C4, IYGB, PAPER U

1. a) $(1+12x)^{\frac{3}{4}} = 1 + \frac{\frac{3}{4}}{1}(12x)^1 + \frac{\frac{3}{4}(-\frac{1}{4})}{1\times 2}(12x)^2 + \frac{\frac{3}{4}(-\frac{1}{4})(-\frac{5}{4})}{1\times 2\times 3}(12x)^3 + O(x^4)$

$= 1 + 9x - \frac{27}{2}x^2 + \frac{135}{2}x^3 + O(x^4)$

b) VALID FOR $|12x| < 1$

$|x| < \frac{1}{12}$ $\quad \therefore -\frac{1}{12} < x < \frac{1}{12}$

c) $1 + 12x = \frac{53}{50}$

$12x = \frac{3}{50}$

$x = \frac{1}{200} = 0.005$

SUB INTO THE EXPANSION

$\left[1 + 12(0.005)\right]^{\frac{3}{4}} \approx 1 + 9(0.005) - \frac{27}{2}(0.005)^2 + \frac{135}{2}(0.005)$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.044670938$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467$

2. $\int_0^1 \frac{10x^4}{2x^{\frac{5}{2}}+1}\,dx = \ldots$ BY SUBSTITUTION

$= \int_1^3 \frac{10x^4}{u}\,\frac{du}{5x^{3/2}} = \int_1^3 \frac{2x^{\frac{5}{2}}}{u}\,du$

$= \int_1^3 \frac{u-1}{u}\,du = \int_1^3 1 - \frac{1}{u}\,du$

$= \left[u - \ln|u|\right]_1^3 = (3 - \ln 3) - (1 - \cancel{\ln 1})$

$= 2 - \ln 3$

$u = 2x^{\frac{5}{2}} + 1$

$\frac{du}{dx} = 5x^{\frac{3}{2}}$

$dx = \frac{du}{5x^{\frac{3}{2}}}$

$x = 0, \; u = 1$

$x = 1, \; u = 3$

$2x^{\frac{5}{2}} = u - 1$

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3. a)

$$\frac{dN}{dt} = 2N - N^2$$

$$\Rightarrow \frac{1}{2N-N^2}\,dN = \int 1\,dt$$

$$\Rightarrow \int \frac{1}{N(2-N)}\,dN = \int 1\,dt$$

N = no of foxes (1000)
t = time (years)
t=0, N=1

BY PARTIAL FRACTIONS

$$\Rightarrow \frac{1}{N(2-N)} \equiv \frac{A}{N} + \frac{B}{2-N}$$

$$\Rightarrow 1 \equiv A(2-N) + BN$$

If $N=0$, $2A=1 \Rightarrow A=\frac{1}{2}$
If $N=2$, $2B=1 \Rightarrow B=\frac{1}{2}$

$$\Rightarrow \int \frac{\frac{1}{2}}{N} + \frac{\frac{1}{2}}{2-N}\,dN = \int 1\,dt$$

$$\Rightarrow \int \frac{1}{N} + \frac{1}{2-N}\,dN = \int 2\,dt$$

$$\Rightarrow \ln|N| - \ln|2-N| = 2t + C$$

$$\Rightarrow \ln\left|\frac{N}{2-N}\right| = 2t + C$$

$$\Rightarrow \frac{N}{2-N} = e^{2t+C} = e^{2t} \times e^{C}$$

$$\Rightarrow \boxed{\frac{N}{2-N} = Ae^{2t}}$$

APPLY $t=0$ $N=1$
$1 = Ae^{0}$
$\boxed{A=1}$

$$\Rightarrow \frac{N}{2-N} = e^{2t}$$

$\Rightarrow N = (2-N)e^{2t}$

$\Rightarrow N = 2e^{2t} - Ne^{2t}$

$\Rightarrow N + Ne^{2t} = 2e^{2t}$

$\Rightarrow N(1+e^{2t}) = 2e^{2t}$

$\Rightarrow N = \dfrac{2e^{2t}}{1+e^{2t}}$

b) $N = \dfrac{2e^{2t}e^{-2t}}{1e^{-2t} + e^{2t}e^{-2t}}$

$N = \dfrac{2e^{0}}{e^{-2t} + e^{0}}$

$N = \dfrac{2}{e^{-2t}+1}$

As $t \to \infty$ $\quad e^{-2t} \to 0$

$N \to 2$

$\therefore$ FOXES REACH A LIMITING FIGURE OF 2000

4. $2^{x}y + 2^{y}x = 6xy$

$\Rightarrow \frac{d}{dx}(2^{x}y) + \frac{d}{dx}(2^{y}x) = \frac{d}{dx}(6xy)$

$\Rightarrow \left[2^{x}\ln 2 \times y + 2^{x} \times 1\frac{dy}{dx}\right] + \left[2^{y}\ln 2\frac{dy}{dx} \times x + 2^{y} \times 1\right] = \left(6y + 6x \times \frac{dy}{dx}\right)$

$\Rightarrow 2^{x}y\ln 2 + 2^{x}\frac{dy}{dx} + 2^{y}x\frac{dy}{dx}\ln 2 + 2^{y} = 6y + 6x\frac{dy}{dx}$

AT $P(4,2)$

$\Rightarrow 32\ln 2 + 16\left.\frac{dy}{dx}\right|_{(4,2)} + 16\ln 2\left.\frac{dy}{dx}\right|_{(4,2)} + 4 = 12 + 24\left.\frac{dy}{dx}\right|_{(4,2)}$

$\Rightarrow (16\ln 2 - 8)\left.\frac{dy}{dx}\right|_{(4,2)} = 8 - 32\ln 2$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(4,2)} = \dfrac{8-32\ln 2}{16\ln 2 - 8}$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(4,2)} = \dfrac{1-4\ln 2}{2\ln 2 - 1}$

$a = 4$

$b = 2$

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5. a) $x = t^2 + t \qquad y = 2t - 1.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t+1}$$

$$\left.\frac{dy}{dx}\right|_{t=p} = \frac{2}{2p+1}$$

$\boxed{P(p^2+p,\ 2p-1)}$

$$\Rightarrow y - y_0 = m(x - x_0)$$

$$\Rightarrow y - (2p-1) = \frac{2}{2p+1}\left(x - (p^2+p)\right)$$

$$\Rightarrow y - (2p-1) = \frac{2}{2p+1}(x - p^2 - p)$$

$$\Rightarrow y(2p+1) - (2p-1)(2p+1) = 2x - 2p^2 - 2p$$

$$\Rightarrow y(2p+1) = 2x - 2p^2 - 2p + (2p-1)(2p+1)$$

$$\Rightarrow y(2p+1) = 2x - 2p^2 - 2p + 4p^2 - 1$$

$$\Rightarrow y(2p+1) = 2x + 2p^2 - 2p - 1$$

AS REQUIRED

b) $(2,1) \Rightarrow t = 1$ ie $p = 1$
$(0,-3) \Rightarrow t = -1$ ie $p = -1$

$T_1:\ 3y = 2x + 2 - 2 - 1$
$T_2:\ -y = 2x + 2 + 2 - 1$

$\Rightarrow$
$3y = 2x - 1$
$-y = 2x + 3$

SUBTRACT $4y = -4$

$\boxed{y = -1}$

$\boxed{x = -1}$

$\therefore Q(-1,-1)$

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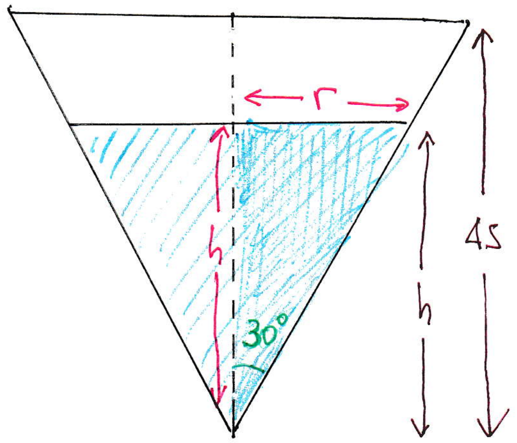
6.

a)



- $\tan 30 = \dfrac{r}{h}$

$\dfrac{\sqrt{3}}{3} = \dfrac{r}{h}$ (or $\dfrac{1}{\sqrt{3}}$)

$3r = \sqrt{3}h$

$\boxed{r = \dfrac{\sqrt{3}}{3}h}$

- $V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{\sqrt{3}}{3}\right)^2 h$

$V = \frac{1}{9}\pi h^3$ // AS REQUIRED

b) I) $\dfrac{dV}{dt} = -80$ (GIVEN)

$\Rightarrow \dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt}$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{3}{\pi h^2} \times (-80)$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{-240}{\pi h^2}$

$\Rightarrow \left.\dfrac{dh}{dt}\right|_{h=20} = \dfrac{-240}{\pi \times 20^2}$

$\Rightarrow \left.\dfrac{dh}{dt}\right|_{h=20} = -\dfrac{3}{5\pi} \simeq -0.191$ //

IT DECREASING AT 0.191 cm s^{-1}

$$\boxed{\begin{aligned} V &= \tfrac{1}{9}\pi h^3 \\ \frac{dV}{dh} &= \tfrac{1}{3}\pi h^2 \\ \frac{dh}{dV} &= \frac{3}{\pi h^2} \end{aligned}}$$

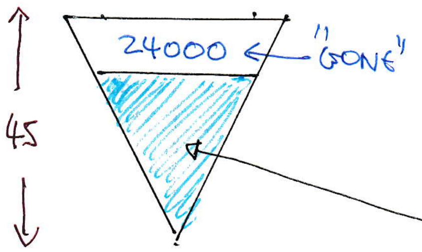
II) • "CONSTANT RATE" OF 80 cm^3 EVERY SECOND

• IT STARTS FROM "FULL"

• 5 MINUTES $= 5 \times 60 = 300$ SECONDS

HENCE

IN FIVE MINUTES $300 \times 80 = 24000 \text{ cm}^3$ HAVE LEAKED OUT



INITIAL VOLUME $= \frac{1}{9}\pi \times 45^3 = \underline{10125\pi} \text{ cm}^3$

$- 24000 \text{ cm}^3$

GIVES $7808.625618\ldots \text{ cm}^3$ LEFT

$$7808.625618\ldots = \frac{1}{9}\pi h^3$$

$$h^3 = 22370.06\ldots$$

$$\boxed{h = 28.1766\ldots}$$

$$\therefore \left.\frac{dh}{dt}\right|_{h=28.1766\ldots} = -\frac{240}{\pi(28.1766\ldots)^2} = -0.0962 \text{ cm s}^{-1}$$

7. a) $$\underline{r}_1 = \begin{pmatrix}4\\10\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\-2\end{pmatrix} = \begin{pmatrix}4\\10\\1\end{pmatrix} + 2w\begin{pmatrix}-1\\1\\-2\end{pmatrix}$$

(LET $\lambda = 2w$)

$$= \begin{pmatrix}4\\10\\1\end{pmatrix} + w\begin{pmatrix}-2\\2\\-4\end{pmatrix} = \begin{pmatrix}4\\10\\1\end{pmatrix} + (-\mu+2)\begin{pmatrix}-2\\2\\-4\end{pmatrix}$$

(LET $w = -\mu+2$)

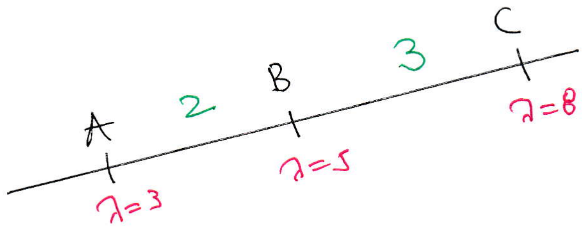
C4, IYGB, PAPER V

$$= \begin{pmatrix}4\\10\\1\end{pmatrix} - \mu\begin{pmatrix}-2\\2\\-4\end{pmatrix} + 2\begin{pmatrix}-2\\2\\-4\end{pmatrix} = \begin{pmatrix}4\\10\\1\end{pmatrix} + \begin{pmatrix}-4\\4\\-8\end{pmatrix} - \mu\begin{pmatrix}-2\\2\\-4\end{pmatrix}$$

$$= \begin{pmatrix}0\\14\\-7\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\4\end{pmatrix} = \underline{r}_2$$

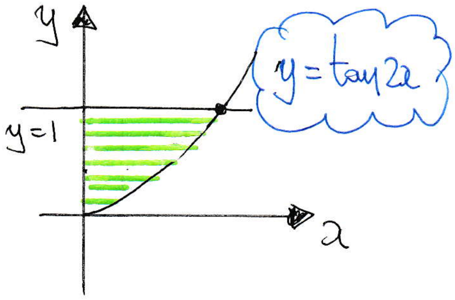
& $\lambda = 2w = 2(-\mu+2) = -2\mu+4$ IE $\lambda = -2\mu+4$

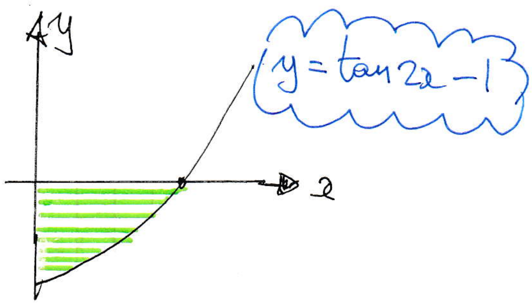
b) $\left.\begin{array}{l}\lambda=3\\ \lambda=5\\ \lambda=8\end{array}\right\} \Rightarrow$



$\therefore AB : BC = 2 : 3$

8.





- TRANSLATE THE REGION "DOWN" BY 1, AND REVOLVE ABOUT THE x AXIS INSTEAD, IE $y = \tan 2x - 1$
- $y=0 \Rightarrow \tan 2x - 1 = 0$

$\tan 2x = 1$

$2x = \frac{\pi}{4}$ (FIRST SOLUTION)

$\boxed{x = \frac{\pi}{8}}$

(P.T.O)

C4, IYGB, PAPER U

$$\therefore V = \pi \int_{x_1}^{x_2} [y(x)]^2 \, dx = \pi \int_0^{\frac{\pi}{8}} (\tan 2x - 1)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{8}} \tan^2 2x - 2\tan 2x + 1 \, dx$$

$$= \pi \int_0^{\frac{\pi}{8}} \sec^2 2x \cancel{-1} - 2\tan 2x \cancel{+1} \, dx$$

$$= \pi \int_0^{\frac{\pi}{8}} \sec^2 2x - 2\tan 2x \, dx$$

$$= \pi \left[\frac{1}{2}\tan 2x - \ln|\sec 2x| \right]_0^{\frac{\pi}{8}}$$

$$= \pi \left[\left(\frac{1}{2} - \ln[\sqrt{2}] \right) - \left(0 \cancel{- \ln 1} \right) \right]$$

$$= \frac{1}{2}\pi \left[1 - 2\ln\sqrt{2} \right]$$

$$= \frac{1}{2}\pi (1 - \ln 2)$$