C3, IYGB, PAPER T

1. a) $$f(x) = \frac{e^x - 1}{e^x + 1}$$ ← This is of course $\tanh\frac{x}{2}$ which of course is odd

I) $$f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} = \text{... MULTIPLY "TOP \& BOTTOM" BY } e^x$$

$$= \frac{e^{-x}e^{x} - e^x}{e^{-x}e^x + e^x} = \frac{1 - e^x}{1 + e^x} = \frac{-(e^x - 1)}{e^x + 1}$$

$$= -\frac{e^x - 1}{e^x + 1} = -f(x)$$, IE ODD // AS REQUIRED

II) $$f'(x) = \frac{d}{dx}\left(\frac{e^x - 1}{e^x + 1}\right) = \frac{(e^x+1)e^x - (e^x - 1)e^x}{(e^x+1)^2}$$

$$= \frac{e^{2x} + e^x - e^{2x} + e^x}{(e^x+1)^2} = \frac{2e^x}{(e^x+1)^2}$$ // AS REQUIRED

b) EITHER $f'(x)$ IS POSITIVE FOR ALL x, SO $f(x)$ IS INCREASING SO IT WILL HAVE AN INVERSE AS IT IS A ONE TO ONE

OR

$f'(x) \neq 0 \Rightarrow$ NO STATIONARY POINTS + ODD $\Big\} \Rightarrow$ ONE TO ONE SO INVERTIBLE

c) LET $y = f(x) = \dfrac{e^x - 1}{e^x + 1}$

$$\Rightarrow ye^x + y = e^x - 1$$

$$\Rightarrow y + 1 = e^x - ye^x$$

$$\Rightarrow y + 1 = e^x(1 - y)$$

$$\Rightarrow e^x = \frac{1+y}{1-y}$$

$$\Rightarrow x = \ln\left(\frac{1+y}{1-y}\right)$$

$$\therefore f^{-1}(x) = \ln\left(\frac{1+x}{1-x}\right)$$ //

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c) $f(g(x)) = \dfrac{x^2+6x+8}{x^2+6x+10}$ $\qquad f(x) = \dfrac{e^x-1}{e^x+1}$

$f^{-1}(f(g(x))) = f^{-1}\left(\dfrac{x^2+6x+8}{x^2+6x+10}\right)$ $\qquad f^{-1}(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

"CANCEL"

$\Rightarrow g(x) = \ln\left(\dfrac{1+\frac{x^2+6x+8}{x^2+6x+10}}{1-\frac{x^2+6x+8}{x^2+6x+10}}\right)$

MULTIPLY "TOP & BOTTOM" IN THE ARGUMENT OF THE LOG BY $x^2+6x+10$

$\Rightarrow g(x) = \ln\left[\dfrac{(x^2+6x+10)+(x^2+6x+8)}{(x^2+6x+10)-(x^2+6x+8)}\right]$

$\Rightarrow g(x) = \ln\left(\dfrac{2x^2+12x+18}{2}\right) = \ln(x^2+6x+9) = \ln(x+3)^2$

$\therefore g(x) = 2\ln(x+3)$

2. I) $\dfrac{d}{dx}\left[\dfrac{x-4}{\sqrt{x}+2}\right] = \dfrac{(x^{\frac{1}{2}}+2)\times 1-(x-4)\times\frac{1}{2}x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}+2)^2}$

$= \dfrac{x^{\frac{1}{2}}+2-\frac{1}{2}x^{\frac{1}{2}}+2x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}+2)^2} = \dfrac{\frac{1}{2}x^{\frac{1}{2}}+2+2x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}+2)^2}$

$= \dfrac{\frac{1}{2}x^{-\frac{1}{2}}\left[x+4x^{\frac{1}{2}}+4\right]}{(x^{\frac{1}{2}}+2)^2} = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}(x+4x^{\frac{1}{2}}+4)}{(x^{\frac{1}{2}})^2+2\times 2\times x^{\frac{1}{2}}+4}$

$= \dfrac{\frac{1}{2}x^{-\frac{1}{2}}(\cancel{x+4x^{\frac{1}{2}}+4})}{\cancel{x+4x^{\frac{1}{2}}+4}} = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{1}{2\sqrt{x}}$

WORKS SLIGHTLY BETTER AS

$\dfrac{d}{dx}\left(\dfrac{x-4}{\sqrt{x}+2}\right) = \dfrac{d}{dx}\left[\dfrac{(x-4)(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}\right] = \dfrac{d}{dx}\left[\dfrac{\cancel{(x-4)}(\sqrt{x}-2)}{\cancel{x-4}}\right]$

$= \dfrac{d}{dx}\left(x^{\frac{1}{2}}-2\right) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

II) $\frac{d}{dx}\left[\frac{4x-8\sqrt{x}+3}{(\sqrt{x}-1)^2}\right] = \frac{d}{dx}\left[\frac{4x-8x^{\frac{1}{2}}+3}{(x^{\frac{1}{2}}-1)^2}\right]$

$$= \frac{(x^{\frac{1}{2}}-1)^{\cancel{2}\,1}(4-4x^{-\frac{1}{2}}) - (4x-8x^{\frac{1}{2}}+3)\times 2\cancel{(x^{\frac{1}{2}}-1)}\times\frac{1}{2}x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^{\cancel{4}\,3}}$$

$$= \frac{(x^{\frac{1}{2}}-1)(4-4x^{-\frac{1}{2}}) - x^{-\frac{1}{2}}(4x-8x^{\frac{1}{2}}+3)}{(x^{\frac{1}{2}}-1)^3}$$

$$= \frac{\cancel{4x^{\frac{1}{2}}}\cancel{-4}\cancel{-4}+4x^{-\frac{1}{2}}\cancel{-4x^{\frac{1}{2}}}\cancel{+8}-3x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^3} = \frac{x^{-\frac{1}{2}}}{(x^{\frac{1}{2}}-1)^3}$$

$$= \frac{1}{\sqrt{x}(\sqrt{x}-1)^3}$$

AS REQUIRED

ALTERNATIVE

$$\frac{d}{dx}\left[\frac{4x-8\sqrt{x}+3}{(\sqrt{x}-1)^2}\right] = \frac{d}{dx}\left[\frac{4[x-2\sqrt{x}+1]-1}{(x-2\sqrt{x}-1)}\right]$$

$$= \frac{d}{dx}\left[4-\frac{1}{(\sqrt{x}-1)^2}\right] = \frac{d}{dx}\left[4-(x^{\frac{1}{2}}-1)^{-2}\right]$$

$$= 2(x^{\frac{1}{2}}-1)^{-3}\times\frac{1}{2}x^{-\frac{1}{2}} = x^{-\frac{1}{2}}(x^{\frac{1}{2}}-1)^{-3} = \frac{1}{\sqrt{x}(\sqrt{x}-1)^3}$$

AS BEFORE

3. LET $\psi = 4\operatorname{arccot}2 + \arctan\frac{24}{7}$

$\qquad\qquad\qquad\theta \qquad\qquad \phi$

$$\Rightarrow \psi = 4\theta + \phi$$

$$\Rightarrow \cos\psi = \cos(4\theta + \phi)$$

$$\Rightarrow \cos\psi = \cos 4\theta\cos\phi - \sin 4\theta\sin\phi$$

$$\Rightarrow \cos\psi = (2\cos^2 2\theta - 1)^2\cos\phi - 2\sin 2\theta\cos 2\theta\sin\phi$$

$$\Rightarrow \cos\psi = \left[2\left[2\cos^2\theta - 1\right]^2 - 1\right]^2\cos\phi - 2(2\sin\theta\cos\theta)(2\cos^2\theta - 1)\sin\phi$$

$$\Rightarrow \cos\psi = \left[2\left[2\times\frac{4}{5} - 1\right]^2 - 1\right]^2 \times \frac{7}{25} - 2\left[2\times\frac{2}{5}\right]\left[2\times\frac{4}{5} - 1\right]\times\frac{24}{25}$$

$$\Rightarrow \cos\psi = \left[-\frac{49}{625}\right] - \frac{576}{625}$$

$$\Rightarrow \cos\psi = -1$$

$$\psi = \ldots -3\pi, -\pi, \pi, 3\pi, \ldots$$

BUT θ & ϕ ARE ACUTE $\Rightarrow 0 < 4\theta + \phi < 5\times\frac{\pi}{2}$

$$\Rightarrow 0 < \psi < \frac{5\pi}{2}$$

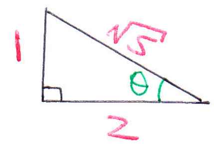
$$\Rightarrow \psi = \pi$$

$$\therefore 4\operatorname{arccot}2 + \arctan\frac{24}{7} = \pi$$

LET $\theta = \operatorname{arccot}2$

$\cot\theta = 2$

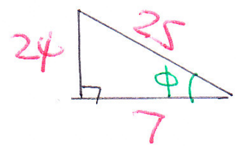
$\tan\theta = \frac{1}{2}$



$\sin\theta = \frac{1}{\sqrt{5}}$

$\cos\theta = \frac{2}{\sqrt{5}}$

LET $\phi = \arctan\frac{24}{7}$



$\sin\phi = \frac{24}{25}$

$\cos\phi = \frac{7}{25}$

ALTERNATIVE BY COMPLEX NUMBERS

FIRSTLY $4\operatorname{arccot}2 + \arctan\frac{24}{7} = 4\arctan\frac{1}{2} + \arctan\frac{24}{7}$

NOW CONSIDER $(2+i)^4(7+24i)$ ← $\arg(7+24i) = \arctan\frac{24}{7}$

$\arg(2+i) = \arctan\frac{1}{2}$

$$(2+i)^2 = 4 + 4i - 1 = 3 + 4i$$

$$(2+i)^4 = (3+4i)^2 = 9 + 24i - 16 = -7 + 24i$$

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HENCE $(2+i)^4(7+24i) = (-7+24i)(7+24i)$
$= -49-576$
$= -625.$

SO $(2+i)^4(7+24i) = -625$

$\arg\left[(2+i)^4(7+24i)\right] = \arg(-625)$

$\arg(2+i)^4 + \arg(7+24i) = \pi$

$4\arg(2+i) + \arg(7+24i) = \pi$

$4\arctan\frac{1}{2} + \arctan\frac{24}{7} = \pi$

$4\text{arcot}\,2 + \arctan\frac{24}{7} = \pi$

4. $3|x+1| - |x-4| \leq 11$

- SKETCH THE GRAPH OF $y = 3|x+1| - |x-4|$
- THE "CRITICAL VALUES" OF THE GRAPH ARE $x=-1$ & $x=4$

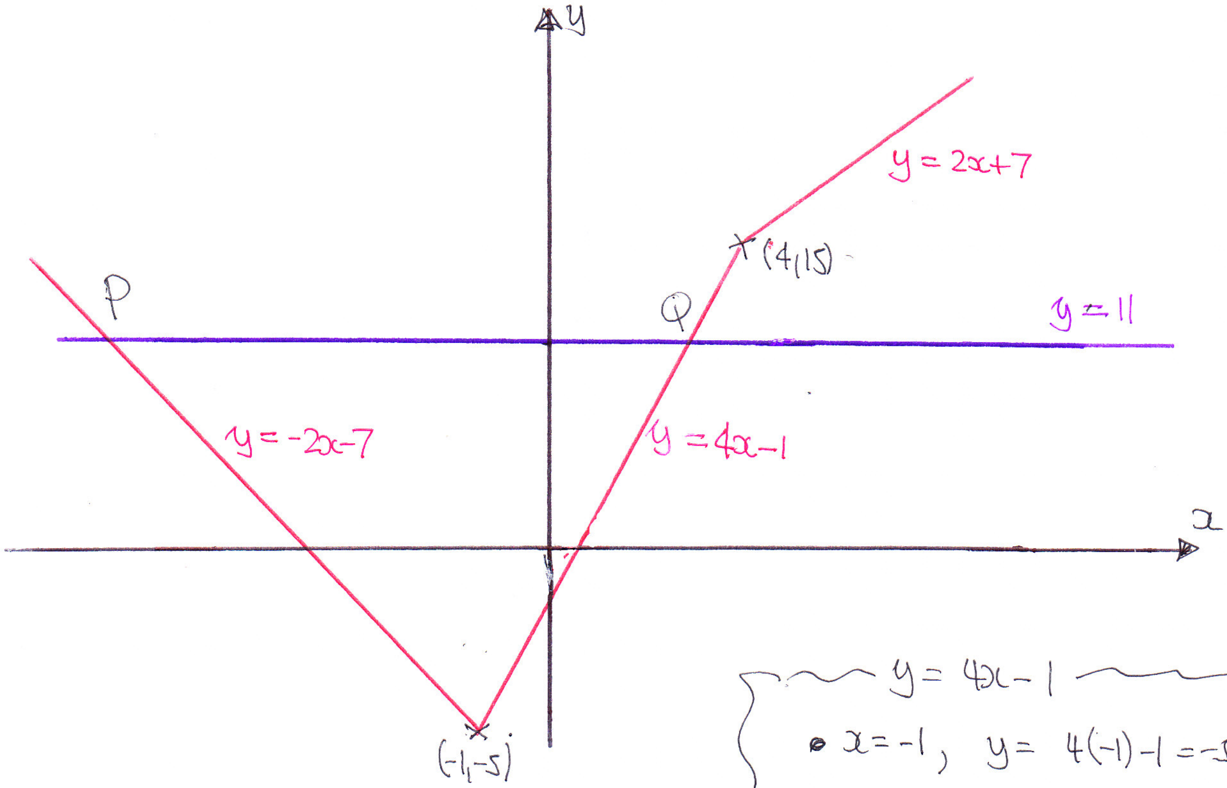
IF $x<-1 \Rightarrow y = 3(-x-1) - (-x+4)$
$\Rightarrow y = -3x-3+x-4$
$\Rightarrow y = -2x-7$

IF $1<x<4 \Rightarrow y = 3(x+1) - (-x+4)$
$\Rightarrow y = 3x+3+x-4$
$\Rightarrow y = 4x-1$

IF $x>4 \Rightarrow y = 3(x+1) - (x-4)$
$\Rightarrow y = 3x+3-x+4$
$\Rightarrow y = 2x+7$

C3, IYGB, PAPER T

SKETCH THE GRAPH



$y = 4x - 1$

- $x = -1, \quad y = 4(-1) - 1 = -5$
  I.E $(-1, -5)$
- $x = 4, \quad y = 4 \times 4 - 1 = 15$
  I.E $(4, 15)$

Thus

$$-2x - 7 = 11 \qquad \& \qquad 4x - 1 = 11$$
$$-2x = 18 \qquad\qquad 4x = 12$$
$$\boxed{x = -9} \leftarrow P \qquad\qquad \boxed{x = 3} \leftarrow Q$$

HENCE FROM "GRAPH"

$$-9 \leq x \leq 3$$

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5. a) $f(x) = e^{ax} + b$

$$\Rightarrow f\left[\ln\tfrac{3}{2}\right] = \frac{13}{4}$$

$$\Rightarrow e^{a\ln\frac{3}{2}} + b = \frac{13}{4}$$

$$\Rightarrow e^{\ln\left(\frac{3}{2}\right)^a} + b = \frac{13}{4}$$

$$\Rightarrow \left[e^{\ln\frac{3}{2}}\right]^a + b = \frac{13}{4}$$

$$\Rightarrow \left(\frac{3}{2}\right)^a + b = \frac{13}{4}$$

$$\Rightarrow b = \frac{13}{4} - \left(\frac{3}{2}\right)^a$$

AS REQUIRED

b) $f\left[\ln\left(\frac{2}{3}\right)\right] = \frac{13}{9}$

$$\Rightarrow e^{a\ln\frac{2}{3}} + b = \frac{13}{9}$$

$$\Rightarrow \left(e^{\ln\frac{2}{3}}\right)^a + b = \frac{13}{9}$$

$$\Rightarrow \left(\frac{2}{3}\right)^a + b = \frac{13}{9}$$

$$\Rightarrow b = \frac{13}{9} - \left(\frac{2}{3}\right)^a$$

- SOLVING SIMULTANEOUSLY

$$\Rightarrow \frac{13}{4} - \left(\frac{3}{2}\right)^a = \frac{13}{9} - \left(\frac{2}{3}\right)^a$$

$$\Rightarrow 0 = \left(\frac{3}{2}\right)^a - \left(\frac{2}{3}\right)^a - \frac{65}{36}$$

$$\Rightarrow 0 = \left(\frac{3}{2}\right)^a - \left(\frac{3}{2}\right)^{-a} - \frac{65}{36}$$

- LET $t = \left(\frac{3}{2}\right)^a$

$$\Rightarrow t - t^{-1} - \frac{65}{36} = 0$$

$$\Rightarrow t - \frac{1}{t} - \frac{65}{36} = 0$$

$$\Rightarrow t^2 - 1 - \frac{65}{36}t = 0$$

$$\Rightarrow 36t^2 - 36 - 65t = 0$$

$$\Rightarrow 36t^2 - 65t - 36 = 0$$

BY QUADRATIC FORMULA
OR
FACTORIZATION

$$\Rightarrow (4t - 9)(9t + 4) = 0$$

$\Rightarrow t = \begin{cases} 9/4 \\ -4/9 \end{cases}$

$\Rightarrow \left(\frac{3}{2}\right)^a = \begin{cases} 9/4 \\ \cancel{-\frac{4}{9}} \end{cases}$

$\Rightarrow a = 2$ (BY INSPECTION)

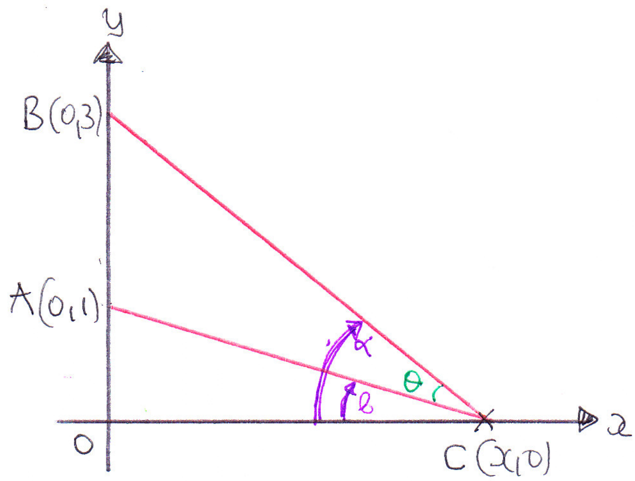
FINALLY

$b = \frac{13}{4} - \left(\frac{3}{2}\right)^a$

$b = \frac{13}{4} - \frac{9}{4}$

$b = 1$

6.



- $\tan\theta = \tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{\frac{|OB|}{|OC|} - \frac{|OA|}{|OC|}}{1 + \frac{|OB|}{|OC|} \times \frac{|OA|}{|OC|}}$

$= \dfrac{\frac{3}{x} - \frac{1}{x}}{1 + \frac{3}{x} \times \frac{1}{x}} = \dfrac{\frac{2}{x}}{1 + \frac{3}{x^2}} = \dfrac{2x}{x^2+3}$

- LET $f(x) = \dfrac{2x}{x^2+3} \qquad x > 0$

$f'(x) = \dfrac{(x^2+3)\times 2 - 2x(2x)}{(x^2+3)^2} = \dfrac{6 - 2x^2}{(x^2+3)^2}$

$f''(x) = \dfrac{(x^2+3)^2(-4x) - (6-2x^2) \times 2(x^2+3)(2x)}{(x^2+3)^4}$

$$\therefore f''(x) = \frac{-4x(x^2+3) - 4x(6-2x^2)}{(x^2+3)^3} = \frac{-4x^3 - 12x - 24x + 8x^3}{(x^2+3)^3}$$

$$= \frac{4x^3 - 36x}{(x^2+3)^3} = \frac{4x(x^2-9)}{(x^2+3)^3}$$

- FOR LOCAL MIN/MAX $f'(x) = 0$

$$6 - 2x^2 = 0$$
$$2x^2 = 6$$
$$x^2 = 3$$
$$x = +\sqrt{3}$$

- CHECK THE NATURE

$$f''(\sqrt{3}) = \frac{4\sqrt{3}(3-9)}{(3+3)^2} = -\frac{\sqrt{3}}{9} < 0 \quad \therefore \text{A LOCAL MAXIMUM}$$

$$f(0) = 0$$

$$\lim_{x \to +\infty} [f(x)] = 0$$

$\therefore$ A TRUE MAXIMUM AT $x = \sqrt{3}$

- $f(\sqrt{3}) = \dfrac{2\sqrt{3}}{3+3} = \dfrac{2\sqrt{3}}{6} = \dfrac{\sqrt{3}}{3}$

BUT $\tan\theta = f(x)$

SO $\tan\theta = \dfrac{\sqrt{3}}{3}$

$$\theta = \pi/6$$

AS REQUIRED

7. $\tan(x+y) = 2\tan(x-y)$

$$\Rightarrow \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{2\tan x - 2\tan y}{1 + \tan x \tan y}$$

$$\Rightarrow (\tan x + \tan y)(1 + \tan x \tan y) - (2\tan x - 2\tan y)(1 - \tan x \tan y) = 0$$

$$\Rightarrow \begin{matrix} \tan x + \tan^2 x \tan y + \tan y + \tan x \tan^2 y \\ - 2\tan x + 2\tan^2 x \tan y + 2\tan y - 2\tan x \tan^2 y \end{matrix} = 0$$

$$\Rightarrow -\tan x + 3\tan^2 x \tan y + 3\tan y - \tan x \tan^2 y = 0$$

$$\Rightarrow 3\tan^2 x \tan y + 3\tan y - \tan x \tan^2 y - \tan x = 0$$

$$\Rightarrow 3\tan y(\tan^2 x + 1) - \tan x(\tan^2 y + 1) = 0$$

$$\Rightarrow 3\tan y \sec^2 x - \tan x \sec^2 y = 0$$

$$\Rightarrow 3\tan y \sec^2 x = \tan x \sec^2 y$$

$$\Rightarrow \frac{3\cancel{\tan y \sec^2 x}}{\cancel{\tan y \sec^2 x}} = \frac{\tan x \sec^2 y}{\tan y \sec^2 x}$$

$$\Rightarrow 3 = \frac{\frac{\sin x}{\cos x}}{\frac{\sin y}{\cos y}} \times \frac{\cos^2 x}{\cos^2 y}$$

$$\Rightarrow 3 = \frac{\sin x \cancel{\cos y}}{\cancel{\cos x} \sin y} \times \frac{\cos^{\cancel{2}} x}{\cos^{\cancel{2}} y}$$

$$\Rightarrow 3 = \frac{\sin x \cos x}{\sin y \cos y}$$

$$\Rightarrow 3 = \frac{2\sin x \cos x}{2\sin y \cos y}$$

$$\Rightarrow \frac{\sin 2x}{\cos 2y} = 3$$

AS REQUIRED