3$ WE OBTAIN $P(3,1,-1)$

b) DOTTING THE DIRECTION VECTORS

$\Rightarrow (1,1,1)\cdot(2,-1,3) = |1,1,1||2,-1,3|\cos\theta$

$\Rightarrow 2-1-3 = \sqrt{1+1+1}\sqrt{4+1+9}\cos\theta$

$\Rightarrow -2 = \sqrt{3}\sqrt{14}\cos\theta$



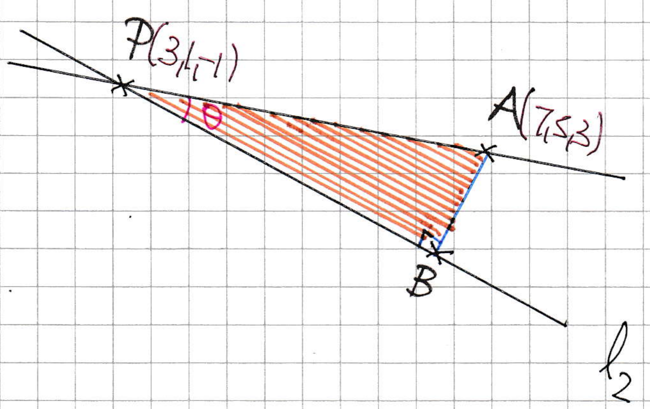
IYGB - FP1 PAPER L - QUESTION 9

$$\Rightarrow -\frac{2}{\sqrt{42}} = \cos\theta$$

$$\Rightarrow \theta \approx 107.9752838\ldots$$

$\therefore$ ACUTE ANGLE $\simeq 72.0°$

c) START WITH A DIAGRAM



- $|\overrightarrow{AP}| = |p - a| = |(3,1,-1)-(7,5,3)| = |-4,-4,-4| = \sqrt{16+16+16} = \sqrt{48}$
- $|AB| = |AP|\sin\theta$
- $|PB| = |AP|\cos\theta$
- $\text{AREA} = \frac{1}{2}|AB||PB| = \frac{1}{2}|AP|\sin\theta \times |AP|\cos\theta$

$$= \frac{1}{4}|AP|^2 \sin 2\theta$$

$$= \frac{1}{4} \times 48 \times \sin(2 \times 72.0\ldots°)$$

$$= 7.045044575\ldots$$

$$\simeq 7.05$$