C3, IYGB, PAPER Z

1. $$\boxed{x = \ln(y^2+9)^{\frac{3}{2}}}$$

$$x = \frac{3}{2}\ln(y^2+9)$$

$$\Rightarrow \frac{dx}{dx} = \frac{3}{2} \times \frac{1}{y^2+9} \times 2y$$

$$\Rightarrow \frac{dx}{dy} = \frac{3y}{y^2+9}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2+9}{3y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{3y} + \frac{9}{3y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{3} + \frac{3}{y}$$

AS REQUIRED

2. METHOD A

$$\text{LHS} = \frac{\sin 2\phi}{\sin\phi} - \frac{\cos 2\phi}{\cos\phi} = \frac{\sin 2\phi\cos\phi - \cos 2\phi\sin\phi}{\sin\phi\cos\phi}$$

$$= \frac{\sin(2\phi - \phi)}{\sin\phi\cos\phi} = \frac{\cancel{\sin\phi}}{\cancel{\sin\phi}\cos\phi} = \frac{1}{\cos\phi} = \sec\phi = \text{RHS}$$

METHOD B

$$\text{LHS} = \frac{\sin 2\phi}{\sin\phi} - \frac{\cos 2\phi}{\cos\phi} = \frac{2\cancel{\sin\phi}\cos\phi}{\cancel{\sin\phi}} - \frac{2\cos^2\phi - 1}{\cos\phi}$$

$$= 2\cos\phi - \left[\frac{2\cos^2\phi}{\cos\phi} - \frac{1}{\cos\phi}\right]$$

$$= 2\cos\phi - \left[2\cos\phi - \sec\phi\right]$$

$$= \cancel{2\cos\phi} - \cancel{2\cos\phi} + \sec\phi$$

$$= \sec\phi$$

$$= \text{RHS}$$

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3. a) $y = \dfrac{x}{1+2\ln x}$

$$\frac{dy}{dx} = \frac{(1+2\ln x)\times 1 - x\left(\frac{2}{x}\right)}{(1+2\ln x)^2} = \frac{1+2\ln x - 2}{(1+2\ln x)^2} = \frac{-1+2\ln x}{(1+2\ln x)^2}$$

SOLVE FOR ZERO

$$\frac{-1+2\ln x}{(1+2\ln x)^2} = 0 \implies -1+2\ln x = 0$$
$$2\ln x = 1$$
$$\ln x = \tfrac{1}{2}$$
$$\underline{x = e^{\frac{1}{2}} = \sqrt{e}}$$

$$y = \frac{x}{1+2\ln x} = \frac{e^{\frac{1}{2}}}{1+1} = \tfrac{1}{2}e^{\frac{1}{2}} = \underline{\tfrac{1}{2}\sqrt{e}}$$

$$\therefore \left(\sqrt{e}, \tfrac{1}{2}\sqrt{e}\right)$$

AS REQUIRED

b) $$\frac{d^2y}{dx^2} = \frac{(1+2\ln x)^2\left(\frac{2}{x}\right) - (-1+2\ln x)\times 2(1+2\ln x)\left(\frac{2}{x}\right)}{(1+2\ln x)^4}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{2}{x}(1+2\ln x) - \frac{4}{x}(-1+2\ln x)}{(1+2\ln x)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\substack{x=\sqrt{e} \\ 2\ln x = 1}} = \frac{\frac{2}{\sqrt{e}}\times 2 - 0}{(1+1)^3} = \tfrac{1}{2}e^{-\frac{1}{2}} > 0.$$

$\therefore$ IT IS A LOCAL MIN

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4. a)

$$y = \arctan x$$

$$\Rightarrow x = \tan y$$

$$\Rightarrow \frac{dx}{dy} = \sec^2 y$$

$$\Rightarrow \frac{dx}{dy} = 1 + \tan^2 y$$

$$\Rightarrow \frac{dx}{dy} = 1 + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

b)

$$y = \arctan x - 4\ln(1+x^2) - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} - 4 \times \frac{1}{1+x^2} \times 2x - 6x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2} - \frac{8x}{1+x^2} - 6x$$

- SOLVE FOR ZERO

$$\Rightarrow 0 = \frac{1-8x}{1+x^2} - 6x$$

$$\Rightarrow 6x = \frac{1-8x}{1+x^2}$$

$$\Rightarrow 6x + 6x^3 = 1 - 8x$$

$$\Rightarrow 6x^3 + 14x - 1 = 0$$

AS REQUIRED

c) LET $f(x) = 6x^3 + 14x - 1$

$f(0) = -1 < 0$

$f(1) = 19 > 0$

AS $f(x)$ IS CONTINUOUS & CHANGES SIGN IN THE INTERVAL, THERE MUST BE AT LEAST ONE SOLUTION IN THIS INTERVAL

d) $x_{n+1} = \dfrac{1 - 6x_n^{\ 3}}{14}$

$x_0 = 0$

$x_1 = 0.071429$

$x_2 = 0.071272$

$x_3 = 0.071273$

e) $\alpha \simeq 0.07127$ (5 d.p)

5. a) $A = k\left(1 - e^{-\frac{1}{12}t}\right)$

$41 = k\left(1 - e^{-\frac{5}{12}}\right)$

$\dfrac{41}{1 - e^{-\frac{5}{12}}} = k$

$k = 120.3195088\ldots$

$k \simeq 120$

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b) $H = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\Rightarrow 90 = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\Rightarrow \frac{3}{4} = 1 - e^{-\frac{1}{12}t}$

$\Rightarrow e^{-\frac{1}{12}t} = \frac{1}{4}$

$\Rightarrow e^{\frac{1}{12}t} = 4$

$\Rightarrow \frac{1}{12}t = \ln 4$

$\Rightarrow \frac{1}{12}t = 2\ln 2$

$\Rightarrow t = 24\ln 2$

c) $H = 120\left(1 - e^{-\frac{1}{12}t}\right)$

$\frac{dH}{dt} = 120\left(0 + \frac{1}{12}e^{-\frac{1}{12}t}\right)$

$\frac{dH}{dt} = 10e^{-\frac{1}{12}t}$

$\vdots$

$\frac{dH}{dt} = 10 - \frac{H}{12}$ AS REQUIRED

$\frac{H}{120} = 1 - e^{-\frac{1}{12}t}$

$e^{-\frac{1}{12}t} = 1 - \frac{H}{120}$

$10e^{-\frac{1}{12}t} = 10 - \frac{H}{12}$

d) $\frac{dH}{dt} = 7.5$

$7.5 = 10 - \frac{H}{12}$

$90 = 120 - H$

$H = 30$

e) As $t \to \infty$, $e^{-\frac{1}{12}t} \to 0$

$\therefore H \to 120$

i.e. 120

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6. a) $f(x) = a + \cos bx \quad 0 \leq x \leq 2\pi$

RANGE $2 \leq f(x) \leq 4$

(FROM GRAPH)

- $-1 \leq \cos bx \leq 1$

$2 \leq a + \cos bx \leq 4$

$\therefore a = 3$

- HALF PERIOD IS 2π

$\therefore b = \frac{1}{2}$

b) LET $y = 3 + \cos\frac{1}{2}x$

$y - 3 = \cos\frac{1}{2}x$

$\frac{1}{2}x = \arccos(y-3)$

$x = 2\arccos(y-3)$

$\therefore f^{-1}(x) = 2\arccos(x-3)$

c)

	$f(x)$	$f^{-1}(x)$
DOMAIN	$0 \leq x \leq 2\pi$	$2 \leq x \leq 4$
RANGE	$2 \leq f(x) \leq 4$	$0 \leq f^{-1}(x) \leq 2\pi$

$\therefore$ DOMAIN $2 \leq x \leq 4$

RANGE $0 \leq f^{-1}(x) \leq 2\pi$

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d) $f(x) = 3 + \cos\frac{1}{2}x$

$f'(x) = -\frac{1}{2}\sin\left(\frac{1}{2}x\right)$

$f'\left(\frac{4\pi}{3}\right) = -\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{4}$

e) RECIPROCAL IE $-\frac{4}{\sqrt{3}}$

7. a) START WITH

$\sin(A+B) = \sin A\cos B + \cos A\sin B$
$\sin(A-B) = \sin A\cos B - \cos A\sin B$

ADD $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

LET $\boxed{A+B=P}$, $\boxed{A-B=Q}$ ADD $2A = P+Q \Rightarrow \boxed{A = \frac{P+Q}{2}}$
SUBTRACT $2B = P-Q \Rightarrow \boxed{B = \frac{P-Q}{2}}$

$\therefore \sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$

AS REQUIRED

b) $\sin\theta - \sin3\theta + \sin5\theta = 0$

$\Rightarrow \sin 5\theta + \sin\theta = \sin3\theta$
P Q

$\Rightarrow 2\sin\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right) = \sin3\theta$

$\Rightarrow 2\sin3\theta\cos2\theta = \sin3\theta$

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$\Longrightarrow 2\sin 3\theta \cos 2\theta - \sin 3\theta = 0$

$\Longrightarrow \sin 3\theta (2\cos 2\theta - 1) = 0$

- $\sin 3\theta = 0$

  $\arcsin(0) = 0$

  $\begin{cases} 3\theta = 0 \pm 360n \\ 3\theta = 180 \pm 360n \end{cases}$ $n = 0,1,2,3,\ldots$

  $\begin{cases} \theta = 0 \pm 120n \\ \theta = 60 \pm 120n \end{cases}$

- $\cos 2\theta = \frac{1}{2}$

  $\arccos(\frac{1}{2}) = 60°$

  $\begin{cases} 2\theta = 60 \pm 360n \\ 2\theta = 300 \pm 360n \end{cases}$ $n = 0,1,2,3\ldots$

  $\begin{cases} \theta = 30° \pm 180n \\ \theta = 150° \pm 180n \end{cases}$

$\theta = 0°, 30°, 60°, 120°, 150°, 180°$

8.

$\sqrt{x} \longmapsto \sqrt{x+1} \longmapsto \sqrt{|x|+1} \longmapsto \sqrt{|x+1|+1}$