C3, IYGB, PAPER I

1. a)

$$4\arccos x = x+1$$

$$4\arccos x - x - 1 = 0$$

LET $f(x) = 4\arccos x - x - 1$

$$f(0.5) = 2.6887\ldots > 0$$
$$f(1) = -2 < 0$$

AS $f(x)$ IS CONTINUOUS AND $f(0.5)f(1) < 0$, $\exists\, \alpha \in (0.5, 1)$: $f(\alpha) = 0$

(MUST BE IN RADIANS)

b)

$$4\arccos x = x+1$$

$$\arccos x = \frac{x+1}{4}$$

$$\cos(\arccos x) = \cos\left(\frac{x+1}{4}\right)$$

$$\boxed{x = \cos\left(\frac{x+1}{4}\right)}$$

$$x_{n+1} = \cos\left(\frac{x_n+1}{4}\right)$$

$x_0 = 1$

$x_1 \simeq 0.87758$

$x_2 \simeq 0.89184$

$x_3 \simeq 0.89022$

$x_4 \simeq 0.89041$

$x_5 \simeq 0.89039$

c) $\alpha = 0.8904$ to 4 d.p

2.

$$f(x) = \frac{4x-1}{2(x-1)} - \frac{3}{2(x-1)(2x-1)} - \frac{2}{1}$$

$$= \frac{(4x-1)(2x-1) - 3 - 2\times 2(x-1)(2x-1)}{2(x-1)(2x-1)}$$

$$= \frac{8x^2 - 4x - 2x + 1 - 3 - 4(2x^2 - x - 2x + 1)}{2(x-1)(2x-1)}$$

$$= \frac{8x^2 - 6x - 2 - 4(2x^2 - 3x + 1)}{2(x-1)(2x-1)} = \frac{\cancel{8x^2} - 6x - 2 - \cancel{8x^2} + 12x - 4}{2(x-1)(2x-1)}$$

$$= \frac{6x-6}{2(x-1)(2x-1)} = \frac{6\cancel{(x-1)}}{2\cancel{(x-1)}(2x-1)} = \frac{6}{2(2x-1)} = \frac{3}{2x-1}$$

$k=3$

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3. $e^{x} - e^{-x} = \frac{3}{2}$

$\Rightarrow e^{x} - \frac{1}{e^{x}} = \frac{3}{2}$

$\Rightarrow y - \frac{1}{y} = \frac{3}{2}$ $\boxed{y = e^{x}}$

$\Rightarrow y^2 - 1 = \frac{3}{2}y$

$\Rightarrow 2y^2 - 2 = 3y$

$\Rightarrow 2y^2 - 3y - 2 = 0$

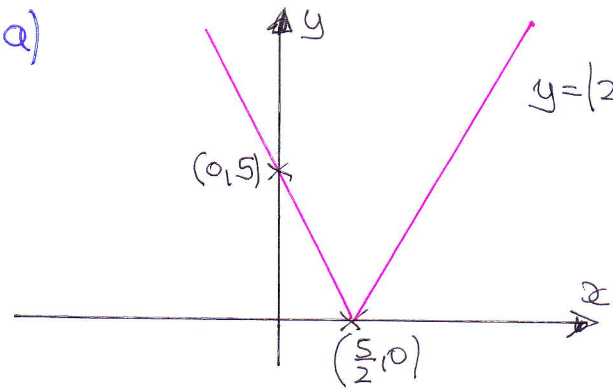
$\Rightarrow (2y+1)(y-2) = 0$

$\Rightarrow y = \begin{cases} 2 \\ -\frac{1}{2} \end{cases}$

$\Rightarrow e^{x} = \begin{cases} 2 \\ \cancel{-\frac{1}{2}} \end{cases}$

$\Rightarrow x = \ln 2$

4. a)



b) $f(x) = x$

$|2x-5| = x$

EITHER $2x-5 = x$ OR $2x-5 = -x$

$x = 5$ OR $3x = 5$

$x = \frac{5}{3}$

$\therefore x = \begin{cases} 5 \\ \frac{5}{3} \end{cases}$ BOTH OK

c) $g(x) = x^2 - x$

$\Rightarrow f(g(x)) = 7$

$\Rightarrow f(x^2 - x) = 7$

$\Rightarrow |2(x^2 - x) - 5| = 7$

$\Rightarrow |2x^2 - 2x - 5| = 7$

EITHER

$\Rightarrow 2x^2 - 2x - 5 = 7$

$\Rightarrow 2x^2 - 2x - 12 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x-3)(x+2) = 0$

$x = \begin{cases} 3 \\ -2 \end{cases}$

OR

$\Rightarrow 2x^2 - 2x - 5 = -7$

$\Rightarrow 2x^2 - 2x + 2 = 0$

$\Rightarrow x^2 - x + 1 = 0$

NO SOLUTIONS $b^2 - 4ac$

$= (-1)^2 - 4 \times 1 \times 1 = -3 < 0$

$\therefore$ IRREDUCIBLE

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5. a) $f(x) = 27x^3 - 9x - 2$

$f\left(-\frac{1}{3}\right) = 27\left(-\frac{1}{3}\right)^3 - 9\left(-\frac{1}{3}\right) - 2 = 27\left(\frac{1}{27}\right) + 3 - 2 = 0$

$\therefore$ INDEED A FACTOR OF $f(x)$

b) $36\cos 2\theta \cos\theta + 9\sin 2\theta \sin\theta = 4$

(SINCE WE ARE ASKED FOR POSSIBLE VALUES OF $\cos\theta$, AIM TO CHANGE INTO COSINES)

$\Rightarrow 36(2\cos^2\theta - 1)\cos\theta + 9(2\sin\theta\cos\theta)\sin\theta = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18\sin^2\theta\cos\theta = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18(1 - \cos^2\theta)\cos\theta = 4$

$\Rightarrow 72\cos^3\theta - 36\cos\theta + 18\cos\theta - 18\cos^3\theta = 4$

$\Rightarrow 54\cos^3\theta - 18\cos\theta - 4 = 0$

$\Rightarrow 27\cos^3\theta - 9\cos\theta - 2 = 0$

LET $\cos\theta = x$

$\Rightarrow 27x^3 - 9x - 2 = 0 \leftarrow$ PART (a)

FACTORIZE BY INSPECTION OR LONG DIVISION

$$\begin{array}{r}
9x^2 - 3x - 2 \\
3x+1 \overline{\smash{)}\,27x^3 + 0x^2 - 9x - 2} \\
-27x^3 - 9x^2 \\
\hline
-9x^2 - 9x - 2 \\
9x^2 + 3x \\
\hline
-6x - 2 \\
6x + 2 \\
\hline
0
\end{array}$$

$\Rightarrow (3x+1)(9x^2 - 3x - 2) = 0$

$\Rightarrow (3x+1)(3x-2)(3x+1) = 0$

$\Rightarrow x = \begin{cases} -\frac{1}{3} & \text{(TWICE)} \\ \frac{2}{3} \end{cases}$

$\Rightarrow \cos\theta = \begin{cases} -\frac{1}{3} \\ \frac{2}{3} \end{cases}$

6. a) $f(x) = \dfrac{1}{1+\tan x} = (1+\tan x)^{-1}$

$f'(x) = -(1+\tan x)^{-2} \times \sec^2 x$

$= -\dfrac{\sec^2 x}{(1+\tan x)^2}$

(COULD HAVE ALSO USED QUOTIENT RULE)

$f'(x) < 0$ SINCE ~~IT~~ THE FRACTION CONTAINS SQUARED QUANTITIES & THERE IS A MINUS IN FRONT

$\therefore f(x)$ IS A DECREASING FUNCTION, SO ONE TO ONE

b) LET $y = \dfrac{1}{1+\tan x}$

$\Rightarrow y + y\tan x = 1$

$\Rightarrow y\tan x = 1-y$

$\Rightarrow \tan x = \dfrac{1-y}{y}$

$\Rightarrow x = \arctan\left(\dfrac{1-y}{y}\right)$

$\therefore f^{-1}(x) = \arctan\left(\dfrac{1-x}{x}\right)$

c) AS THE FUNCTION IS DECREASING THE "ENDPOINTS" OF THE DOMAIN WILL DETERMINE THE RANGE

$\therefore f(0) = \dfrac{1}{1+0} = 1$

$f\left(\frac{\pi}{2}\right) = \dfrac{1}{1+\text{"}\infty\text{"}} = 0$

$\therefore 0 < f(x) \leqslant 1$

7. a) $x = \sec\left(\frac{1}{2}y\right)$

$\Rightarrow \dfrac{dx}{dy} = \frac{1}{2}\sec\left(\frac{1}{2}y\right)\tan\left(\frac{1}{2}y\right)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\frac{1}{2}\sec\left(\frac{1}{2}y\right)\tan\left(\frac{1}{2}y\right)}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{\sec\left(\frac{1}{2}y\right)\tan\left(\frac{1}{2}y\right)}$

NOW

$1 + \tan^2\frac{1}{2}y = \sec^2\frac{1}{2}y$

$\tan^2\frac{1}{2}y = \sec^2\frac{1}{2}y - 1$

$\tan\frac{1}{2}y = \pm\sqrt{\sec^2\frac{1}{2}y - 1}$

BUT $0 \leqslant y < \pi$

$0 \leqslant \frac{1}{2}y < \frac{\pi}{2}$

$\tan\frac{1}{2}y > 0$

$\therefore \tan\frac{1}{2}y = \sqrt{\sec^2\frac{1}{2}y - 1}$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec\frac{1}{2}y\sqrt{\sec^2\frac{1}{2}y - 1}}$$

BUT $x = \sec\frac{1}{2}y$

$$\therefore \frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}}$$ AS REQUIRED

b) $\frac{dy}{dx} = \sqrt{2}$

$$\Rightarrow \frac{2}{x\sqrt{x^2-1}} = \sqrt{2}$$

$$\Rightarrow \left(\frac{2}{x\sqrt{x^2-1}}\right)^2 = \left(\sqrt{2}\right)^2$$

$$\Rightarrow \frac{4}{x^2(x^2-1)} = 2$$

$$\Rightarrow \frac{4}{x^4-x^2} = 2$$

$$\Rightarrow 4 = 2x^4 - 2x^2$$

$$\Rightarrow 2x^4 - 2x^2 - 4 = 0$$

$$\Rightarrow x^4 - x^2 - 2 = 0$$

$$\Rightarrow (x^2+1)(x^2-2) = 0$$

$$\Rightarrow x^2 = \cancel{-1},\ 2$$

$$\Rightarrow x = \sqrt{2},\ -\sqrt{2}$$

IF $x = \sqrt{2}$

$$\sqrt{2} = \sec\frac{y}{2}$$

$$\frac{1}{\sqrt{2}} = \cos\frac{y}{2}$$

$$\frac{y}{2} = \frac{\pi}{4}$$ (ONLY VALUE IN RANGE)

$$y = \frac{\pi}{2}$$

IF $x = -\sqrt{2}$

$$-\sqrt{2} = \sec\frac{y}{2}$$

$$-\frac{1}{\sqrt{2}} = \cos\frac{y}{2}$$

$$\frac{y}{2} = \frac{3\pi}{4}$$

$y = \dots$ too big, NO SOLUTION

$\therefore \left(\sqrt{2}, \frac{\pi}{2}\right)$

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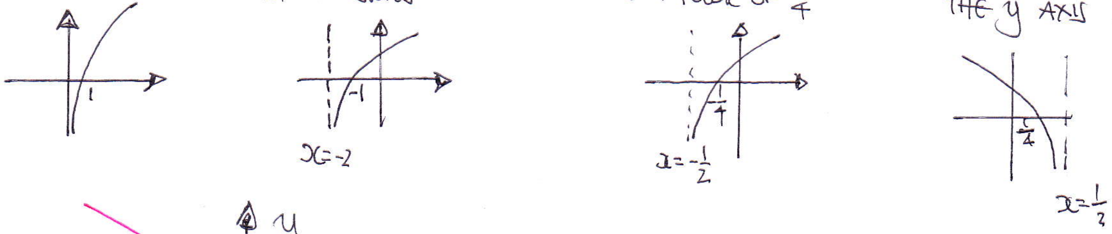
8.

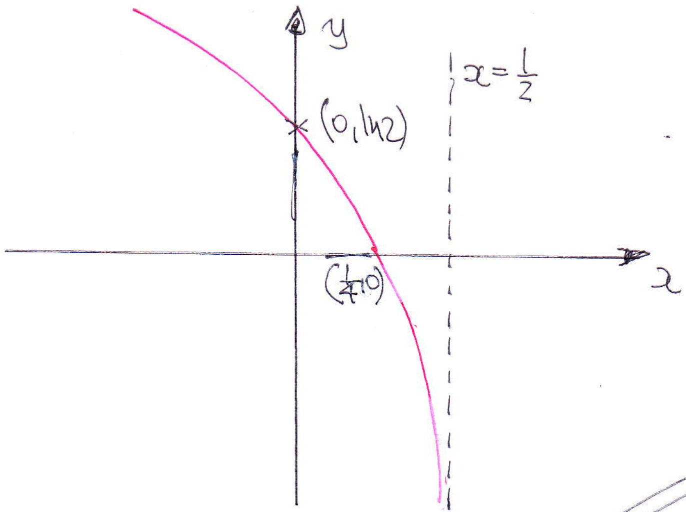
$\ln x \longmapsto \ln(x+2) \longmapsto \ln(4x+2) \longmapsto \ln(4(-x)+2)$

TRANSLATION "LEFT" BY 2 UNITS

HORIZONTAL STRETCH BY FACTOR OF $\frac{1}{4}$

REFLECTION IN THE y AXIS





$y = \ln(2-4x)$
when $x=0$
$y = \ln 2$

9. METHOD A

$A(\pi, -8) \Rightarrow -8 = P + Q\sec\pi$
$B(2\pi, -2) \Rightarrow -2 = P + Q\sec 2\pi$

$\Rightarrow \begin{cases} -8 = P - Q \\ -2 = P + Q \end{cases} \Rightarrow$ ADD $2P = -10$

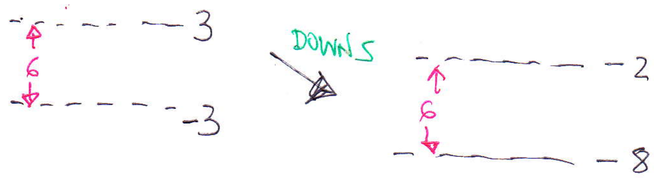
$P = -5$

$\therefore Q = 3$

METHOD B BY INSPECTION & TRANSFORMATIONS

THE GRAPH OF $\sec x$ "LIVES" ABOVE 1 & BELOW −1 IE THERE IS A GAP OF 2 WHERE THERE IS NO GRAPH — HERE THE GAP IS FROM −8 to −2 IE 6 $\therefore$ STRETCHING OF S.F 3 $\Longrightarrow Q = 3$

BUT THERE IS ALSO A TRANSLATION DOWN BY 5 IE $P = -5$



10. $y = 10e^{-kx}$

$\Rightarrow \frac{dy}{dx} = -10ke^{-kx}$ (SUBSTITUTE ONLY ONE OF THE "k"s IN)

$\Rightarrow \ln\frac{\sqrt{2}}{2} = -10 \times \frac{1}{5}\ln 2\, e^{-kx}$

$\Rightarrow \ln\frac{\sqrt{2}}{2} = -2\ln 2\, e^{-kx}$

$\Rightarrow \ln\left(\frac{1}{\sqrt{2}}\right) = -2\ln 2\, e^{-kx}$

NOTE $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\Rightarrow \ln 2^{-\frac{1}{2}} = -2\ln 2\, e^{-kx}$

$\Rightarrow -\frac{1}{2}\cancel{\ln 2} = -2\cancel{\ln 2}\, e^{-kx}$

$\Rightarrow e^{-kx} = \frac{1}{4}$

$\Rightarrow e^{kx} = 4$

$\Rightarrow kx = \ln 4$

$\Rightarrow \left(\frac{1}{5}\cancel{\ln 2}\right)x = 2\cancel{\ln 2}$

$\Rightarrow \frac{1}{5}x = 2$

$\Rightarrow x = 10$