C4, IYGB, PAPER D

1\. a) $\displaystyle\int_0^3 \frac{4}{2x+3}\,dx = \Big[2\ln|2x+3|\Big]_0^3 = 2\ln 9 - 2\ln 3$

$= 2\left[\ln 9 - \ln 3\right] = 2\ln 3$ or $\ln 9$

b) $\displaystyle\int_0^{\frac{\pi}{6}} \sin\left(4x+\frac{\pi}{6}\right)dx = \left[-\frac{1}{4}\cos\left(4x+\frac{\pi}{6}\right)\right]_0^{\frac{\pi}{6}} = \frac{1}{4}\left[\cos\left(4x+\frac{\pi}{6}\right)\right]_{\frac{\pi}{6}}^{0}$

$= \frac{1}{4}\left[\cos\frac{\pi}{6} - \cos\frac{5\pi}{6}\right] = \frac{1}{4}\left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right]$

$= \frac{1}{4}\sqrt{3}$

2\. a) $\dfrac{16}{(1-x)(2-x)^2} \equiv \dfrac{A}{1-x} + \dfrac{B}{(2-x)^2} + \dfrac{C}{2-x}$

$16 \equiv A(2-x)^2 + B(1-x) + C(1-x)(2-x)$

If $x=2 \Rightarrow 16 = -B \Rightarrow \boxed{B=-16}$

If $x=1 \Rightarrow 16 = A \Rightarrow \boxed{A=16}$

If $x=0 \Rightarrow 16 = 4A + B + 2C$

$16 = 64 - 16 + 2C$

$-32 = 2C$

$\boxed{C=-16}$

$\therefore A=16$
$B=-16$
$C=-16$

b) $\dfrac{16}{(1-x)(2-x)^2} = \dfrac{16}{1-x} - \dfrac{16}{(2-x)^2} - \dfrac{16}{2-x}$

- $16(1-x)^{-1} = 16\left[1 + \frac{-1}{1}(-x)^1 + \frac{-1(-2)}{1\times2}(-x)^2 + O(x^3)\right]$

$= 16\left[1 + x + x^2 + O(x^3)\right]$

$= 16 + 16x + 16x^2 + O(x^3)$

- $-16(2-x)^{-2} = -16\times2^{-2}\left(1-\frac{1}{2}x\right)^{-2} = -4\left(1-\frac{1}{2}x\right)^{-2}$

$= -4\left[1 + \frac{-2}{1}\left(-\frac{1}{2}x\right) + \frac{-2(-3)}{1\times2}\left(-\frac{1}{2}x\right)^2 + O(x^3)\right]$

$= -4\left[1 + x + \frac{3}{4}x^2 + O(x^3)\right]$

$= -4 - 4x - 3x^2 + O(x^3)$

C4, IYGB, PAPER D

- $-16(2-x)^{-1} = -16\times 2^{-1}\left(1-\frac{1}{2}x\right)^{-1} = -8\left(1-\frac{1}{2}x\right)^{-1}$

$$= -8\left[1 + \frac{-1}{1}\left(-\frac{1}{2}x\right)^{1} + \frac{(-1)(-2)}{1\times 2}\left(-\frac{1}{2}x\right)^{2} + O(x^3)\right]$$

$$= -8\left[1 + \frac{1}{2}x + \frac{1}{4}x^2 + O(x^3)\right]$$

$$= -8 - 4x - 2x^2$$

$$\therefore \frac{16}{(1-x)(2-x)^2} = \begin{array}{l} 16 + 16x + 16x^2 + O(x^3) \\ -4 - 4x - 3x^2 + O(x^3) \\ -8 - 4x - 2x^2 + O(x^3) \\ \hline 4 + 8x + 11x^2 + O(x^3) \end{array}$$

3. a) $\underline{r}_1 = (4,5,0) + t(-2,4,1) = (4-2t, 4t+5, t)$
$\underline{r}_2 = (-4,-1,3) + s(5,1,-2) = (5s-4, s-1, 3-2s)$

- EQUATE $\underline{j}$ & $\underline{k}$

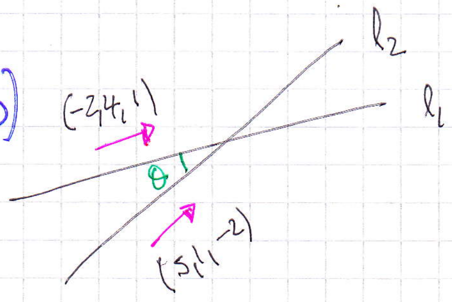
$$\left.\begin{array}{l} \underline{j}: 4t+5 = s-1 \\ \underline{k}: t = 3-2s \end{array}\right\} \Rightarrow \begin{array}{l} 4(3-2s)+5 = s-1 \\ 12-8s+5 = s-1 \\ 18 = 9s \\ \boxed{s=2} \\ \boxed{t=-1} \end{array}$$

- CHECK $\underline{i}$

$4-2t = 4-2(-1) = 6$
$5s-4 = 5\times 2-4 = 6$

AS ALL 3 COMPONENTS AGREE THE LINES INTERSECT

USING $s=2$, $A(6,1,-1)$

b) 

DOTTING DIRECTION VECTORS

$(-2,4,1)\cdot(5,1,-2) = |-2,4,1|\,|5,1,-2|\cos\theta$

$-10+4-2 = \sqrt{4+16+1}\sqrt{25+1+4}\cos\theta$

C4, IYGB, PAPER D

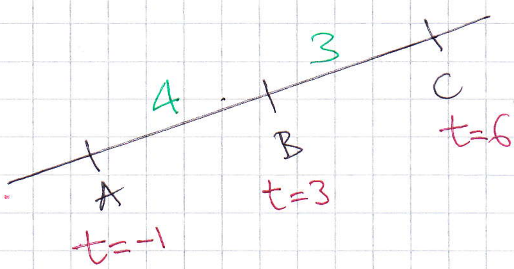
$-8 = \sqrt{21}\sqrt{30}\cos\theta$

$\cos\theta = -\frac{8}{\sqrt{21\times30}}$

$\theta \simeq 108.6°$

∴ ACUTE ANGLE $71.4°$ (1.25^c)

c)



$|AB| : |BC|$

$4 : 3$

4. a)

$I = \int (x-1)(4-x)^{\frac{1}{2}}\,dx = \ldots$ SUBSTITUTION

$u = (4-x)^{\frac{1}{2}}$

$u^2 = 4-x$

$x = 4-u^2$

$\frac{dx}{du} = -2u$

$dx = -2u\,du$

$I = \int (4-u^2-1)u\,(-2u\,du)$

$I = \int -2u^2(3-u^2)\,du$

$I = \int 2u^4-6u^2\,du$

$I = \frac{2}{5}u^5-2u^3+C$

$I = \frac{2}{5}(4-x)^{\frac{5}{2}}-2(4-x)^{\frac{3}{2}}+C$

b)

$I = \frac{2}{5}(4-x)^{\frac{3}{2}}\left[(4-x)-5\right]+C$ (FACTORIZATION)

$I = \frac{2}{5}(4-x)^{\frac{3}{2}}(-x-1)+C$

$I = -\frac{2}{5}(x+1)(4-x)^{\frac{3}{2}}+C$

AS REQUIRED

C4, IYGB, PAPER D

c) $I = \int (x-1)(4-x)^{\frac{1}{2}}\, dx$

$x-1$	1
$-\frac{2}{3}(4-x)^{\frac{3}{2}}$	$(4-x)^{\frac{1}{2}}$

$$I = -\frac{2}{3}(x-1)(4-x)^{\frac{3}{2}} - \int -\frac{2}{3}(4-x)^{\frac{3}{2}}\, dx$$

$$I = -\frac{2}{3}(x-1)(4-x)^{\frac{3}{2}} + \int \frac{2}{3}(4-x)^{\frac{3}{2}}\, dx$$

$$I = -\frac{2}{3}(x-1)(4-x)^{\frac{3}{2}} - \frac{4}{15}(4-x)^{\frac{5}{2}} + C$$

$$I = \frac{-10}{15}(x-1)(4-x)^{\frac{3}{2}} - \frac{4}{15}(4-x)^{\frac{5}{2}} + C$$

$$I = -\frac{2}{15}(4-x)^{\frac{3}{2}}\left[5(x-1) + 2(4-x)\right] + C$$

$$I = -\frac{2}{15}(4-x)^{\frac{3}{2}}(3x+3) + C$$

$$I = -\frac{2}{5}(x+1)(4-x)^{\frac{3}{2}} + C$$

5. a) $x^2 - 8y + 4y^2 = 0$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(8y) + \frac{d}{dx}(4y^2) = 0$$

$$2x - 8\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$2x = (8-8y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{8-8y}$$

$$\frac{dy}{dx} = \frac{x}{4-4y}$$

$$\frac{dy}{dx} = \frac{x}{4(1-y)}$$

AS REQUIRED

b) • $\frac{dy}{dx} = 0$

$x = 0$

$-8y + 4y^2 = 0$

$4y(y-2) = 0$

$y = \begin{cases} 0 \\ 2 \end{cases}$

• $\frac{dy}{dx} = \infty$

INFINITE GRADIENT IMPLIES THAT THE DENOMINATOR IS ZERO.

$1 - y = 0$

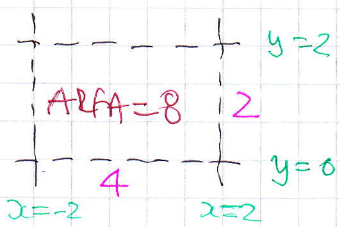
$y = 1$

$x^2 - 8 + 4 = 0$

$x^2 = 4$

$x = \pm 2$

$\therefore$



C4, IYGB, PAPER D

6. a) $x = 6\tan\theta \qquad y = \sin 2\theta \qquad 0 \le \theta < \frac{\pi}{2}$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{6\sec^2\theta} = \frac{1}{3}\cos^2\theta\cos 2\theta$$

SOLVE FOR ZERO

EITHER $\cos^2\theta = 0$ OR $\cos 2\theta = 0$

$\cos\theta = 0 \qquad 2\theta = \frac{\pi}{2}$

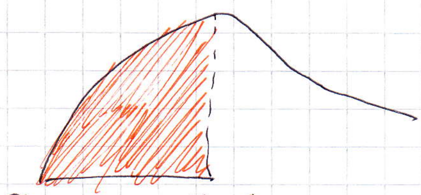
NO SOLUTIONS IN RANGE $\qquad \theta = \frac{\pi}{4}$

ONLY SOLUTION IN RANGE

$\therefore P\left(6\tan\frac{\pi}{4}, \sin\frac{\pi}{2}\right)$

$P(6, 1)$

b) $$V = \pi\int_{x_1}^{x_2}[y(x)]^2\,dx = \pi\int_{\theta_1}^{\theta_2}[y(\theta)]^2\frac{dx}{d\theta}\,d\theta$$



$$\Rightarrow V = \pi\int_0^{\frac{\pi}{4}}(\sin 2\theta)^2\,6\sec^2\theta\,d\theta$$

$$\Rightarrow V = \pi\int_0^{\frac{\pi}{4}}(2\sin\theta\cos\theta)^2 \times 6\sec^2\theta\,d\theta$$

$$\Rightarrow V = \pi\int_0^{\frac{\pi}{4}}4\sin^2\theta\cancel{\cos^2\theta} \times \frac{6}{\cancel{\cos^2\theta}}\,d\theta$$

$$\Rightarrow V = \pi\int_0^{\frac{\pi}{4}}24\sin^2\theta\,d\theta$$ AS REQUIRED

c) $$\Rightarrow V = \pi\int_0^{\frac{\pi}{4}}24\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = \pi\int_0^{\frac{\pi}{4}}12 - 12\cos 2\theta\,d\theta$$

$$= \pi\Big[12\theta - 6\sin 2\theta\Big]_0^{\frac{\pi}{4}} = \pi\Big[(3\pi - 6) - (0)\Big]$$

$$= 3\pi^2 - 6\pi \quad \text{OR} \quad 3\pi(\pi - 2)$$

7\. a)

$$\boxed{\frac{dV}{dt} = +k \times \frac{1}{V}}$$

$$\frac{dV}{dt} = \frac{k}{V}$$

$$\frac{dV}{dr} \times \frac{dr}{dt} = \frac{k}{V}$$

$$4\pi r^2 \times \frac{dr}{dt} = \frac{k}{\frac{4}{3}\pi r^3}$$

$$4\pi r^2 \times \frac{dr}{dt} = \frac{3k}{4\pi r^3}$$

$$\frac{dr}{dt} = \frac{3k}{16\pi^2 r^5}$$

$$\frac{dr}{dt} = \frac{A}{r^5} \quad \left(A = \frac{3k}{16\pi^2}\right)$$

AS REQUIRED

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

b)

$$r^5\,dr = A\,dt$$

$$\Rightarrow \int r^5\,dr = \int A\,dt$$

$$\Rightarrow \frac{1}{6}r^6 = At + C$$

$$\Rightarrow \boxed{r^6 = Bt + D}$$

APPLY CONDITIONS

- $t=0 \quad r=2 \Rightarrow 64 = D$

$$\boxed{r^6 = Bt + 64}$$

- $t=1 \quad r=3 \Rightarrow 729 = 1\times B + 64$

$$B = 665$$

$$\therefore r^6 = 665t + 64$$

AS REQUIRED

c) $t=6$

$$r^6 = 665\times 6 + 64$$

$$r^6 = 4054$$

$$r = \sqrt[6]{4054}$$

$$r \simeq 3.993\ldots$$

$$r \simeq 4.0 \text{ cm}$$

8. $\displaystyle\int_0^1 \frac{8}{(1+x^2)^2}\,dx = \ldots$ substitution

$x = \tan\theta$

$\frac{dx}{d\theta} = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$x=0$, $\tan\theta = 0$, $\theta = 0$

$x=1$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{8\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} 8\cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} 8\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right)d\theta = \int_0^{\frac{\pi}{4}} 4+4\cos 2\theta\,d\theta$$

$$= \Big[4\theta + 2\sin 2\theta\Big]_0^{\frac{\pi}{4}} = (\pi+2)-(0) = \pi+2$$