C4, IYGB, PAPER C

1. a) $(1+x)^{\frac{1}{2}} = 1 + \frac{\frac{1}{2}}{1}(x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times 2}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times 2\times 3}x^3 + O(x^4)$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + O(x^4)$

b) $\sqrt{4+2x} = (4+2x)^{\frac{1}{2}} = 4^{\frac{1}{2}}(1+\frac{1}{2}x)^{\frac{1}{2}} = 2(1+\frac{1}{2}x)^{\frac{1}{2}}$

$= 2\left[1 + \frac{1}{2}(\frac{1}{2}x) - \frac{1}{8}(\frac{1}{2}x)^2 + \frac{1}{16}(\frac{1}{2}x)^3 + O(x^4)\right]$

$= 2\left[1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + O(x^4)\right]$

$= 2 + \frac{1}{2}x - \frac{1}{16}x^2 + \frac{1}{64}x^3 + O(x^4)$

c) VALID FOR $|\frac{1}{2}x| < 1$

$|x| < 2$ $\quad\therefore\quad -2 < x < 2$

2. $x^3 + 2xy = e^y$

$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(2xy) = \frac{d}{dx}(e^y)$

$\Rightarrow 3x^2 + 2y + 2x\frac{dy}{dx} = e^y\frac{dy}{dx}$

$\Rightarrow 3x^2 + 2y = (e^y - 2x)\frac{dy}{dx}$

$\Rightarrow \boxed{\frac{3x^2+2y}{e^y - 2x} = \frac{dy}{dx}}$

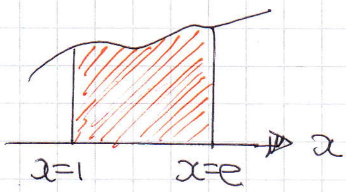
BUT $e^y = x^3 + 2xy$

$\therefore \frac{dy}{dx} = \frac{3x^2+2y}{(x^3+2xy)-2x}$

$\frac{dy}{dx} = \frac{3x^2+2y}{x^3+2xy-2x}$

AS REQUIRED

3.



$$V = \pi \int_{x_1}^{x_2} \left(y(x)\right)^2 \, dx$$

IN THIS CASE

$$V = \pi \int_1^e \left[x^{\frac{3}{2}} \sqrt{\ln x}\right]^2 dx$$

THUS

$$V = \pi \int_1^e x^3 \ln x \, dx = \ldots$$ BY PARTS AND IGNORING THE LIMITS AND π FOR THE TIME BEING...

$\ln x$	$\frac{1}{x}$
$\frac{1}{4}x^4$	x^3

$$\ldots = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \times \frac{1}{x} \, dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

REINTRODUCE π & LIMITS

$$= \pi \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^e$$

$$= \pi \left[\left(\frac{1}{4}e^4 \ln e - \frac{1}{16}e^4\right) - \left(\cancel{\frac{1}{4} \times 1^4 \times \ln 1} - \frac{1}{16}\right)\right]$$

$$= \pi \left[\frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16}\right]$$

$$= \pi \left[\frac{3}{16}e^4 + \frac{1}{16}\right]$$

$$= \frac{1}{16}\pi\left(3e^4 + 1\right)$$

AS REQUIRED

C4, IYGB, PAPER C

4. a) • WHEN $x=0$

$0=t^2-8t+12$

$0=(t-2)(t-6)$

$t=\begin{cases}2\\6\end{cases}$ $y=\begin{cases}-2\\2\end{cases}$

$\therefore (0,-2)\ (0,2)$

• WHEN $y=0$

$0=t-4$

$t=4$

$x=4^2-8\times4+12$

$x=16-32+12$

$x=-4$

$\therefore (-4,0)$

b) $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{1}{2t-8}$

AT $P(-3,1)$

$\uparrow$

$t=5$ BY INSPECTION OF $y=t-4$

$\left.\frac{dy}{dx}\right|_{t=5}=\frac{1}{2\times5-8}=\frac{1}{2}$

NORMAL GRADIENT $=-2$

$y-y_0=m(x-x_0)$

$y-1=-2(x+3)$

$y-1=-2x-6$

$y+2x+5=0$

AS REQUIRED

c) $y=t-4$

$\boxed{t=y+4}$

SUB INTO THE OTHER

$x=(y+4)^2-8(y+4)+12$

$x=y^2+\cancel{8y}+16-\cancel{8y}-32+12$

$x=y^2-4$

$y^2=x+4$

C4, IYGB, PAPER C

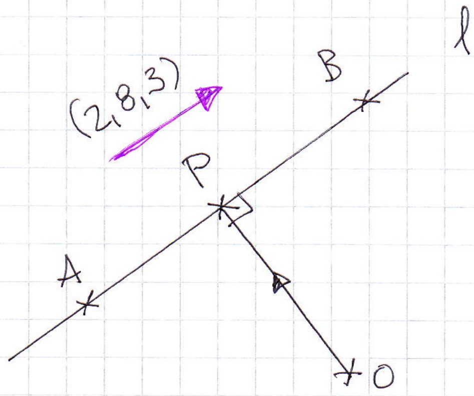
5. a) $\overrightarrow{AB} = \underline{b} - \underline{a} = (13,23,7) - (11,15,4) = (2,8,3)$

$\underline{r}$ = FIXED POINT + λ (DIRECTION)

$\underline{r} = (11,15,4) + \lambda(2,8,3)$

$\underline{r} = (2\lambda+11,\ 8\lambda+15,\ 3\lambda+4)$

b)



- LET $P(x,y,z)$

  OR $\underline{p} = (x,y,z)$

- $\overrightarrow{OP} \perp l$

  $(x,y,z)\cdot(2,8,3) = 0$

  $\boxed{2x + 8y + 3z = 0}$

- P LIES ON l

  $\boxed{\begin{aligned} x &= 2\lambda+11 \\ y &= 8\lambda+15 \\ z &= 3\lambda+4 \end{aligned}}$

THUS $2(2\lambda+11) + 8(8\lambda+15) + 3(3\lambda+4) = 0$

$4\lambda + 22 + 64\lambda + 120 + 9\lambda + 12 = 0$

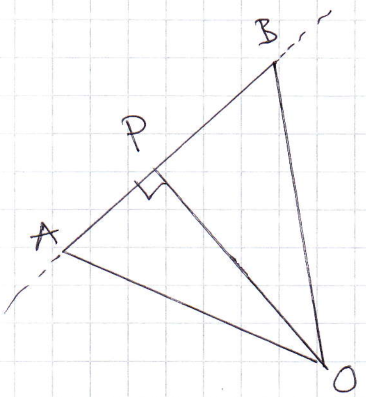
$77\lambda = -154$

$\boxed{\lambda = -2}$

$\therefore$ USING $\underline{r} = (2\lambda+11,\ 8\lambda+15,\ 3\lambda+4)$

GIVES $P(2(-2)+11,\ 8(-2)+15,\ 3(-2)+4)$

$P(7,-1,-2)$ AS REQUIRED



- $|\overrightarrow{AB}| = |2,8,3| = \sqrt{4+64+9} = \sqrt{77}$
- $|\overrightarrow{OP}| = |7,-1,-2| = \sqrt{49+1+4} = \sqrt{54}$

$\text{AREA} = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{OP}|$

$= \frac{1}{2}\sqrt{77}\sqrt{54} = \frac{3}{2}\sqrt{462}$

$\simeq 32.24$

6. a) "STEADY RATE" OF $1.5\ \text{cm}^2$ PER (EVERY) SECOND $\Longrightarrow$

IN 4 SECONDS $A = 4 \times 1.5$

$$A = 6\ \text{cm}^2$$

$$\pi r^2 = 6$$

$$r^2 = \frac{6}{\pi}$$

$$r = 1.38\ \text{cm}$$

b) $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dr}{dA} = \frac{1}{2\pi r}$$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \times 1.5$$

$$\left.\frac{dr}{dt}\right|_{\substack{r=1.38\ldots \\ (t=4)}} = \frac{1}{2\pi \times 1.38\ldots} \times 1.5$$

$$\left.\frac{dr}{dt}\right|_{t=4} \simeq 0.173\ \text{cm}\,\text{s}^{-1}$$

7. a) $\boxed{\frac{dm}{dt} = -k(m-10)}$

$\Longrightarrow dm = -k(m-10)\,dt$

$\Longrightarrow \frac{1}{m-10}\,dm = -k\,dt$

$\Longrightarrow \int \frac{1}{m-10}\,dm = \int -k\,dt$

$\Longrightarrow \ln|m-10| = -kt + C$

$\Longrightarrow m - 10 = e^{-kt+C}$

$\Longrightarrow m - 10 = Ae^{-kt}$ $(A = e^C)$

$\Longrightarrow m = 10 + Ae^{-kt}$

AS REQUIRED

b) • $t = 0$ $m = 120$

$$120 = 10 + Ae^0$$

$$120 = 10 + A$$

$$\boxed{A = 110}$$

$$\boxed{m = 10 + 110e^{-kt}}$$

• $60 = 10 + 110e^{-k \times 3}$

$$50 = 110e^{-3k}$$

$$\frac{5}{11} = e^{-3k}$$

$$\frac{11}{5} = e^{3k}$$

$$3k = \ln\frac{11}{5}$$

$$k = \frac{1}{3}\ln\frac{11}{5}$$

C4 IYGB, PAPER C

c) $\boxed{m = 10 + 110e^{-0.2628\ldots t}}$ $\quad \frac{1}{3}\ln\frac{11}{5}$

with $t=6$

$m = 10 + 110e^{-0.2628\ldots \times 6}$

$m \simeq 32.7\text{g}$ $= \frac{360}{11}$

8. a) $$\frac{2u^2}{(u-1)(u+1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}$$

$$\boxed{2u^2 \equiv A(u+1)(u-1) + B(u-1) + C(u+1)}$$

- If $u=1$, $2 = 2C \implies C=1$
- If $u=-1$, $2 = -2B \implies B=-1$
- If $u=0$, $0 = -A - B + C$

$A = 1-(-1)$

$A = 2$

b) $$\int_3^8 \frac{\sqrt{x+1}}{x}\,dx = \ldots \text{SUBSTITUTION}\ldots$$

$$= \int_2^3 \frac{u}{x}\,2u\,du = \int_2^3 \frac{2u^2}{x}\,du$$

$$= \int_2^3 \frac{2u^2}{u^2-1}\,du = \int_2^3 \frac{2u^2}{(u-1)(u+1)}\,du$$

$$= \int_2^3 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du \quad \text{(PART a)}$$

$$= \Big[2u + \ln|u-1| - \ln|u+1|\Big]_2^3$$

$u^2 = x+1$

$u = (x+1)^{\frac{1}{2}}$

$2u\frac{du}{dx} = 1$

$2u\,du = dx$

$x=3,\ u=2$

$x=8,\ u=3$

$x = u^2-1$

$$= \left[6 + \ln 2 - \ln 4\right] - \left[4 + \cancel{\ln 1} - \ln 3\right]$$

$$= 2 + \ln 2 - \ln 4 + \ln 3$$

$$= \left(2 + \ln \tfrac{3}{2}\right)$$

c)

x	3	4	5	6	7	8
y	0.6667	0.5590	0.4899	0.4410	0.4041	0.3750

d)

$$\int_3^8 \frac{\sqrt{x+1}}{x}\,dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\simeq \frac{1}{2}\left[0.6667 + 0.3750 + 2(0.5590 + 0.4899 + 0.4410 + 0.4041)\right]$$

$$\approx 2.41479\ldots$$

$$\simeq 2.415$$

e)

- EXACT VALUE $= 2 + \ln\frac{3}{2} = 2 + \ln 2 - \ln 4 + \ln 3 \simeq 2.405$
- TRAPEZIUM RULE $= 2.415$

DIFFERENCE OF $2.415 - 2.405 = 0.01$

TRAPEZIUM OVERESTIMATES