IYGB - FP1 PAPER K - QUESTION 1

a) OBTAIN DIRECTION VECTOR FROM $A(5,1,6)$ & $B(2,2,1)$

$\vec{AB} = \underline{b} - \underline{a} = (2,2,1) - (5,1,6) = (-3,1,-5)$

"SCALE IT" TO HAVE LESS NEGATIVES

$\vec{AB} = (3,-1,5)$

$\Rightarrow \underline{r} = (5,1,6) + \lambda(3,-1,5)$

$\Rightarrow \underline{r} = (3\lambda+5, 1-\lambda, 5\lambda+6)$

b) REWRITE THE TWO LINES INCLUDING $\underline{r}_2 = (6,6,-4) + \mu(4,-2,3)$

- $\underline{r}_1 = (3\lambda+5, 1-\lambda, 5\lambda+6)$
- $\underline{r}_2 = (4\mu+6, 6-2\mu, 3\mu-4)$

$\left.\begin{array}{ll} \underline{i} & 3\lambda+5 = 4\mu+6 \\ \underline{j} & 1-\lambda = 6-2\mu \end{array}\right\} \Rightarrow \quad \lambda = 2\mu-5$

$\therefore \; 3(2\mu-5)+5 = 4\mu+6$

$6\mu - 15 + 5 = 4\mu + 6$

$2\mu = 16$

$\mu = 8$

$\lambda = 2\times8-5$

$\lambda = 11$

CHECKING $\underline{k}$ FOR THESE VALUES

$5\lambda+6 = 5\times11+6 = 61$

$3\mu+4 = 3\times8-4 = 20$

$61 \neq 20$

LINES NOT PARALLEL & NOT INTERSECTING, SO SKEW

IYGB - FP1 PAPER K - QUESTION 2

- IF THE TRANSFORMATION IS NOT INVERTIBLE, THE MATRIX $\underline{A}$ IS NOT INVERTIBLE, SO $\det \underline{A} = 0$ — EXPAND BY THE FIRST ROW

$$\Rightarrow |\underline{A}| = 0$$

$$\Rightarrow \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} a & 1 & 2 \\ 2 & -1 & a \\ 3 & a & 4 \end{vmatrix} = 0$$

$$\Rightarrow a\begin{vmatrix} -1 & a \\ a & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & a \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 3 & a \end{vmatrix} = 0$$

$$\Rightarrow a(-4-a^2) - (8-3a) + 2(2a+3) = 0$$

$$\Rightarrow \cancel{-4a} - a^3 - 8 + 3a + \cancel{4a} + 6 = 0$$

$$\Rightarrow 0 = a^3 - 3a + 2$$

- BY INSPECTION $a=1$ IS A SOLUTION — BY LONG DIVISION (OR MANIPULATIO[N]

$$\Rightarrow a^2(a-1) + a(a-1) - 2(a-1) = 0$$

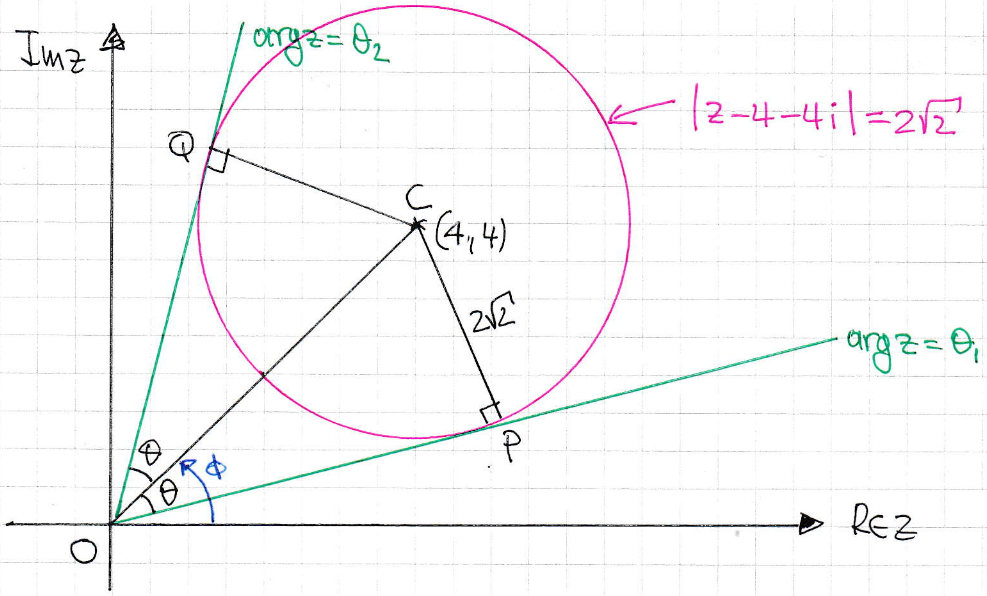
$$\Rightarrow (a-1)(a^2+a-2) = 0$$

$$\Rightarrow (a-1)(a-1)(a+2) = 0$$

$$\Rightarrow a = \begin{cases} 1 \\ -2 \end{cases}$$ (REPEATED)

IYGB – FP1 PAPER K – QUESTION 3

STARTING WITH THE SKETCH



- $|OC| = \sqrt{4^2+4^2} = \sqrt{32} = 4\sqrt{2}$
- $\sin\theta = \frac{2\sqrt{2}}{4\sqrt{2}} = \frac{1}{2}$
- $\theta = \pi/6$
- $\sin\phi = \frac{4}{4\sqrt{2}} = \frac{1}{\sqrt{2}}$
- $\phi = \frac{\pi}{4}$

HENCE WE NOW HAVE

$$\theta_1 = \phi - \theta = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$$

$$\theta_2 = \phi + \theta = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$$

$$\therefore \quad \frac{\pi}{12} \leqslant \arg z \leqslant \frac{5\pi}{12}$$

IYGB – FP1 PAPER K – QUESTION 4

SPLIT THE SUM INTO INDIVIDUAL COMPONENTS

$$\sum_{n=1}^{k}\left(18n^2+28n+5\right) = \sum_{n=1}^{k}(18n^2) + \sum_{n=1}^{k}(28n) + \sum_{n=1}^{k}5$$

$$= 18\sum_{n=1}^{k}n^2 + 28\sum_{n=1}^{k}n + 5\sum_{n=1}^{k}1$$

USING STANDARD RESULTS

- $\sum_{n=1}^{k}n^2 = \frac{1}{6}k(k+1)(2k+1)$
- $\sum_{n=1}^{k}n = \frac{1}{2}k(k+1)$
- $\sum_{n=1}^{k}1 = k$

SIMPLIFYING

$$\dots = 18\times\frac{1}{6}k(k+1)(2k+1) + 28\times\frac{1}{2}k(k+1) + 5\times k$$

$$= 3k(k+1)(2k+1) + 14k(k+1) + 5k$$

$$= k\left[3(k+1)(2k+1) + 14(k+1) + 5\right]$$

$$= k\left[6k^2+9k+3+14k+14+5\right]$$

$$= k\left[6k^2+23k+22\right]$$

$$= k(k+2)(6k+11)$$

IYGB - FP1 - PAPER K - QUESTION 5

WRITE THE PLANE IN "PARAMETRIC COMPACT FORM"

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2\lambda+\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix}$$

TRANSFORM THE VECTOR VIA THE MATRIX $\underline{A}$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1+2\lambda+\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix} = \begin{pmatrix} 1+2\lambda+\mu+2+4\mu \\ 1+2\lambda+\mu+2+\lambda \\ 1+2\lambda+\mu+4+2\lambda+1+2\mu \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 3+2\lambda+5\mu \\ 3+3\lambda+\mu \\ 6+4\lambda+3\mu \end{pmatrix}$$

ELIMINATE THE PARAMETERS λ & μ — USE $\underline{1}$: $\mu = Y-3-3\lambda$ AND SUBSTITUTE INTO THE OTHER TWO

$$\left\{\begin{array}{l} X = 3+2\lambda+5(Y-3-3\lambda) \\ Z = 6+4\lambda+3(Y-3-3\lambda) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} X = 3+2\lambda+5Y-15-15\lambda \\ Z = 6+4\lambda+3Y-9-9\lambda \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} X = -12-13\lambda+5Y \\ Z = -3-5\lambda+3Y \end{array}\right\} \begin{array}{l} \times 5 \\ \times(-13) \end{array}$$

$$\Rightarrow \left\{\begin{array}{l} 5X = -60-65\lambda+25Y \\ -13Z = 39+65\lambda-39Y \end{array}\right\}$$

ADDING THE EQUATIONS

$$5X-13Z = -21-14Y$$

$$\therefore \ 5X+14Y-13Z = -21$$

OR $\quad 5x+14y-13z+21=0$

IYGB - FP1 PAPER J - QUESTION 5

ALTERNATIVE METHOD

FIND 3 POINTS ON THE PLANE

$$\begin{pmatrix} 1+2\lambda+2\mu \\ 2+\lambda \\ 1+2\mu \end{pmatrix}$$

$\lambda=0, \mu=0 \quad A(1,2,1)$

$\lambda=0, \mu=1 \quad B(3,2,3)$

$\lambda=1, \mu=0 \quad C(3,3,1)$

TRANSFORM THESE THREE POINTS

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}\begin{array}{c}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \\ A \quad B \quad C\end{array} = \begin{array}{c}\begin{pmatrix} 3 & 8 & 5 \\ 3 & 4 & 6 \\ 6 & 9 & 10 \end{pmatrix} \\ A' \quad B' \quad C'\end{array}$$

FIND TWO VECTORS WHICH LIE ON THE TRANSFORMED PLANE

$\overrightarrow{A'B'} = \underline{b}' - \underline{a}' = (8,4,9) - (3,3,6) = (5,1,3)$

$\overrightarrow{A'C'} = \underline{c}' - \underline{a}' = (5,6,10) - (3,3,6) = (2,3,4)$

FIND THE PERPENDICULAR (NORMAL) BY CROSS PRODUCT

$$\underline{n} = \overrightarrow{A'B'} \wedge \overrightarrow{A'C'} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = (-5, -14, 13)$$

EQUATION OF THE TRANSFORMED PLANE IS

$$-5x - 14y + 13z = \text{constant}$$

USING ONE OF THE TRANSFORMED POINTS SAY $A'(3,3,6)$ TO FIND THE CONSTANT

$$(-5\times3) - 14\times3 + 13\times6 = 21$$

$$\therefore \quad -5x - 14y + 13z = 21$$

$$5x + 14y - 13z = -21$$

$$5x + 14y - 13z + 21 = 0$$

AS BEFORE

IYGB – FP1 PAPER K – QUESTION 6

LET $z = a+bi$, WHERE $a \in \mathbb{R}$, $b \in \mathbb{R}$

$$\Rightarrow z^2 = 21 - 20i$$
$$\Rightarrow (a+bi)^2 = 21 - 20i$$
$$\Rightarrow a^2 + 2abi - b^2 = 21 - 20i$$

EQUATE REAL AND IMAGINARY PARTS

$$\left.\begin{array}{l} a^2 - b^2 = 21 \\ 2ab = -20 \end{array}\right\} \Rightarrow \boxed{b = -\frac{10}{a}}$$

$$\Rightarrow a^2 - \left(-\frac{10}{a}\right)^2 = 21$$
$$\Rightarrow a^2 - \frac{100}{a^2} = 21$$
$$\Rightarrow a^4 - 100 = 21a^2$$
$$\Rightarrow a^4 - 21a^2 - 100 = 0$$
$$\Rightarrow (a^2+4)(a^2-25) = 0$$
$$\Rightarrow a^2 = \begin{cases} 25 \\ \cancel{-4} \end{cases} \quad a \in \mathbb{R}$$
$$\Rightarrow a = \begin{cases} 5 \\ -5 \end{cases} \quad \& \quad b = \begin{cases} -2 \\ 2 \end{cases}$$

$$\therefore z = \begin{cases} 5 - 2i \\ -5 + 2i \end{cases}$$

IYGB - FP1 PAPER K - QUESTION 7

LET $f(n) = 5^{n-1} + 11^{n}$, $n \in \mathbb{N}$

ESTABLISH A BASE CASE

$f(1) = 5^{0} + 11^{1} = 1 + 11 = 12 = 2 \times 6$

I.E THE RESULT HOLDS FOR $n=1$

SUPPOSE THAT THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, IE $f(k) = 6A$, $A \in \mathbb{N}$

$\Rightarrow f(k+1) - f(k) = \left[5^{(k+1)-1} + 11^{k+1}\right] - \left[5^{k-1} + 11^{k}\right]$

$\Rightarrow f(k+1) - 6A = 5^{k} + 11^{k+1} - 5^{k-1} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 5^{1} \times 5^{k-1} + 11 \times 11^{k} - 5^{k-1} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 5 \times 5^{k-1} - 5^{k-1} + 11 \times 11^{k} - 11^{k}$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10 \times 11^{k}$

BUT WE ALSO HAVE
$f(k) = 6A$
$5^{k-1} + 11^{k} = 6A$
$11^{k} = 6A - 5^{k-1}$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 10\left[6A - 5^{k-1}\right]$

$\Rightarrow f(k+1) - 6A = 4 \times 5^{k-1} + 60A - 10 \times 5^{k-1}$

$\Rightarrow f(k+1) = 66A - 6 \times 5^{k-1}$

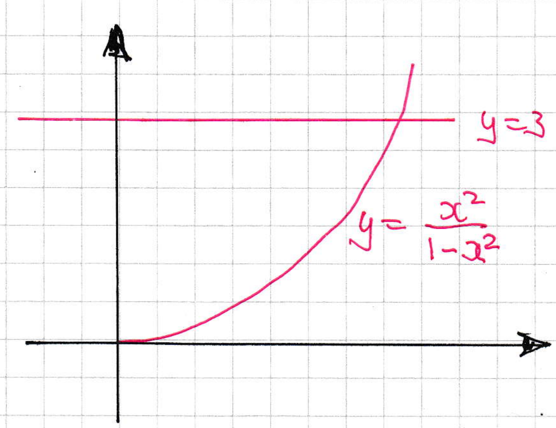
$\Rightarrow f(k+1) = 6\left[11A - 5^{k-1}\right]$

IF THE RESULT HOLDS FOR $n = k \in \mathbb{N}$, THEN IT MUST ALSO HOLD FOR $n = k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

IYGB – FP1 PAPER K – QUESTION 8

LOOKING AT THE DIAGRAM



$$y=\frac{x^2}{1-x^2}$$
$$y-x^2y=x^2$$
$$y=x^2+x^2y$$
$$y=x^2(1+y)$$
$$x^2=\frac{y}{1+y}$$

SETTING UP A VOLUME INTEGRAL ABOUT y

$$\Longrightarrow V=\pi\int_{y_1}^{y_2}\left[x(y)\right]^2\,dy$$

$$\Longrightarrow V=\pi\int_0^3\frac{y}{1+y}\,dy$$

OR THE SUBSTITUTION $u=1+y$

$$\Longrightarrow V=\pi\int_0^3\frac{(1+y)-1}{1+y}\,dy$$

$$\Longrightarrow V=\pi\int_0^3 1-\frac{1}{y+1}\,dy$$

$$\Longrightarrow V=\pi\left[y-\ln|y+1|\right]_0^3$$

$$\Longrightarrow V=\pi\left[(3-\ln 4)-(0-\cancel{\ln 1})\right]$$

$$\Longrightarrow V=\pi(3-\ln 4)$$

IYGB - FP1 PAPER K - QUESTION 9

LET THE "BEFORE" CO-ORDINATES BE (x,y) AND THE "AFTER" CO-ORDINATES BE DENOTED BY (X,Y)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{ie } \underline{X} = \underline{A}\,\underline{x}$$

$$\underline{A}^{-1}\underline{X} = \underline{A}^{-1}\underline{A}\,\underline{x}$$

$$\underline{A}^{-1}\underline{X} = \underline{x}$$

$$\underline{A}^{-1} = \frac{1}{(4\times2-2\times3)}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}$$

HENCE WE NOW HAVE

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4X-3Y \\ -2X+2Y \end{pmatrix} = \begin{pmatrix} 2X-\frac{3}{2}Y \\ -X+Y \end{pmatrix}$$

SUBSTITUTE INTO THE CIRCLE & TIDY

$$\Rightarrow x^2+y^2 = 4$$

$$\Rightarrow \left(2X-\tfrac{3}{2}Y\right)^2+(Y-X)^2 = 4$$

$$\left.\begin{array}{l}\Rightarrow 4X^2-6XY+\frac{9}{4}Y^2 \\ \Rightarrow X^2-2XY+Y^2\end{array}\right\} = 4$$

$$\Rightarrow 5X^2-8XY+\tfrac{13}{4}Y^2 = 4$$

$$\Rightarrow 20X^2-32XY+13Y^2 = 16$$

$$\text{ie } 20x^2-32xy+13y^2 = 16$$

IYGB – FP1 PAPER K – QUESTION 10

FIRSTLY IN THE USUAL NOTATION FOR THE GIVEN CUBIC

$$\alpha+\beta+\gamma = -\frac{b}{a} = 4$$
$$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} = 2$$
$$\alpha\beta\gamma = -5$$

NOW FOR THE REQUIRED CUBIC

$$A+B+C = \frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}+\frac{\alpha\beta}{\gamma} = \frac{\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2}{\alpha\beta\gamma} = \frac{(\beta\gamma)^2+(\gamma\alpha)^2+(\alpha\beta)^2}{\alpha\beta\gamma}$$

NOW WE HAVE FOR THE NUMERATOR

$$(\beta\gamma+\gamma\alpha+\alpha\beta)^2 = (\beta\gamma)^2+(\gamma\alpha)^2+(\alpha\beta)^2+2\left[\beta\gamma^2\alpha+\alpha^2\beta\gamma+\alpha\beta^2\gamma\right]$$
$$2^2 = (\beta\gamma)^2+(\gamma\alpha)^2+(\alpha\beta)^2+2\alpha\beta\gamma(\gamma+\alpha+\beta)$$
$$4 = (\beta\gamma)^2+(\gamma\alpha)^2+(\alpha\beta)^2+2(-5)\times 4$$
$$(\beta\gamma)^2+(\gamma\alpha)^2+(\alpha\beta)^2 = -36$$

$$\therefore\ A+B+C = \frac{-36}{5}$$

NOW THE SUM IN PAIRS

$$AB+BC+CA = \frac{\alpha\beta\gamma^2}{\alpha\beta}+\frac{\alpha^2\beta\gamma}{\beta\gamma}+\frac{\alpha\beta^2\gamma}{\alpha\gamma} = \gamma^2+\alpha^2+\beta^2$$
$$= (\alpha+\beta+\gamma)^2-\left[2\alpha\beta+2\beta\gamma+2\gamma\alpha\right]$$
$$= (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$= 16-2\times 2$$
$$= 12$$

$$\therefore\ AB+BC+CA = 12$$

FINALLY THE PRODUCT OF THREE

$$ABC = \frac{\alpha^2\beta^2\gamma^2}{\alpha\beta\gamma} = \alpha\beta\gamma = -5$$

$$\therefore\ ABC = -5$$

IYGB - FP1 PAPER K - QUESTION 10

HENCE THE REQUIRED CUBIC IS

$$x^3 - (A+B+C)x^2 + (AB+BC+CA)x - (ABC) = 0$$

$$x^3 - \left(-\frac{36}{5}x^2\right) + 12x - 5 = 0$$

$$x^3 + \frac{36}{5}x^2 + 12x - 5 = 0$$

$$5x^3 + 36x^2 + 60x - 25 = 0$$