C4, MGB, PAPER G

1. a) $\frac{x-5}{x^2+5x+4} = \frac{x-5}{(x+1)(x+4)} \equiv \frac{A}{x+1} + \frac{B}{x+4}$

$\therefore \boxed{x-5 \equiv A(x+4)+B(x+1)}$

If $x=-1 \Rightarrow -6 = 3A \Rightarrow A=-2$

If $x=-4 \Rightarrow -9 = -3B \Rightarrow B=3$

$\therefore f(x) = \frac{3}{x+4} - \frac{2}{x+1}$

b) $\int_0^2 f(x)\,dx = \int_0^2 \frac{3}{x+4} - \frac{2}{x+1}\,dx = \Big[3\ln|x+4| - 2\ln|x+1|\Big]_0^2$

$= (3\ln 6 - 2\ln 3) - (3\ln 4 - 2\cancel{\ln 1})$

$= \ln 216 - \ln 9 - \ln 64 = \ln\left(\frac{216}{9\times 64}\right) = \ln\left(\frac{3}{8}\right)$

2. a) $f(x) = 2x(1+2x)^{-3}$

$= 2x\left[1 + \frac{-3}{1}(2x) + \frac{-3(-4)}{1\times 2}(2x)^2 + \frac{-3(-4)(-5)}{1\times2\times3}(2x)^3 + \dots\right]$

$= 2x\left[1 - 6x + 24x^2 - 80x^3 + \dots\right]$

$= 2x - 12x^2 + 48x^3 - 160x^4 + \dots$

b) VALID FOR $|2x| < 1$

$|x| < \frac{1}{2}$

$-\frac{1}{2} < x < \frac{1}{2}$

3. $\frac{dy}{dx}\cos^2 4x = y$

$\Rightarrow \frac{1}{y}\,dy = \frac{1}{\cos^2 4x}\,dx$

$\Rightarrow \int \frac{1}{y}\,dy = \int \sec^2 4x\,dx$

$\Rightarrow \ln|y| = \frac{1}{4}\tan 4x + C$

$\Rightarrow y = e^{\frac{1}{4}\tan 4x + C}$

$\Rightarrow y = e^{\frac{1}{4}\tan 4x} \times e^{C}$

$\Rightarrow y = Ae^{\frac{1}{4}\tan 4x} \quad (A = e^C)$

APPLY CONDITION

$x = \frac{\pi}{16} \quad y = e^3$

$\Rightarrow e^{3} = A e^{\frac{1}{4}\tan\left(4\times\frac{\pi}{16}\right)}$

$\Rightarrow e^{3} = Ae^{\frac{1}{4}}$

$\Rightarrow \frac{e^{3}}{e^{\frac{1}{4}}} = A$

$\Rightarrow A = e^{\frac{11}{4}}$

Thus

$y = e^{\frac{11}{4}} \times e^{\frac{1}{4}\tan 4x}$

$y = e^{\frac{1}{4}\tan 4x + \frac{11}{4}}$

$y = e^{\frac{1}{4}(11 + \tan 4x)}$ //

AS REQUIRED

4. $\int_{0}^{1} \frac{x}{\sqrt{16-7x^2}}\, dx = \ldots$ by substitution

$u = \sqrt{16-7x^2}$

$u^2 = 16 - 7x^2$

$2u\frac{du}{dx} = -14x$

$2u\,du = -14x\,dx$

$dx = -\frac{2u\,du}{14x}$

$dx = -\frac{u}{7x}du$

$x=0,\ u=4$

$x=1,\ u=3$

$= \int_{4}^{3} \frac{\cancel{x}}{\cancel{u}}\left(\frac{-\cancel{u}}{7\cancel{x}}du\right)$ (USE THE MINUS TO REVERSE THE LIMITS)

$= \int_{3}^{4} \frac{1}{7}\, du = \left[\frac{1}{7}u\right]_{3}^{4}$

$= \frac{4}{7} - \frac{3}{7} = \frac{1}{7}$ //

5. a) $\underline{a} = (7,4,0)$

$\underline{b} = (0,-3,7)$

$\vec{AB} = \underline{b} - \underline{a} = (0,-3,7) - (7,4,0) = (-7,-7,7)$

USE $(1,1,-1)$ AS A DIRECTION VECTOR

$\underline{r} = (7,4,0) + \lambda(1,1,-1)$

$\underline{r} = (\lambda+7,\ \lambda+4,\ -\lambda)$ //

b) $\underline{r}_2 = (3,-2,-4) + \mu(1,2,3) = (\mu+3,\ 2\mu-2,\ 3\mu-4)$

$\underline{r}_1 = \text{- - - - - - - - - - -} = (\lambda+7,\ \lambda+4,\ -\lambda)$

• EQUATE $\underline{i}$ & $\underline{k}$

$\mu + 3 = \lambda + 7$

$3\mu - 4 = -\lambda$

Add equations $\Rightarrow 4\mu - 1 = 7$

$\boxed{\mu = 2}$

$\lambda = 4 - 3\mu$

& $\boxed{\lambda = -2}$

CHECK ⊥ : $2\mu - 2 = 2\times2-2 = 2$
$\lambda + 4 = -2+2 = 2$

AS ALL 3 COMPONENTS AGREE WITH $\lambda=-2$ & $\mu=2$ THE LINES INTERSECT

USING $\mu=2$ INTO $(\mu+3, 2\mu-2, 3\mu-4)$ WE OBTAIN THE INTERSECTION
$C(5,2,2)$

c) DOTTING THE DIRECTION VECTORS OF THE TWO LINES
$(1,1,-1)\cdot(1,2,3) = 1+2-3 = 0$

$\therefore$ INDEED PERPENDICULAR

d) BY INSPECTION IF $\mu=1$ $(\mu+3, 2\mu-2, 3\mu-4)$ BECOMES $(4,0,-1)$

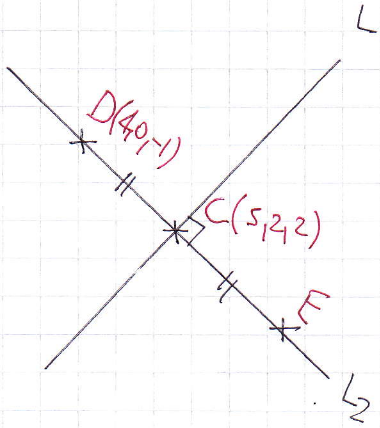
$\therefore D$ IS ON L_2

ALTERNATIVE

$\mu+3 = 4 \Rightarrow \mu=1$
$2\mu-2=0 \Rightarrow 2\mu=2$ SO $\mu=1$
$3\mu-4=-1 \Rightarrow 3\mu=3$ SO $\mu=1$

$\therefore D$ IS ON L_2

e)



• THE POINT C MUST BE THE MIDPOINT OF DE

BY INSPECTION

$E(6,4,5)$

C4, IYGB, PAPER G

6. a) $y = 2^{\sin 2x}$

$\Rightarrow \ln y = \ln 2^{\sin 2x}$

$\Rightarrow \ln y = (\sin 2x)\ln 2$

$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\left((\sin 2x)\ln 2\right)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = 2\cos 2x \times \ln 2$

$\Rightarrow \frac{dy}{dx} = 2y\ln 2\cos 2x$

$\Rightarrow \frac{dy}{dx} = 2 \times 2^{\sin 2x} \times \ln 2 \times \cos 2x$

ALTERNATIVE

If $y = a^x$

$\frac{dy}{dx} = a^x \times \ln a$

THUS

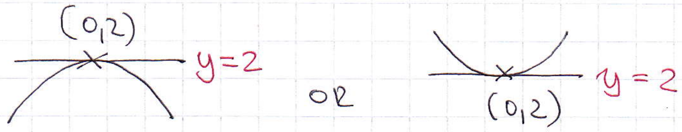
If $y = 2^{\sin 2x}$

$\frac{dy}{dx} = 2^{\sin 2x} \times \ln 2 \times 2\cos 2x$

b) $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}} = 2^{\sin\frac{\pi}{2}} \times \ln 2 \times \cancel{2\cos\left(\frac{\pi}{2}\right)} = 0$

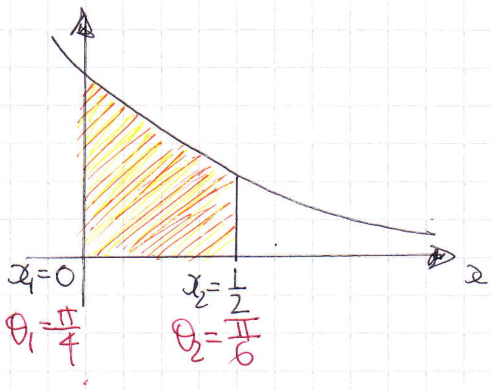
$y = 2^{\sin\frac{\pi}{2}} = 2^1 = 2 \qquad \therefore (0,2)$

THUS



EITHER WAY $y = 2$

7. a)



$x = \cos 2\theta$

$y = \sec\theta \qquad 0 < \theta < \frac{\pi}{2}$

$0 = \cos 2\theta$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

(ONLY SOLUTION IN RANGE)

$\frac{1}{2} = \cos 2\theta$

$2\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{6}$

(ONLY SOLUTION IN RANGE)

C4, IYGB, PAPER G

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \underbrace{(\sec\theta)}_{y(\theta)}\underbrace{(-2\sin 2\theta)}_{\frac{dx}{d\theta}}\,d\theta$$

USE THE MINUS TO REVERSE THE LIMITS

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cancel{\cos\theta}} \times 2(2\sin\theta\cancel{\cos\theta})\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin\theta\,d\theta$$

AS REQUIRED

b) INTEGRATE... $= \left[-4\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 4\left[\cos\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{6}}$

$$= 4\left[\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\right] = 4\left[\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right]$$

$$= 2(\sqrt{3} - \sqrt{2}) = 2\sqrt{3} - 2\sqrt{2}$$

c) $V = \pi\int_{x_1}^{x_2}\left[y(x)\right]^2 dx = \pi\int_{\theta_1}^{\theta_2}\left[y(\theta)\right]^2\frac{dx}{d\theta}\,d\theta$

$$= \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \underbrace{\sec^2\theta}_{(y(\theta))^2}\ \underbrace{(-2\sin 2\theta)}_{\frac{dx}{d\theta}}\,d\theta = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\cos^{\cancel{2}}\theta} \times 2(2\sin\theta\cancel{\cos\theta})\,d\theta$$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{4\sin\theta}{\cos\theta}\,d\theta = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\tan\theta\,d\theta$$

$$= \pi\left[4\ln|\sec\theta|\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = 4\pi\left[\ln\sec\frac{\pi}{4} - \ln\sec\frac{\pi}{6}\right]$$

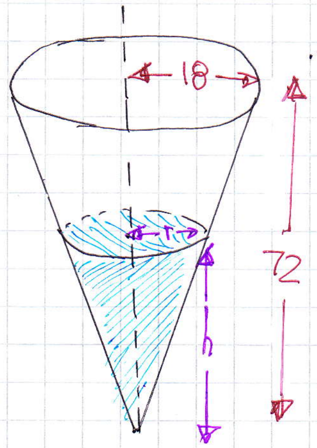
$$= 4\pi\left[\ln\sqrt{2} - \ln\left(\tfrac{2}{3}\sqrt{3}\right)\right]$$

OR TIDY FURTHER $= 4\pi\left[\ln\left(\dfrac{\sqrt{2}}{\frac{2}{3}\sqrt{3}}\right)\right]$

$$= 4\pi\ln\left(\frac{\sqrt{6}}{2}\right) = 2\pi \times 2\ln\left(\frac{\sqrt{6}}{2}\right) = 2\pi\ln\left(\frac{\sqrt{6}}{2}\right)^2$$

$$= 2\pi\ln\frac{3}{2}$$

8. a)



BY SIMILAR TRIANGLES

$$\frac{r}{h} = \frac{18}{72}$$

$$\frac{r}{h} = \frac{1}{4}$$

$$\boxed{r = \frac{1}{4}h}$$

VOLUME OF A CONE

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h\right)^2 h$$

$$V = \frac{1}{48}\pi h^3$$

AS REQUIRED

b)

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$V = \frac{1}{48}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{16}\pi h^2$$

$$\frac{dh}{dV} = \frac{16}{\pi h^2}$$

$$\frac{dh}{dt} = \frac{16}{\cancel{\pi} h^2} \times 6\cancel{\pi}$$

$$\frac{dh}{dt} = \frac{96}{h^2}$$

$$\left.\frac{dh}{dt}\right|_{h=4} = \frac{96}{4^2} = 6 \text{ cm s}^{-1}$$

c)

"CONSTANT RATE OF 6π cm^3 EVERY SECOND"

$12\frac{1}{2}$ MINUTES $= 12.5 \times 60$
$= 750$ SECONDS

VOLUME OF WATER AFTER $12\frac{1}{2}$ MIN IS $750 \times 6\pi = 4500\pi$

USING $V = \frac{1}{48}\pi h^3$

$$4500\cancel{\pi} = \frac{1}{48}\cancel{\pi} h^3$$

$$h^3 = 216000$$

$$h = 60$$

$$\left.\frac{dh}{dt}\right|_{t=12\frac{1}{2}\text{ MINUTES}}$$

$$= \left.\frac{dh}{dt}\right|_{t=750 \text{ SECONDS}}$$

$$= \left.\frac{dh}{dt}\right|_{h=60}$$

$$= \frac{96}{60^2} = \frac{2}{75} \simeq 0.0267 \text{ cm s}^{-1}$$