C3, IYGB, PAPER W

1. a) $y = \dfrac{kx^2 - a}{kx^2 + a}$

$$\frac{dy}{dx} = \frac{(kx^2+a)(2kx) - (kx^2-a)(2kx)}{(kx^2+a)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{2k^2x^3} + 2akx - \cancel{2k^2x^3} + 2akx}{(kx^2+a)^2}$$

$$\frac{dy}{dx} = \frac{4akx}{(kx^2+a)^2}$$

b) $\dfrac{dy}{dx} = 0$ YIELDS $4akx = 0$

$\Longrightarrow x = 0$ FOR ALL a, k

& $y = \dfrac{-a}{a} = -1$ $\therefore (0, -1)$

2. a) $LHS = \dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\tan x}{\sec^2 x} = 2\tan x \cos^2 x$

$= 2 \times \dfrac{\sin x}{\cos x} \times \cos^2 x = 2\sin x \cos x = \sin 2x$

$= RHS$

b) USE $x = 15°$ IN THE ABOVE IDENTITY

$$\frac{2\tan 15}{1+\tan^2 15} = \sin 30$$

$$\Longrightarrow \frac{2T}{1+T^2} = \frac{1}{2} \qquad (T = \tan 15)$$

$\Rightarrow 1+T^2 = 4T$

$\Rightarrow T^2 - 4T + 1 = 0$

$\Rightarrow (T-2)^2 - 4 + 1 = 0$

$\Rightarrow (T-2)^2 = 3$

$\Rightarrow T - 2 = \pm\sqrt{3}$

$\Rightarrow T = 2 \pm \sqrt{3}$

$\Rightarrow \tan 15 = \begin{cases} \cancel{2+\sqrt{3}} > 1 \\ 2-\sqrt{3} < 1 \end{cases}$

$\tan 45 = 1$ & $\tan x$ is AN INCREASING FUNCTION

3. $7\sin^2 x + \sin x \cos x = 6$

$\Rightarrow \frac{7\sin^2 x}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} = \frac{6}{\cos^2 x}$

$\Rightarrow 7\tan^2 x + \tan x = 6\sec^2 x$

$\Rightarrow 7\tan^2 x + \tan x = 6(1 + \tan^2 x)$

$\Rightarrow 7\tan^2 x + \tan x = 6 + 6\tan^2 x$

$\Rightarrow \tan^2 x + \tan x - 6 = 0$

$\Rightarrow (\tan x - 2)(\tan x + 3) = 0$

$\tan x = \begin{cases} 2 \\ -3 \end{cases}$

- $\arctan(2) \simeq 63.4°$

  $x = 63.4° \pm 180n$

  $n = 0, 1, 2, 3, \ldots$

- $\arctan(-3) = -71.57°$

  $x = -71.6° \pm 180n$

  $n = 0, 1, 2, 3, \ldots$

$x = 63.4°, 243.4°, 108.4°, 288.4°$

C3, IYGB, PAPER W

ALTERNATIVE

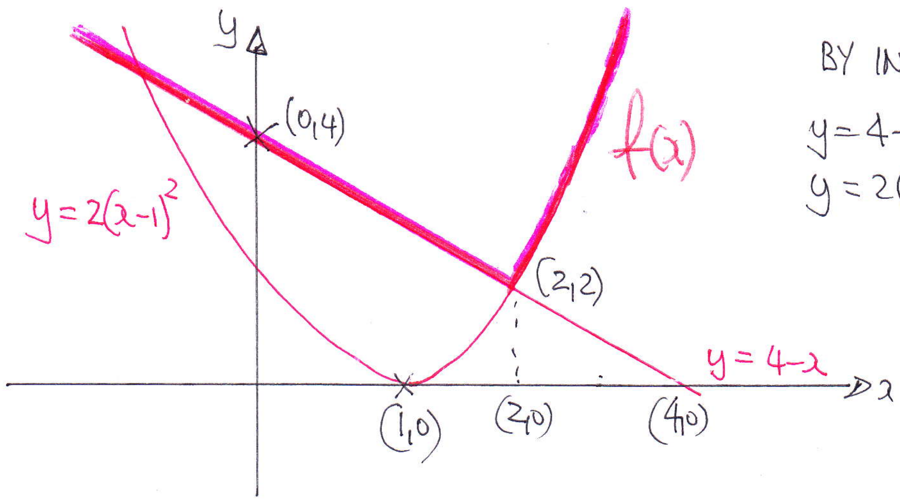
$$7\sin^2 x + \sin x\cos x = 6$$
$$7\left(\tfrac{1}{2}-\tfrac{1}{2}\cos 2x\right) + \tfrac{1}{2}(2\sin x\cos x) = 6$$
$$\tfrac{7}{2} - \tfrac{7}{2}\cos 2x + \tfrac{1}{2}\sin 2x = 6$$
$$7 - 7\cos 2x + \sin 2x = 12$$
$$\sin 2x - 7\cos 2x = 5$$

$\Downarrow$

WRITE AS $R\sin(2x-\alpha) = 5$

ETC ETC

4. a)



BY INSPECTION

$y = 4 - 2 = 2$

$y = 2(2-1)^2 = 2$

b) $f(x) \geq 2$

c) • $f(x) = 18$

$4 - x = 18$

$x = -14$

• $f(x) = 18$

$2(x-1)^2 = 18$

$(x-1)^2 = 9$

$x - 1 = \begin{cases} 3 \\ -3 \end{cases}$

$x = \begin{cases} 4 \\ \cancel{-2} \end{cases}$

5. a) IF $f(x) = |f(x)| \Rightarrow f(x) \geqslant 0$

- SO EITHER

$$f(x) = 9\left[x^2 + \tfrac{2}{3}x + \tfrac{2}{9}\right]$$
$$= 9\left[\left(x+\tfrac{1}{3}\right)^2 - \tfrac{1}{9} + \tfrac{2}{9}\right]$$
$$= 9\left[\left(x+\tfrac{1}{3}\right)^2 + \tfrac{1}{9}\right]$$
$$= 9\left(x+\tfrac{1}{3}\right)^2 + 1 \geqslant 1 > 0$$
$$\therefore f(x) = |f(x)|$$

- OR

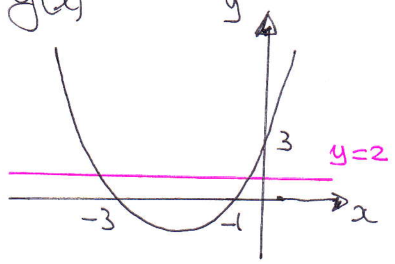
USE DISCRIMINANT

$$b^2 - 4ac$$
$$= 6^2 - 4 \times 9 \times 2$$
$$= 36 - 72 = -36 < 0$$

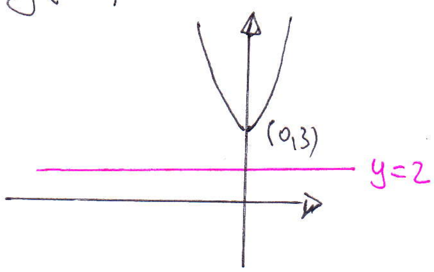
IE GRAPH LIES ENTIRELY ABOVE x AXIS

b) GRAPHICALLY

$g(x)$



$g(|x|)$



THE MINIMUM VALUE OF $g(|x|)$ IS 3, SO NO INTERSECTIONS & HENCE NO SOLUTIONS TO $g(|x|) = 2$

ALGEBRAICALLY

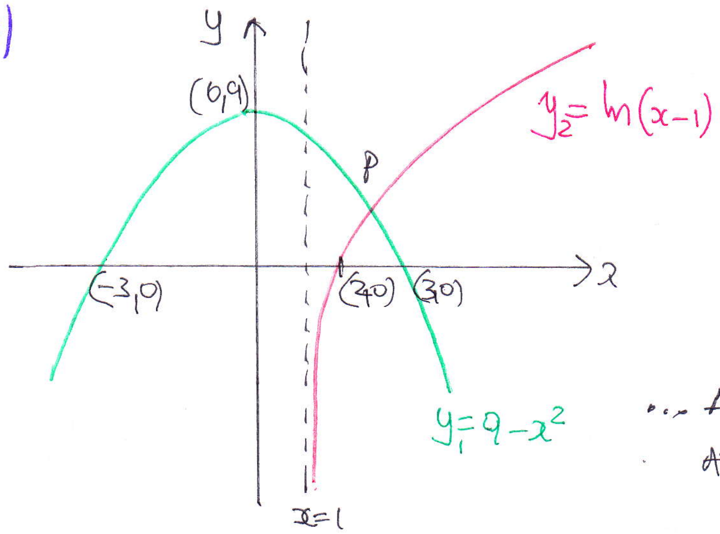
$$g(x) = (x+1)(x+3)$$
$$g(x) = x^2 + 4x + 3$$
$$g(|x|) = |x|^2 + 4|x| + 3$$

AS $|x| \geqslant 0$

$$g(|x|) \geqslant 3$$

& SIMILAR ARGUMENT FOLLOWS

6. a)



∴ AS THE GRAPHS INTERSECT AT ONE POINT ONLY

b) AS y_2 INTERCEPT AT (2,0) & y_1 INTERCEPTS AT (3,0) P MUST LIE BETWEEN THESE VALUES, SHOWN BY THE CONFIGURATION OF THE GRAPHS

c) $$\boxed{x_{n+1} = \sqrt{9 - \ln(x_n - 1)}}$$

$x_1 = 2.5$

$x_2 = 2.93164$

$x_3 = 2.88819$

$x_4 = 2.89212$

$x_5 = 2.89176$

$x_6 = 2.89180$

$x_7 = 2.89179$

ETC

$\therefore x \simeq 2.892$

d) $y \approx 9 - 2.89179...^2 \simeq 0.6375...$ $P(2.892, 0.638)$

TRANSLATION "LEFT" BY 1

FOLLOWED BY VERTICAL STRETCH BY SCALE FACTOR 3

$$P(2.892, 0.638) \xrightarrow{-1} \xrightarrow{\times 3} P'(1.89, 1.91)$$

7. a) $x = y(9-4y^2)^{\frac{1}{2}}$

$$\Rightarrow \frac{dx}{dy} = 1(9-4y^2)^{\frac{1}{2}} + y(9-4y^2)^{-\frac{1}{2}} \times \frac{1}{2} \times (-8y)$$

$$\Rightarrow \frac{dx}{dy} = (9-4y^2)^{\frac{1}{2}} - 4y^2(9-4y^2)^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dx}{dy} = (9-4y^2)^{-\frac{1}{2}}\left[(9-4y^2)^1 - 4y^2\right]$$

$$\Rightarrow \frac{dx}{dy} = \frac{9-8y^2}{(9-4y^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(9-4y^2)^{\frac{1}{2}}}{9-8y^2}$$

AS REQUIRED

b) FOR INFINITE GRADIENT DENOMINATOR MUST EQUAL ZERO

$$9 - 8y^2 = 0$$

$$8y^2 = 9$$

$$y^2 = \frac{9}{8}$$

$$y^2 = \frac{18}{16}$$

$$\therefore y = \begin{cases} \frac{3}{4}\sqrt{2} \\ -\frac{3}{4}\sqrt{2} \end{cases}$$

C3, IYGB, PAPER W

$$x = \begin{cases} \frac{3}{4}\sqrt{2}\sqrt{9-4\times\frac{9}{8}} = \frac{9}{4} \\ -\frac{3}{4}\sqrt{2}\sqrt{9-4\times\frac{9}{8}} = -\frac{9}{4} \end{cases}$$

$\therefore \left(\frac{9}{4}, \frac{3}{4}\sqrt{2}\right), \left(-\frac{9}{4}, -\frac{3}{4}\sqrt{2}\right)$

8.

$$\boxed{N = \frac{600}{1+e^{-0.25t}}, \quad t \geqslant 0}$$

a) $t=0$, $N = \frac{600}{1+e^0} = 300$

b) $N = 455$

$$455 = \frac{600}{1+e^{-0.25t}}$$

$$\Rightarrow 1+e^{-0.25t} = \frac{600}{455}$$

$$\Rightarrow e^{-0.25t} = \frac{29}{91}$$

$$\Rightarrow e^{0.25t} = \frac{91}{29}$$

$$\Rightarrow 0.25t = \ln\left(\frac{91}{29}\right)$$

$$\Rightarrow t = 4\ln\left(\frac{91}{29}\right) \simeq 4.57$$

(P.T.O)

c) $$\boxed{N = 600\left(1+e^{-0.25t}\right)^{-1}}$$

$$\Rightarrow \frac{dN}{dt} = -600\left(1+e^{-0.25t}\right)^{-2} \times \left(-0.25e^{-0.25t}\right)$$

$$\Rightarrow \frac{dN}{dt} = \frac{150\,e^{-0.25t}}{\left(1+e^{-0.25t}\right)^2}$$

BUT
$$\boxed{\begin{aligned} 1+e^{-0.25t} &= \frac{600}{N} \\ e^{-0.25t} &= \frac{600}{N} - 1 \end{aligned}}$$

$$\Rightarrow \frac{dN}{dt} = \frac{150\left(\frac{600}{N}-1\right)}{\frac{360000}{N^2}}$$

MULTIPLY TOP/BOTTOM BY N^2

$$\Rightarrow \frac{dN}{dt} = \frac{150\left(600N - N^2\right)}{360000}$$

$$\Rightarrow \frac{dN}{dt} = \frac{90000N}{360000} - \frac{150N^2}{360000}$$

$$\Rightarrow \frac{dN}{dt} = \frac{1}{4}N - \frac{1}{2400}N^2$$

AS REQUIRED

d) LET $\frac{dN}{dt} = f(N)$

RATE OF GROWTH

$$\therefore \boxed{f(N) = \frac{1}{4}N - \frac{1}{2400}N^2}$$

C3, IYGB, PAPER W

$$f'(N) = \frac{1}{4} - \frac{1}{1200}N$$

$$0 = \frac{1}{4} - \frac{1}{1200}N$$

$$N = 300$$

$\therefore$ MAX GROWTH AT $t=0$