C4, IYGB, PAPER E

1. $\displaystyle\int_0^{\frac{1}{3}} xe^{3x}\,dx$ = ... BY PARTS & IGNORING LIMITS ...

x	1
$\frac{1}{3}e^{3x}$	e^{3x}

$$= \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

... LIMITS ...

$$= \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^{\frac{1}{3}} = \left(\cancel{\frac{1}{9}e^1 - \frac{1}{9}e^1}\right) - \left(0 - \frac{1}{9}\right) = \frac{1}{9}$$

AS REQUIRED

2. $\boxed{\frac{dr}{dt} = 2.5}$

$$\boxed{V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2}$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times 2.5$$

$$\Rightarrow \frac{dV}{dt} = 10\pi r^2$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{r=8} = 10\pi \times 8^2 = 640\pi \simeq 2011 \text{ cm}^3\text{s}^{-1}$$

3. a) $x^2 - 4xy + y^2 = 13$

$$\Rightarrow \frac{d}{dx}(x^2) - \frac{d}{dx}(4xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(13)$$

$$\Rightarrow 2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (2y - 4x)\frac{dy}{dx} = 4y - 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y - 2x}{2y - 4x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y - x}{y - 2x}$$

MULTIPLY TOP/BOTTOM BY −1

$$\Rightarrow \frac{dy}{dx} = \frac{x - 2y}{2x - y}$$

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b) IF $x=2 \implies 2^2 - 4\times2\times y + y^2 = 13$

$$4 - 8y + y^2 = 13$$
$$y^2 - 8y - 9 = 0$$
$$(y+1)(y-9) = 0$$
$$y = \begin{cases} -1 \\ 9 \end{cases}$$

c) $\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{2-2(-1)}{2\times2-(-1)} = \frac{4}{5}$

$\left.\frac{dy}{dx}\right|_{(2,9)} = \frac{2-2\times9}{2\times2-9} = \frac{2-18}{4-9} = \frac{-16}{-5} = \frac{16}{5}$

- $y+1 = \frac{4}{5}(x-2)$
- $y-9 = \frac{16}{5}(x-2)$

SUBTRACT $10 = -\frac{12}{5}(x-2)$

$$50 = -12(x-2)$$
$$50 = -12x+24$$
$$12x = -26$$
$$6x = -13$$
$$x = -\frac{13}{6}$$

$$\left.\begin{aligned} 4y+4 &= \frac{16}{5}(x-2) \\ y-9 &= \frac{16}{5}(x-2) \end{aligned}\right\} \implies 4y+4 = y-9$$
$$3y = -13$$
$$y = -\frac{13}{3}$$
$$\therefore P\left(-\frac{13}{6}, -\frac{13}{3}\right)$$

4. a) $(1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times2\times3}(ax)^3 + \frac{n(n-1)(n-2)(n-3)}{1\times2\times3\times4}(ax)^4 + O(x^5)$

$$= 1 + \underbrace{\boxed{na}}_{15}x + \boxed{\tfrac{1}{2}n(n-1)a^2}x^2 + \boxed{\tfrac{1}{6}n(n-1)(n-2)a^3}x^3 + \tfrac{1}{24}n(n-1)(n-2)(n-3)a^4x^4 + \dots$$

EQUAL

$\therefore \boxed{an = 15}$

$$\frac{1}{2}\cancel{n}\cancel{(n-1)}a^2 = \frac{1}{6}\cancel{n}\cancel{(n-1)}(n-2)a^3$$
$$\frac{1}{2}a^2 = \frac{1}{6}(n-2)a^3$$
$$3a^2 = (n-2)a^3$$
$$3 = (n-2)a$$
$$3 = an - 2a$$
$$3 = 15 - 2a$$
$$2a = 12$$
$$a = 6$$

b) $\therefore \quad 6n = 15$
$$n = \frac{5}{2}$$

c) coeff of $x^4 = \frac{1}{24} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \left(-\frac{1}{2}\right) \times 6^4 = -\frac{405}{8}$

5. $$\frac{dy}{dx} = \frac{5y}{(2+x)(1-2x)}$$
$$\Rightarrow \frac{1}{y}\,dy = \frac{5}{(2+x)(1-2x)}\,dx$$
$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{5}{(2+x)(1-2x)}\,dx$$

BY PARTIAL FRACTIONS

$$\frac{5}{(2+x)(1-2x)} \equiv \frac{A}{2+x} + \frac{B}{1-2x}$$
$$\boxed{5 \equiv A(1-2x) + B(2+x)}$$
If $x = -2 \Rightarrow 5 = 5A \Rightarrow A = 1$

If $x = \frac{1}{2} \Rightarrow 5 = \frac{5}{2}B \Rightarrow B = 2$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{2+x} + \frac{2}{1-2x}\,dx$$
$$\Rightarrow \ln|y| = \ln|x+2| - \ln|1-2x| + \ln A$$
C

$$\Rightarrow \ln|y| = \ln\left|\frac{A(x+2)}{1-2x}\right|$$
$$\Rightarrow \boxed{y = \frac{A(x+2)}{1-2x}}$$

when $x=0 \quad y=2$
$$2 = \frac{2A}{1}$$
$$A = 1$$
$$y = \frac{x+2}{1-2x}$$

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6. a) $\vec{AB} = \underline{b} - \underline{a} = (9,-2,14) - (8,0,12) = (1,-2,2)$

$\underline{r} = (8,0,12) + \lambda(1,-2,2)$

$\underline{r} = (\lambda+8, -2\lambda, 2\lambda+12)$

b) DOTTING THEIR DIRECTION VECTORS

$(1,-2,2)\cdot(2,1,0) = 2-2+0 = 0$

$\therefore$ INDEED PERPENDICULAR

c) $\underline{r}_1 = (\lambda+8, -2\lambda, 2\lambda+12)$

$\underline{r}_2 = (2\mu+1, \mu+9, 2)$

- EQUATE $\underline{k}$ $\Rightarrow$ $2\lambda+12 = 2$, $2\lambda = -10$, $\boxed{\lambda = -5}$
- EQUATE $\underline{j}$ $\Rightarrow$ $-2\lambda = \mu+9$, $10 = \mu+9$, $\boxed{\mu = 1}$
- CHECK $\underline{i}$

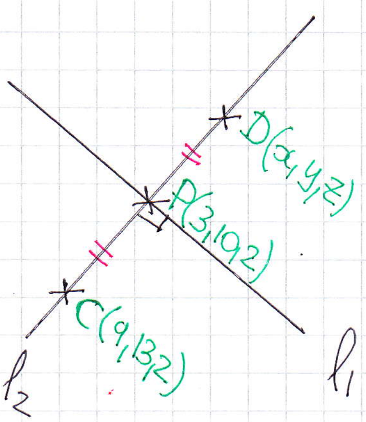
$\lambda+8 = -5+8 = 3$

$2\mu+1 = 2\times1+1 = 3$

AS ALL 3 COMPONENTS AGREE THE TWO LINES INTERSECT

USING $\mu=1$ into $(2\mu+1, \mu+9, 2)$ WE OBTAIN $P(3,10,2)$

d)



EVIDENTLY P MUST BE THE MIDPOINT OF CD

BY INSPECTION $D(-3,7,2)$

7. a)

x	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$	2π
y	0	0.2031	0.8602	0.8602	0.2031	0

b) $$\int_0^{2\pi} \sin^3\left(\tfrac{1}{2}x\right)dx = \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$= \frac{2\pi/5}{2}\left[0+0+2(0.2031+0.8602+\dots+0.2031)\right]$$

$$\simeq 2.672\dots$$

$$\simeq 2.67$$

c) $$\int_0^{2\pi} \sin^3\left(\tfrac{1}{2}x\right)dx = \dots \text{by substitution}$$

$$u = \cos\left(\tfrac{1}{2}x\right)$$

$$\frac{du}{dx} = -\frac{1}{2}\sin\left(\tfrac{1}{2}x\right)$$

$$-2\frac{du}{dx} = \sin\left(\tfrac{1}{2}x\right)$$

$$-2\,du = \sin\left(\tfrac{1}{2}x\right)dx$$

$$dx = -\frac{2}{\sin\left(\tfrac{1}{2}x\right)}\,du$$

$x=0 \quad u=1$

$x=2\pi \quad u=-1$

$$= \int_1^{-1} \sin^{\cancel{3}\,2}\left(\tfrac{1}{2}x\right) \times \frac{-2}{\cancel{\sin\left(\tfrac{1}{2}x\right)}}\,du$$

USED THE MINUS TO REVERSE THE LIMITS

$$= \int_{-1}^{1} 2\sin^2\left(\tfrac{1}{2}x\right)du$$

$$= \int_{-1}^{1} 2\left[1-\cos^2\left(\tfrac{1}{2}x\right)\right]du$$

$$= \int_{-1}^{1} 2 - 2\cos^2\left(\tfrac{1}{2}x\right)du$$

$$= \int_{-1}^{1} 2 - 2u^2\,du$$

$$= \left[2u - \tfrac{2}{3}u^3 + C\right]_{-1}^{1}$$

$$= \left(2-\tfrac{2}{3}\right) - \left(-2+\tfrac{2}{3}\right)$$

$$= \tfrac{4}{3} - \left(-\tfrac{4}{3}\right) = \tfrac{8}{3}$$

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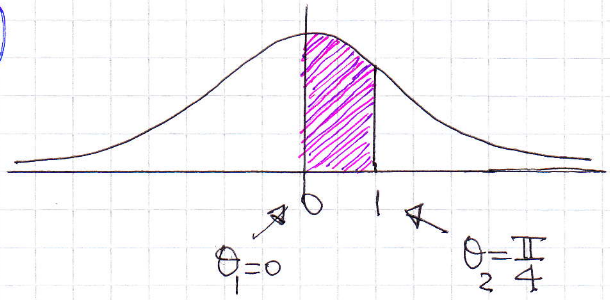
8. a)



- $x = 0$, $\tan\theta = 0$, $\theta = 0$
- $x = 1$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

- AREA $= \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \cos^2\theta(\sec^2\theta)\,d\theta = \int_0^{\frac{\pi}{4}} \cancel{\cos^2\theta} \times \frac{1}{\cancel{\cos^2\theta}}\,d\theta$

$= \int_0^{\frac{\pi}{4}} 1\,d\theta = [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

b) • $V = \pi\int_{x_1}^{x_2} (y(x))^2\,dx = \pi\int_{\theta_1}^{\theta_2} (y(\theta))^2\frac{dx}{d\theta}\,d\theta$

$= \pi\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2(\sec^2\theta)\,d\theta = \pi\int_0^{\frac{\pi}{4}} \cos^4\theta \times \frac{1}{\cos^2\theta}\,d\theta$

$= \pi\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta = \pi\int_0^{\frac{\pi}{4}} \frac{1}{2} + \frac{1}{2}\cos 2\theta\,d\theta$

$= \pi\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \pi\left[\left(\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin\frac{\pi}{2}\right) - \left(0 + \cancel{\tfrac{1}{4}\sin 0}\right)\right]$

$= \pi\left[\frac{\pi}{8} + \frac{1}{4}\right] = \frac{1}{8}\pi(\pi + 2)$

c) $x = \tan\theta$ $\quad$ $y = \cos^2\theta$

$x^2 = \tan^2\theta$ $\quad$ $\frac{1}{y} = \sec^2\theta$

BUT $1 + \tan^2\theta = \sec^2\theta$

$1 + x^2 = \frac{1}{y}$

$\frac{1}{1+x^2} = y$

ie $y = \frac{1}{x^2+1}$