C4, IYGB, PAPER A

1. a) $(1+x)^{-2} = 1 + \frac{-2}{1}(x) + \frac{-2(-3)}{1\times2}(x)^2 + \frac{-2(-3)(-4)}{1\times2\times3}(x)^3 + O(x^4)$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + O(x^4)$

b) $(1+2x)^{-2} = 1 - 2(2x) + 3(2x)^2 - 4(2x)^3 + O(x^4)$

$= 1 - 4x + 12x^2 - 32x^3 + O(x^4)$

VALID FOR $|2x| < 1$

$|x| < \frac{1}{2}$ $\qquad \therefore -\frac{1}{2} < x < \frac{1}{2}$

2. $y^2 + 3xy + x^2 = 20$

$\Rightarrow \frac{d}{dx}(y^2) + \frac{d}{dx}(3xy) + \frac{d}{dx}(x^2) = \frac{d}{dx}(20)$

$\Rightarrow 2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} + 2x = 0$

AT $(2,2)$

$\Rightarrow 2\times2\left.\frac{dy}{dx}\right|_{(2,2)} + 3\times2 + 3\times2\times\left.\frac{dy}{dx}\right|_{(2,2)} + 2\times2 = 0$

$\Rightarrow -10\left.\frac{dy}{dx}\right|_{(2,2)} = -10$

$\Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = -1$

$\therefore \quad y - y_0 = m(x - x_0)$

$y - 2 = -(x-2)$

$y - 2 = -x + 2$

$y = 4 - x$

3. $3y^2\frac{dy}{dx}+2x=1$

$\Rightarrow 3y^2\frac{dy}{dx}=1-2x$

$\Rightarrow 3y^2\,dy=(1-2x)\,dx$

$\Rightarrow \int 3y^2\,dy=\int 1-2x\,dx$

$\Rightarrow y^3=x-x^2+C$

4. a) $\frac{5x+13}{(2x+1)(x+4)} \equiv \frac{A}{2x+1}+\frac{B}{x+4}$

$5x+13 \equiv A(x+4)+B(2x+1)$

If $x=-4 \Rightarrow -7=-7B \Rightarrow B=1$

If $x=-\frac{1}{2} \Rightarrow \frac{21}{2}=\frac{7}{2}A \Rightarrow A=3$

b) $\int_0^4 \frac{5x+13}{(2x+1)(x+4)}\,dx = \int_0^4 \frac{3}{2x+1}+\frac{1}{x+4}\,dx$

$=\left[\frac{3}{2}\ln|2x+1| + \ln|x+4|\right]_0^4 = \left(\frac{3}{2}\ln 9+\ln 8\right)-\left(\cancel{\frac{3}{2}\ln 1}+\ln 4\right)$

$=\ln 9^{\frac{3}{2}}+\ln 8-\ln 4 = \ln 27+\ln 8-\ln 4=\ln 54$

5. $\frac{dS}{dt}=512$

$S=4\pi r^2$

$\frac{dS}{dr}=8\pi r$

$\frac{dr}{dS}=\frac{1}{8\pi r}$

$\frac{dr}{dt}=\frac{dr}{dS}\times\frac{dS}{dt}$

$\frac{dr}{dt}=\frac{1}{8\pi r}\times 512$

$\frac{dr}{dt}=\frac{64}{\pi r}$

$\therefore \left.\frac{dr}{dt}\right|_{r=8}=\frac{64}{8\pi}=\frac{8}{\pi}\approx 2.55 \text{ cm s}^{-1}$

6. $\int \frac{4}{x(1+4\ln x)^2}\,dx = \ldots$ by SUBSTITUTION ...

$$u = 1 + 4\ln x$$
$$\frac{du}{dx} = \frac{4}{x}$$
$$4\,dx = x\,du$$
$$dx = \frac{x}{4}\,du$$

$$= \int \frac{\cancel{4}}{\cancel{x}u^2} \times \frac{\cancel{x}}{\cancel{4}}\,du = \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du = -u^{-1} + C$$

$$= -\frac{1}{u} + C = -\frac{1}{1+4\ln x} + C$$

7. $$V = \pi \int_{x_1}^{x_2} (y(x))^2\,dx = \pi \int_{-1}^{3} \left(\frac{6}{x+3}\right)^2 dx$$

$$= \pi \int_{-1}^{3} \frac{36}{(x+3)^2}\,dx = \pi \int_{-1}^{3} 36(x+3)^{-2}\,dx$$

$$= \pi \left[-36(x+3)^{-1}\right]_{-1}^{3} = 36\pi \left[\frac{1}{x+3}\right]_{3}^{-1}$$

$$= 36\pi \left[\frac{1}{2} - \frac{1}{6}\right] = 36\pi \times \frac{1}{3} = 12\pi$$

8. a) $\vec{AB} = \underline{b} - \underline{a} = (0,15,12) - (2,10,7) = (-2,5,5)$

$$\underline{r}_1 = (2,10,7) + \lambda(-2,5,5)$$
$$\underline{r}_1 = (2-2\lambda,\ 5\lambda+10,\ 5\lambda+7)$$

b) $$\underline{r}_2 = (4,1,-6) + \mu(2,-1,3)$$
$$\underline{r}_2 = (2\mu+4,\ 1-\mu,\ 3\mu-6)$$

EQUATE $\underline{j}$ & $\underline{k}$:

$$\left.\begin{array}{l} \underline{j}: \ 5\lambda+10 = 1-\mu \\ \underline{k}: \ 5\lambda+7 = 3\mu-6 \end{array}\right\} \text{SUBTRACT}$$

$\Rightarrow 3 = 7 - 4\mu$

$\Rightarrow 4\mu = 4$

$\boxed{\mu = 1}$

$\text{k} \quad 5\lambda + 10 = 1 - \mu$

$5\lambda + 10 = 0$

$5\lambda = -10$

$\boxed{\lambda = -2}$

CHECK $\underline{i}$ $\quad 2 - 2\lambda = 2 - 2(-2) = 6$

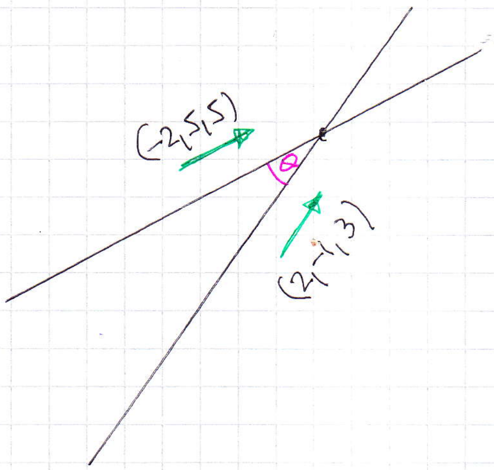
$2\mu + 4 = 2 \times 1 + 4 = 6$

AS ALL 3 COMPONENTS AGREE THE LINES INTERSECT

USING $\mu = 1$ INTO $(2\mu + 4,\ 1 - \mu,\ 3\mu - 6)$

WE OBTAIN $P(6, 0, -3)$

c)



DOTTING THE DIRECTION VECTORS

$(-2,5,5) \cdot (2,-1,3) = |-2,5,5|\,|2,-1,3| \cos\theta$

$-4 - 5 + 15 = \sqrt{4+25+25}\sqrt{4+1+9}\cos\theta$

$6 = \sqrt{54}\sqrt{14}\cos\theta$

$\cos\theta = \dfrac{6}{\sqrt{54}\sqrt{14}}$

$\theta \simeq 77.4°$

9. a)

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
	0	0.1309	0.4534	0.7854	0.9069	0.6545

$\frac{\pi}{3}\sin\left(2 \times \frac{\pi}{3}\right)$

b)

$$\int_0^{\frac{5\pi}{12}} x \sin 2x \, dx \simeq \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2 \times \text{REST}\right]$$

$$\simeq \frac{\pi/12}{2}\left[0 + 0.6545 + 2(0.1309 + \dots + 0.9069)\right]$$

$$\simeq 0.68168\dots$$

$$\simeq 0.682$$

c) $\int_0^{\frac{5\pi}{12}} x\sin 2x \, dx$ = IGNORING LIMITS ....

x	1
$-\frac{1}{2}\cos 2x$	$\sin 2x$

$$= -\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x \, dx$$

$$= -\frac{1}{2}x\cos 2x + \int \frac{1}{2}\cos 2x \, dx$$

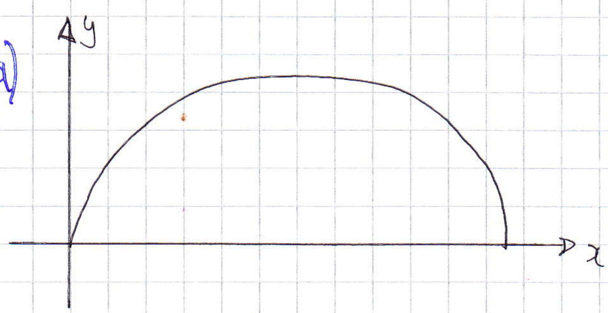
$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$$

... LIMITS ... $= \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{5\pi}{12}}$

$$= \left[-\frac{5\pi}{24}\cos\frac{5\pi}{6} + \frac{1}{4}\sin\frac{5\pi}{6}\right] - \left[0 + \cancel{\frac{1}{4}\sin 0}\right]$$

$$= -\frac{5\pi}{24}\left(-\frac{\sqrt{3}}{2}\right) + \frac{1}{4} \times \frac{1}{2} = \frac{1}{8} + \frac{5\pi\sqrt{3}}{48}$$

10. a)



$x = \theta - \sin\theta$

$y = 1 - \cos\theta$

$0 \leq \theta \leq 2\pi$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\, dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\, d\theta$$

$$= \int_0^{2\pi} \underbrace{(1-\cos\theta)}_{y(\theta)}\underbrace{(1-\cos\theta)}_{\frac{dx}{d\theta}}\, d\theta = \int_0^{2\pi}(1-\cos\theta)^2\, d\theta$$

AS REQUIRED

<u>LIMITS</u> $\theta = 0$ PRODUCES $x=0$, $y=0$ IE $(0,0)$

$\theta = 2\pi$ PRODUCES $x = 2\pi$, $y=0$ IE $(2\pi, 0)$

b) $\displaystyle\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta \;d\theta$

USING $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

$$= \int_0^{2\pi} 1 - 2\cos\theta + \left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \int_0^{2\pi} \frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta \;d\theta$$

$$= \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \left[\frac{3}{2}(2\pi) - \cancel{2\sin(2\pi)} + \cancel{\tfrac{1}{2}\sin 4\pi}\right] - \left[0 - \cancel{2\sin 0} + \cancel{\tfrac{1}{4}\sin 0}\right]$$

$$= 3\pi$$