C4, IYGB, PAPER S

1. $xy + x^3y + ay = 1$

Diff w.r.t x

$\Rightarrow \left[1xy + x\frac{dy}{dx}\right] + \left[3x^2y + x^3\frac{dy}{dx}\right] + a\frac{dy}{dx} = 0$

$\Rightarrow y + x\frac{dy}{dx} + 3x^2y + x^3\frac{dy}{dx} + a\frac{dy}{dx}$

$\Rightarrow (x+x^3+a)\frac{dy}{dx} = -y - 3x^2y$

$\Rightarrow \frac{dy}{dx} = -\frac{y+3x^2y}{x+x^3+a}$

$\Rightarrow \frac{dy}{dx} = -\frac{y(y+3x^2y)}{y(x+x^3+a)}$

EVIDENTLY $y \neq 0$ FROM EQUATION OF THE CURVE OTHERWISE $0=1$

$\Rightarrow \frac{dy}{dx} = -\frac{y^2+3x^2y^2}{\underbrace{yx+yx^3+ay}_{=1}}$

$\Rightarrow \frac{dy}{dx} = -(y^2+3x^2y^2) < 0$

FOR ALL x & y

ALTERNATIVE WITHOUT IMPLICIT

$xy + x^3y + ay = 1$

$y(x+x^3+a) = 1$

$y = \frac{1}{x+x^3+a}$

$y = (x^3+x+a)^{-1}$

$\frac{dy}{dx} = -(x^3+x+a)^{-2}(3x^2+1)$

$\frac{dy}{dx} = -\frac{3x^2+1}{(x^3+x+a)^2}$ ← AT LEAST 1 ← NON NEGATIVE

$\therefore \frac{dy}{dx} < 0$

C4, IYGB, PAPER S

2. a) $\left(1+\frac{4}{7}nx\right)^n = 1+\frac{n}{1}\left(\frac{4}{7}nx\right)+\frac{n(n-1)}{1\times 2}\left(\frac{4}{7}nx\right)^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}\left(\frac{4}{7}nx\right)^3+O(x^4)$

$= 1+\frac{4}{7}n^2x+\boxed{\frac{8}{49}n^3(n-1)}x^2+\boxed{\frac{32}{1029}n^4(n-1)(n-2)}x^3+O(x^4)$

NOW $\frac{8}{49}n^3(n-1) = \frac{32}{1029}n^4(n-1)(n-2)$ $\quad n\neq 0,\ n\neq 1$

$\Rightarrow \frac{8}{49} = \frac{32}{1029}n(n-2)$

$\Rightarrow \frac{21}{4} = n(n-2)$

$\Rightarrow 21 = 4n(n-2)$

$\Rightarrow 0 = 4n^2-8n-21$

$\Rightarrow 0 = (2n+3)(2n-7)$

$\Rightarrow n = \begin{cases} 7/2 \\ -3/2 \end{cases}$

b) • IF $n=\frac{7}{2}$

$\frac{4}{7}\times\frac{7}{2}\times 1 = 2 > 1$

$n\neq\frac{7}{2}$

• IF $n=-\frac{3}{2}$

$\frac{4}{7}\left(-\frac{3}{2}\right)\times 1 = -\frac{6}{7}$

AS $-1 < -\frac{6}{7} < 1$

$n=-\frac{3}{2}$ IS THE ONLY VALUE

(P.T.O)

3. a) $x = 4\cos 3\theta$, $y = 4\sin\theta$, $0 \le \theta \le \frac{\pi}{6}$

- $x = 0 \implies \cos 3\theta = 0$

$\implies 3\theta = \pm\frac{\pi}{2} \pm 2n\pi \quad n = 0,1,2,3,\ldots$

$\implies \theta = \pm\frac{\pi}{6} \pm \frac{2}{3}n\pi$

ONLY VALUE IS $\theta = \frac{\pi}{6}$

- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-12\sin 3\theta} = -\frac{\cos\theta}{3\sin 3\theta}$

- $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = -\frac{\cos\pi/6}{3\sin\frac{\pi}{2}} = -\frac{\sqrt{3}/2}{3\times 1} = -\frac{\sqrt{3}}{6}$

- EQUATION OF TANGENT

$y = -\frac{\sqrt{3}}{6}x + 2 \implies 6y = -\sqrt{3}x + 12$

$6y + \sqrt{3}x = 12$

AS REQUIRED

b) - FIRSTLY BY INSPECTION OF $x = 4\cos 3\theta \implies Q(4,0)$

$\theta = 0$ AT Q

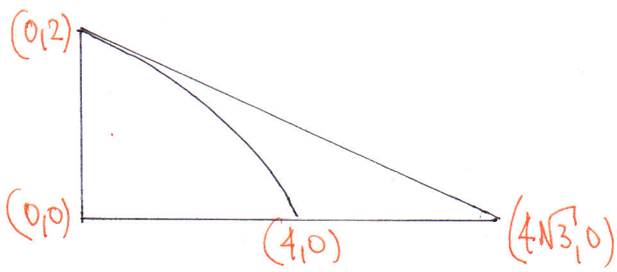
- SECONDLY THE TANGENT MEETS THE x AXIS AT $\cancel{6\times 0} + \sqrt{3}x = 12$

$3x = 12\sqrt{3}$

$x = 4\sqrt{3}$

$\therefore (4\sqrt{3}, 0)$

- SO "TRIANGLE" AREA $= \frac{1}{2} \times 2 \times 4\sqrt{3} = 4\sqrt{3}$



AREA UNDER PARAMETRIC CURVE (LIMITS FOUND EARLIER)

$$A = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_{\frac{\pi}{6}}^{0} 4\sin\theta(-12\sin 3\theta)\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{0} -48\sin 3\theta \sin\theta\,d\theta = \int_{0}^{\frac{\pi}{6}} 48\sin 3\theta\sin\theta\,d\theta$$

BY PARTS TWICE OR TRIGONOMETRIC IDENTITIES

$$\cos 4\theta = \cos(3\theta+\theta) = \cos 3\theta\cos\theta - \sin 3\theta\sin\theta$$
$$\cos 2\theta = \cos(3\theta-\theta) = \cos 3\theta\cos\theta + \sin 3\theta\sin\theta$$

SUBTRACT: $\cos 2\theta - \cos 4\theta = 2\sin 3\theta\sin\theta$

$$24\cos 2\theta - 24\cos 4\theta = 48\sin 3\theta\sin\theta$$

$$\dots = \int_0^{\frac{\pi}{6}} 24\cos 2\theta - 24\cos 4\theta\,d\theta = \Big[12\sin 2\theta - 6\sin 4\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \left(12\sin\frac{\pi}{3} - 6\sin\frac{2\pi}{3}\right) - (0)$$

$$= 12\left(\frac{\sqrt{3}}{2}\right) - 6\left(\frac{\sqrt{3}}{2}\right)$$

$$= 6\sqrt{3} - 3\sqrt{3}$$

$$= 3\sqrt{3}$$

$\therefore$ REQUIRED AREA $= 4\sqrt{3} - 3\sqrt{3} = \sqrt{3}$

AS REQUIRED

4. a) I) $x^2+x+2=(u-x)^2$

$\cancel{x^2}+x+2=u^2-2ux+\cancel{x^2}$

$2ux+x=u^2-2$

$x(2u+1)=u^2-2$

$x=\dfrac{u^2-2}{2u+1}$ // AS REQUIRED

II) $\dfrac{dx}{du}=\dfrac{(2u+1)(2u)-(u^2-2)\times 2}{(2u+1)^2}=\dfrac{4u^2+u-2u^2+4}{(2u+1)^2}$

$=\dfrac{2u^2+2u+4}{(2u+1)^2}=\dfrac{2(u^2+u+2)}{(2u+1)^2}$ // AS REQUIRED

b) $\displaystyle\int \frac{1}{x\sqrt{x^2+x+2}}\,dx$

$=\displaystyle\int \frac{1}{\left(\frac{u^2-2}{2u+1}\right)(u-x)}\frac{dx}{du}\,du$

$=\displaystyle\int \frac{2u+1}{(u^2-2)(u-x)}\times\frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$=\displaystyle\int \frac{2u+1}{(u^2-2)\left(\frac{u^2+u+2}{2u+1}\right)}\times\frac{2(u^2+u+2)}{(2u+1)^2}\,du$

$=\displaystyle\int \frac{\cancel{(2u+1)^2}}{(u^2-2)\cancel{(u^2+u+2)}}\times\frac{2\cancel{(u^2+u+2)}}{\cancel{(2u+1)^2}}\,du$

$=\displaystyle\int \frac{2}{u^2-2}\,du$

LET $\sqrt{x^2+x+2}=u-x$

$x^2+x+2=(u-x)^2$

$x=\dfrac{u^2-2}{2u+1}.$

$\dfrac{dx}{du}=\dfrac{2(u^2+u+2)}{(2u+1)^2}$

$u-x=u-\dfrac{u^2-2}{2u+1}$

$u-x=\dfrac{2u^2+u-u^2+2}{2u+1}$

$u-x=\dfrac{u^2+u+2}{2u+1}$

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$$= \int \frac{2}{(u-\sqrt{2})(u+\sqrt{2})}\, du$$

BY PARTIAL FRACTIONS (OR STANDARD QUOTED RESULTS)

$$\frac{2}{(u-\sqrt{2})(u+\sqrt{2})} \equiv \frac{A}{u-\sqrt{2}} + \frac{B}{u+\sqrt{2}}$$

$$2 \equiv A(u+\sqrt{2}) + B(u-\sqrt{2})$$

- If $u=\sqrt{2} \implies 2 = 2\sqrt{2}A \implies 1 = \sqrt{2}A \implies \boxed{A = \frac{1}{\sqrt{2}}}$
- If $u=-\sqrt{2} \implies 2 = -2\sqrt{2}B \implies -1 = \sqrt{2}B \implies \boxed{B = -\frac{1}{\sqrt{2}}}$

$$= \int \frac{\frac{1}{\sqrt{2}}}{u-\sqrt{2}} - \frac{\frac{1}{\sqrt{2}}}{u+\sqrt{2}}\, du = \frac{1}{\sqrt{2}}\ln|u-\sqrt{2}| - \frac{1}{\sqrt{2}}|u+\sqrt{2}| + C$$

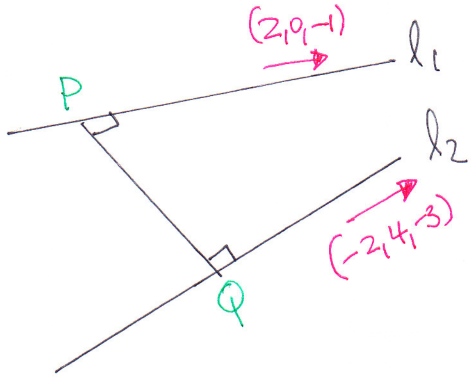
$$= \frac{1}{\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + C$$

BUT $u = x + \sqrt{x^2+x+2}$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{x+\sqrt{x^2+x+2}-\sqrt{2}}{x+\sqrt{x^2+x+2}-\sqrt{2}}\right| + C$$

(P.T.O)

5.



- FIRST FIND A "COMMON" PERPENDICULAR

$$(x,y,z)\cdot(2,0,-1)=0$$
$$(x,y,z)\cdot(-2,4,-3)=0$$

$$\left.\begin{array}{l}2x-z=0\\-2x+4y-3z=0\end{array}\right\}\quad \text{LET } \boxed{z=2}$$

THEN $2x-2=0$, $2x=2$, $\boxed{x=1}$

$-2x+4y-3z=0$
$-2+4y-6=0$
$4y=8$
$\boxed{y=2}$

HENCE THE COMMON PERPENDICULAR IS $(1,2,2)$

(NATURALLY THE CROSS PRODUCT CAN BE USED HERE IF KNOWN)

- NOW $\underline{r}_1=(7,1,2)+\lambda(2,0,-1)=(2\lambda+7,1,2-\lambda)$

$\underline{r}_2=(14,19,3)+\mu(-2,4,-3)=(14-2\mu,4\mu+19,3-3\mu)$

NOW LET $P(2\lambda+7,1,2-\lambda)$ & $Q(14-2\mu,4\mu+19,3-3\mu)$

- $\overrightarrow{PQ}=\underline{q}-\underline{p}=(14-2\mu,4\mu+19,3-3\mu)-(2\lambda+7,1,2-\lambda)$

$=(7-2\mu-2\lambda,\ 18+4\mu,\ 1-3\mu+\lambda)$

- $\overrightarrow{PQ}=k(1,2,2)$ FOR SOME k

$$\text{If}\quad \left.\begin{array}{l}7-2\mu-2\lambda=k\\18+4\mu=2k\\1-3\mu+\lambda=2k\end{array}\right\}\Rightarrow\left.\begin{array}{l}18+4\mu=2(7-2\mu-2\lambda)\\1-3\mu+\lambda=2(7-2\mu-2\lambda)\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}18+4\mu=14-4\mu-4\lambda\\1-3\mu+\lambda=14-4\mu-4\lambda\end{array}\right\}\Rightarrow\left.\begin{array}{l}4\lambda+8\mu=-4\\5\lambda+\mu=13\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{l}\lambda + 2\mu = -1 \\ 5\lambda + \mu = 13\end{array}\right\} \Rightarrow \boxed{\mu = 13 - 5\lambda}$$

$$\Rightarrow \lambda + 2(13 - 5\lambda) = -1$$

$$\Rightarrow \lambda + 26 - 10\lambda = -1$$

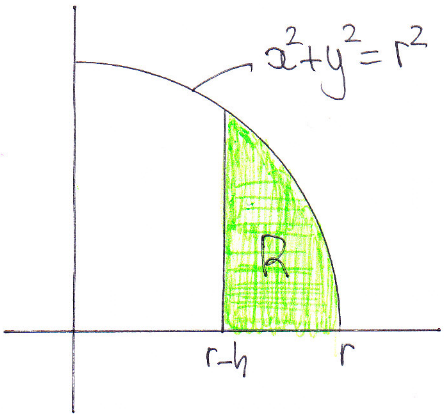
$$\Rightarrow -9\lambda = -27$$

$$\Rightarrow \boxed{\lambda = 3}$$

$$\therefore \boxed{\mu = -2}$$

& Thus $P(13, 1, -1)$ & $Q(18, 11, 9)$

6.



$$V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx$$

$$V = \pi \int_{r-h}^{r} r^2 - x^2 \, dx$$

$$V = \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{r-h}^{r}$$

$$V = \pi \left[\left(r^3 - \frac{1}{3}r^3 \right) - \left[r^2(r-h) - \frac{1}{3}(r-h)^3 \right] \right]$$

$$\Rightarrow V = \pi\left[r^3 - \frac{1}{3}r^3 - r^3 + r^2h + \frac{1}{3}\left(r^3 - 3r^2h + 3rh^2 - h^3\right)\right]$$

$$\Rightarrow V = \pi\left[\cancel{r^3} - \cancel{\frac{1}{3}r^3} - \cancel{r^3} + \cancel{r^2h} + \cancel{\frac{1}{3}r^3} - \cancel{r^2h} + rh^2 - \frac{1}{3}h^3\right]$$

$$\Rightarrow V = \pi\left[rh^2 - \frac{1}{3}h^3\right]$$

$$\Rightarrow V = \frac{1}{3}\pi h^2(3r - h)$$

AS REQUIRED

7. a) $\boxed{V = \dfrac{k}{D-h}}$

$D = 800$

$V = \dfrac{k}{800-h}$

- $h = 350 \quad V = 8$

$8 = \dfrac{k}{800-350}$

$8 = \dfrac{k}{450}$

$k = 3600$

- $h = 400 \quad V = ?$

$V = \dfrac{3600}{800-400}$

$V = \dfrac{3600}{400}$

$V = 9$

AS REQUIRED

b) $$\frac{dV}{dh} = \frac{d}{dh}\left[\frac{k}{800-h}\right] = \frac{d}{dh}\left[k(800-h)^{-1}\right]$$

$$= k(800-h)^{-2} = \frac{k}{(800-h)^2} = \frac{3600}{(800-h)^2}$$

$$\left.\frac{dV}{dh}\right|_{h=400} = \frac{3600}{(800-400)^2} = \frac{3600}{400^2} = \frac{3600}{160000} = \frac{9}{400}$$

i.e. $\frac{9}{400}$ cm^3 per cm risen

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c) $$\frac{dr}{dh} = \frac{dr}{dV} \times \frac{dV}{dh}$$

$$\frac{dr}{dh} = \frac{1}{4\pi r^2} \times \frac{3600}{(800-h)^2}$$

$$\frac{dr}{dh} = \frac{900}{\pi r^2 (800-h)^2}$$

$$\left.\frac{dr}{dh}\right|_{h=400} = \frac{900}{\pi \times \frac{9}{(4\pi)^{\frac{2}{3}}} \times 400^2}$$

$$\left(r^2 = \frac{9}{(4\pi)^{\frac{2}{3}}} \right)$$

$$\left.\frac{dr}{dh}\right|_{h=400} = \frac{4 \times 900}{4\pi \times \frac{9}{(4\pi)^{2/3}} \times 160000}$$

$$\left.\frac{dr}{dh}\right|_{h=400} = \frac{\cancel{4}^{1} \times \cancel{900}^{100}}{\cancel{9} \times (4\pi)^{\frac{1}{3}} \times \cancel{160000}_{400}}$$

$$\left.\frac{dr}{dh}\right|_{h=400} = \frac{1}{400\sqrt[3]{4\pi}}$$

AS REQUIRED

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

when $h = 400$
$V = 9$

$$9 = \frac{4}{3}\pi r^3$$

$$\frac{27}{4\pi} = r^3$$

$$(r^3)^{\frac{2}{3}} \quad \left(\frac{27}{4\pi}\right)^{\frac{2}{3}}$$

$$r^2 = \frac{9}{(4\pi)^{\frac{2}{3}}}$$

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8. a) $\frac{dx}{dt} \propto \frac{8-t}{x}$

$\Rightarrow \frac{dx}{dt} = \frac{k(8-t)}{x}$

- APPLY CONDITION

$72 = \frac{k \times 6}{336}$

$6k = 72 \times 336$

$k = 4032$

$\Rightarrow \frac{dx}{dt} = \frac{4032(8-t)}{x}$

$\Rightarrow x\frac{dx}{dt} = 4032(8-t)$

AS REQUIRED

x = SALES
t = TIME OPEN

$t = 2$
$x = 336$
$\frac{dx}{dt} = 72$

b) $\Rightarrow x\,dx = 4032(8-t)\,dt$

$\Rightarrow \int_{x=336}^{x} x\,dx = 4032\int_{t=2}^{t} 8-t\,dt$

EASIER IF WE ASSUME $t=0, x=0$

$\Rightarrow \left[\frac{1}{2}x^2\right]_{336}^{x} = 4032\left[8t-\frac{1}{2}t^2\right]_{2}^{t}$

$\Rightarrow \left[x^2\right]_{336}^{x} = 4032\left[16t-t^2\right]_{2}^{t}$

$\Rightarrow x^2 - 336^2 = 4032\left[(16t-t^2)-(32-4)\right]$

$\Rightarrow x^2 - 336^2 = 4032\left[16t-t^2-28\right]$

$\Rightarrow x^2 - \cancel{112896} = 4032(16t-t^2) - \cancel{112896}$

$\Rightarrow x^2 = 4032t(16-t)$

AS REQUIRED

C4, IYGB, PAPER S

c) when $t=8$ $\Rightarrow$ $x^2 = 4032 \times 8 \times (16-8)$

$x^2 = 4032 \times 64$

$x^2 = 258048$

$x \simeq 507.98\ldots$

ft $\simeq 507.98$

d) $x\frac{dx}{dt} = 4032(8-t)$

$x^2\left(\frac{dx}{dt}\right)^2 = 4032^2(8-t)$

now $\frac{dx}{dt} = 24$

$\Rightarrow \cancel{4032}t(16-t) \times 24^2 = 4032^{\cancel{2}}(8-t)^2$

$\Rightarrow t(16-t) \times 576 = (8-t)^2 \times 4032$

$\Rightarrow t(16-t) = (8-t)^2 \times 7$

$\Rightarrow 16t - t^2 = (64-16t+t^2) \times 7$

$\Rightarrow 16t - t^2 = 448 - 112t + 7t^2$

$\Rightarrow 0 = 8t^2 - 128t + 448$

$\Rightarrow 0 = t^2 - 16t + 56$

$\Rightarrow (t-8)^2 - 64 + 56 = 0$

$\Rightarrow (t-8)^2 = 8$

$\Rightarrow t = 8 \pm 2\sqrt{2}$

ft $8 - 2\sqrt{2}$

$\simeq 5.171\ldots$

$\simeq 14:10$

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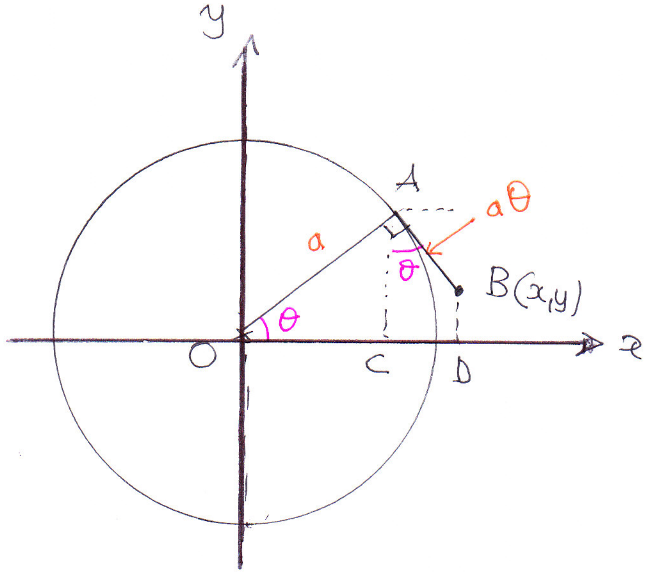
9.



BY GEOMETRY

- $x = |OD| = |OC| + |CD|$
  $x = a\cos\theta + a\theta\sin\theta$
  $x = a(\cos\theta + \theta\sin\theta)$

- $y = |BD|$
  $y = |AC| - |AB|\cos\theta$
  $y = a\sin\theta - a\theta\cos\theta$
  $y = a(\sin\theta - \theta\cos\theta)$

$\therefore \; x = a(\cos\theta + \theta\sin\theta)$
$y = a(\sin\theta - \theta\cos\theta)$