VECTOR DIFFERENTIAL EQUATIONS

First Order

Question 1 (**+)

A particle P is moving on the Cartesian plane so that its position vector $\mathbf{r}$ m at time t s satisfies the differential equation

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}.$$

When $t = 0$, $\mathbf{r} \cdot \mathbf{i} = 0$ and $\mathbf{r} \wedge \mathbf{i} = 2\mathbf{j} - \mathbf{k}$.

Express $\mathbf{r}$ in terms of t.

$$\boxed{\mathbf{r} = (\mathbf{j} + 2\mathbf{k})\mathrm{e}^{t}}$$

$\frac{d\underline{r}}{dt} = \underline{r}$ "STANDARD EXPONENTIAL GROWTH"

$\boxed{\underline{r} = \underline{A}e^{t}}$ LET $\underline{A} = (a,b,c)$

$t=0$ $\underline{r} \cdot (1,0,0) = 0$

$\underline{A} \cdot (1,0,0) = 0$

$(a,b,c) \cdot (1,0,0) = 0$

$\therefore \boxed{a=0}$

$\underline{r} \wedge (1,0,0) = (0,2,-1)$

$\underline{A} \wedge (1,0,0) = (0,2,-1)$

$\begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (0,2,-1)$

$(0,c,-b) = (0,2,-1)$

$\therefore c=2$, $b=1$

$\therefore \underline{r} = (\underline{j} + 2\underline{k})e^{t}$

Question 2 (**+)

A particle moves in a plane so that its position vector, $\mathbf{r}$ m at time t s, satisfies the differential equation

$$\frac{d\mathbf{r}}{dt} + \mathbf{r} = 2t\mathbf{i} - \mathrm{e}^{-t}\,\mathbf{j}.$$

When $t = 0$ the particle is at the point with position vector $(\mathbf{i} - 2\mathbf{j})$ m.

Express $\mathbf{r}$ in terms of t.

$\boxed{}$, $\boxed{\mathbf{r} = \left(2t - 2 + 3\mathrm{e}^{-t}\right)\mathbf{i} - \left(t\,\mathrm{e}^{-t} + 2\,\mathrm{e}^{-t}\right)\mathbf{j}}$

DEALING WITH EACH COMPONENT SEPARATELY — LET $\underline{r} = (x, y)$

$\frac{dx}{dt} + x = 2t$ $\qquad$ $\frac{dy}{dt} + y = -e^{-t}$

THE INTEGRATING FACTOR FOR BOTH SIDES IS

$\text{I.F} = e^{\int 1\,dt} = e^{t}$

HENCE WE OBTAIN FOR EACH COMPONENT

$\Rightarrow \frac{d}{dt}(xe^{t}) = 2te^{t}$

$\Rightarrow xe^{t} = \int 2te^{t}\,dt$ (BY PARTS)

$\Rightarrow xe^{2t} = 2te^{t} - 2e^{t} + C$

$\Rightarrow \boxed{x = 2t - 2 + Ce^{-2t}}$

$t = 0,\ x = 1$

$1 = -2 + C$

$C = 3$

$\Rightarrow \frac{d}{dt}(ye^{t}) = -e^{-t}e^{t}$

$\Rightarrow \frac{d}{dt}(ye^{t}) = -1$

$\Rightarrow ye^{t} = -t + k$

$\Rightarrow \boxed{y = -te^{-t} + ke^{-t}}$

$t = 0 \quad y = -2$

$-2 = 0 + k$

$k = -2$

COLLECTING THE RESULTS

$\underline{r} = (x, y) = \left(2t - 2 + 3e^{-2t},\ -te^{-t} - 2e^{-t}\right)$ //

Question 3 (***)

A particle moves in a plane so that its position vector, $\mathbf{r}$ m at time t s, satisfies the differential equation

$$\frac{d\mathbf{r}}{dt} + (\tan t)\mathbf{r} = \left(2\sin t \cos^2 t\right)\mathbf{i} + \left(\cos^2 t\right)\mathbf{j}, \qquad 0 \le t < \frac{\pi}{2}.$$

When $t = 0$ the particle is at the point with position vector $\mathbf{j}$ m.

Express $\mathbf{r}$ in terms of t.

$\boxed{}$, $\boxed{\mathbf{r} = \left(\sin^2 t \cos t\right)\mathbf{i} + \left(\sin t \cos t + \cos t\right)\mathbf{j}}$

SEPARATE THE O.D.E INTO COMPONENTS — LET $\underline{r} = (x, y)$

$\frac{dx}{dt} + x\tan t = 2\sin t\cos^2 t$ $\qquad$ $\frac{dy}{dt} + y\tan t = \cos^2 t$

INTEGRATING FACTOR FOR BOTH O.D.Es IS

$\text{IF} = e^{\int \tan t\, dt} = e^{\ln|\sec t|} = \sec t = \frac{1}{\cos t}$

HENCE WE OBTAIN FOR EACH O.D.E

$\frac{d}{dt}(x\sec t) = 2\sin t\cos^2 t\sec t$

$\frac{d}{dt}(x\sec t) = 2\sin t\cos t$

$x\sec t = \int 2\sin t\cos t\, dt$

$\frac{x}{\cos t} = \sin^2 t + A$

$x = \sin^2 t\cos t + A\cos t$

$t=0, x=0$

$0 = 0 + A$

$A = 0$

$\therefore x = \sin^2 t\cos t$

$\frac{d}{dt}(y\sec t) = \cos^2 t\sec t$

$\frac{d}{dt}(y\sec t) = \cos t$

$y\sec t = \int \cos t\, dt$

$\frac{y}{\cos t} = \sin t + B$

$y = \sin t\cos t + B\cos t$

$t=0, y=1$

$1 = 0 + B$

$B = 1$

$\therefore y = \sin t\cos t + \cos t$

$\therefore \underline{r} = (x, y) = \left[\sin^2 t\cos t, \sin t\cos t + \cos t\right]$

Second Order

Question 1 (**)

A particle P is moving on the Cartesian plane so that its position vector $\mathbf{r}$ m at time t s satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{r}}{dt}.$$

When $t=0$, P has position vector $(\mathbf{i}+\mathbf{j})$ m and moving with velocity $(2\mathbf{i}-\mathbf{j})\,\text{ms}^{-1}$.

Express $\mathbf{r}$ in terms of t.

$$\boxed{\mathbf{r} = (-\mathbf{i}+2\mathbf{j}) + (2\mathbf{i}-\mathbf{j})e^t \quad \text{or} \quad \mathbf{r} = \left(2e^t - 1\right)\mathbf{i} + \left(2 - e^t\right)\mathbf{j}}$$

$$\frac{d^2\underline{r}}{dt^2} = \frac{d\underline{r}}{dt}$$

$$\frac{d^2\underline{r}}{dt^2} - \frac{d\underline{r}}{dt} = \underline{0}$$

AUXILIARY EQUATION

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = \begin{cases} 0 \\ 1 \end{cases}$$

$$\therefore \boxed{\underline{r} = \underline{A} + \underline{B}e^t}$$

$$\boxed{\dot{\underline{r}} = \underline{v} = \underline{B}e^t}$$

APPLY CONDITIONS

- $t=0$, $\underline{r} = (1,1) \Rightarrow (1,1) = \underline{A} + \underline{B}$
- $t=0$, $\underline{v} = (2,-1) \Rightarrow (2,-1) = \underline{B}$

$$\therefore \underline{A} = (-1,2)$$

$$\therefore \underline{r} = (-1,2) + (2,-1)e^t$$

OR

$$\underline{r} = \left(2e^t - 1,\ 2 - e^t\right)$$

Question 2 (**+)

A particle P is moving on the Cartesian plane so that its position vector $\mathbf{r}$ m at time t s satisfies the differential equation

$$\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} = 6(\mathbf{r} + t\mathbf{i} - 2\mathbf{j}).$$

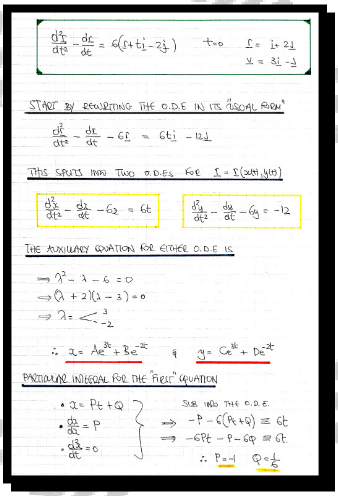
When $t = 0$, P has position vector $(\mathbf{i} + 2\mathbf{j})$ m and moving with velocity $(3\mathbf{i} - \mathbf{j})$ ms^{-1}.

Express $\mathbf{r}$ in terms of t.

$\boxed{}$, $\boxed{\mathbf{r} = \frac{1}{15}e^{3t}(17\mathbf{i} - 3\mathbf{j}) - \frac{1}{10}e^{-2t}(3\mathbf{i} - 2\mathbf{j})e^{t} + \frac{1}{6}(\mathbf{i} + 12\mathbf{j}) - t\mathbf{i}}$

or

$\mathbf{r} = \left(\frac{17}{15}e^{3t} - \frac{3}{10}e^{-2t} - t + \frac{1}{6}\right)\mathbf{i} + \left(-\frac{1}{5}e^{3t} + \frac{1}{5}e^{-2t} + 2\right)\mathbf{j}$



$\boxed{\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} = 6(\mathbf{r} + t\mathbf{i} - 2\mathbf{j}) \qquad t=0 \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j}, \quad \mathbf{v} = 3\mathbf{i} - \mathbf{j}}$

START BY REWRITING THE O.D.E IN ITS "USUAL FORM"

$\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} - 6\mathbf{r} = 6t\mathbf{i} - 12\mathbf{j}$

THIS SPLITS INTO TWO O.D.Es FOR $\mathbf{r} = \mathbf{r}(x(t), y(t))$

$\boxed{\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 6t}$ $\boxed{\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = -12}$

THE AUXILIARY EQUATION FOR EITHER O.D.E IS

$\Rightarrow \lambda^2 - \lambda - 6 = 0$

$\Rightarrow (\lambda + 2)(\lambda - 3) = 0$

$\Rightarrow \lambda = <^{3}_{-2}$

$\therefore\ x = Ae^{3t} + Be^{-2t}$ & $y = Ce^{3t} + De^{-2t}$

PARTICULAR INTEGRAL FOR THE "FIRST" EQUATION

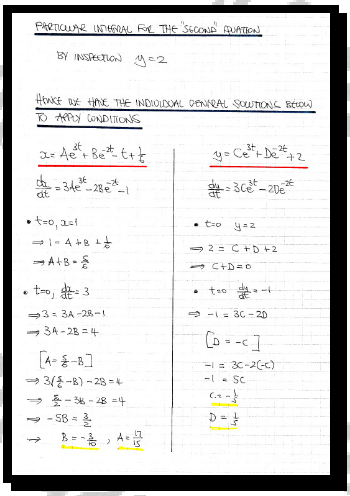
- $x = Pt + Q$
- $\frac{dx}{dx} = P$
- $\frac{d^2x}{dt} = 0$

SUB INTO THE O.D.E.

$\Rightarrow -P - 6(Pt + Q) \equiv 6t$

$\Rightarrow -6Pt - P - 6Q \equiv 6t$

$\therefore\ P = -1 \quad Q = \frac{1}{6}$



PARTICULAR INTEGRAL FOR THE "SECOND" EQUATION

BY INSPECTION $y = 2$

HENCE WE HAVE THE INDIVIDUAL GENERAL SOLUTIONS BELOW TO APPLY CONDITIONS

$x = Ae^{3t} + Be^{-2t} - t + \frac{1}{6}$

$\frac{dx}{dt} = 3Ae^{3t} - 2Be^{-2t} - 1$

- $t=0,\ x=1$

$\Rightarrow 1 = A + B + \frac{1}{6}$

$\Rightarrow A + B = \frac{5}{6}$

- $t=0,\ \frac{dx}{dt} = 3$

$\Rightarrow 3 = 3A - 2B - 1$

$\Rightarrow 3A - 2B = 4$

$[A = \frac{5}{6} - B]$

$\Rightarrow 3(\frac{5}{6} - B) - 2B = 4$

$\Rightarrow \frac{5}{2} - 3B - 2B = 4$

$\Rightarrow -5B = \frac{3}{2}$

$\Rightarrow B = -\frac{3}{10},\ A = \frac{17}{15}$

$y = Ce^{3t} + De^{-2t} + 2$

$\frac{dy}{dt} = 3Ce^{3t} - 2De^{-2t}$

- $t=0 \quad y=2$

$\Rightarrow 2 = C + D + 2$

$\Rightarrow C + D = 0$

- $t=0 \quad \frac{dy}{dt} = -1$

$\Rightarrow -1 = 3C - 2D$

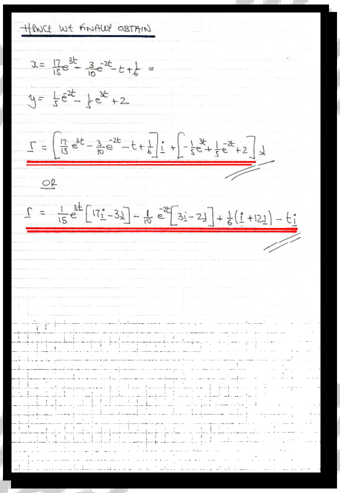
$[D = -C]$

$-1 = 3C - 2(-C)$

$-1 = 5C$

$C = -\frac{1}{5}$

$D = \frac{1}{5}$



HENCE WE FINALLY OBTAIN

$x = \frac{17}{15}e^{3t} - \frac{3}{10}e^{-2t} - t + \frac{1}{6} =$

$y = \frac{1}{5}e^{-2t} - \frac{1}{5}e^{3t} + 2$

$\mathbf{r} = \left[\frac{17}{15}e^{3t} - \frac{3}{10}e^{-2t} - t + \frac{1}{6}\right]\mathbf{i} + \left[-\frac{1}{5}e^{3t} + \frac{1}{5}e^{-2t} + 2\right]\mathbf{j}$

OR

$\mathbf{r} = \frac{1}{15}e^{3t}[17\mathbf{i} - 3\mathbf{j}] - \frac{1}{10}e^{-2t}[3\mathbf{i} - 2\mathbf{j}] + \frac{1}{6}(\mathbf{i} + 12\mathbf{j}) - t\mathbf{i}$