

Created by T. Madas

# ROTATIONAL MOTION

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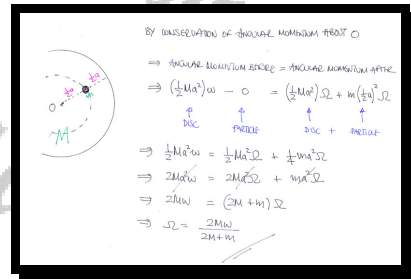
**Question 1 (\*\*)**

A uniform disc, of mass  $M$  and radius  $a$ , is rotating with constant angular velocity  $\omega$ , in a horizontal plane, about a fixed smooth vertical axis  $L$ , which is perpendicular to the disc and passes through its centre  $O$ .

A particle of mass  $m$  is gently lowered on to the disc at a distance  $\frac{1}{2}a$  from  $O$ , and as soon as it touches the disc it adheres to the disc.

Determine the new angular speed of the disc in terms of  $m$ ,  $M$  and  $\omega$ .

$$\Omega = \frac{2M\omega}{2M+m}$$



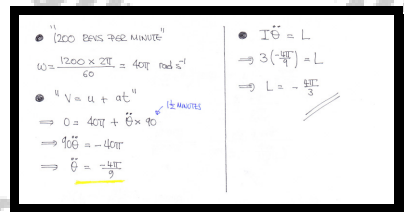
**Question 2 (\*\*)**

A flywheel, of moment of inertia  $M$  and radius  $a$ , is rotating at 1200 revolutions per minute, when the source which was maintaining this rotation is switched off.

The flywheel comes to rest after  $1\frac{1}{2}$  minutes due to a resistive couple  $L$ .

Determine the exact value of  $L$ .

$$L = -\frac{4}{3}\pi$$



## Question 3 (\*\*)

The centre of mass of a rigid body  $B$ , of mass  $m$  kg, lies at the origin  $O$ .

The point  $A$  with coordinates  $(3, -4, 1)$  lies on  $B$ .

When a force  $\mathbf{F} = (27\mathbf{i} + 16\mathbf{j} - 17\mathbf{k})$  N acts on  $A$ , it causes  $B$  an angular acceleration of  $22.75 \text{ s}^{-2}$ , about  $O$ .

Determine the moment of inertia of  $B$  about  $O$ .

$$I_O = 8 \text{ kg m}^2$$

Handwritten solution showing the calculation of the moment of inertia  $I$  from the torque  $\mathbf{G}$  and angular acceleration  $\ddot{\theta}$ .

$$\mathbf{F} = (27, 16, -17)$$

$$\mathbf{r} = (3, -4, 1)$$

$$\mathbf{G} = \mathbf{r} \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ 27 & 16 & -17 \end{vmatrix} = (52, 78, 156)$$

$$|\mathbf{G}| = |52, 78, 156|$$

$$= 26 \sqrt{2^2 + 3^2 + 6^2} = 26 \sqrt{4 + 9 + 36}$$

$$= 26 \times 7 = 182$$

using  $\mathbf{L} = I \ddot{\theta}$

$$182 = I \times 22.75$$

$$I = 8 \text{ kg m}^2$$

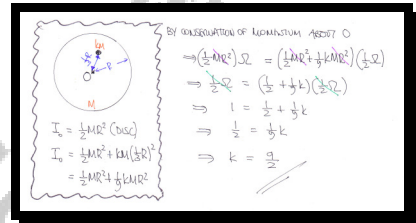
**Question 4** (\*\*)

A uniform circular disc, of mass  $M$  and radius  $R$ , is rotating with constant angular velocity in a horizontal plane about a vertical axis through its centre  $O$ .

A particle of mass  $kM$ , where  $k$  is a positive constant is gently placed on the disc at a distance  $\frac{1}{3}R$  from  $O$ . The particle becomes instantly attached to the disc.

Given that the disc now rotates with half its original angular velocity. Determine the value of  $k$ .

$$k = \frac{9}{2}$$



**Question 5 (\*\*)**

A uniform rod  $AB$ , of mass  $m$  and length  $2a$ , is free to rotate in a horizontal plane about a fixed smooth vertical axis  $L$ , which is perpendicular to the rod and passes through  $A$ .

The rod has angular speed  $\omega$  when it strikes a stationary particle  $P$  of mass  $m$ , which adheres to the rod.

Just before  $P$  adheres to the rod,  $P$  is at a distance  $x$  from  $A$ .

Given that after  $P$  adheres to the rod, the angular speed of the rod reduces to  $\frac{3}{4}\omega$ , express  $x$  in terms of  $a$ .

$$x = \frac{2}{3}a$$

The image shows a handwritten solution for the problem. It includes two diagrams of a rod of length  $2a$  pivoted at  $A$ . The first diagram shows the rod rotating with angular speed  $\omega$  before a particle of mass  $m$  strikes it at a distance  $x$  from  $A$ . The second diagram shows the rod and particle rotating together with angular speed  $\frac{3}{4}\omega$  after the collision.

By conservation of angular momentum about  $A$ :

$$\frac{1}{2}m\omega^2 \times 2a - (4m\omega^2 + m\omega^2) = \frac{3}{4}\omega$$

$$\Rightarrow \frac{1}{2}m\omega^2 = 4m\omega^2 + \frac{3}{4}m\omega^2$$

$$\Rightarrow \frac{1}{2}a^2 = a^2 + \frac{3}{4}a^2$$

$$2a^2 = 4a^2 + 3a^2$$

$$2 = \frac{7}{3}a$$

● MOMENT OF INERTIA OF ROD ABOUT  $A$   
 $\frac{1}{2}m(2a)^2 + m\omega^2 = \frac{1}{2}m(2a)^2$  (PRE-COLLISION)

● MOMENT OF INERTIA OF ROD + PARTICLE ABOUT  $A$   
 $\frac{1}{2}m(2a)^2 + m\omega^2$

**Question 6** (\*\*)

A uniform rod  $AB$ , of mass  $m$  and length  $2a$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to the rod and passes through  $A$ .

When the rod is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period  $\tau$ .

Find the length, in terms of  $a$ , of a simple pendulum whose period of small amplitude oscillations is also  $\tau$ .

$$l = \frac{4}{3}a$$

A diagram shows a rod of length  $2a$  pivoted at point  $A$ . The center of mass  $G$  is at a distance  $a$  from  $A$ . The rod is displaced by an angle  $\theta$  from the vertical. Forces shown are weight  $mg$  acting downwards from  $G$ , and reaction forces  $R$  and  $S$  at the pivot  $A$ .

EQUATION OF MOTION  
 $\rightarrow I \ddot{\theta} = L$   
 $\Rightarrow \frac{1}{2} m (2a)^2 \ddot{\theta} = -mg \sin \theta \times a$   
 $\Rightarrow 4a^2 \ddot{\theta} = -2mg \sin \theta$   
 $\Rightarrow \ddot{\theta} = -\frac{3g}{4a} \sin \theta$   
 For  $|\theta| \ll 1$   
 $\Rightarrow \ddot{\theta} = -\frac{3g}{4a} \theta$   
 I.E. SIMPLE HARM. MOTION  $\omega^2 = \frac{3g}{4a}$   
 PERIOD  $= T = 2\pi \sqrt{\frac{4a}{3g}}$   
 NEW SIMPLE PENDULUM HAS PERIOD  $T = 2\pi \sqrt{\frac{l}{g}}$   
 COMPARING  $l = \frac{4}{3}a$

**Question 7 (\*\*)**

A uniform rod  $AB$ , of mass  $3m$  and length  $2a$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to the rod and passes through  $A$ . A particle of mass  $2m$  is attached to  $B$ .

When the rod is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period  $T$ .

Show that

$$T \approx 2\pi \sqrt{\frac{12a}{7g}}$$

proof

• MOMENT OF INERTIA ABOUT A  
 $\frac{1}{2}(3m)a^2 + 2m(2a)^2 = 12ma^2$   
 (rod) (particle)

• EQUATION OF MOTION  
 $\Rightarrow I\ddot{\theta} = L$   
 $\Rightarrow (12ma^2)\ddot{\theta} = -2mg \sin \theta \cdot 2a$   
 $\quad \quad \quad - 3mg \sin \theta \cdot a$   
 $\Rightarrow 12ma^2\ddot{\theta} = -4mg \sin \theta \cdot 2a$   
 $\Rightarrow 12ma^2\ddot{\theta} = -7mg \sin \theta$   
 $\Rightarrow 12a\ddot{\theta} = -7g \sin \theta$   
 $\Rightarrow \ddot{\theta} = -\frac{7g}{12a} \sin \theta$

FOR SMALL AMPLITUDE OSCILLATIONS  
 $\sin \theta \approx \theta$   
 $\Rightarrow \ddot{\theta} = -\frac{7g}{12a} \theta$

IE SHM WITH PERIOD  $T = 2\pi \sqrt{\frac{I}{k}}$   
 IE  $T = 2\pi \sqrt{\frac{12a}{7g}}$

**Question 8 (\*\*+)**

A pulley is in the shape of a disc of radius  $a$  and mass  $M$ .

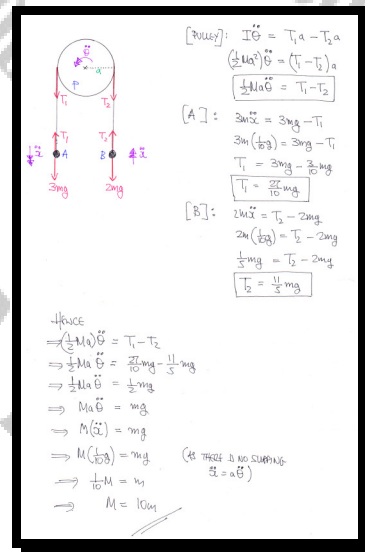
It is free to rotate in a vertical plane about a fixed smooth horizontal axis through its centre  $O$ . A light inextensible string passes over the pulley and has a particle of mass  $3m$  attached at one of its ends and a particle of mass  $2m$  attached at the other end.

The particles are held initially at rest, at the same horizontal level and at the same vertical plane as the pulley, with the string taut.

The system is released from rest and the particles begin to move with the string not slipping on the pulley.

Given that each the particles experience an acceleration of  $\frac{1}{10}g$ , express  $M$  in terms of  $m$ .

**$M = 10m$**





**Question 9** (\*\*\*)

A uniform circular disc, with centre  $C$ , has mass  $2m$  and radius  $a$ .

A particle of mass  $m$  is attached at the point  $B$  on the circumference of the disc. The disc is free to rotate about a smooth fixed horizontal axis  $L$ , which is perpendicular to the plane of the disc and passes through the point  $A$  on the circumference of the disc.

Given that the straight line  $ACB$  is a diameter of the disc, show that if the disc is slightly disturbed from its position of stable equilibrium, its subsequent motion will be approximately simple harmonic.

proof

$\bullet$  FIND THE MOMENT OF INERTIA ABOUT  $A$   
 DISC:  $\frac{1}{2}(2m)a^2 + (2m)a^2 = 3ma^2$   
 PARTICLE:  $m(2a)^2 = 4ma^2$   
 TOTAL:  $7ma^2$

$\bullet$  EQUATION OF MOTION  
 $\Rightarrow I\ddot{\theta} = L$   
 $\Rightarrow 7ma^2\ddot{\theta} = (-2mg\sin\theta) \times a + (-mg\sin\theta) \times 2a$   
 $\Rightarrow 7ma^2\ddot{\theta} = -4mg\sin\theta$   
 $\Rightarrow 7a\ddot{\theta} = -4g\sin\theta$   
 $\Rightarrow \ddot{\theta} = -\frac{4g}{7a}\sin\theta$

FOR SMALL ANGLES OSCILLATIONS  
 $\sin\theta \approx \theta$   
 $\Rightarrow \ddot{\theta} = -\frac{4g}{7a}\theta$   
 $\therefore \omega^2 = \frac{4g}{7a}$

**Question 10 (\*\*+)**

A pulley is in the shape of a disc of radius  $a$  and mass  $3m$ .

It is free to rotate in a vertical plane about a fixed smooth horizontal axis through its centre  $O$ . A light inextensible string has one end attached to a point on the rim of the pulley and is wound several times around the rim of the pulley. The portion of the string not wound on the pulley has length  $8a$  and has a particle of mass  $m$  attached to its free end.

The particle is held at the same level as  $O$ , close to the rim of the still pulley and is released from rest.

Determine, in terms of  $a$  and  $g$ , the angular velocity of the pulley immediately after the string becomes taut.

$$\omega = \frac{8}{5} \sqrt{\frac{g}{a}} = \sqrt{\frac{64g}{25a}}$$

**Energy by kinematics**

$u = 0$        $v^2 = u^2 + 2as$   
 $a = g$        $v^2 = 2g(8a)$   
 $\frac{1}{2} = \frac{8a}{a}$        $v^2 = 16ag$   
 $v = ?$        $v = 4\sqrt{ag}$

**Moment of inertia of pulley**

$I = \frac{1}{2}(3m)a^2 = \frac{3}{2}ma^2$

**By conservation of angular momentum about O**  
 BEFORE (JUST BEFORE IT GETS TAUT) = AFTER (TAUT)

$\Rightarrow (mv) \times a + 0 = I\omega + mVa$   
(MOMENT OF PARTICLE)      (MOMENT OF PARTICLE)

$\Rightarrow m(4\sqrt{ag})a + 0 = (\frac{3}{2}ma^2)\omega + mV a$   
 $\Rightarrow 4\sqrt{ag} = \frac{3}{2}a\omega + V$       BUT  $V = a\omega$   
 $\Rightarrow 4\sqrt{ag} = \frac{3}{2}a\omega + a\omega$   
 $\Rightarrow 8\sqrt{ag} = 3a\omega + 2a\omega$   
 $\Rightarrow 5a\omega = 8\sqrt{ag}$   
 $\Rightarrow \omega = \frac{8}{5} \sqrt{\frac{g}{a}}$

**Question 11** (\*\*\*)

Four uniform rods, each of mass  $m$  and length  $2\sqrt{2}a$ , are rigidly joined together to form a square framework  $ABCD$ .

The framework is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to plane of the framework and passes through  $A$ .

When the framework is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period  $\tau$ .

Find the length, in terms of  $a$ , of a simple pendulum whose period of small amplitude oscillations is also  $\tau$ .

$$l = \frac{37}{12}a$$

$I_G = -4mg \sin \theta \times 2a$   
 $\frac{1}{2} m \dot{\theta}^2 = -8mg \sin \theta$   
 $\frac{1}{2} m \dot{\theta}^2 = -8g \sin \theta$   
 $\dot{\theta} = -\frac{12g}{5a} \sin \theta$   
 for  $|\theta| \ll 1$   
 $\dot{\theta} = -\frac{12g}{5a} \theta$   
 $T = 2\pi \sqrt{\frac{5a}{12g}}$   
 $T = 2\pi \sqrt{\frac{l}{g}}$  (for a simple pendulum)  
 $\therefore l = \frac{37}{12}a$

Moment of inertia of each rod about the axis through  $A$ , perpendicular to the plane of the framework:  
 (AB):  $\frac{1}{3} m (2\sqrt{2}a)^2 = \frac{8}{3} ma^2$   
 (BC):  $\frac{1}{2} m (2\sqrt{2}a)^2 + m (2a)^2 = \frac{8}{3} ma^2 + 4ma^2 = \frac{20}{3} ma^2$  (by parallel axis)  
 AND SIMILARLY THE OTHER TWO  
 MOMENT OF INERTIA =  $2 \times (\frac{8}{3} ma^2 + \frac{20}{3} ma^2) = \frac{76}{3} ma^2$

$|AG|^2 + |BG|^2 = |AB|^2$   
 $2|AG|^2 = (2\sqrt{2}a)^2$   
 $|AG|^2 = 4a^2$   
 $|AG| = 2a$

**Question 12** (\*\*\*)

A uniform rod  $AB$ , of mass  $3m$  and length  $2l$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$ , which is perpendicular to the rod and passes through  $A$ .

A particle of mass  $m$  is attached to the rod at  $B$ .

The loaded rod is held in a horizontal position and is released from rest.

Find, in terms of  $g$  and  $l$ , the speed of the particle when the rod is first vertical.

$$v = \sqrt{5gl}$$

The image shows a handwritten solution for Question 12. It includes two diagrams: one of a horizontal rod AB of length 2l pivoted at A, and another of the rod AB vertical with a particle of mass m at B. The solution uses conservation of energy, equating initial potential energy to final kinetic energy (rotational and translational). The final speed v is found to be  $\sqrt{5gl}$ .

$\bullet$  THE MOMENT OF INERTIA OF THE UNIFORM ROD ABOUT  $A$   
 $I_A = \frac{1}{3}(3m)(2l)^2 = 8ml^2$

$\bullet$  THE LOCATION OF THE CENTRE OF MASS BY INSPECTION IS  $l + \frac{1}{2}l = \frac{3}{2}l$  FROM  $A$ . (RATIO OF MASSES 3:1)

$\bullet$  BY CONSERVING TAKING THE LEVEL OF  $A$  AS THE ZERO POTENTIAL LEVEL

$\Rightarrow K.E_{rot} + P.E_{part} = K.E_{rot} + P.E_{part}$   
 $\Rightarrow 0 + 0 = \frac{1}{2}I\omega^2 + 4mg(\frac{3}{2}l)$   
 $\Rightarrow 5mgl = \frac{1}{2}(8ml^2)\omega^2$   
 $\Rightarrow 10g = 8l\omega^2$   
 $\Rightarrow \omega^2 = \frac{5g}{4l}$

$\bullet$  FINALLY USING  $v = \omega r$  TO FIND THE LINEAR SPEED OF  $B$   
 $v = (\frac{\omega}{2}) \times 2l$   
 $v = \sqrt{\frac{5g}{4l}} \times 2l$   
 $v = \sqrt{5gl}$

**Question 13** (\*\*+)

A uniform rectangular lamina  $ABCD$ , where  $|AB| = 2a$  and  $|BC| = a$ , has mass  $2m$ .

The lamina is rotating with angular speed  $\omega$ , in a horizontal plane about a smooth fixed vertical axis which passes through the centre of the lamina. A particle of mass  $m$  is held at rest just above the surface of the lamina when it adheres to the corner  $B$ .

Find, in terms of  $\omega$ , the new angular speed of the now loaded lamina.

$$\Omega = \frac{2}{5} \omega$$

• BY STEADY STATE BALANCE  
 • MOMENT OF INERTIA ABOUT  $I_2 = \frac{1}{2}(2m)(2a)^2 = 4ma^2$   
 • MOMENT OF INERTIA ABOUT  $I_3 = \frac{1}{2}(2m)a^2 = ma^2$   
 • BY THE PERPENDICULAR AXIS THEOREM, THE MOMENT OF INERTIA ABOUT A PERPENDICULAR AXIS TO THE PLANE OF THE LAMINA, THROUGH  $G$   
 $I_G = \frac{1}{2}ma^2 + 3ma^2 = \frac{7}{2}ma^2$   
 • THE MOMENT OF INERTIA OF THE PARTICLE ABOUT  $G$   
 $I = m|BG|^2 = m\sqrt{a^2 + (3/2a)^2}^2 = m \times \frac{25}{4}a^2 = \frac{25}{4}ma^2$   
 • BY CONSERVATION OF ANGULAR MOMENTUM ABOUT  $G$   
 $I_G \omega + 0 = I' \Omega$   
 $\frac{7}{2}ma^2 \omega = (\frac{7}{2}ma^2 + \frac{25}{4}ma^2) \Omega$   
 $\frac{7}{2} \omega = \frac{37}{4} \Omega$   
 $\Omega = \frac{14}{37} \omega$

**Question 14** (\*\*\*)

A uniform rod  $AB$ , of mass  $m$  and length  $2a$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$ , which is perpendicular to the rod and passes through the point  $O$  of the rod, where  $OA = \frac{1}{3}a$ .

- a) Find the moment of inertia of the rod about  $L$ .

The rod is held at rest with  $B$  vertically above  $O$  and is slightly displaced.

- b) Determine, when  $OB$  makes an angle  $\theta$  with the upward vertical, ...

i. ... the angular speed of the rod.

ii. ... the magnitude of the angular acceleration of the rod.

- c) Given that the length of the rod is 2 m, calculate the angular speed of the rod when the force acting on the rod at  $O$  is perpendicular to the rod.

$$I_O = \frac{7}{9}ma^2, \quad \dot{\theta} = \sqrt{\frac{12g}{7a}(1 - \cos\theta)}, \quad \ddot{\theta} = \frac{6g}{7a}\sin\theta, \quad \omega = 2.8 \text{ rad s}^{-1}$$

Handwritten solution for Question 14:

a)  $I_O = \frac{1}{3}ma^2$   
 $I_O = \frac{1}{3}m(\frac{2}{3}a)^2 + m(\frac{1}{3}a)^2$   
 $I_O = \frac{7}{9}ma^2$

b) BY METHOD TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL.  
 $\rightarrow KE_{rot} + PE_{cm} = KE_{rot} + PE_B$   
 $\rightarrow 0 + mg(\frac{2}{3}a) = \frac{1}{2}I_O\dot{\theta}^2 + mg(\frac{2}{3}a\cos\theta)$   
 $\rightarrow \frac{2}{3}mg = \frac{1}{2}(\frac{7}{9}m\dot{\theta}^2) + \frac{2}{3}mg\cos\theta$   
 $\times 6$   
 $12mg = 7m\dot{\theta}^2 + 12mg\cos\theta$   
 $12g = 7\dot{\theta}^2 + 12g\cos\theta$   
 $7\dot{\theta}^2 = 12g(1 - \cos\theta)$   
 $\dot{\theta} = \sqrt{\frac{12g}{7a}(1 - \cos\theta)}$   
 $7a\dot{\theta}^2 = 12g(1 - \cos\theta)$   
 $14a\dot{\theta}\ddot{\theta} = 12g\sin\theta \times \dot{\theta}$   
 $7a\ddot{\theta} = 6g\sin\theta$   
 $\ddot{\theta} = \frac{6g}{7a}\sin\theta$

c) We require  $2=0$   
 Method 1:  
 $m(-\frac{2}{3}a\dot{\theta}^2) = -mg\cos\theta$   
 $-\frac{2}{3}m\dot{\theta}^2 = -mg\cos\theta$   
 $\frac{2}{3}\dot{\theta}^2 = g\cos\theta$   
 $\dot{\theta}^2 = \frac{3g}{2}\cos\theta$   
 OR  $7a\dot{\theta}^2 = 12g(1 - \cos\theta)$   
 $\rightarrow 7a\dot{\theta}^2 = 12g - 12g\cos\theta$   
 $\rightarrow 7a\dot{\theta}^2 = 12g - 12g(\frac{2}{3}\dot{\theta}^2)$   
 $\rightarrow 7a\dot{\theta}^2 = 12g - 8a\dot{\theta}^2$   
 $\rightarrow 15a\dot{\theta}^2 = 12g$   
 $\dot{\theta}^2 = \frac{4g}{5a}$   
 $\dot{\theta} = \sqrt{\frac{4g}{5a}}$   
 $\dot{\theta} = 2.8 \text{ rad s}^{-1}$

**Question 15** (\*\*\*)

A uniform rectangular lamina  $ABCD$ , where  $AB = a$  and  $BC = 4a$ , has mass  $4m$ . The lamina is free to rotate about the edge  $AB$ , which is fixed and vertical.

A particle, of mass  $m$ , is moving horizontally with speed  $u$  in a direction which is perpendicular to the lamina. The lamina is at rest when it is struck by the particle at  $C$ .

The coefficient of restitution between the particle and the lamina is 0.75.

Find the angular speed of the lamina immediately after the impact.

$$\omega = \frac{3u}{16a}$$

The handwritten solution includes the following elements:

- Diagram (Before):** A rectangle  $ABCD$  with height  $a$  and width  $4a$ . Edge  $AB$  is vertical and fixed. A particle of mass  $m$  moves horizontally with speed  $u$  towards edge  $CD$  at point  $C$ .
- Diagram (After):** The rectangle is rotated counter-clockwise with angular speed  $\omega$ . The particle has a new velocity  $v$  after impact.
- Calculation 1:** Moment of inertia of the lamina about  $AB$ :  $\frac{1}{2} \times 4a \times a^2 = \frac{2}{3} (4m)(a^2) = \frac{8}{3} ma^2$ .
- Calculation 2:** Conservation of angular momentum about  $AB$ :  $(mu) \times 4a + 0 = (mv) \times 4a + I\omega$ . This leads to  $4mu = 4mv + \frac{8}{3} ma^2 \omega$  and  $4u = 4v + \frac{8}{3} a\omega$ .
- Calculation 3:** Coefficient of restitution  $e = \frac{v - u}{u - 0} = 0.75$ . This leads to  $\frac{v}{u} = \frac{3(4u - v) - Y}{4} = \frac{3u - 4v}{4}$ .
- Final Result:** After solving the system of equations,  $\omega = \frac{3u}{16a}$ .


**Question 16** (\*\*\*)

A uniform square lamina  $ABCD$  has side length  $a$  and mass  $m$ . The lamina is free to rotate in a vertical plane about a fixed horizontal axis, which is perpendicular to the plane of the lamina and passes through  $O$ , the centre of the lamina. Two particles, each of mass  $m$ , are attached to the vertices  $A$  and  $B$ . The system is released from rest with  $AB$  vertical.

Find, in terms of  $a$  and  $g$ , the angular velocity of the system when  $AB$  is horizontal.

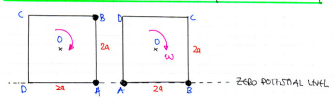
,  $\omega = \sqrt{\frac{6g}{7a}}$

START BY DETERMINING THE MOMENT OF INERTIA OF THE LAMINA, ABOUT O



- BY THE "STEENHOUT" BACKWARDS  $I_x = I_y = \frac{1}{3}ma^2$   
AS THE LAMINA COMPLETES TO A END OF LENGTH  $2a$  & MASS  $m$ , WITH ICE CHARGE AT O
- BY THE PERPENDICULAR AXIS THEOREM,  $I_o$ , PERPENDICULAR TO THE PLANE OF THE LAMINA IS GIVEN BY  $I_o = \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$
- $|oa| = |ob| = \sqrt{2}a$  (BY PYTHAGORAS)
- CONTRIBUTION OF EACH PARTICLE TO THE MOMENT OF INERTIA ABOUT O, IS GIVEN BY  $m(2\sqrt{2}a)^2 = 2ma^2$
- TOTAL MOMENT OF INERTIA ABOUT O  $I_{tot} = \frac{2}{3}ma^2 + 2(2ma^2) = \frac{14}{3}ma^2$

LOOKING AT THE DIAGRAM BELOW AND BY CONSIDERING ENERGY



NOTE THAT THE CENTRE OF MASS OF THE LAMINA (MASS  $m$ ) DOES NOT MOVE AND THEREFORE WE MAY IGNORE IT

$\Rightarrow KE_{\text{initial}} + PE_{\text{initial}} = KE_{\text{final}} + PE_{\text{final}}$

$\Rightarrow 0 + mg(2a) = \frac{1}{2}I_{tot}\omega^2 + 0$

$\Rightarrow 2mga = \frac{1}{2}(\frac{14}{3}m a^2)\omega^2$

$\Rightarrow 2mg a = \frac{7}{3}m a^2 \omega^2$

$\Rightarrow 2g = \frac{7}{3}a \omega^2$

$\Rightarrow \frac{6a}{7a} = \omega^2$

$\Rightarrow \omega = \sqrt{\frac{6g}{7a}}$



**Question 17 (\*\*\*)**

A uniform circular disc, with centre  $C$ , has mass  $5m$  and radius  $a$ .

The straight line  $AB$  is a diameter of the disc.

A particle of mass  $m$  is attached to the disc at the point  $M$ , where  $M$  is the midpoint of  $AC$ . The disc is free to rotate about a smooth horizontal axis  $L$ , which lies in the plane of the disc and is a tangent to the disc at  $B$ .

- a) Find the moment of inertia of the loaded disc about  $L$ .

The loaded disc is released from rest with  $AB$  at an angle of  $45^\circ$  to the upward vertical. When  $A$  is vertically below  $B$ , the loaded disc has angular speed  $\Omega$ .

- b) Show that

$$\Omega^2 = \frac{26g(1+\sqrt{2})}{17a}$$

$$I = \frac{17}{2}ma^2$$

4)

MASS OF DISC IS  $5m$   
MASS OF PARTICLE IS  $m$

MOMENT OF INERTIA OF DISC...  
ABOUT AN AXIS THROUGH C AND PERPENDICULAR TO THE PLANE OF THE DISC IS  
 $\frac{1}{2}Mk^2 = \frac{1}{2}(5m)a^2 = \frac{5}{2}ma^2$

BY PERPENDICULAR AXIS THEOREM  
 $I_B = I_C + 5m \cdot a^2$   
 $I_B = \frac{5}{2}ma^2 + 5ma^2 = \frac{15}{2}ma^2$

BY PARALLEL AXIS THEOREM  
 $I = I_B + m \cdot (\frac{3}{2}a)^2$   
 $I = \frac{15}{2}ma^2 + \frac{9}{4}ma^2 = \frac{33}{4}ma^2$

• NOW ADD THE PARTICLE OF  $m$  LOCATED AT  $M$   
TOTAL =  $\frac{33}{4}ma^2 + m(\frac{3}{2}a)^2$   
 $= \frac{33}{4}ma^2 + \frac{9}{4}ma^2$   
 $= \frac{42}{4}ma^2 = \frac{21}{2}ma^2$

b)

LOOK AT THE LOCATION OF THE CENTER FROM B (TANGENT AX)

MASS	DIST	$\frac{5m}{2}$	$\frac{m}{2}$	$\frac{5m}{2}$
DISTANCE FROM B		$a$	$\frac{3}{2}a$	$\frac{3}{2}a$

$G_{DISC} = 5m \cdot a$   
 $G_C = \frac{5m}{2} \cdot \frac{3}{2}a = \frac{15}{4}ma$   
 $G = \frac{13}{4}ma$  ← FROM B

• NOW DRAWING THE DISC AS A CROSS SECTION, FROM B TO G

BY FINDEES TAKING THE LEVEL OF B AS THE ZERO POTENTIAL LEVEL.  
 $K.E._{INITIAL} + P.E._{INITIAL} = K.E._{FINAL} + P.E._{FINAL}$   
 $0 + (5m)g(\frac{3}{2}a \cos \theta) = \frac{1}{2}I\Omega^2 + (5m)g(\frac{3}{2}a)$   
 $\frac{13}{4}mg \cdot \frac{3}{2}a = \frac{1}{2}(\frac{21}{2}ma^2)\Omega^2 - (5m)g(\frac{3}{2}a)$   
 $\frac{39}{8}mg \cdot \frac{3}{2}a = \frac{21}{4}ma^2\Omega^2 - \frac{15}{4}mg \cdot \frac{3}{2}a$   
 $26 \cdot \frac{3}{8}mg \cdot \frac{3}{2}a = \frac{21}{4}ma^2\Omega^2 - 26g \cdot \frac{3}{4}a$   
 $17a \cdot \frac{3}{8}g = \frac{21}{4}a\Omega^2 - 26g$   
 $\Omega^2 = \frac{26g(1+\sqrt{2})}{17a}$

**Question 18 (\*\*\*)**

Two uniform spheres, each of mass  $5m$  and radius  $r$ , are attached to each of the ends of a thin uniform rod  $AB$ , of mass  $m$  and length  $6r$ . The centres of the spheres are collinear with  $AB$ , and are located  $8r$  apart.

The above described system is free to rotate about a fixed smooth horizontal axis, perpendicular to  $AB$ , and passing through a point on the rod  $O$ , where  $|AO| = r$ .

The system is slightly disturbed from rest with  $B$  vertically above  $A$ .

Determine the angular velocity of the system when  $A$  vertically above  $B$ .

$$|\omega| = \sqrt{\frac{88g}{211r}}$$

The diagram shows a horizontal rod AB of length 6r. A pivot O is located at a distance r from A. Two spheres of mass 5m and radius r are attached to the ends of the rod. The distance between the centers of the spheres is 8r. The system is shown in two positions: initially with B vertically above A, and finally with A vertically above B.

**MOMENT OF INERTIA ABOUT O**

$$I_0 = \underbrace{\frac{1}{2}(5m)(2r)^2 + 5m(2r)^2}_{\text{SPHERE A + PARALLEL AXES}} + \underbrace{\frac{1}{2}m(6r)^2 + m(2r)^2}_{\text{ROD}} + \underbrace{\frac{1}{2}(5m)(2r)^2 + 5m(2r)^2}_{\text{SPHERE B + PARALLEL AXES}}$$

$$I_0 = 20mr^2 + 20mr^2 + 36mr^2 + 4mr^2 + 20mr^2 + 20mr^2 = 211mr^2$$

**THE LOCATION OF THE CENTRE OF MASS OF THE SYSTEM IS 2r FROM A (BY SYMMETRY)**

**BY ENERGIES**

EG-GROUND = F.E. LOST

$$\frac{1}{2}I\omega^2 = m_{\text{TOT}}gh$$

$$\frac{1}{2}(211mr^2)\omega^2 = (11m)g(4r)$$

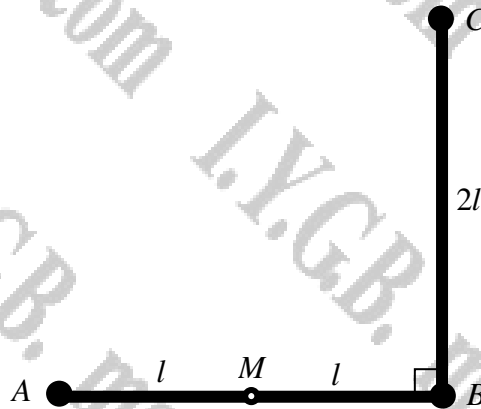
$$\frac{211}{2}r\omega^2 = 44mgr$$

$$\frac{211}{2}r\omega^2 = 44g$$

$$\omega^2 = \frac{88g}{211r}$$

$$|\omega| = \sqrt{\frac{88g}{211r}}$$

Question 19 (\*\*\*)



Two identical uniform rods  $AB$  and  $BC$ , each of mass  $m$  and length  $l$  are rigidly joined at  $B$ , so that  $\angle ABC = 90^\circ$ . Three particles of masses  $m$ ,  $2m$  and  $3m$  are fixed at  $A$ ,  $B$  and  $C$ , respectively. The system of the two rods and the three particles can rotate freely in a vertical plane about a horizontal axis through  $M$ , where  $M$  is the midpoint of  $AB$ .

- a) Show clearly that the moment of inertia of the system about an axis through  $M$  and perpendicular to the plane  $ABC$  is  $\frac{62}{3}ml^2$ .

The system is released from rest with  $AB$  horizontal and  $C$  vertically above  $B$ .

- b) Determine, in terms of  $g$  and  $l$ , the angular velocity of the system when  $BC$  is horizontal and  $B$  is vertically below  $A$ .

$$\omega = \sqrt{\frac{31g}{32l}}$$

(b) BY CONSIDERING TAKING THE LEVEL OF  $M$  AS THE ZERO POTENTIAL LEVEL AND TREATING EACH OBJECT SEPARATELY

PE LOSS = KE GAIN

$$\Rightarrow 0 + 3mg \cdot \frac{1}{2} + 2mg \cdot \frac{1}{2} + 3mg \cdot \frac{3}{2} - 3mg \cdot \frac{1}{2} = \frac{1}{2} I \omega^2$$

$\uparrow$  ROD AB     $\uparrow$  ROD BC     $\uparrow$  3m     $\uparrow$  2m     $\uparrow$  3m     $\uparrow$  3m  
(ANNO BECAUSE IT GAINED POTENTIAL ENERGY)

$$\Rightarrow 11g = \frac{1}{2} \times \frac{62}{3} \omega^2$$

$$\Rightarrow 11g = \frac{31}{3} \omega^2$$

$$\Rightarrow 33g = 31 \omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{33g}{31}}$$

**Question 20 (\*\*\*)**

A compound pendulum consists of a thin uniform rod  $OC$  of length  $8a$  and mass  $m$  is rigidly attached at  $C$  to the centre of a thin uniform circular disc of radius  $a$  and mass  $4m$ . The rod is in the same vertical plane as the disc.

The pendulum is free to rotate in this vertical plane, through a smooth horizontal axis through  $O$ , perpendicular to the plane of the disc.

- a) Show that the moment of inertia of the pendulum about the above described axis is  $\frac{835}{3}ma^2$ .
- b) Show further that the period of small amplitude oscillations of the pendulum, about the position of the stable equilibrium is  $2\pi\sqrt{\frac{835a}{108g}}$ .

proof

**a)**

- Moment of inertia about  $O$   
 $DOC: \frac{1}{3}m(8a)^2 = \frac{64}{3}ma^2$
- Moment of inertia of the disc about  $O$   
 ABOUT AN AXIS THROUGH  $C$ , PERPENDICULAR TO THE PLANE OF THE DISC  
 $\frac{1}{2}(4m)a^2 = 2ma^2$
- BY PERPENDICULAR AXES THEOREM SINCE THE MOMENT OF INERTIA ABOUT A DIAMETER IS  $\frac{1}{2}ma^2$
- BY PARALLEL AXES THEOREM, THE MOMENT OF INERTIA THROUGH  $O$  IS  
 $ma^2 + (4m)(8a)^2 = 257ma^2$

∴ MOMENT OF INERTIA TOTAL =  $257ma^2 + \frac{64}{3}ma^2 = \frac{835}{3}ma^2$

**b)**

$\frac{1}{2}I\ddot{\theta} = L$   
 $\Rightarrow \left(\frac{835}{3}ma^2\right)\ddot{\theta} = (-mg\sin\theta)(8a) + (-4mg\sin\theta)(8a)$   
 $\Rightarrow \frac{835}{3}m\ddot{\theta} = -36mg\sin\theta$   
 $\Rightarrow \ddot{\theta} = -\frac{108g\sin\theta}{835a}$   
 $\Rightarrow \ddot{\theta} \approx -\frac{108g}{835a}\theta$   
 $\therefore T = 2\pi\sqrt{\frac{835a}{108g}}$

**Question 21** (\*\*\*)

A thin uniform rod  $AB$ , of length  $2a$  and mass  $m$ , is free to rotate in a vertical plane, about a fixed smooth horizontal axis through  $A$ .

The rod is hanging in equilibrium, with  $B$  vertically below  $A$ , when it receives a horizontal impulse of magnitude  $m\sqrt{ag}$ , in a direction perpendicular to the axis through  $A$ .

Find the angle by which the rod turns before coming to instantaneous rest.

120°

$I_A = \frac{1}{3}m(2a)^2$  (STANDARD RESULT)  
 • AMOUNT OF IMPULSE = CHANGE OF ANGULAR MOMENTUM  
 $\Rightarrow \frac{1}{2}(\sqrt{ag}) \times 2a = \frac{1}{3}m(2a)^2 \times \omega$   
 $\Rightarrow a\sqrt{ag} = \frac{4}{3}ma^2\omega$   
 $\Rightarrow \omega = \frac{3\sqrt{ag}}{4a}$   
 $\Rightarrow \omega = \frac{3}{4}\sqrt{\frac{g}{a}}$

• BY TAKING THE LINE OF  $A$  AS THE ZERO POTENTIAL LEVEL, WE OBTAIN  
 $\Rightarrow KE_{\text{rot}} + PE_{\text{rot}} = KE_{\text{rot}} + PE_{\text{rot}}$   
 $\Rightarrow \frac{1}{2}I\omega^2 + mg(-a) = \frac{1}{2}I\omega^2 + mg(a\cos\theta)$   
 $\Rightarrow \frac{1}{2} \times \frac{4}{3}m(2a)^2 \times \frac{9}{16} \times \frac{g}{a} - mga = -mga\cos\theta$   
 $\Rightarrow \frac{3}{2}mag - mga = -mga\cos\theta$   
 $\Rightarrow \frac{1}{2}mag = -mga\cos\theta$   
 $\Rightarrow \cos\theta = -\frac{1}{2}$   
 $\Rightarrow \theta = 120^\circ$

**Question 22 (\*\*\*)**

A thin uniform rod  $AB$  of length  $2a$  and mass  $2m$  is free to rotate in a vertical plane, through a fixed smooth horizontal axis through  $A$ . The rod is hanging in equilibrium with  $B$  vertically below  $A$ . A particle of mass  $m$ , moving horizontally with speed  $u$  in a vertical plane perpendicular to the axis through  $A$ , strikes the rod at the point  $C$  and adheres to it.

Given that the speed of the particle immediately after it adheres to the rod is  $\frac{2}{5}u$ , determine the distance  $AC$

$$|AC| = \frac{3}{2}a$$

FIRSTLY THE MOMENT OF INERTIA OF THE ROD ABOUT ITS CENTRE IS:  
 $\frac{1}{12}(2m)a^2 = \frac{1}{6}ma^2$   
 MOMENT OF INERTIA OF THE ROD PLUS PARTICLE IS FOUND BY:  
 $\frac{1}{6}ma^2 + mx^2$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT A:  
 $\Rightarrow (2mu)x + 0 = \left[ \frac{1}{6}ma^2 + mx^2 \right] \Omega$   
 $\Rightarrow ux = \left[ \frac{1}{6}a^2 + x^2 \right] \Omega$   
 BUT " $v = \omega r$ "  
 $\frac{2}{5}u = \Omega x$   
 $\Omega = \frac{2u}{5x}$   
 $\Rightarrow ux = \left[ \frac{1}{6}a^2 + x^2 \right] \frac{2u}{5x}$   
 $\Rightarrow \frac{5}{2}x^2 = \frac{1}{6}a^2 + x^2$   
 $\Rightarrow \frac{3}{2}x^2 = \frac{1}{6}a^2$   
 $\Rightarrow x = \frac{3}{2}a$

**Question 23 (\*\*\*)**

A uniform square lamina has side length  $2a$ . The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$ , which is perpendicular to the lamina and passes through one of the vertices of the lamina.

The lamina is suspended through  $L$ , and hanging in stable equilibrium when it is slightly displaced from that position.

Determine the period of small oscillation about this position, in terms of  $\pi, a$  and  $g$ .

$$T = 2\pi \sqrt{\frac{8a}{3\sqrt{2}g}}$$

The handwritten solution includes the following components:

- Diagram 1:** A square lamina ABCD with side length  $2a$ . The center of mass G is at the intersection of the diagonals. A horizontal axis L is shown passing through vertex A.
- Diagram 2:** The lamina is shown in a vertical position, suspended from axis L at vertex A. The center of mass G is at a distance  $\sqrt{2}a$  from A. The lamina makes an angle  $\theta$  with the vertical.
- Calculations:**
  - Let  $M$  be the mass of the lamina.
  - $I_A = \frac{1}{3}Ma^2$  (Parallel Axis Theorem)
  - $I_G = \frac{1}{12}Ma^2$
  - $I_C = \frac{1}{3}Ma^2$  (Perpendicular Axis Theorem)
  - $I_A = \frac{1}{3}Ma^2 + M(a\sqrt{2})^2 = \frac{5}{3}Ma^2$  (By Parallel-Axis Theorem)
- Equation of Motion:**
  - $\tau = -Mg \sin\theta \times \sqrt{2}a$
  - $\Rightarrow \frac{5}{3}M\omega^2 \theta = -Mg\sqrt{2} \sin\theta$
  - $\Rightarrow \ddot{\theta} = -\frac{3g\sqrt{2}}{5a} \sin\theta$
  - But if  $|\theta| \ll 1$ ,  $\sin\theta \approx \theta$
  - $\Rightarrow \ddot{\theta} = -\frac{3g\sqrt{2}}{5a} \theta$
  - $\therefore \sin\theta \approx \theta$ ,  $\omega^2 = \frac{3g\sqrt{2}}{5a}$
  - $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{5a}{3\sqrt{2}g}}$

**Question 24 (\*\*\*)**

A thin uniform rod  $AB$  of length  $4l$  and mass  $m$  is free to rotate in a vertical plane, through a fixed smooth horizontal axis through a point  $O$  on the rod, which is at a distance  $l$  from  $A$ . The rod is released from rest in a horizontal position and when the rod is vertical for the first time its angular velocity is  $\omega$ .

Show that when the rod is first vertical, the magnitude of the force acting on the rod at  $O$  is  $\frac{13}{7}mg$ .

proof

The handwritten solution is divided into two parts:

- Energy Conservation:**
  - Diagram: A horizontal rod of length  $4l$  pivoted at point  $O$ , which is at distance  $l$  from end  $A$ . The center of mass  $G$  is at distance  $3l$  from  $O$ . The rod is released from rest.
  - Equation:  $I_A = \frac{1}{12}m(4l)^2 + ml^2$  (parallel axis theorem)
  - Equation:  $I_O = \frac{1}{3}ml^2$
  - Text: "BY CONSERVING ENERGY THE LEVEL OF  $G$  AS THE ZERO POTENTIAL LEVEL WE OBTAIN"
  - Equation:  $\Rightarrow KE + PE = KE + PE$
  - Equation:  $\Rightarrow 0 + 0 = \frac{1}{2}I_O\omega^2 - mgl$
  - Equation:  $\Rightarrow mgl = \frac{1}{2}(\frac{1}{3}ml^2)\omega^2$
  - Equation:  $\Rightarrow \frac{6}{l^2} = \omega^2$
- Force Analysis:**
  - Text: "ALSO IN THE VERTICAL POSITION, THERE IS NO WORKING AS THE FORCES (WEIGHT AND REACTION) PASS THROUGH  $O$  - THEN  $\Sigma \tau = 0$ "
  - Diagram: A vertical rod pivoted at  $O$ . The center of mass  $G$  is at distance  $3l$  from  $O$ . Forces shown are weight  $mg$  acting downwards at  $G$  and reaction force  $Y$  acting upwards at  $O$ .
  - Text: "EQUATION OF MOTION ORBALLY"
  - Equation:  $\Rightarrow m\vec{r} = mg - Y$
  - Equation:  $\Rightarrow Y = mg - m\vec{r}$
  - Equation:  $\Rightarrow Y = mg - m(-\omega^2 \times 3l)$  (DIRECTION OF  $\vec{r}$  FROM  $O$ )
  - Equation:  $\Rightarrow Y = mg + m(\frac{6}{l} \times 3l)$
  - Equation:  $\Rightarrow Y = mg + 6mg$
  - Equation:  $\Rightarrow Y = \frac{13}{7}mg$



**Question 25 (\*\*\*)**

A cartwheel, consisting of 8 spokes and a circular rim, is placed over a well. Each of the 8 spokes is modelled as a uniform rod of mass  $\frac{1}{2}m$  and length  $a$ . The rim is modelled as a uniform circular hoop of mass  $3m$ .

The 8 spokes are equally spaced on the rim and they meet at the centre of the hoop  $O$ . The cartwheel is modelled as a two dimensional rigid structure. A bucket of mass  $m$ , which is modelled as a particle, is attached to one end of a light inextensible string of length  $8a$ . The other end of the string is attached to a point  $P$  on the rim of the cartwheel, so that  $OP$  is horizontal.

The bucket is held next to  $P$  and released from rest with string slack.

Determine, in terms of  $a$  and  $g$ , the angular velocity of the cartwheel just after the instant the string becomes taut.

$$\omega = \sqrt{\frac{9a}{16g}}$$

**MOMENT OF INERTIA OF THE 8 RODS OF THE WHEEL**  
 $I = 8 \times \left(\frac{1}{2}ma\right)^2 + 3ma^2 = \frac{13}{2}ma^2$

**BY MOMENTUM ON THE BUCKET UNTIL THE STRING BECOMES TAUT**  
 $\begin{cases} u = 0 \\ s = 8a \\ v = ? \end{cases} \quad \begin{cases} v = u + at \\ s = ut + \frac{1}{2}at^2 \\ v^2 = u^2 + 2as \end{cases}$   
 $v^2 = 0 + 2g(8a)$   
 $v = 4\sqrt{2g}$   
 $|v| = 4\sqrt{2g}$

**BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O, BEFORE & AFTER THE STRING BECOMES TAUT, WE OBTAIN**

**ANGULAR MOMENTUM BEFORE = ANGULAR MOMENTUM AFTER**

$\Rightarrow (mv) \times a = (mU) \times a + I\Omega$

$\Rightarrow mva = mUa + \frac{13}{2}m a^2 \Omega$

**But  $v = 4\sqrt{2g}$**

$\Rightarrow mva = m a \Omega + \frac{13}{2} m a^2 \Omega$

$\Rightarrow v = a \Omega + \frac{13}{2} a \Omega$

$\Rightarrow 4\sqrt{2g} = \frac{15}{2} a \Omega$

$\Rightarrow \frac{2 \times 4\sqrt{2g}}{15} = \Omega$

$\Rightarrow \Omega = \frac{8\sqrt{2g}}{15}$

**Question 26 (\*\*\*)**

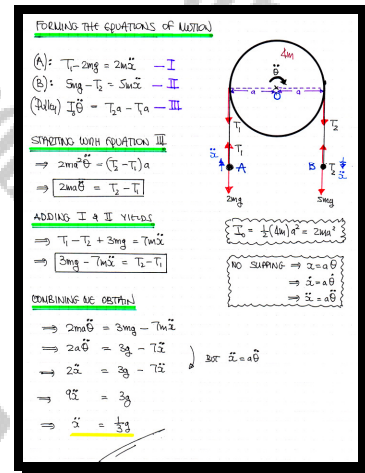
A pulley is modelled as a uniform circular disc of mass  $4m$  and radius  $a$ .

The pulley is free to rotate about a fixed smooth horizontal axis through its centre and perpendicular to its plane. A light inextensible string passes over the pulley and two particles  $A$  and  $B$ , of respective mass  $2m$  and  $5m$  are attached to the two ends of the string.

The particles are released from rest with the string taut.

Assuming further that there is no slipping between the string and the pulley find, in terms of  $g$ , the acceleration of the particles.

,  $\ddot{x} = \frac{1}{3}g$



**Question 27** (\*\*\*)

A thin uniform rod of mass  $2m$  and length  $2a$  is freely pivoted at  $A$  and is hanging at rest in a vertical position with  $B$  below  $A$ .

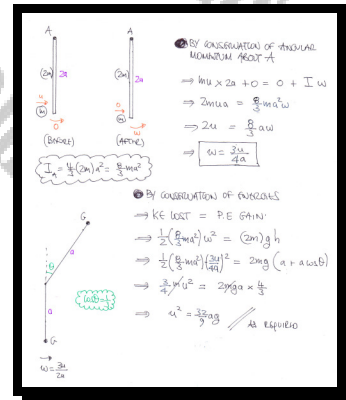
A particle of mass  $m$ , travelling horizontally with speed  $u$ , strikes the rod at  $B$ .

After the impact the particle remains at rest and the rod begins to rotate coming to instantaneous rest when  $AB$  is at  $\arccos \frac{1}{3}$  to the upward vertical through  $A$ .

Using a clear method, show that

$$u^2 = \frac{32}{9} ag.$$

proof



**Question 28** (\*\*\*)

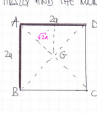
A uniform square lamina  $ABCD$ , of mass  $m$  and side  $2a$ , is free to rotate in a vertical plane about a fixed, smooth, horizontal axis  $L$ , which passes through  $A$  and is perpendicular to the plane of the lamina.

The lamina is released from rest with  $AC$  horizontal.

Determine, in terms of  $mg$ , the magnitude of the component of the force exerted by the lamina on  $L$ , along  $AC$ , when  $AC$  is vertical with  $C$  below  $A$ .


$\frac{5}{2}mg$

• FIRST FIND THE MOMENT OF INERTIA ABOUT THE REQUIRED AXIS



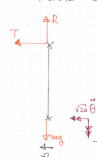
"SHIFT" TO  $AC \Rightarrow I_A = \frac{1}{3}m(2a)^2$   
 "SHIFT" TO  $BD \Rightarrow I_G = \frac{1}{12}m(2a)^2$   
 BY THE PERPENDICULAR AXIS THEOREM  $I_A = I_G + m(AG)^2$   
 PERPENDICULAR TO THE PLANE OF THE LAMINA IS  $\frac{1}{3}m(2a)^2 + \frac{1}{12}m(2a)^2 = \frac{5}{4}ma^2$

• NEXT DRAW THE LAMINA AS THE ROD  $AG$



BY ENERGY  
 I.E.  $\Delta K = \Delta U$   
 $\frac{1}{2}I\omega^2 = mg(a\sqrt{2})$   
 $\frac{1}{2}(\frac{5}{4}ma^2)\omega^2 = mga\sqrt{2}$   
 $\frac{5}{8}a\omega^2 = g\sqrt{2}$   
 $a\omega^2 = \frac{3\sqrt{2}}{4}g$

• NEXT CONSIDERING FORCES NOTING THAT IN THE REQUIRED POSITION THERE IS NO ANGULAR ACCELERATION (NO MOMENT PRECISIT) FORCE



• TANGENTIAL  $m(\sqrt{2}a\dot{\theta}) = T$   
 $T = 0$   
 • RADIAL  $m(-\sqrt{2}a\dot{\theta}^2) = mg - R$   
 $R = mg + m(\frac{3\sqrt{2}}{4}g)a$   
 $R = mg + \frac{3}{4}mg$   
 $R = \frac{5}{4}mg$

**Question 29** (\*\*\*)

A thin uniform rod of mass  $m$  and length  $2a$  has a particle of mass  $m$  attached at  $B$ .

The rod is freely pivoted at  $A$  and is hanging at rest in a vertical position, with  $B$

below  $A$ , when it is given an angular velocity about  $A$  of magnitude  $\sqrt{\frac{4g}{a}}$ .

Determine, in terms of  $m$  and  $g$ , the horizontal and vertical components of the force exerted on the axis of rotation when  $AB$  is in a horizontal position.

$$F_{hor} = \frac{69}{8}mg, \quad F_{ver} = \frac{5}{16}mg$$

**Diagram:** A vertical rod of length  $2a$  is pivoted at point  $A$  at the top. A particle of mass  $m$  is attached at point  $B$  at the bottom. The rod is initially vertical and is given an initial angular velocity  $\omega = \sqrt{\frac{4g}{a}}$  at  $A$ . The rod is then shown in a horizontal position.

**Calculations:**

- Moment of Inertia about A:**

$$I = \frac{1}{3}m(2a)^2 + m(2a)^2 = \frac{10}{3}ma^2$$
- By Conservation of Energy (Take initial position as the zero potential level):**

$$\frac{1}{2}I\omega^2 = mga + mg(2a) + \frac{1}{2}I\Omega^2$$

$$\frac{1}{2}\left(\frac{10}{3}m\right)\left(\frac{4g}{a}\right) = 3mga + \frac{1}{2}\left(\frac{10}{3}m\right)\Omega^2$$

$$\frac{20}{3}mga = 3mga + \frac{5}{3}m\Omega^2 a^2$$

$$\frac{10}{3}mga = \frac{5}{3}m\Omega^2 a^2$$

$$\Rightarrow 2ga = \Omega^2 a^2$$

$$\Rightarrow \Omega^2 = \frac{2g}{a}$$
- Reaction at A (Acc:  $-r\dot{\theta}^2$ ):**

$$m(-a\Omega^2) + m(-2a\Omega^2) = -Y$$

$$Y = 3ma\Omega^2$$

$$Y = 3ma\left(\frac{2g}{a}\right)$$

$$Y = \frac{69}{8}mg$$
- Tension at A (Acc:  $r\ddot{\theta}$ ):**

$$T\ddot{\theta} = L$$

$$\frac{1}{2}m(2a)\ddot{\theta} = mg(2a) - mg(a)$$

$$\ddot{\theta} = -\frac{g}{4a}$$

Thus:

$$m(a\ddot{\theta}) + m(2a\ddot{\theta}) = X - 2mg$$

$$3ma\left(-\frac{g}{4a}\right) = X - 2mg$$

$$-\frac{3}{4}mg = X - 2mg$$

$$X = \frac{5}{16}mg$$

**Question 30 (\*\*\*)**

A uniform disc, of mass  $4m$  and radius  $a$ , is free to rotate about a smooth, fixed horizontal axis which is tangential to a point on the rim of the disc and lies on the plane of the disc.

The disc is hanging in stable equilibrium when it is struck by a particle of mass  $m$  moving with speed  $u$  in a direction perpendicular to the plane of the disc. The particle hits the disc at the lowest point of the disc and immediately adheres to it.

Given that in the subsequent motion the disc performs full revolutions about its axis, show that

$$u^2 \geq \frac{54}{5} ag.$$

proof

**1. FIRSTLY, SOME MOMENT OF INERTIA CONSIDERATIONS**

MOMENT OF INERTIA THROUGH AN AXIS THROUGH O PERPENDICULAR TO THE PLANE OF THE DISC IS  $\frac{1}{2}(4m)a^2 = 2ma^2$

BY THE PERPENDICULAR AXIS THEOREM SINCE WE ARE THE MOMENT OF INERTIA ABOUT A DIAMETER IS  $I_{xx} = I_{yy} = ma^2$

BY PARALLEL AXIS THEOREM  $I_p = ma^2 + (4m)a^2 = 5ma^2$

**2. NOW, LOOKING AT THE CROSS-SECTION OF THE DISC**

BY CONSERVATION OF ANGLE MOMENTUM ABOUT THE POINT O

$$\Rightarrow (4m) \times 2a \times 0 = (I_p + I_{cm}) \omega$$

$$\Rightarrow 2mua = (5ma^2 + m(2a)^2) \omega$$

$$\Rightarrow 2mua = 9ma^2 \omega$$

$$\Rightarrow \omega = \frac{2u}{9a}$$

**3. NEXT, THE LOCATION OF THE CENTER OF MASS FROM THE AXIS**

$$\frac{4m \times 0}{a} + \frac{m \times 2a}{2a} \Rightarrow \bar{x} = \frac{4a + 2a}{2} = 3a$$

(FROM THE ORIGINAL LEVEL OF CENTER OF MASS AS ZERO POSITION)

**4. BY ENERGIES — WE COMPUTE INITIAL KE — KE FINAL >= 0**

(FROM THE ORIGINAL LEVEL OF CENTER OF MASS AS ZERO POSITION)

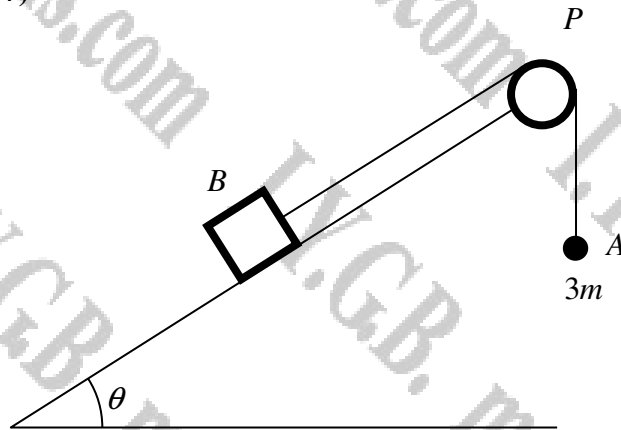
$$\frac{1}{2} I_p \omega^2 + 0 \geq m g \left( \frac{2a}{2} \times 2 \right)$$

$$\frac{1}{2} (9ma^2) \left( \frac{2u}{9a} \right)^2 \geq mg(2a)$$

$$\frac{2}{9} m u^2 \geq \frac{2}{9} m g a$$

$$u^2 \geq \frac{54}{5} ag$$

Question 31 (\*\*\*)



A small box  $B$  and a particle  $A$ , of mass  $m$  and  $3m$  respectively, are attached to each of the ends of a light inextensible string.

The string passes over a pulley  $P$ , at the top of a fixed rough plane, inclined at an angle  $\theta$  to the horizontal, where  $\tan \theta = \frac{3}{4}$ . The pulley is modelled as a uniform disc of mass  $2m$  and radius  $a$ , rotating about a smooth fixed horizontal axis.

The small box  $B$  is placed at rest on the incline plane while  $A$  is hanging freely at the end of the incline plane vertically below  $P$ , as shown in the figure above. It is further given that  $A$ ,  $B$ ,  $P$  and the string lie in a vertical plane parallel to the line of greatest slope of the incline plane.

The system is released from rest with the string taut. The box  $B$  begins to move up the incline plane, where the coefficient between  $B$  and the plane is  $0.5$ . Ignoring air resistance, find the acceleration of  $A$ , immediately after the system is set in motion.

$$\ddot{x} = \frac{2}{5}g = 3.92 \text{ ms}^{-2}$$

Free body diagrams and equations for motion:

$(A): 3mg - T_1 = 3m\ddot{x}$   
 $(B): T_1 - mg \cos \theta - \mu R = m\ddot{x}$   
 $(P): (T_2 - T_1) = I\ddot{\theta}$

$\Rightarrow \begin{cases} 3mg - T_1 = 3m\ddot{x} \\ T_1 - mg \cos \theta - \mu R = m\ddot{x} \end{cases} \Rightarrow \begin{cases} 3mg - T_1 = 3m\ddot{x} \\ T_1 - \frac{3}{5}mg - \frac{1}{2}(mg \cos \theta) = m\ddot{x} \end{cases} \Rightarrow$   
 $\Rightarrow \begin{cases} 3mg - T_1 = 3m\ddot{x} \\ T_1 - T_2 = m\ddot{x} \end{cases} \Rightarrow \begin{cases} T_2 = 3mg - 3m\ddot{x} \\ m\ddot{x} = T_1 - T_2 \end{cases} \Rightarrow$   
 $\Rightarrow m\ddot{x} = (3mg - 3m\ddot{x}) - (mg + \mu R)$   
 $\Rightarrow m\ddot{x} = 2mg - 4\mu R$   
 $\Rightarrow \ddot{x} = 2g - 4\mu g$   
 $\Rightarrow \ddot{x} = \frac{2}{5}g$   
 $(3.92 \text{ ms}^{-2})$

$\tan \theta = \frac{3}{4}$   
 $\cos \theta = \frac{4}{5}$   
 $\mu = \frac{1}{2}$

The string is not slipping on the pulley  $\Rightarrow \ddot{x} = a\ddot{\theta}$   
 $\Rightarrow \ddot{x} = 2g - 4\mu g$   
 $\Rightarrow \ddot{x} = \frac{2}{5}g$   
 $(3.92 \text{ ms}^{-2})$

**Question 32** (\*\*\*)

A uniform circular disc, of mass  $m$  and radius  $a$ , is free to rotate about a fixed smooth horizontal axis  $L$ , tangential to a point  $A$  on the circumference of the disc.

The centre  $O$  of the disc moves in a vertical plane that is perpendicular to  $L$ .

The disc is held with its plane horizontal and released from rest.

Determine the magnitude of each of the components, in the radial and transverse directions to the motion of the disc, of the force on  $L$ , when the disc has turned through an angle of  $60^\circ$ .

$$F_R = \frac{13\sqrt{2}}{10} mg, \quad F_T = \frac{1}{10} mg$$

**MOMENTS ABOUT POINT A**

$$I_A = \frac{1}{2} ma^2$$

$$I_L = \frac{1}{2} ma^2 + ma^2$$

$$I_L = \frac{3}{2} ma^2$$

**LOOKING AT A SIDE VIEW OF THE DISC**

•  $I \ddot{\theta} = 1$   
 $(\frac{3}{2} ma^2) \ddot{\theta} = (mg \cos \theta) \times a$   
 $\frac{3}{2} a \ddot{\theta} = g \cos \theta$   
 $a \ddot{\theta} = \frac{2}{3} g \cos \theta$

• ALSO BY ENERGY — POTENTIAL ENERGY LOST = K.E GAINED  
 (CHANGING THE LEVEL OF A, AS THE DISC ROTATES)

$$mg(a \sin \theta) = \frac{1}{2} I \dot{\theta}^2$$

$$mg a \sin \theta = \frac{1}{2} (\frac{3}{2} ma^2) \dot{\theta}^2$$

$$\frac{2}{3} g \sin \theta = a \dot{\theta}^2$$

**• RADIAL**

$$m(a \ddot{\theta}) = mg \sin \theta - R$$

$$R = mg \sin \theta + ma \ddot{\theta}$$

$$R = mg \sin \theta + m (\frac{2}{3} g \sin \theta)$$

$$R = \frac{5}{3} mg \sin \theta$$

$$R = \frac{13\sqrt{2}}{10} mg$$

**• TRANSVERSE**

$$m(a \dot{\theta}) = mg \cos \theta - T$$

$$T = mg \cos \theta - ma \dot{\theta}$$

$$T = mg \cos \theta - m (\frac{2}{3} g \sin \theta)$$

$$T = \frac{1}{3} mg \cos \theta$$

$$T = \frac{1}{10} mg$$



**Question 33** (\*\*\*)

A thin uniform rod  $AB$  of length  $2a$  and mass  $m$  is free to rotate in a vertical plane, about a smooth horizontal axis through  $A$ .

The rod is held at  $\frac{\pi}{4}$  to the upward vertical through  $A$ , and released from rest.

Determine, in terms of  $m$  and  $g$ , the magnitude and direction of the force exerted on the axis at  $A$ , when  $B$  is vertically below  $A$ .

$$\frac{1}{4}mg [10 + 3\sqrt{2}], \text{ radially inwards}$$

• BY ENERGY METHOD: THE LEVEL OF  $\frac{2a}{2}$  AT THE ZERO POTENTIAL LEVEL  
 $\rightarrow K.E_{rot} + P.E_{rot} = K.E_{rot} + P.E_{trans}$   
 $\rightarrow 0 + mgs \cos \frac{\pi}{4} = \frac{1}{2} I \omega^2 + mgs$   
 $\rightarrow \frac{1}{2} (2ma^2) \omega^2 = \frac{1}{2} mgs + mgs$   
 $\rightarrow \frac{1}{2} \omega^2 = \frac{2gs}{2a^2}$   
 $\rightarrow \omega^2 = \frac{2gs}{a^2}$   
 $\rightarrow \omega = \frac{\sqrt{2gs}}{a}$

$I_A = \frac{1}{3} m(2a)^2$

• BY INSPECTION  $X=0$   
 (AS TAKING MOMENTS ABOUT A, THERE IS NO OTHER FORCE TENDENCY)

• DYNAMICALLY  
 $\rightarrow M \ddot{\theta} = mg - Y$   
 $\rightarrow m(-a\dot{\theta}^2) = mg - Y$   
 $\rightarrow Y = mg + ma\dot{\theta}^2$   
 $\rightarrow Y = mg + m \times \frac{2gs}{a^2} (2a)^2$   
 $\rightarrow Y = mg + 8mg$   
 $\rightarrow Y = 9mg$   
 (Note: The handwritten solution shows a different path to the final answer, likely including a component of the reaction force.)

$\rightarrow Y = \frac{1}{4} mg (10 + 3\sqrt{2})$   
 RADIALLY IN

**Question 34 (\*\*\*)**

A thin uniform rod  $AB$  of length  $2a$  and mass  $m$  is free to rotate in a vertical plane, about a smooth horizontal axis through  $O$ , where  $|AO| = \frac{2}{3}a$ .

When the rod is vertical with  $B$  below  $O$ , the rod has angular velocity  $\sqrt{\frac{9g}{2a}}$ .

Determine, in terms of  $m$  and  $g$ , the magnitude and direction of the force exerted on the axis at  $O$ , when  $AB$  is horizontal

$\frac{5}{4}mg$

**• MOMENT OF INERTIA OF THE ROD ABOUT O IS**  
 $\frac{1}{3}m(2a)^2 + m\left(\frac{2}{3}a\right)^2 = \frac{10}{3}ma^2$

**• EQUATION OF MOTION WHEN THE ROD IS HORIZONTAL**  
 $\tau = L$   
 $\left(\frac{10}{3}ma^2\right)\ddot{\theta} = -mg\left(\frac{1}{2}a\right)$   
 $4a\ddot{\theta} = -3g$   
 $a\ddot{\theta} = -\frac{3}{4}g$

**• BY USING, TAKING THE VERTICAL POSITION AS THE ZERO POTENTIAL LEVEL**  
 $\frac{1}{2}I\omega^2 + 0 = \frac{1}{2}I\dot{\theta}^2 + mg\left(\frac{1}{2}a\right)$   
 $\frac{1}{2}\left(\frac{10}{3}ma^2\right)\left(\frac{9g}{2a}\right) = \frac{1}{2}\left(\frac{10}{3}ma^2\right)\dot{\theta}^2 + \frac{1}{2}mga$   
 $mga = \frac{5}{3}ma^2\dot{\theta}^2 + \frac{1}{2}mga$   
 $\frac{5}{6}g = \frac{5}{3}a\dot{\theta}^2$   
 $a\dot{\theta}^2 = \frac{3}{4}g$

**• NEXT CONSIDER THE FORCES AT THE POINT**

IN POINT  
 $a = (r \cdot \dot{\theta}) \hat{e} = \frac{10}{3}m\dot{\theta} \hat{e}$   
 here  $r = \frac{1}{2}a$   
 $F = F_{\text{net}}$   
 $a = \left(-\frac{1}{2}\dot{\theta}^2\right) \hat{e} + \frac{1}{3}a\ddot{\theta} \hat{e}$

**• FINCE THE EQUATION OF MOTION YIELDS**  
 $\Rightarrow T - mg = m\left(\frac{1}{2}a\dot{\theta}^2\right) \Rightarrow -2 = m\left(-\frac{1}{2}a\dot{\theta}^2\right)$   
 $\Rightarrow T - mg = m \times \frac{1}{2}\left(-\frac{3}{4}g\right) \Rightarrow 2 = \frac{1}{2}ma\dot{\theta}^2$   
 $\Rightarrow T - mg = -\frac{1}{4}mg \Rightarrow 2 = \frac{1}{2}m \times \frac{3}{4}g$   
 $\Rightarrow T = \frac{5}{4}mg$

**• MAGNITUDE OF THE REACTION FORCE AT THE POINT**  
 $|F| = \sqrt{T^2 + R^2}$   
 $= \sqrt{\left(\frac{5}{4}mg\right)^2 + \left(\frac{3}{4}mg\right)^2}$   
 $= \frac{5}{4}mg$

**Question 35** (\*\*\*)

A uniform rod of mass 5 kg and length 3 m is free to rotate in a vertical plane about a fixed horizontal axis through one of the two ends of the rod.

The rod is released from rest in a horizontal position. A constant frictional couple of magnitude 36.75 Nm opposes the motion.

- Find the initial angular acceleration of the rod.
- Determine the angle that the rod makes with the horizontal when its angular acceleration is zero.
- Calculate the greatest angular speed of the rod.

$$\ddot{\theta} = 2.45 \text{ rad s}^{-2}, \quad \theta = 60^\circ, \quad \dot{\theta} \approx 1.83 \text{ rad s}^{-1}$$

The handwritten solution is divided into three parts:

- a) INITIAL ANGULAR ACCELERATION:**
  - Diagram: A horizontal rod of length 3m pivoted at the left end. Gravity acts downwards at the center (1.5m from pivot).
  - Equation:  $\tau = I\ddot{\theta}$
  - Calculation:  $15g = 5g \times 1.5 = 36.75$
  - Result:  $\ddot{\theta} = 2.45 \text{ rad s}^{-2}$
- b) ANGLE WHEN ANGULAR ACCELERATION IS ZERO:**
  - Diagram: The rod is at an angle  $\theta$  to the horizontal. Gravity acts downwards at the center. A frictional couple  $\tau = 36.75$  Nm acts counter-clockwise.
  - Equation:  $\tau = I\ddot{\theta}$
  - Calculation:  $36.75 = 5g \cos\theta \times 1.5 = 36.75$
  - Result:  $\cos\theta = \frac{1}{2}$ ,  $\theta = 60^\circ$
- c) MAXIMUM ANGULAR SPEED:**
  - Diagram: The rod is vertical at its lowest point.
  - Equation:  $kE_{\text{rot}} + P.E_{\text{rot}} + P.E_{\text{cm}} = kE_{\text{rot}} + P.E_{\text{cm}}$
  - Calculation:  $-36.75 \times \frac{3}{2} = \frac{1}{2} \times 5 \times 3^2 \dot{\theta}^2 - 5g \times 1.5 \times \frac{3}{2}$
  - Result:  $7.5\dot{\theta}^2 = 25.125$ ,  $\dot{\theta} \approx 1.83 \text{ rad s}^{-1}$

**Question 36** (\*\*\*)

A plane shape  $S$  of mass  $m$  is formed by removing a circular disc with centre  $O$  and radius  $a$  from a uniform circular disc with centre  $O$  and radius  $3a$ .

$S$  is free to rotate about a fixed smooth horizontal axis  $L$ , which passes through  $O$  and lies in the plane of  $S$ . Initially  $S$  is at rest in a horizontal plane when a particle of mass  $2m$  falls vertically and strikes  $S$  at the point  $P$ , where  $OP = 2a$  and  $OP$  is perpendicular to  $L$ . Immediately before the particle strikes  $P$  the speed of the particle is  $u$ . The particle adheres to  $S$  at  $P$ .

Find, in terms of  $m$  and  $u$ , the loss in kinetic energy due to the impact.

$$\frac{5}{21}mu^2$$

**AREA OF DISCS**  
 AREA OF  $\odot = \pi r^2$   
 AREA OF  $\odot = \pi(3a)^2 = 9\pi a^2$   
 AREA OF  $\odot = \pi a^2$   
 MASS OF  $\odot = \rho \pi r^2 t$   
 MASS OF  $\odot = \rho \pi a^2 t$   
 $\therefore$  MASS OF  $\odot = \frac{1}{9}M$   
 MASS OF  $\odot = \frac{8}{9}M$

**BY ADDITION RULE (UNO CONCENTRADO) ABOUT THE DISC**  
 $\odot + \odot = \odot$   
 $\frac{1}{2}(9M)a^2 + I = \frac{1}{2}(\frac{9}{8}M)(3a)^2$   
 $\frac{1}{2}9Ma^2 + I = \frac{1}{2}(\frac{9}{8})9Ma^2$   
 $I = \frac{5}{8}Ma^2$

**BY PERPENDICULAR AXES THEOREM (UNO CONCENTRADO)**  
 $I_x + I_y = I_z$   
 $2I_x = I_z$   
 $2I_x = \frac{5}{8}Ma^2$   
 $I_x = \frac{5}{16}Ma^2$

**THE MOMENT OF INERTIA OF THE LAMINA WITH THE PARTICLE STRUCK AT P**  
 $I = \frac{5}{16}Ma^2 + (2m)(2a)^2 = \frac{17}{8}Mu^2$

**BY CONSERVATION OF ANGULAR MOMENTUM ABOUT L**  
 $(2Mu) \times 2a + 0 = I\omega$   
 $4Mu = \frac{17}{8}Mu\omega$   
 $8u = 17a\omega$   
 $\omega = \frac{8u}{17a}$

**K.E BEFORE** =  $\frac{1}{2}(2m)u^2 = Mu^2$   
**K.E AFTER** =  $\frac{1}{2}I\omega^2 = \frac{1}{2}(\frac{17}{8}Mu^2)(\frac{8u}{17a})^2 = \frac{16}{17}Mu^2$   
 $\therefore$  A LOSS OF  $\frac{1}{17}Mu^2$

**Question 37** (\*\*\*)

The points  $A$ ,  $B$ ,  $C$  and  $D$  lie on the circumference of a circular hoop of mass  $m$  and radius  $a$ , so that  $AC$  and  $BD$  are two perpendicular diameters of the hoop.

Two particles, each of mass  $M$ , are attached to  $A$  and  $B$ .

The loaded hoop is free to rotate in a vertical plane, about a fixed smooth horizontal axis through  $D$ .

The system is released from rest, with  $AC$  in a vertical position,  $A$  uppermost.

When  $AC$  is in a horizontal position the angular velocity of the system is  $\omega$ .

Show that

$$\omega^2 = \frac{(m + 4M)g}{(m + 3M)a},$$

and hence deduce  $\omega$  if the mass of the hoop is insignificant compared to that of the two particles.

$$|\omega| = \sqrt{\frac{4g}{3a}}$$

The handwritten solution includes two diagrams of the circular hoop. The top diagram shows the hoop with diameter  $AC$  vertical ( $A$  at the top,  $C$  at the bottom) and diameter  $BD$  horizontal ( $B$  on the left,  $D$  on the right). The bottom diagram shows the hoop with diameter  $AC$  horizontal ( $A$  on the left,  $C$  on the right) and diameter  $BD$  vertical ( $B$  at the top,  $D$  at the bottom). The calculations are as follows:

- Moment of Inertia about  $D$ :**

$$I_D = I_C + M(2a)^2 + M(2a)^2 = 2ma^2 + 2Ma^2 + 2Ma^2 = (m + 4M)a^2$$
- By Symmetry:** When  $BD$  is vertical, the system would have lost potential energy equal to:
 
$$g \left[ \frac{m(2a)}{2} + \frac{M(2a)}{1} + \frac{M(2a)}{1} \right] = g(ma + 4Ma) = (m + 4M)ag$$
- Energy Conservation:** Kinetic Energy (KE) = Potential Energy (PE) lost.
 
$$\frac{1}{2} I_D \omega^2 = (m + 4M)ag$$

$$\frac{1}{2} [(m + 4M)a^2] \omega^2 = (m + 4M)ag$$

$$\omega^2 = \frac{(m + 4M)g}{(m + 4M)a}$$
- Final Result:**

$$\omega^2 = \frac{g}{a}$$

$$|\omega| = \sqrt{\frac{g}{a}}$$

**Question 38** (\*\*\*)

A uniform rod  $AB$ , of mass  $m$  and length  $2l$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$ , which is perpendicular to the rod and passes through  $A$ . The rod is released from rest in a horizontal position and when the rod first becomes vertical it hits a smooth peg at a distance  $l$  vertically below  $A$ .

The peg exerts an impulse  $J$  on the rod and the rod next comes to instantaneous rest at  $\arccos x$  to the downward vertical through  $A$ .

Determine the value of  $x$  given that  $J = 2m\sqrt{\frac{3}{2}gl}$ .

$$k = \frac{3}{4}$$

The handwritten solution includes the following steps:

- Initial state:**  $I_A = \frac{1}{3}ml^2$  (using parallel axis theorem).  
By conservation of energy:  $PE_{initial} = KE_{final}$   
 $\frac{1}{2}I\omega^2 = mgl$   
 $\frac{1}{2}(\frac{1}{3}ml^2)\omega^2 = mgl$   
 $\frac{1}{3}ml^2\omega^2 = 2mgl$   
 $\frac{1}{3}l^2\omega^2 = 2gl$   
 $\omega^2 = \frac{6g}{l}$
- Impulse at the peg:**  $J = 2m\sqrt{\frac{3}{2}gl}$   
Momentum of impulse = change in angular momentum  
 $J \times l = I(\Omega - \omega)$   
 $\sqrt{\frac{3}{2}} \times l = \frac{1}{3}ml^2[\Omega - \sqrt{\frac{6g}{l}}]$   
 $\sqrt{\frac{3}{2}} = \frac{1}{3}l[\Omega + \sqrt{\frac{6g}{l}}]$   
 $\frac{3}{2}\sqrt{\frac{3}{2}} = \Omega + \sqrt{\frac{6g}{l}}$   
 $\Omega = \frac{3}{2}\sqrt{\frac{3}{2}} - \sqrt{\frac{6g}{l}}$
- Energy conservation after impulse:**  $KE_{final} = PE_{final}$   
 $\frac{1}{2}I\Omega^2 = mgh$   
 $\frac{1}{2}(\frac{1}{3}ml^2)(\frac{3}{2}\sqrt{\frac{3}{2}} - \sqrt{\frac{6g}{l}})^2 = mgh$   
 $\frac{1}{2}mgl = mgh$   
 $h = \frac{1}{2}l$
- Final state:**  $\cos \theta = \frac{h}{l} = \frac{1}{2}$   
 $\theta = \arccos(\frac{1}{2})$

**Question 39** (\*\*\*)

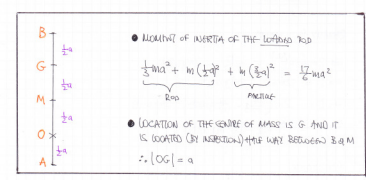
A particle of mass  $m$  is attached to the point  $B$  of a uniform rod  $AB$ , of mass  $m$  and length  $2a$ .

The loaded rod is free to rotate about a smooth, horizontal axis through the point  $O$  on the rod, where  $|OA| = \frac{1}{6}a$ .

The rod is held in a vertical position with  $B$  above  $O$  and is slightly disturbed. When the rod has turned by an angle  $\theta$  from the upward vertical the magnitude of the force exerted by the rod on the axis is  $F$ .

Determine an expression for  $F$ , in terms of  $m$ ,  $g$  and  $\theta$ , and hence determine in terms of  $m$  and  $g$ , an expression for  $F$  when  $\cos \theta = \frac{1}{6}$

$$F = \frac{2}{17}mg\sqrt{601 - 1968\cos\theta + 1656\cos^2\theta}, \quad F = \frac{2}{17}mg\sqrt{319}$$



- MOMENT OF INERTIA OF THE LOADED ROD
- LOCATION OF THE CENTRE OF MASS IS G AND IT IS LOCATED (BY INSPECTION) THE WAY RESOLVED TO G.M.  $\therefore |OG| = a$

$$\frac{1}{2}ma^2 + m\left(\frac{1}{6}a\right)^2 + m\left(\frac{5}{6}a\right)^2 = \frac{17}{12}ma^2$$

- FROM THE EQUATION OF MOTION
- $I\ddot{\theta} = L$
- $\int \frac{17}{12}m\ddot{\theta} = (2mg\sin\theta) \times a$
- $\frac{17}{12}m\dot{\theta} = 2mg\sin\theta$
- $17a\dot{\theta} = 12g\sin\theta$
- $17a(2\dot{\theta}) = 24g\dot{\theta}\sin\theta$
- INTEGRATE W.R.T  $\theta$ , SUBJECT TO  $\dot{\theta} = 0$  AT  $\theta = 0$
- $\int 17a\dot{\theta}^2 = \int 24g\dot{\theta}\sin\theta$
- $17a\dot{\theta}^2 = 24g\cos\theta$

SIMILY (OUTWARD)

$$2m(-a\dot{\theta}^2) = -R - 2mg\cos\theta$$

$$R = 2ma\dot{\theta}^2 - 2mg\cos\theta$$

$$R = 2m\left(\frac{24}{17}g\right) - 2mg\cos\theta$$

$$R = \frac{48}{17}mg - 2mg\cos\theta$$

TECHNIQUE

$$2m(a\ddot{\theta}) = 2mg\sin\theta - T$$

$$T = 2mg\sin\theta - 2ma\ddot{\theta}$$

$$T = 2mg\sin\theta - 2m\left(\frac{12}{17}g\sin\theta\right)$$

$$T = \frac{10}{17}mg\sin\theta$$

METHOD

$$R = \frac{2}{17}mg(24 - 41\cos\theta)$$

$$T = \frac{2}{17}mg(5\sin\theta)$$

$$\therefore \text{MAGNITUDE} = \frac{2}{17}mg\sqrt{(24 - 41\cos\theta)^2 + 25\sin^2\theta}$$

$$\text{MAGNITUDE} = \frac{2}{17}mg\sqrt{601 - 1968\cos\theta + 1656\cos^2\theta}$$

$$\text{WHEN } \cos\theta = \frac{1}{6}$$

$$\text{MAGNITUDE} = \frac{2}{17}mg\sqrt{601 - 1968 \times \frac{1}{6} + 1656 \times \frac{1}{36}}$$

$$\text{MAGNITUDE} = \frac{2}{17}\sqrt{319}mg$$

**Question 40** (\*\*\*)

A bucket of mass  $3m$  is attached to one end of rope and moves in a vertical line.

The rope passes vertically up from the bucket and is wrapped several times around a cylindrical drum of mass  $2m$  and radius  $a$ .

The drum is free to rotate about its axis of symmetry which remains in a fixed horizontal position.

The bucket is released from rest, with the rope taut, and begins to move vertically downwards.

The bucket is modelled as a particle, the drum as a uniform cylinder rotating about its fixed smooth axis, the rope as a light inextensible string.

Ignoring that air resistance show that

a) ... the angular acceleration of the drum is  $\frac{3g}{4a}$ .

b) ... the time it takes the drum to complete 9 full revolutions is  $4\sqrt{\frac{3\pi a}{g}}$ .

proof

The handwritten solution is divided into two parts, a) and b).

**Part a):** A free-body diagram shows a bucket of mass  $3m$  with forces  $3mg$  (down) and  $T$  (up). A drum of mass  $2m$  and radius  $a$  is shown with tension  $T$  applied at the top edge, causing a clockwise angular acceleration  $\alpha$ . The moment of inertia about the center is  $\frac{1}{2}(2m)a^2 = ma^2$ . The equations of motion are:
 
$$3m\ddot{x} = 3mg - T$$

$$ma\ddot{\theta} = Ta$$
 Equating the tensions  $T = 3mg - 3m\ddot{x}$  and  $T = ma\ddot{\theta}$  gives:
 
$$3m\ddot{x} = 3mg - 3m\ddot{\theta}a$$

$$\Rightarrow \ddot{\theta} = \frac{3g}{4a}$$

**Part b):** Since the acceleration is constant, the constant acceleration equations are used:
 
$$u = 0$$

$$a = \ddot{\theta} = \frac{3g}{4a}$$

$$s = \theta = 18\pi \leftarrow 9 \text{ revolutions}$$

$$v = \dot{\theta} = \omega$$

$$v^2 = \omega^2 = ?$$
 Hence:
 
$$s = ut + \frac{1}{2}at^2$$

$$\Rightarrow \theta = 18\pi = \frac{1}{2} \left(\frac{3g}{4a}\right) t^2$$

$$\Rightarrow 18\pi = \frac{3g}{8a} t^2$$

$$\Rightarrow \frac{48\pi a}{3g} = t^2$$

$$\Rightarrow t = 4\sqrt{\frac{3\pi a}{g}}$$



**Question 41** (\*\*\*)

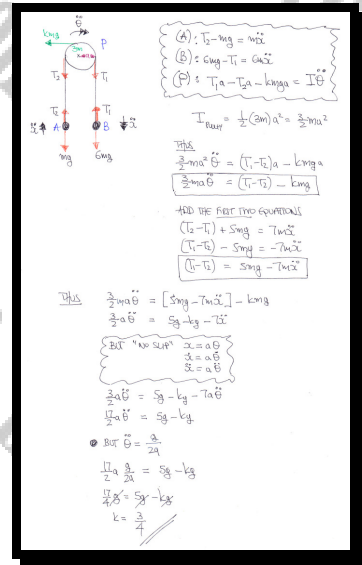
A light inextensible string has a particle of mass  $m$  attached to one end and a particle of mass  $6m$  attached to the other end. The string passes over a rough pulley, which is modelled as a uniform disc of mass  $3m$  and radius  $a$ .

The pulley rotates in a vertical plane through a fixed smooth horizontal axis which passes through the centre of the pulley and is perpendicular to the plane of the pulley.

The system is released from rest with the string taut and the parts of the string not in contact with the pulley vertical. The string does not slip on the pulley during the consequent motion. The pulley is experiencing a constant frictional couple of magnitude  $kmg a$ , where  $k$  is a positive constant.

Given that the angular acceleration of the pulley is  $\frac{g}{2a}$ , determine the value of  $k$ .

$k = \frac{3}{4}$



Question 42 (\*\*\*)



A particle  $A$  of mass  $m$  is connected to small box  $B$  of mass  $2m$  by a light inextensible string. The string passes over a pulley  $P$ , which is located at the end of a smooth horizontal table. The box is held on the table with the particle hanging vertically at the end of the table, as shown in the figure above. The pulley is modelled as a disc of mass  $4m$  and radius  $a$ , rotating about a smooth horizontal axis through its centre. The system is released from rest with the string taut.

In the subsequent motion ...

- ... the string does not slip on the pulley.
- ... the section of the string  $PB$  not in contact with the pulley remains horizontal at all times and the section of the string  $AP$  not in contact with the pulley remains vertical at all times.

Find the acceleration of the system and hence show that while the system is in motion the force exerted on the pulley has magnitude

$$\frac{10}{3}\sqrt{2}mg$$

$$\ddot{x} = \frac{1}{3}g$$

$T_1 - T_2 = m\ddot{x} - mg$   
 $T_1 - T_2 = -\frac{1}{2}I\ddot{\theta}$   
 $m\ddot{x} - mg = -\frac{1}{2}I\ddot{\theta}$   
 $m\ddot{x} - mg = -2m(a\ddot{\theta})$   
 $m\ddot{x} - mg = -2m\ddot{x}$   
 $3m\ddot{x} = mg$   
 $\ddot{x} = \frac{1}{3}g$

Now  $T_1 = 2m\ddot{x} = 2m(\frac{1}{3}g) = \frac{2}{3}mg$   
 $T_2 = mg - m\ddot{x} = mg - m(\frac{1}{3}g) = \frac{2}{3}mg$

The magnitude  
 $= mg\sqrt{(\frac{1}{3})^2 + (\frac{1}{3})^2}$   
 $= \frac{10}{3}\sqrt{2}mg$

**Question 43** (\*\*\*)

A pendulum is modelled as a uniform rod  $AB$ , of mass  $3m$  and length  $2a$ , which has a particle of mass  $2m$  attached at  $B$ . The pendulum is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$  which passes through  $A$ . The vertical plane is perpendicular to  $L$ .

The pendulum is hanging at rest in a vertical position, with  $B$  below  $A$ , when it receives a horizontal impulse of magnitude  $J$ . The impulse acts at  $B$  in a vertical plane which is perpendicular to  $L$ .

Given that the pendulum turns through an angle of  $60^\circ$  before first coming to instantaneous rest show that  $J = m\sqrt{21ag}$ .

proof

**1. FIND THE MOMENT OF INERTIA AT THE POSITION OF THE CENTRE OF MASS**

PARALLEL AXIS	2nd PARALLEL AXIS
MASS	3
DISTANCE FROM A	a
	2a
	2a
	2a

$I_A = \frac{1}{12}(3m)a^2 + 2m(2a)^2$   
 $I_A = 6ma^2 + 8ma^2$   
 $I_A = 14ma^2$

$3a + 2(2a) = 5a$   
 $5a = 7a$   
 $R = \frac{7}{5}a$

**2. NEXT BY CONSERVATION OF ENERGY - KE (INIT) = PE (FINAL)**

$\frac{1}{2}I\omega^2 = \text{Sang} \left( \frac{3}{5}a - \frac{7}{5}a \cos 60^\circ \right)$   
 $\frac{1}{2}(14ma^2)\omega^2 = 7mga(1 - \cos 60^\circ)$   
 $6ma^2\omega^2 = \frac{7}{2}ga$   
 $\omega^2 = \frac{7g}{12a}$

**3. FINALLY MOMENT OF MOMENTUM = CHANGE OF ANGLE OF MOMENTUM (ABOUT A)**

$J \times 2a = \omega I$   
 $J = \frac{\omega I}{2a}$   
 $J = \frac{\sqrt{\frac{7g}{12a}} \times 14ma^2}{2a}$   
 $J = \frac{\sqrt{7g}}{\sqrt{12a}} \times 7ma = \sqrt{\frac{7g}{12a}} \times 7ma$   
 $J = m\sqrt{21ag}$

**Question 44** (\*\*\*)

A pulley is modelled as a uniform circular disc of mass  $16m$  and radius  $a$ . The pulley is free to rotate about a fixed horizontal axis through its centre and perpendicular to its plane. A light inextensible string passes over the pulley and two particles  $A$  and  $B$ , of respective mass  $2m$  and  $5m$  are attached to the two ends of the string.

The particles are released from rest with the string taut.

A constant couple of magnitude of  $mga$  resists the rotation of the pulley about its axis.

In the consequent motion there is no slipping between the string and the pulley.

Determine, in terms of  $mg$ , the tension in each of the two sections of the string to which the two particles are attached.

$$\boxed{\phantom{000000}}, \quad T_A = \frac{34}{15}mg, \quad T_B = \frac{13}{3}mg$$

**● FINDING THE EQUATION OF MOTION FOR EACH COMPONENT OF THE SYSTEM**

**LOOKING AT A**  
 $T_1 - 2mg = 2m\ddot{x}$

**LOOKING AT B**  
 $5mg - T_2 = 5m\ddot{x}$

**LOOKING AT PULLEY**  
 $T_2 - T_1 - mga = I\ddot{\theta}$   
 $(T_2 - T_1) - mga = 8ma\ddot{\theta}$   
 $T_2 - T_1 - mg = 8m\ddot{\theta}$

- MOMENT OF INERTIA OF PULLEY IS  $\frac{1}{2}(16m)a^2 = 8ma^2$
- POSSIBLE COUPLE ON PULLEY  $mga$
- NO SLIP  $\Rightarrow 2a\ddot{\theta} = \ddot{x}$

**● SUBTRACTING THE FIRST TWO EQUATIONS GIVES**  
 $\Rightarrow T_2 - T_1 + 3mg = 7m\ddot{x}$   
 $\Rightarrow 3mg - 7m\ddot{x} = T_2 - T_1$

**● SUBSTITUTING INTO THE THIRD EQUATION**  
 $\Rightarrow (T_2 - T_1) - mga = 8ma\ddot{\theta}$   
 $\Rightarrow (3mg - 7m\ddot{x}) - mga = 8ma\ddot{\theta}$   
 $\Rightarrow 2mg - 7m\ddot{x} = 8m\ddot{\theta}$   
 $\Rightarrow 15\ddot{x} = 2g$   
 $\Rightarrow \ddot{x} = \frac{2g}{15}$

**● HENCE WE OBTAIN**  
 $T_1 - 2mg = 2m\ddot{x}$   
 $T_1 - 2mg = 2m(\frac{2g}{15})$   
 $T_1 - 2mg = \frac{4}{15}mg$   
 $T_1 = \frac{34}{15}mg$

$5mg - T_2 = 5m\ddot{x}$   
 $5mg - T_2 = 5m(\frac{2g}{15})$   
 $5mg - T_2 = \frac{2}{3}mg$   
 $T_2 = \frac{13}{3}mg$

**Question 45 (\*\*\*)**

A uniform rod  $AB$ , of mass  $m$ , is free to rotate about a smooth fixed horizontal axis  $L$ , which passes through  $A$ .

The rotation of the rod takes place in a vertical plane.

The rod is held so that  $AB$  makes an angle of  $60^\circ$  with the upward vertical and released from rest.

- a) Given that the moment of inertia of the rod about  $L$  is  $12ma^2$ , show that in the subsequent motion

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{8a}(1 - 2\cos\theta),$$

where  $\theta$  is the angle that  $AB$  makes with the upward vertical.

- b) Determine, in terms of  $m$ ,  $g$  and  $\theta$ , the magnitude and direction of the radial force exerted on  $L$  by the rod.

,  $\frac{1}{2}mg(26\cos\theta - 9)$ , radially outwards

a) SOMETHING WITH A DIVISOR.  
UNLESS U SHOW THE RADIAL  
OF THE ROD.

$I_A = \frac{1}{3}ml^2$   
 $12ma^2 = \frac{1}{3}ml^2$   
 $l^2 = 4a^2$   
 $l = 2a$

BY CHOOSING THE LEVEL OF 'A' AS THE ZERO POTENTIAL LINE

$\Rightarrow PE_{\text{cm}} + PE_B = KE + PE_A$

$\Rightarrow mg(2a)\cos\theta = \frac{1}{2}I\dot{\theta}^2 + mg(2a)\cos\theta$

$\Rightarrow mg(2a)\cos\theta = \frac{1}{2}(12ma^2)\dot{\theta}^2 + mg(2a)\cos\theta$

$\Rightarrow \frac{1}{2}mg(2a)\cos\theta = 6m\dot{\theta}^2 + mg(2a)\cos\theta$

$\Rightarrow 2a\dot{\theta}^2 = 2g\cos\theta$

$\Rightarrow \dot{\theta}^2 = \frac{2g}{2a}(1 - 2\cos\theta)$

b) LOOKING IN THE RADIAL DIRECTION

$\rightarrow$  "net" = resultant force

$\rightarrow m(-3a\dot{\theta}^2) = -2 - mg\cos\theta$

$\rightarrow R = 3ma\dot{\theta}^2 - mg\cos\theta$

$\rightarrow R = 3ma\left(\frac{2g}{2a}(1 - 2\cos\theta)\right) - mg\cos\theta$

$\rightarrow R = \frac{3}{2}mg(1 - 2\cos\theta) - mg\cos\theta$

$\rightarrow R = \frac{1}{2}mg(9 - 18\cos\theta - 8\cos\theta)$

$\rightarrow R = \frac{1}{2}mg(9 - 26\cos\theta)$

HENCE THE REQUIRED COMPONENT IS

$\frac{1}{2}mg(9 - 26\cos\theta)$ , radially inwards  
 or  
 $\frac{1}{2}mg(26\cos\theta - 9)$ , radially outwards

**Question 46 (\*\*\*\*)**

The point  $O$  lies on a uniform rod  $AB$  so that the ratio  $|AO| : |OB|$  is  $3 : 5$ .

The rod is held in a horizontal position on a rough horizontal table so that  $AB$  is perpendicular to the straight edge of the table.

The part of the rod  $AO$  is in contact with the table and the part  $OB$  overhangs the edge of the table.

The rod is released from rest and begins to rotate about  $O$ .

When the rod has turned by an angle  $\alpha$  it begins to slip.

If the coefficient of friction between the rod and the table is  $\lambda$ , show that

$$16\lambda = 25 \tan \alpha.$$

, proof

START WITH ALLOCATING SIZE TO THE MASSES, ABOUT  $O$

$\bullet$  LET THE LENGTH BE  $m$   
 $\bullet$  LET  $AO = 3a$   
 $\bullet$  LET  $OB = 5a$

$I = \frac{1}{2} M (AO)^2 = \frac{9}{2} Ma^2$   
 $I_0 = \frac{1}{2} M (AO)^2 + M (OB)^2 = \frac{9}{2} Ma^2 + 25 Ma^2 = \frac{59}{2} Ma^2$  (PARALLEL AXES THEOREM)

NOW CONSIDER THE ROD IN AN ARBITRARY POSITION BEFORE IT SLIPS

BY EQUATING (CONSERVATION)

$K.E \text{ GAINED} = P.E \text{ LOST}$   
 $\frac{1}{2} I \dot{\theta}^2 = m g (a \sin \theta)$   
 $\frac{1}{2} (\frac{59}{2} M a^2) \dot{\theta}^2 = m g a \sin \theta$   
 $\frac{59}{4} a \dot{\theta}^2 = g \sin \theta$   
 $\dot{\theta}^2 = \frac{4g \sin \theta}{59a}$

DIFFERENTIATE (WRT TIME)

$\frac{d}{dt} (\dot{\theta}^2) = \frac{d}{dt} (\frac{4g \sin \theta}{59a})$   
 $2 \dot{\theta} \ddot{\theta} = \frac{4g \cos \theta}{59a} \dot{\theta}$   
 $2 \ddot{\theta} = \frac{4g \cos \theta}{59a}$   
 $\ddot{\theta} = \frac{2g \cos \theta}{59a}$

NOW CONSIDER THE FORCES AT THE INSTANT OF SLIPPING

$a \dot{\theta}^2 = \frac{4g \sin \theta}{59}$   
 $a \ddot{\theta} = \frac{2g \cos \theta}{59}$

RESOLVE  $\Sigma$  TRANSVERSELY

$m(-a \dot{\theta}^2) = m g \sin \theta - 2R$   
 $m(a \ddot{\theta}) = m g \cos \theta - R$

$2R = m g \sin \theta + \frac{4}{59} m g \sin \theta$   
 $R = m g \cos \theta - \frac{2}{59} m g \cos \theta$

$\frac{2R}{R} = \frac{m g \sin \theta + \frac{4}{59} m g \sin \theta}{m g \cos \theta - \frac{2}{59} m g \cos \theta}$   
 $\Rightarrow \lambda = \frac{2g \sin \theta}{g \cos \theta}$   
 $\Rightarrow 16\lambda = 25 \tan \theta$  // AT THE INSTANT

**Question 47 (\*\*\*\*)**

Four identical rods, each of mass  $m$  and length  $2a$  are joined together to form a square rigid framework  $ABCD$ .

A fifth rod  $AC$ , of mass  $3m$ , is added to the framework for extra support.

The 5 rod framework is free to rotate about a smooth fixed horizontal axis  $L$ , which passes through  $A$ , so that the rotation of the framework takes place in a vertical plane.

The framework is held so that  $D$  is vertically above  $A$  and released from rest.

On the subsequent rotation, when  $B$  is vertically below  $A$ , a stationary particle of mass  $M$  adheres to  $B$ .

Given that the angular speed of the framework, after the particle has adhered to it, is

$$\frac{2}{9} \sqrt{\frac{21g}{a}}$$

determine  $M$  in terms of  $m$ .

$$\boxed{M = \frac{2}{3}m}$$

START BY A DIAGRAM

- Length of  $AC$  is  $2\sqrt{2}a$
- Length of  $AG$  is  $\sqrt{2}a$

MOMENT OF INERTIA OF THE ROD  $AB$ ,  $BC$ ,  $CD$ ,  $AD$  ABOUT  $A$

$$I = \frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$$

MOMENT OF INERTIA OF THE ROD  $BC$ ,  $DC$  ABOUT  $A$

$$I = \frac{1}{2}m(2a)^2 + m(\sqrt{2}a)^2 = \frac{5}{2}ma^2 + 2ma^2 = \frac{9}{2}ma^2$$

MOMENT OF INERTIA OF THE ROD  $AC$  ABOUT  $A$

$$I = \frac{1}{3}(3m)(2\sqrt{2}a)^2 + 3m(\sqrt{2}a)^2 = 24ma^2 + 6ma^2 = 30ma^2$$

ADDING TOGETHER THE MOMENT OF INERTIA OF ALL THE RODS GIVES

$$I_{TOTAL} = \frac{4}{3}ma^2 + \frac{9}{2}ma^2 + \frac{9}{2}ma^2 + \frac{9}{2}ma^2 + 30ma^2$$

$$I_{TOTAL} = \frac{64}{3}ma^2$$

WHEN  $B$  IS VERTICALLY BELOW  $A$ , THE CENTER OF MASS  $G$  OF THE SYSTEM WOULD HAVE "DROPPED" FROM A HEIGHT THE LEVEL OF  $A$  TO  $\sqrt{2}a$  BECAUSE  $A$  IS BELOW THE LINE OF  $A$

BY CONSERVATION

$$\Rightarrow \frac{1}{2}I\omega^2 = 7mg(2a)$$

$$\Rightarrow \frac{1}{2}(\frac{64}{3}m)a^2\omega^2 = 14mg(2a)$$

$$\Rightarrow \frac{32}{3}m\omega^2 = 14g$$

$$\Rightarrow \omega^2 = \frac{21g}{16a}$$

$$\Rightarrow \omega = \frac{1}{4}\sqrt{\frac{21g}{a}}$$

NEW BY CONSERVATION OF ANGULAR MOMENTUM ABOUT  $A$

- $I_{NEW TOTAL} = \frac{64}{3}ma^2 + M(2a)^2$
- $= \frac{64}{3}ma^2 + 4Ma^2$

$$I_{TOTAL} \times \omega = I_{NEW TOTAL} \times \omega$$

$$(\frac{64}{3}ma^2) \times \frac{1}{4}\sqrt{\frac{21g}{a}} = (\frac{64}{3}ma^2 + 4Ma^2) \times \frac{1}{4}\sqrt{\frac{21g}{a}}$$

$$\Rightarrow \frac{16}{3}\sqrt{\frac{21g}{a}}ma^2 = \sqrt{\frac{21g}{a}}(\frac{64}{3}m + 4M)a^2$$

$$\Rightarrow \frac{16}{3}m = \frac{64}{3}m + 4M$$

$$\Rightarrow \frac{16}{3}m = \frac{128}{3}m + 4M \quad \div 8$$

$$\Rightarrow \frac{2}{3}m = \frac{16}{3}m + \frac{1}{2}M \quad \times 6$$

$$\Rightarrow 4m = 32m + 3M$$

$$\Rightarrow 2M = -28m$$

$$\Rightarrow M = -\frac{14}{3}m$$

**Question 48 (\*\*\*\*)**

A uniform circular disc, of radius  $3a$  and mass  $2m$ , is free to rotate about a smooth fixed horizontal axis  $L$ , which is perpendicular to the plane of the disc and is passing through the point  $A$ , which lies at the circumference of the disc. The disc is held with its centre  $O$  at the same horizontal level as  $A$ , and released from rest.

Show that the horizontal component of the force exerted on  $L$  has magnitude

$$\frac{2}{3}mg\sqrt{1+48\sin^2\theta},$$

where  $\theta$  is the angle that  $AO$  makes with the horizontal.

,  proof

TREAT THE MOMENT OF INERTIA OF THE DISC ABOUT A IS GIVEN BY

$$\frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2$$

$$= \frac{3}{2}(2m)(3a)^2$$

$$= 27ma^2$$

BY CONSERVING THE INITIAL POSITION AS THE ZERO QUANTITATIVE POTENTIAL LEVEL

$$KE_{rot} + PE_{var} = KE_{rot} + PE_{var}$$

$$0 + 0 = \frac{1}{2}I\dot{\theta}^2 - Mgh$$

$$0 = \frac{1}{2}(27ma^2)\dot{\theta}^2 - (2m)g(3a\sin\theta)$$

$$0 = \frac{27}{2}a^2\dot{\theta}^2 - 6ga\sin\theta$$

$$\frac{27}{2}a^2\dot{\theta}^2 = 6ga\sin\theta$$

$$\dot{\theta}^2 = \frac{4}{9}\sin\theta$$

EQUATION OF ROTATIONAL MOTION AND YIELD

$$I\ddot{\theta} = L$$

$$(27ma^2)\ddot{\theta} = (2mg\cos\theta) \times 3a$$

$$27a^2\ddot{\theta} = 6g\cos\theta$$

$$\ddot{\theta} = \frac{2}{9}\cos\theta$$

THE EQUATION OF MOTION GIVES

ROTATION

$$\Rightarrow 2mg\cos\theta - R = 2m(-3a\ddot{\theta})$$

$$\Rightarrow 2mg\cos\theta + 6ma\ddot{\theta} = R$$

$$\Rightarrow R = 2mg\cos\theta + 6ma\left(\frac{4}{9}\sin\theta\right)$$

$$\Rightarrow R = 2mg\cos\theta + \frac{8}{3}mg\sin\theta$$

$$\Rightarrow R = \frac{2}{3}mg\sqrt{1+48\sin^2\theta}$$

TRANSLATION

$$\Rightarrow 2mg\sin\theta - T = 2m(3a\ddot{\theta})$$

$$\Rightarrow 2mg\sin\theta - 6ma\ddot{\theta} = T$$

$$\Rightarrow T = 2mg\sin\theta - 6ma\left(\frac{4}{9}\sin\theta\right)$$

$$\Rightarrow T = 2mg\sin\theta - \frac{8}{3}mg\sin\theta$$

$$\Rightarrow T = \frac{2}{3}mg\cos\theta$$

FINALLY THE MAGNITUDE OF THIS FORCE IS GIVEN BY

$$F = \sqrt{R^2 + T^2}$$

$$F = \frac{2}{3}mg\sqrt{(7\sin\theta)^2 + \cos^2\theta}$$

$$F = \frac{2}{3}mg\sqrt{49\sin^2\theta + \cos^2\theta}$$

$$F = \frac{2}{3}mg\sqrt{1+48\sin^2\theta}$$



**Question 49** (\*\*\*\*)

A uniform rod  $AB$ , of mass  $m$  and length  $4a$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$  passing through the point  $O$ , where  $|AO| = a$ . The vertical plane is perpendicular to  $L$ .

The rod is hanging at rest in a vertical position, with  $B$  below  $A$ , when it is struck at its midpoint by a particle  $P$  of mass  $3m$ , travelling horizontally with speed  $u$ . The path of  $P$  on impact with the rod is in a vertical plane which is perpendicular to  $L$ .

Given that  $P$  attaches itself at the midpoint of the rod on impact, determine, in terms of  $m$  and  $g$ , the magnitude of the force acting on the rod at  $L$ , when the rod first comes to instantaneous rest.

$$\frac{1}{2}mg\sqrt{19}$$

• MOMENT OF INERTIA OF THE ROD ABOUT  $O$   
 $I = \frac{1}{12}m(4a)^2 + ma^2 = \frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$

• MOMENT OF INERTIA OF THE LOADED ROD ABOUT  $O$   
 $I_{tot} = I_0 + 3ma^2 = \frac{4}{3}ma^2 + 3ma^2 = \frac{13}{3}ma^2$

• BY CONSERVATION OF ANGULAR MOMENTUM ABOUT  $O$   
 $\Rightarrow 3mu \times a + 0 = I_{tot} \times \omega$

MOMENTUM OF IMPULSE OF PARTICLE BEFORE AFTER  
 $\Rightarrow 3mu = \frac{13}{3}ma\omega$   
 $\Rightarrow 9\mu = 13ma\omega$   
 $\Rightarrow \omega = \frac{9\mu}{13a}$

• BY CONSERVATION OF ENERGY  
 $\Rightarrow$  GAIN IN P.E. = LOSS IN K.E.  
 $\Rightarrow (4mg)(\frac{3a}{2}) = \frac{1}{2}I_{tot}\omega^2$   
 $\Rightarrow 4mg(\frac{3a}{2}) = \frac{1}{2}(\frac{13}{3}ma^2)(\frac{81\mu^2}{169a^2})$   
 $\Rightarrow 6\mu ga = \frac{36}{13}m\mu^2$   
 $\Rightarrow \frac{6ga}{13} = \mu$

$\rightarrow u = \frac{9}{13}\sqrt{19}a$

At THE POSITION OF  $P$   
 $\vec{v} = 0$  &  $I\dot{\theta} = -4mg \sin \theta \times a$   
 $\frac{13}{3}ma^2\dot{\theta} = -4mg \sin \theta \times a$   
 $\frac{13}{3}a\dot{\theta} = -4g \sin \theta$   
 $\frac{13}{3}\dot{\theta} = -\frac{4g}{a} \sin \theta$   
 $a\dot{\theta} = -\frac{12g}{13} \sin \theta$

THEN WE NOW HAVE  
 •  $4m(-a\dot{\theta}) = X - 4mg \cos \theta$  (RADIAL)  
 $0 = X - 2mg$   
 $X = 2mg$

•  $4m(a\ddot{\theta}) = Y - 4mg \sin \theta$  (TANGENTIAL)  
 $4m(\frac{12g}{13} \sin \theta) = Y - 2g \sin \theta$   
 $Y = \frac{1}{2} \sqrt{19} mg$

• FINALLY THE MAGNITUDE IS  
 $\sqrt{X^2 + Y^2} = mg \sqrt{4 + \frac{19}{4}} = \frac{1}{2} \sqrt{19} mg$

**Question 50** (\*\*\*\*)

A uniform rod  $AB$ , of mass  $m$  and length  $2a$ , is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$  passing through  $A$ . The vertical plane is perpendicular to  $L$ .

The rod is hanging at rest in a vertical position, with  $B$  below  $A$ , when it receives a horizontal impulse of magnitude  $m\sqrt{ag}$ . The impulse acts at  $B$  in a vertical plane which is perpendicular to  $L$ .

Determine, in terms of  $m$  and  $g$ , the magnitude of the force acting on the rod at  $L$ , when the rod first comes to instantaneous rest.

$$\frac{1}{2}mg\sqrt{163}$$

The image shows two pages of handwritten work for Question 50. The left page contains the following text and equations:

- MOMENT OF INERTIA OF THE ROD ABOUT A**  
 $I_A = \frac{1}{3}m(2a)^2$
- MOMENT OF IMPULSE = CHANGE OF ANGULAR MOMENTUM (ABOUT A)**  
 $\rightarrow J \times 2a = I(\omega - 0)$   
 $\rightarrow 2Ja = \frac{1}{3}m(2a)^2\omega$   
 $\rightarrow J = \frac{2}{3}m a \omega$   
 $\rightarrow \omega = \frac{3J}{2ma} \quad (J = m\sqrt{ag})$
- BY ENERGY TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL**  
 $\rightarrow \frac{1}{2}I\omega^2 - mga = \frac{1}{2}I\Omega^2 - mga \cos \theta$   
 $\rightarrow mga \cos \theta = mga - \frac{1}{2}I\omega^2$   
 $\rightarrow mga \cos \theta = mga - \frac{1}{2}(\frac{4}{3}m a^2)\omega^2$   
 $\rightarrow mga \cos \theta = mga - \frac{2}{3}m a^2 (\frac{9J^2}{4m^2 a^2})$   
 $\rightarrow mga \cos \theta = mga - \frac{3}{2} \frac{J^2}{m a}$   
 $\rightarrow mga \cos \theta = mga - \frac{3}{2} \frac{m^2 ag}{m a}$   
 $\Rightarrow \cos \theta = -\frac{1}{2}$   
 $\Rightarrow \theta = \frac{2\pi}{3}$

The right page contains the following text and equations:

- VELOCITY**  
 $\vec{v} = (v \cos \theta)\hat{i} + (v \sin \theta)\hat{j}$   
 BUT  $v \cos \theta = 0$   
 $v \sin \theta = v$   
 $\vec{v} = (0)\hat{i} + (v)\hat{j}$   
 $\vec{v} = (0)\hat{i} + a\omega\hat{j}$   
 $\vec{v} = (0)\hat{i} + a\Omega\hat{j}$
- WHEN  $\theta = \frac{2\pi}{3}$ ,  $I\dot{\theta} = L$  AND  $\dot{\theta} = 0$**   
 $\dot{\theta}(\frac{1}{3}m(2a)^2) = (mg \sin \frac{2\pi}{3})a$   
 $\frac{4}{3}m a \dot{\theta} = 3m \sin \frac{2\pi}{3}$   
 $\frac{4}{3}a \dot{\theta} = 3 \frac{\sqrt{3}}{2}$   
 $a \dot{\theta} = \frac{9\sqrt{3}}{8}$
- EQUILIBRIUM**  
 $m(-a\dot{\theta}) = -mg \cos \theta - R$   
 $0 = -R - \frac{1}{2}mg$   
 $R = \frac{1}{2}mg$   
 (OPPOSITE DIRECTION TO THAT IN THE DIAGRAM)
- TENSION**  
 $m(a\ddot{\theta}) = T - mg \sin \theta$   
 $m(\frac{3\sqrt{3}}{8}a) = T - \frac{\sqrt{3}}{2}mg$   
 $T = \frac{15\sqrt{3}}{8}mg$
- MAGNITUDE**  
 $F = mg \sqrt{(\frac{1}{2})^2 + (\frac{15\sqrt{3}}{8})^2}$   
 $F = \frac{1}{8}mg \sqrt{163}$

**Question 51 (\*\*\*\*)**

A uniform circular disc, of radius  $a$  and mass  $m$ , is free to rotate about a smooth fixed horizontal axis  $L$ , which is perpendicular to the plane of the disc and is passing through the point  $A$ , which lies at the circumference of the disc. The disc is held with its centre  $O$  at the same horizontal level as  $A$ , and released from rest.

Show that the horizontal component of the force exerted on  $L$  has magnitude

$$mg |\sin 2\theta|,$$

where  $\theta$  is the angle that  $AO$  makes with the horizontal.

proof

• PIVOT BY THE PARALLEL AXIS THEOREM  
 $I_A = \frac{1}{2}ma^2 + ma^2 = \frac{3}{2}ma^2$

• BY CONSERVATION  
 $\frac{1}{2}I\omega^2 = mgh$   
 $\Rightarrow \frac{1}{2}(\frac{3}{2}ma^2)\omega^2 = mg(a\cos\theta)$   
 $\Rightarrow \frac{3}{4}a\omega^2 = g\cos\theta$   
 $\Rightarrow \omega^2 = \frac{4g\cos\theta}{3a}$

• DIFFERENTIATING w.r.t.  $t$   
 $\Rightarrow 2\omega = \frac{4g}{3a} \sin\theta \frac{dt}{dt}$   
 $\Rightarrow \omega = \frac{2g}{3a} \sin\theta$

• TENSIONALLY  
 $\Rightarrow m(a\ddot{\theta}) = mg\cos\theta - T$   
 $\Rightarrow T = mg\cos\theta - m\ddot{\theta}$   
 $\Rightarrow T = mg\cos\theta - ma(\frac{2g}{3a}\cos\theta)$   
 $\Rightarrow T = mg\cos\theta - \frac{2}{3}mg\cos\theta$   
 $\Rightarrow T = \frac{1}{3}mg\cos\theta$

• FINALLY  
 $\Rightarrow m(-a\ddot{\theta}) = mg\sin\theta - R$   
 $\Rightarrow R = mg\sin\theta + ma\ddot{\theta}$   
 $\Rightarrow R = mg\sin\theta + m(\frac{2g}{3a}\sin\theta)$   
 $\Rightarrow R = mg\sin\theta + \frac{2}{3}mg\sin\theta$   
 $\Rightarrow R = \frac{5}{3}mg\sin\theta$

• FINALLY THE HORIZONTAL REACTION IS GIVEN BY  
 $|T\cos\theta - R\sin\theta| = |\frac{1}{3}mg\cos\theta\sin\theta - \frac{5}{3}mg\sin\theta\cos\theta|$   
 $= |1 - 2mg\cos\theta\sin\theta|$   
 $= |1 - mg\sin 2\theta|$   
 $= mg\sin 2\theta$

**Question 52** (\*\*\*\*)

A circular flywheel  $F$ , of radius 0.2 m, is free to rotate about an axis  $L$ , which is perpendicular to the plane of the flywheel and through its centre. The motion of  $F$  is smooth.

The flywheel receives a tangential impulse, in the plane of  $F$ , of 180 N s.

- a) Given that the moment of inertia of  $F$  about  $L$  is  $6 \text{ kg m}^2$ , determine the angular speed of  $F$  after it receives the impulse.

A resistive couple  $C \text{ Nm}$  is then applied to  $F$  whose magnitude is given by

$$\begin{cases} 2\dot{\theta}^2 & 0 \leq t \leq 0.5 \\ 7.5 & 0.5 < t \leq T \end{cases}$$

where  $\dot{\theta}$  is the angular speed of  $F$  at time  $t \text{ s}$ , bringing  $F$  to rest in time  $T \text{ s}$ .

- b) Form and solve a differential equation, in  $\dot{\theta}$ , to find the angular speed of  $F$  when  $t = 0.5 \text{ s}$ .
- c) Calculate the value of  $T$ .

$$\omega = 6 \text{ rad s}^{-1}, \quad \dot{\theta}|_{t=0.5} = 3 \text{ rad s}^{-1}, \quad T = 2.9 \text{ s}$$

Handwritten solution for Question 52:

a)  $I = 6$ ,  $r = 0.2$ ,  $J = 180$

● MOMENT OF IMPULSE = CHANGE OF ANG. MOMENTUM

$$180 \times 0.2 = I(\omega - 0)$$

$$36 = 6\omega$$

$$\omega = 6 \text{ rad s}^{-1}$$

b)  $I\ddot{\theta} = -C$  (so  $\omega = 6$ )

$$\Rightarrow 6\ddot{\theta} = -2\dot{\theta}^2$$

$$\Rightarrow 3\frac{d\dot{\theta}}{dt} = -\dot{\theta}^2$$

$$\Rightarrow -\frac{1}{\dot{\theta}} d\dot{\theta} = \frac{1}{3} dt$$

$$\Rightarrow \int_{\dot{\theta}}^{\dot{\theta}} \frac{1}{\dot{\theta}} d\dot{\theta} = \int_{0.5}^{0.5} \frac{1}{3} dt$$

$$\Rightarrow \left[ -\frac{1}{\dot{\theta}} \right]_6 = \left[ \frac{1}{3} t \right]_{0.5}^{0.5}$$

$$\Rightarrow -\frac{1}{6} - \left(-\frac{1}{\dot{\theta}}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{1}{\dot{\theta}} = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$\Rightarrow \dot{\theta} = 2 \text{ rad s}^{-1}$$

c) COUPLE IS NOW CONSTANT  
SO ANGLE ACCELERATION MUST BE CONSTANT

$$I\ddot{\theta} = -C$$

$$6\ddot{\theta} = -7.5$$

$$\ddot{\theta} = -1.25$$

where  $\omega = 4 + at^2$

$$0 = 3 - 1.25t^2$$

$$1.25t^2 = 3$$

$$t = 2.4 \text{ s}$$

$\therefore T = 2.4 + 0.5 = 2.9 \text{ s}$

**Question 53** (\*\*\*\*)

A uniform circular disc with centre  $O$  has mass  $m$  and radius  $a$ .

The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis through a point  $A$  on the disc, where  $OA = \frac{1}{2}a$ .

The disc is held at rest in a position with  $O$  vertically above  $A$ . The disc is then released and begins to rotate about  $O$ . The angle between  $OA$  and the upward vertical is denoted by  $\theta$ .

- a) Find, in terms of  $a$ ,  $g$  and  $\theta$ , ...
  - i. ... the angular speed of the disc.
  - ii. ... the angular acceleration of the disc.
- b) Determine, in terms of  $m$ ,  $g$  and  $\theta$ , the radial and tangential component of the force acting at  $A$ .
- c) Calculate, in terms of  $mg$ , the magnitude of the force acting at  $A$ , when the radial component of the force is zero.

$$\dot{\theta} = \sqrt{\frac{4g(1-\cos\theta)}{3a}}, \quad \ddot{\theta} = \frac{2g \sin\theta}{3a}, \quad R = \frac{1}{3}mg(5\cos\theta - 2), \quad T = \frac{2}{3}mg \sin\theta,$$

$$F = \frac{2\sqrt{21}}{15}mg$$

The handwritten solution is divided into several parts:

- Part 1:** Finds the moment of inertia  $I$  about point  $A$  using the parallel axis theorem:  $I = \frac{1}{2}ma^2 + m(\frac{1}{2}a)^2 = \frac{3}{4}ma^2$ .
- Part 2:** Uses energy conservation to find angular speed  $\dot{\theta}$ . It equates the loss in potential energy  $mg \cdot \frac{1}{2}a(1-\cos\theta)$  to the gain in kinetic energy  $\frac{1}{2}I\dot{\theta}^2$ , leading to  $\dot{\theta} = \sqrt{\frac{4g(1-\cos\theta)}{3a}}$ .
- Part 3:** Finds angular acceleration  $\ddot{\theta}$  by taking moments about  $O$ . The torque is  $mg \cdot \frac{1}{2}a \sin\theta$  and the moment of inertia about  $O$  is  $\frac{1}{2}ma^2$ , giving  $\ddot{\theta} = \frac{2g \sin\theta}{3a}$ .
- Part 4:** Finds the radial force  $R$  and tangential force  $T$  at point  $A$ . It uses Newton's second law for the center of mass  $O$ . Radially:  $R - mg \cos\theta = m(-\frac{1}{2}a\dot{\theta}^2)$ . Tangentially:  $T - mg \sin\theta = m(\frac{1}{2}a\ddot{\theta})$ . Solving these gives  $R = \frac{1}{3}mg(5\cos\theta - 2)$  and  $T = \frac{2}{3}mg \sin\theta$ .
- Part 5:** Finds the magnitude of the force  $F$  at  $A$  when the radial component is zero ( $R=0$ ). This occurs when  $\cos\theta = \frac{2}{5}$ , leading to  $T = \frac{2\sqrt{21}}{15}mg$ .

**Question 54** (\*\*\*\*)

A uniform equilateral triangular lamina  $ABC$  has mass  $m$  and side length of  $\sqrt{3}a$ .

- a) Show, by integration, that the moment of inertia of the lamina about an axis through one of its vertices and perpendicular to the plane of the lamina is

$$\frac{5}{4}ma^2.$$

[In this proof, you may assume standard results for the moment of inertia of uniform rods.]

The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis  $L$ , which passes through  $A$  and is perpendicular to the lamina. The midpoint of  $BC$  is the point  $M$ .

The lamina is held with  $AM$  making an angle of  $60^\circ$  with the upward vertical through  $A$  and is projected with angular speed  $\sqrt{\frac{3g}{a}}$ .

- b) Find, in terms of  $a$  and  $g$ , the speed of  $M$  when  $M$  is vertically below  $A$ .

$$v = \sqrt{\frac{99}{20}ag}$$

**a)**  $y$   $x$   $g = \frac{3g}{2} \frac{y}{2a}$   
 $\{ |AM| = \frac{1}{2} \sqrt{3}a \}$   
 $\{ |AM| = \frac{1}{2} \sqrt{3}a \}$   
 LET THE AXIS BE IN  
 $|AM| = \frac{1}{2} \sqrt{3}a$   $\Rightarrow$   $\frac{1}{2} \sqrt{3}a$   
 $\rho = \frac{m}{\text{Area}} = \frac{3m}{4a^2}$   $\leftarrow$  MASS PER UNIT AREA  
 MASS OF INFINITESIMAL STRIP =  $2ay\rho dy$   
 MOMENT OF INFINITESIMAL ABOUT A AXIS =  $\frac{1}{12}(2ay)(y^2)$   
 MOMENT OF INFINITESIMAL ABOUT O =  $\frac{1}{12}(2ay)(y^2) + (2ay\rho dy) \left(\frac{y}{2}\right)^2$   
 SIMILAR CE  
 $I = \int_0^{2a} \left( \frac{1}{6}ay^3 + \frac{2}{3}ay^3 \right) dy = \int_0^{2a} \frac{5}{6}ay^3 dy$   
 $I = \frac{5}{6}a \left[ \frac{y^4}{4} \right]_0^{2a} = \frac{5}{6}a \left[ \frac{16a^4}{4} \right] = \frac{5}{6}a \cdot 4a^4 = \frac{20}{6}a^5 = \frac{10}{3}a^5$   
 $I = \frac{5}{4}ma^2$   
 AS REQUEST

**b)** LET G BE THE CENTRE OF MASS OF THE LAMINA  
 BY CHOOSING TAKING THE LEVEL OF G AS THE ZERO POTENTIAL LEVEL  
 $\{ KE_{\text{START}} + PE_{\text{START}} = KE_{\text{END}} + PE_{\text{END}} \}$   
 $\frac{1}{2}I\omega^2 + mgh_{\text{START}} = \frac{1}{2}I\omega'^2 + mgh_{\text{END}}$   
 USE THE INCLINATION OF THE INITIAL POSITION. VELOCITY IS NOT HORIZONTAL SO SINCE AN THREE AXES PERPENDICULAR COORDS  
 THIS  $\frac{1}{2}(20a^5)\left(\frac{3g}{2a}\right)^2 + \frac{1}{2}mgy = \frac{1}{2}(20a^5)\omega'^2 - mgy$   
 $\frac{15}{2}mga + \frac{1}{2}mgy = \frac{10}{3}mga\omega'^2 - mgy$   
 $\frac{15}{2}mga + \frac{1}{2}mgy = \frac{10}{3}mga\omega'^2 - mgy$   
 $5a\omega'^2 = \frac{15g}{2} + \frac{1}{2}g$   
 $\omega'^2 = \frac{16g}{20a}$   
 $\omega' = \sqrt{\frac{4g}{5a}}$   
 SPEED OF M, USING  $v = \omega r$  OR  $\omega = \frac{v}{r}$   
 $v = \sqrt{\frac{4g}{5a}} \cdot |AM| = \sqrt{\frac{4g}{5a}} \cdot \frac{1}{2} \sqrt{3}a = \sqrt{\frac{3 \cdot 4g \cdot a}{5 \cdot 4}} = \sqrt{\frac{3ga}{5}}$

**Question 55** (\*\*\*)

A uniform circular disc with centre at  $O$ , has radius  $a$  and mass  $m$ . The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis. This axis is perpendicular to the plane of the disc and passes through a point  $P$ , at the circumference of the disc.

The disc is held at rest with  $PQ$  horizontal, where  $PQ$  is a diameter of the disc, and released from rest. At time  $t$  after release, the diameter  $PQ$  makes an angle  $\theta$  below the horizontal, where  $\theta$  is acute.

- a) Find expressions, in terms of  $m$ ,  $g$  and  $\theta$ , for ...
- ... the radial component of the force exerted on the disc by the axis.
  - ... the transverse component of the force exerted on the disc by the axis.

When  $PQ$  is vertical the disc is brought to instantaneous rest by a horizontal impulse  $J$ , acting through  $O$ .

- b) Show clearly that

$$J = m\sqrt{3ag}$$

$$R_{\text{radial}} = \frac{7}{3}mg \sin \theta, \quad R_{\text{transverse}} = \frac{1}{3}mg \cos \theta$$

**(a)**

• FIND MOMENT OF INERTIA ABOUT  $P$   
 $I_p = \frac{1}{2}ma^2 + ma^2 = \frac{3}{2}ma^2$

• BY ENERGY (TAKE  $O$  AS ZERO POTENTIAL LEVEL)  
 $\frac{1}{2}I_p \omega^2 + PE_{\text{CM}} = KE_{\text{CM}} + PE_p$   
 $0 = \frac{1}{2}I_p \omega^2 - mgh$   
 $0 = \frac{1}{2}(\frac{3}{2}ma^2)\omega^2 - mg(a \sin \theta)$   
 $0 = \frac{3}{4}m a^2 \omega^2 - mg a \sin \theta$   
 $0 = \frac{3}{4}a \omega^2 - g \sin \theta$   
 $\omega^2 = \frac{4g \sin \theta}{3a}$

• NEXT DIFF WRT  $t$   
 $\frac{d}{dt}(\omega^2) = \frac{d}{dt}(\frac{4g \sin \theta}{3a})$   
 $2\omega \dot{\omega} = \frac{4g \cos \theta}{3a} \dot{\theta}$   
 $\dot{\omega} = \frac{2g \cos \theta}{3a} \dot{\theta}$

• NOW EQUATE  
 $m(-a\dot{\omega}^2) = mg \sin \theta - R$   
 $\Rightarrow R = mg \sin \theta + ma\dot{\omega}^2$   
 $\Rightarrow R = mg \sin \theta + ma(\frac{4g \sin \theta}{3a})$   
 $\Rightarrow R = \frac{7}{3}mg \sin \theta$

• NOW TRANSVERSE  
 $m(a\dot{\omega}) = mg \cos \theta - T$   
 $\Rightarrow T = mg \cos \theta - ma\dot{\omega}$   
 $\Rightarrow T = mg \cos \theta - m\sqrt{\frac{4g \sin \theta}{3a}}$   
 $\Rightarrow T = \frac{1}{3}mg \cos \theta$

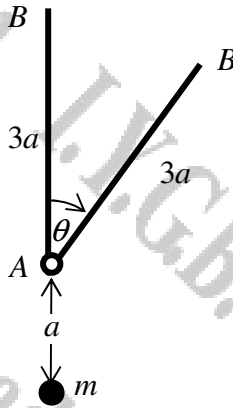
**(b)**

BY ANGULAR MOMENTUM - IMPULSE  
 MOMENT OF IMPULSE = CHANGE IN ANGULAR MOMENTUM  
 $-J \times a = 0 - I\omega$  (CLOCKWISE POSITIVE)  
 $Ja = \frac{3}{2}ma^2 \times \sqrt{\frac{4g \sin \theta}{3a}}$   
 $J = \frac{3}{2}m a \sqrt{\frac{4g \sin \theta}{3a}}$   
 $J = m \sqrt{\frac{9g \sin \theta}{3a}}$   
 $J = m\sqrt{3ag}$  (AS REQUIRED)

• VARIATIONS FOR CORNISH  
 $I_O = L$   
 $\frac{3}{2}m a \dot{\omega} = (mg \cos \theta) \times a$   
 $\frac{3}{2}m \dot{\omega} = mg \cos \theta$   
 $\dot{\omega} = \frac{2g \cos \theta}{3a}$  (AS BEFORE)

• NOW USE EITHER TWO O.D.E  
 $2\dot{\omega} = \frac{d}{dt}(\frac{2g \cos \theta}{3a}) \times a$   
 $\frac{d}{dt}(\dot{\omega}^2) = \frac{d}{dt}(\frac{4g \cos^2 \theta}{3a})$   
 $(\dot{\omega}^2)^2 = (\frac{4g \cos^2 \theta}{3a})^2$   
 $\dot{\omega}^2 = (\frac{4g \cos^2 \theta}{3a}) - (A)$   
 $\dot{\omega}^2 = \frac{4g \cos^2 \theta}{3a}$  (AS BEFORE)

Question 56 (\*\*\*\*)



A uniform rod  $AB$  of mass  $m$  and length  $3a$  is free to rotate in a vertical plane about a horizontal axis through  $A$ . The rod is held at rest with  $B$  vertically above  $A$  and is released from rest. At time  $t$  after release the rod makes an angle  $\theta$  with the upward vertical.

- a) Determine expressions, in terms of  $m$ ,  $g$  and  $\theta$ , for the magnitudes of the components of the reaction at  $A$ , parallel to  $AB$  and perpendicular to  $AB$ .

When  $B$  gets vertically below  $A$ , the rod collides with a particle of mass  $m$  which was at rest at a distance  $a$  vertically below  $A$ . The particle attaches to the rod and the resulting system continues to move until the rod  $AB$  makes an angle  $\phi$  with the downward vertical.

- b) Calculate the value of  $\phi$ .

$$R_{\perp AB} = \left| \frac{1}{2} mg \sin \theta \right|, \quad R_{\parallel AB} = \left| \frac{1}{2} mg (5 \cos \theta - 3) \right|, \quad \phi = \arccos \frac{1}{10} \approx 84.26^\circ$$

The image shows two pages of handwritten student work. The left page (a) shows a free-body diagram of the rod at an angle  $\theta$  from the vertical. It lists the following steps:

- $I_A = \frac{1}{2} m \left(\frac{3a}{2}\right)^2 = \frac{9}{4} ma^2$
- $I_B = L$
- $\Rightarrow 2ma^2 \dot{\theta} = mg \sin \theta \times \frac{3a}{2}$
- $\Rightarrow 2a\dot{\theta} = \frac{3g \sin \theta}{2}$
- $\Rightarrow \dot{\theta} = \frac{3g \sin \theta}{4a}$

It then uses the transverse direction  $(\hat{\theta})$  to find the reaction force  $T$ :

- $\Rightarrow m(r\ddot{\theta}) = mg \sin \theta - T$
- $\Rightarrow T = mg \sin \theta - m r \ddot{\theta}$
- $\Rightarrow T = mg \sin \theta - m \left(\frac{3}{2}\right) \left(\frac{3g \sin \theta}{2a}\right)$
- $\Rightarrow T = mg \sin \theta - \frac{9}{4} mg \sin \theta$
- $\Rightarrow T = \frac{1}{4} mg \sin \theta$

The right page (b) shows the rod at the bottom position ( $\theta = \pi$ ) with  $\dot{\theta} = \frac{3g}{4a}$ . It lists the following steps:

- $I_{\text{rod}} = \frac{1}{2} m \left(\frac{3a}{2}\right)^2 = \frac{9}{4} ma^2$
- $I_{\text{particle}} = ma^2$
- $I_{\text{total}} = \frac{13}{4} ma^2$
- $\Rightarrow 2ma^2 \dot{\omega} = mg \sin \theta \times \frac{3a}{2}$
- $\Rightarrow 2a\dot{\omega} = \frac{3g \sin \theta}{2}$
- $\Rightarrow \dot{\omega} = \frac{3g \sin \theta}{4a}$

It then uses the radial direction  $(\hat{r})$  to find the reaction force  $R$ :

- $\Rightarrow m(-r\dot{\omega}^2) = R - mg \cos \theta$
- $\Rightarrow mg \cos \theta - m r \dot{\omega}^2 = R$
- $\Rightarrow R = mg \cos \theta - m \left(\frac{3}{2}\right) \times \left(\frac{3g \sin \theta}{2a}\right)^2$
- $\Rightarrow R = mg \cos \theta - \frac{27}{8} mg \frac{\sin^3 \theta}{a}$
- $\Rightarrow R = \frac{1}{8} mg (5 \cos \theta - 3)$

For part (b), it uses conservation of angular momentum about A:

- $I_{\text{rod}} \dot{\omega} + I_{\text{particle}} \dot{\omega} = I_{\text{rod}} \dot{\omega}_0 + I_{\text{particle}} \dot{\omega}_0$
- $\Rightarrow \frac{13}{4} ma^2 \dot{\omega} = \frac{9}{4} ma^2 \dot{\omega}_0 + ma^2 \dot{\omega}_0$
- $\Rightarrow \dot{\omega} = \frac{1}{10} \dot{\omega}_0$
- $\Rightarrow \phi \approx 84.26^\circ$



**Question 57** (\*\*\*)

A composite body consists of a thin uniform rod  $AB$ , of mass  $m$  and length  $3a$ , with the end  $B$  rigidly attached to the centre  $O$  of a uniform circular lamina, of radius  $2a$  and mass  $m$ . The rod is perpendicular to the plane of the lamina. The body is free to rotate in a vertical plane about a fixed smooth horizontal axis through  $A$ , and perpendicular to  $AB$ .

- a) Find the moment of inertia of the body about the above described axis.

The body is released from rest with  $AB$  making an angle  $\alpha$  with the downward vertical through  $A$ .

- b) Determine simplified expression for the transverse and radial components of the force acting on the axis, when  $AB$  is making an angle  $\theta$  with the downward vertical through  $A$ , where  $\theta < \alpha$ .

$$I = 16ma^2, \quad R_{\text{radial}} = \frac{3}{2}mg(3\cos\alpha - \cos\theta), \quad R_{\text{transverse}} = \frac{3}{4}mg\sin\theta$$

**a)**  $I = I_{\text{rod}} + I_{\text{disc}}$   
 $I_{\text{rod}} = \frac{1}{3}m(3a)^2 = 3ma^2$   
 $I_{\text{disc}} = \frac{1}{2}(2m)(2a)^2 = 4ma^2$   
 $I = 3ma^2 + 4ma^2 = 7ma^2$

**b)** **DIFFERENTIAL W.R.T t**  
 $\frac{d}{dt}(3m\dot{\theta}) = -3mg\sin\theta$   
 $3m\ddot{\theta} = -3mg\sin\theta$   
 $\ddot{\theta} = -g\sin\theta$

**ENERGY METHOD**  
 $\frac{1}{2}I\dot{\theta}^2 = 3mg(3a)\cos\theta - 3mg(3a)\cos\alpha$   
 $\frac{1}{2}(7ma^2)\dot{\theta}^2 = 9mga(\cos\theta - \cos\alpha)$   
 $\dot{\theta}^2 = \frac{18g}{7a}(\cos\alpha - \cos\theta)$

**FORCES AT ANGLE theta**  
 $T - 3mg\cos\theta = 3m\ddot{\theta}$   
 $T - 3mg\cos\theta = -3mg\sin\theta$   
 $T = 3mg(\cos\theta - \sin\theta)$

**TRANSVERSELY**  
 $3m(2a\ddot{\theta}) = T - 3mg\sin\theta$   
 $6ma\ddot{\theta} = T - 3mg\sin\theta$   
 $T = 3mg\sin\theta + 6ma\ddot{\theta}$   
 $T = 3mg\sin\theta + \frac{36m}{7}g(\cos\alpha - \cos\theta)$   
 $T = 3mg\sin\theta + \frac{36m}{7}g\cos\alpha - \frac{36m}{7}g\cos\theta$   
 $T = \frac{36m}{7}g\cos\alpha - \frac{33m}{7}g\cos\theta + 3mg\sin\theta$

**Question 58** (\*\*\*\*+)

A uniform circular disc with centre at  $O$ , has radius  $r$  and mass  $m$ .

The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis. This axis is perpendicular to the plane of the disc and passes through a point  $P$ , which is  $\frac{3}{4}r$  from  $O$ .

The disc is initially at rest with  $O$  vertically below  $P$ .

A horizontal impulse of magnitude  $\frac{2m}{35}\sqrt{255gr}$  is applied at the lowest point on the circumference of the disc and in the plane of the disc.

a) Show clearly that the disc ...

i. ... begins to move with angular velocity  $\frac{8}{85}\sqrt{\frac{255g}{r}}$ .

ii. ... first comes to rest when  $PO$  is inclined at  $\arcsin\frac{3}{5}$  above the horizontal.

b) Determine the magnitude of the force exerted on the disc by the axis, when the disc first comes to rest.

$$F = \frac{\sqrt{145}}{17}mg$$

(3CA)

$I = \frac{1}{2}mr^2$   
 $I_P = \frac{1}{2}mr^2 + m(\frac{3}{4}r)^2$   
 $I_P = \frac{13}{8}mr^2$

MOMENT OF IMPULSE AT  $P =$  CHANGE IN ANG. MOMENTUM ABOUT  $P$

$$\frac{2m}{35}\sqrt{255gr} \times \frac{3}{4}r = I\omega - I\omega_0$$

$$\frac{3}{20}\sqrt{255gr} = \frac{13}{8}mr^2 \omega$$

$$\omega = \frac{8}{130}\sqrt{255gr} = \frac{8}{85}\sqrt{255gr}$$

AS REQUIRED

BY ENERGY

KE LOST = PE GAINED

$$\frac{1}{2}I\omega^2 = mgh$$

$$\frac{1}{2}(\frac{13}{8}mr^2)(\frac{8}{85}\sqrt{255gr})^2 = mgh$$

$\Rightarrow mgh = \frac{5}{2}mgr$   
 $\Rightarrow h = \frac{5}{4}r$   
 hence  $\cos\theta = \frac{5}{4}r - \frac{3}{4}r = \frac{2}{4}r$   
 so  $\sin\theta = \frac{3}{4}$   
 $\therefore \theta = \arcsin\frac{3}{4}$  AS REQUIRED

NEW EQUATION

$$m(\frac{3}{4}r\dot{\theta}) = -mg\sin\theta - R$$

$$0 = -mg\sin\theta - R$$

$$R = -mg\sin\theta$$

$$R = -mg(\frac{3}{4})$$

IE IT USES "COMBINED" FORCE TO COMBINE

$$\therefore |R| = \frac{3}{4}mg$$

THENCEWE CAN

$$m(\frac{3}{4}r\ddot{\theta}) = T - mg\cos\theta$$

$$T = -mg\cos\theta - \frac{3}{4}mr\ddot{\theta}$$

$$T = -mg\cos\theta - \frac{3}{4}mr(-\frac{8g}{85})$$

$$T = -mg\cos\theta + \frac{6}{85}mg$$

DIRECTION OPPOSITE TO THAT INDICATED IN DIAGRAM

$$\therefore |T|_{\text{GROSS}} = \sqrt{(\frac{6}{85}mg)^2 + (\frac{3}{4}mg)^2}$$

$$= \frac{1}{17}mg\sqrt{145}$$

**Question 59** (\*\*\*\*+)

A system consists of a rod  $AB$  of length  $8a$  and mass  $m$  and a particle of mass  $3m$  attached at  $B$ . The system is freely hinged at the midpoint of the rod and can rotate in a vertical plane.

The system is held in a horizontal position and released from rest.

When the system has turned by an angle  $\theta$ , the magnitude of the reaction force at the axis of rotation is  $F$ , where  $F = f(m, g, \theta)$ .

Determine an expression for  $F$ .

$$F = \frac{2}{13} mg \sqrt{196 + 4293 \sin^2 \theta}$$

**Diagram 1:** A horizontal rod of length  $8a$  is shown with a hinge at the midpoint  $M$ . The left end is  $A$  and the right end is  $B$ . The distance from  $M$  to  $A$  is  $4a$  and from  $M$  to  $B$  is  $4a$ . A particle of mass  $3m$  is attached at  $B$ . The center of mass of the rod is at  $G$ , which is  $2a$  to the right of  $M$ .

**Diagram 2:** The rod is shown at an angle  $\theta$  below the horizontal. The hinge is at  $M$ . The center of mass of the rod is at  $G$ . The particle of mass  $3m$  is at  $B$ . Forces shown are reaction force  $R$  at  $M$ , tension  $T$  at  $B$ , weight  $mg$  at  $G$ , and weight  $3mg$  at  $B$ .

**Calculations:**

- MASS RATIO:**

MASS	$m$	$3m$
SPACE FROM M	$4a$	$8a$

 $\rightarrow 4a \times m = 12ma$   
 $\rightarrow 8a \times 3m = 24ma$   
 $\rightarrow 24ma$
- MOMENT OF INERTIA ABOUT M:**

$$I = \frac{1}{3} m(4a)^2 + 3m(8a)^2 = \frac{20}{3} ma^2$$
- BY EQUILIBRIUM:**
  - $\Sigma \text{moments} = 0$  (clockwise +)
  - $\Sigma \text{forces} = 0$  (up +)
- Final Equations:**
  - $4mg \sin \theta + 12mg \sin \theta = R$
  - $R = 16mg \sin \theta$
  - $T = 12mg \cos \theta$
- Magnitude of  $R$ :**

$$R^2 = (16mg \sin \theta)^2 + (12mg \cos \theta)^2$$

$$R = 4mg \sqrt{16 \sin^2 \theta + 9 \cos^2 \theta}$$

$$R = \frac{4}{13} mg \sqrt{196 + 4293 \sin^2 \theta}$$

**Question 60** (\*\*\*\*+)

A uniform circular disc, of radius  $2r$  and mass  $m$ , is free to rotate about a smooth fixed horizontal axis  $L$ , which is perpendicular to the plane of the disc and is at a distance  $r$  from the centre of the disc  $C$ .

The disc is held at rest with  $C$  vertically above  $L$ . The disc is slightly disturbed and from its position of rest and begins to rotate about  $C$ .

Determine, in terms of  $g$  and  $r$ , the angular velocity of the disc at the two positions where the magnitude of the force exerted on the axis has magnitude  $\frac{2}{3}mg$ .

$$\dot{\theta} = \sqrt{\frac{2g}{9r}}, \quad \dot{\theta} = \sqrt{\frac{10g}{21r}}$$

$I_C = \frac{1}{2}m(2r)^2 + mr^2$   
 $I_C = 3mr^2$   
 ACCELERATION  $\ddot{x} = (r - r\dot{\theta}^2)\ddot{\theta} + (2r\dot{\theta} + r\ddot{\theta})\dot{\theta}$   
 HERE  $r = r \cos \theta \Rightarrow \dot{r} = -r\dot{\theta} \sin \theta$   
 so  $\ddot{r} = -r\ddot{\theta} \sin \theta + r\dot{\theta}^2 \cos \theta$

BY CHOOSING TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL  
 $K.E. + P.E. = K.E. + P.E.$   
 $0 + mgr = \frac{1}{2}I\dot{\theta}^2 + mg(r \cos \theta)$   
 $mgr = \frac{1}{2}(3mr^2)\dot{\theta}^2 + mgr \cos \theta$   
 $gr = \frac{3}{2}r\dot{\theta}^2 + g \cos \theta$   
 $r\dot{\theta}^2 = \frac{2}{3}g(1 - \cos \theta)$

NEXT USING  $L = J\dot{\theta}$   
 $(mg \sin \theta) \times r = 3mr^2 \ddot{\theta}$   
 $mg \sin \theta = 3mr \ddot{\theta}$   
 $g \sin \theta = 3r \ddot{\theta}$   
 $\dot{\theta} = \frac{1}{3}g \sin \theta$

NOW THE EQUATION OF MOTION  
 RADIALY  
 $m(-r\dot{\theta}^2) = -R - mg \cos \theta$   
 $\Rightarrow mr\dot{\theta}^2 = R + mg \cos \theta$   
 $\Rightarrow m\left(\frac{2}{3}g(1 - \cos \theta)\right) = R + mg \cos \theta$   
 $\Rightarrow \frac{2}{3}mg(1 - \cos \theta) = R + mg \cos \theta$   
 $\Rightarrow R = \frac{2}{3}mg(1 - \cos \theta) - mg \cos \theta$   
 $\Rightarrow R = \frac{1}{3}mg[2 - 2\cos \theta - 3\cos \theta]$   
 $\Rightarrow R = \frac{1}{3}mg(2 - 5\cos \theta)$

TRANSVERSELY  
 $mr\ddot{\theta} = mg \sin \theta - T$   
 $\Rightarrow T = mg \sin \theta - mr\ddot{\theta}$   
 $\Rightarrow T = mg \sin \theta - m\left(\frac{1}{3}g \sin \theta\right)$   
 $\Rightarrow T = \frac{2}{3}mg \sin \theta$

MAGNITUDE =  $\sqrt{T^2 + R^2}$   
 $= mg \sqrt{\left(\frac{1}{3}(2 - 5\cos \theta)\right)^2 + \left(\frac{2}{3} \sin \theta\right)^2}$   
 $= mg \times \frac{1}{3} \sqrt{(2 - 5\cos \theta)^2 + 4 \sin^2 \theta}$   
 $= mg \times \frac{1}{3} \sqrt{4 - 20\cos \theta + 25\cos^2 \theta + 4\sin^2 \theta}$   
 $= \frac{1}{3}mg \sqrt{4 - 20\cos \theta + 21\cos^2 \theta + 4(1 - \cos^2 \theta)}$   
 $= \frac{1}{3}mg \sqrt{8 - 20\cos \theta + 21\cos^2 \theta}$

Now  $\frac{2}{3}mg = \frac{1}{3}mg \sqrt{8 - 20\cos \theta + 21\cos^2 \theta}$   
 $2 = \sqrt{8 - 20\cos \theta + 21\cos^2 \theta}$   
 $4 = 8 - 20\cos \theta + 21\cos^2 \theta$

$\Rightarrow 21\cos^2 \theta - 20\cos \theta + 4 = 0$   
 $\Rightarrow (3\cos \theta - 2)(7\cos \theta - 2) = 0$   
 $\Rightarrow \cos \theta = \frac{2}{3}$   
 $\Rightarrow \cos \theta = \frac{2}{7}$

NOW  $r\dot{\theta}^2 = \frac{2}{3}g(1 - \cos \theta)$   
 $r\dot{\theta}^2 = \frac{2}{3}g \times \frac{1}{3}$   
 $r\dot{\theta}^2 = \frac{2}{9}g$   
 $\dot{\theta} = \sqrt{\frac{2g}{9r}}$

**Question 61** (\*\*\*\*+)

A pulley is in the shape of a disc of radius  $a$  and mass  $4m$ .

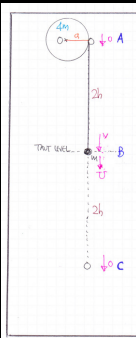
The pulley is free to rotate in a vertical plane about a rough horizontal axis through its centre  $O$ . The rotation of the pulley is opposed by a couple of magnitude  $C$ .

A light inextensible string has one end attached to a point on the rim of the pulley and is wound several times around the rim of the pulley. The portion of the string not wound on the pulley has length  $2h$  and has a particle of mass  $m$  attached to its free end. The particle is held at the same level as  $O$ , close to the rim of the still pulley and is released from rest.

The particle comes to rest at a vertical distance  $4h$  below the level of its release.

Determine, in terms of  $m$ ,  $g$  and  $a$ , the magnitude of  $C$ .

$$C = \frac{4}{3}mga$$



$\bullet$  DIRECTOR  $I_c = \frac{1}{2}(4m)a^2 = 2ma^2$   
 $\bullet$  KINETICS OR ENERGY  
 $\rightarrow v^2 = u^2 + 2as$   
 $\rightarrow v^2 = 2g(2h)$   
 $\rightarrow v^2 = 4gh$   
 $\rightarrow v = 2\sqrt{gh}$

$\bullet$  BY CONSERVATION OF TOTAL MOMENTUM ABOUT O  
 $\Rightarrow (mv)v + 0 = I\omega + (mU)a$   
particle mass times velocity = pulley moment of inertia times angular velocity  
 $\rightarrow 2\sqrt{gh} = 2a\omega + U$   
 $\Rightarrow 2\sqrt{gh} = 2a\omega + U$   
 BUT  $U = a\omega$   
 $\Rightarrow 2\sqrt{gh} = 2a\omega + a\omega$   
 $\Rightarrow 2\sqrt{gh} = 3a\omega$

$\bullet$  FINALLY BY ENERGY TAKING THE LEVEL OF C AS THE ZERO POTENTIAL LEVEL  
 $\Rightarrow KE_s + PE_s + W_{nc} - W_{mf} = KE_f + PE_f$   
 $\Rightarrow (\frac{1}{2}mU^2 + \frac{1}{2}I\omega^2) + mg(2h) - C\theta = 0$   
 $\Rightarrow \frac{1}{2}mU^2 + \frac{1}{2}(2ma^2)\omega^2 + 2mgh - C\theta = 0$   
 $\Rightarrow mU^2 + 2ma^2\omega^2 + 4mgh - 2C\theta = 0$

$\Rightarrow 2C\theta = mU^2 + 2ma^2\omega^2 + 4mgh$   
 BUT AS THE PULLEY ROTATES  
 $2h = a\theta$   
 $\theta = \frac{2h}{a}$

$\bullet$  SUBSTITUTING ALSO  $U^2$  IN THE ABOVE EQUATION  
 $2C(\frac{2h}{a}) = m(a\omega)^2 + 2ma^2\omega^2 + 4mgh$   
 $\frac{4Ch}{a} = ma^2\omega^2 + 2ma^2\omega^2 + 4mgh$   
 $\frac{4Ch}{a} = 3ma^2\omega^2 + 4mgh$   
 $\frac{4Ch}{a} = 3m(\frac{2\sqrt{gh}}{3a})^2 + 4mgh$   
 $\frac{4Ch}{a} = 3m \times \frac{4}{9}gh + 4mgh$   
 $\frac{4Ch}{a} = \frac{4}{3}mgh + 4mgh$   
 $\frac{C}{a} = \frac{1}{3}mg + mg$   
 $\frac{C}{a} = \frac{4}{3}mg$   
 $C = \frac{4}{3}mga$

**Question 62** (\*\*\*\*+)

A heavy pulley is modelled as a uniform circular disc of radius  $a$ , free to rotate through a horizontal axis passing through the centre of the disc and perpendicular to the plane of the disc.

A light inextensible string passes over the rough rim of the pulley.

Two particles of mass  $m$  and  $2m$  are attached to each of the two ends of the string and hang vertically with the string taut until the moment that are gently released from rest, from the same level above horizontal ground.

After the two particles are released, the string does not slip in the pulley, both particles are moving in vertical directions and neither particle reaches the ground or the pulley.

In the subsequent motion the ratio of the tension in the section of the string that the particle of mass  $m$  is attached, to that of the tension in the section of the string that the particle of mass  $2m$  is attached, is  $2 : 3$ .

When the angular velocity of the pulley reaches  $\omega$  the string suddenly breaks.

A couple of constant magnitude brings the pulley to rest.

If the pulley covers an angle  $\pi$  since the couple was applied, show that the magnitude of this couple is  $\frac{2}{\pi} m \omega^2 a^2$

,  proof

START WITH THE STANDARD DIAGRAM - LET THE MASS OF THE PULLEY BE 'M'  
AND THE TENSION ON THE STRING OF THE HEAVIER PARTICLE BE 'T'

$\frac{M}{2} = \frac{2m}{2} \Rightarrow M = 2m$

EQUATIONS OF MOTION FOR PARTICLES

For mass  $2m$ :  $2mg - T = 2m\ddot{x}$   
 For mass  $m$ :  $T - mg = m\ddot{x}$

$2mg - T = 2m\ddot{x}$   
 $T - mg = m\ddot{x}$  (x2)  
 $2mg - T = 2m\ddot{x}$   
 $4T - 2mg = 2m\ddot{x}$   
 $\frac{3}{2}T - 2mg = 2mg - T$   
 $\frac{3}{2}T = 4mg - T$   
 $T = \frac{8}{5}mg$

USE ONE OF THE EQUATIONS TO FIND  $\ddot{x}$

$\frac{3}{2}T - mg = m\ddot{x}$   
 $\Rightarrow \frac{3}{2} \cdot \frac{8}{5}mg - mg = m\ddot{x}$   
 $\Rightarrow \ddot{x} = \frac{1}{5}g$

WRITE THE EQUATION OF MOTION OF THE PULLEY

$\Rightarrow T_1 - \frac{3}{2}T_1 = I\ddot{\theta}$   
 $\Rightarrow \frac{1}{2}T_1 = \frac{1}{2}M a^2 \ddot{\theta}$

BUT THERE IS NO SLIPPING, IE  $\ddot{x} = a\ddot{\theta}$

$\Rightarrow \frac{1}{2}T = \frac{1}{2}M a \ddot{\theta}$   
 $\Rightarrow \frac{1}{2}(\frac{8}{5}mg) = \frac{1}{2}M \ddot{\theta}$   
 $\Rightarrow \frac{4}{5}mg = \frac{1}{2}M \ddot{\theta}$   
 $\Rightarrow \frac{4}{5}mg = \frac{1}{2} \cdot 2m \ddot{\theta}$   
 $\Rightarrow M = 2m$

NOW 'KINEMATICS'

$v^2 = u^2 + 2a s$   
 $\ddot{\theta} = \omega^2 + 2\ddot{\theta} \theta$   
 $0 = \omega^2 + 2\ddot{\theta} \cdot \pi$   
 $\ddot{\theta} = -\frac{\omega^2}{2\pi}$

FINALLY  $L = I\ddot{\theta}$

$L = \frac{1}{2}M a^2 \times (-\frac{\omega^2}{2\pi})$   
 $L = -\frac{1}{4} \frac{M a^2 \omega^2}{\pi}$   
 $L = -\frac{2m a^2 \omega^2}{4\pi}$   
 $L = -\frac{m a^2 \omega^2}{2\pi}$

MAKING  $\frac{2}{\pi} m \omega^2 a^2$

**Question 63 (\*\*\*\*+)**

A uniform circular disc, of radius  $a$  and mass  $m$ , is free to rotate about a smooth fixed horizontal axis  $L$ , which is coplanar to the disc and tangential to a point  $A$  at its circumference.

When the centre of the disc,  $O$ , is vertically below  $A$ , the angular velocity of the disc is  $\sqrt{\frac{a}{g}}$ .

The angle  $OA$  makes with the downward vertical through  $A$  is denoted by  $\theta$ .

When  $\cos \theta = k$  the magnitude of the resultant force on the axis is  $\frac{2\sqrt{29}}{5} mg$ .

Determine the exact value of  $k$ .

$$k = \frac{53}{67}$$

• FIND THE MOMENT OF INERTIA OF THE DISC ABOUT THE GIVEN AXIS (BY PARALLEL AXIS THEOREM) TAKING INTO ACCOUNT THE MASS OF THE DISC  
 $\frac{1}{2}ma^2 + ma^2 = \frac{3}{2}ma^2$   
 • DRAW THE DISC IN A CROSS SECTION UP TO ITS CENTRE O  
  
 • USE THE PRINCIPLE OF ENERGY (SEE ZERO POTENTIAL WITH DISC AT HORIZONTAL POSITION)  
 $\frac{1}{2}I\omega^2 + PE_{\text{center}} = KE_{\text{center}} + PE_{\text{center}}$   
 $\frac{1}{2}I\left(\frac{v}{a}\right)^2 + mg(1-\cos\theta) = \frac{1}{2}mv^2 + mg(1-\cos\theta)$   
 $\frac{1}{2}I\left(\frac{v}{a}\right)^2 = \frac{1}{2}mv^2 + 2mga(1-\cos\theta)$

$\Rightarrow \frac{3}{2}ma^2\left(\frac{v}{a}\right) = \frac{1}{2}ma^2v^2 + 2mga(1-\cos\theta)$   
 $\Rightarrow 3v = v^2 + 4g(1-\cos\theta)$   
 $\Rightarrow 3a\dot{\theta} = v^2 + 4g(1-\cos\theta)$   
 $\Rightarrow 3a\dot{\theta}^2 = -2g + 4g\cos\theta$

• DIFFERENTIATE W.R.T t  
 $\Rightarrow 10a\dot{\theta}\ddot{\theta} = 4g(-\sin\theta)\dot{\theta}$   
 $\Rightarrow 5a\ddot{\theta} = -4g\sin\theta$

• NOW RADIAL ACCELERATION IN O (CENTRE) IS  $-a\dot{\theta}^2$   
 & TANGENTIAL ACCELERATION IN O (CENTRE) IS  $a\ddot{\theta}$

• RADIALLY  $m(-a\dot{\theta}^2) = mg\cos\theta + R$   
 $\Rightarrow R = -ma\dot{\theta}^2 - mg\cos\theta$   
 $\Rightarrow R = -\frac{1}{2}m[3a + 4g\cos\theta]$   
 $\Rightarrow R = -\frac{1}{2}m[-3a + 4g\cos\theta + 4g\cos\theta]$   
 $\Rightarrow R = -\frac{1}{2}m[-3a + 8g\cos\theta]$

• FINALLY THE RESULTANT IS  $\frac{2\sqrt{29}}{5}mg$   
 $\Rightarrow 2\sqrt{29} = \sqrt{10 - 78\cos\theta + 240\cos^2\theta}$   
 $\Rightarrow 116 = 10 - 78\cos\theta + 240\cos^2\theta$   
 $\Rightarrow 0 = 230\cos^2\theta - 78\cos\theta - 106$   
 $\Rightarrow 130\cos^2\theta - 39\cos\theta - 53 = 0$

• QUADRATIC FORMULA  
 $\cos\theta = \frac{39 \pm 113}{260}$   
 $\cos\theta = \frac{20}{13}$   
 NOT EXACT ANSWER IF  $\cos\theta = -\frac{1}{2}$   
 FROM THE EQUATION  
 $5a\dot{\theta}^2 = -2g + 4g\cos\theta$

**Question 64 (\*\*\*\*)**

A uniform rod, of mass  $m$  and length  $2a$ , lies at rest on a smooth horizontal surface and is free to rotate about a smooth vertical axis through its centre  $O$ .

A particle of mass  $m$ , moving on the surface with speed  $U$ , strikes the rod at right angles at the point  $C$  on the rod, so that  $|OC| = \frac{1}{2}a$ .

Given that the collision is perfectly elastic, determine whether there is another collision between the particle and the rod.

there is another collision

$I_O = \frac{1}{3} m a^2$

BEFORE  
AFTER

● FIRSTLY WORKING AT THE POINT  $C$ , & CONSIDERING DECELERATION

$$e = \frac{SEP}{APP}$$

$$1 = \frac{V + \omega \times \frac{1}{2}a}{U}$$

$$U = V + \frac{1}{2}a\omega$$

$$V = U - \frac{1}{2}a\omega$$

● BY CONSERVATION OF ANGULAR MOMENTUM ABOUT  $O$ , THEN'S CONSERVE AS @

$$\Rightarrow 0 + (mU) \times \frac{1}{2}a = \left(\frac{1}{3}m a^2\right)\omega + (mV) \times \frac{1}{2}a$$

↑ Particle before (MOMENT OF MOMENTUM)  
↑ Rod after  
↑ Particle after (MOMENT OF MOMENTUM) (MOMENT AS IF THIS COLLIDES POINT)

$$\Rightarrow \frac{1}{2}U = \frac{1}{3}a\omega + \frac{1}{2}V$$

$$\Rightarrow 3U = 2a\omega + 3V$$

● NOW FIND  $\omega$  &  $V$  IN TERMS OF  $a$  &  $U$

$$2U - a\omega = 2V \quad \times 2 \quad 4U - 2a\omega = 4V \quad \times 3 \quad 12U - 6a\omega = 12V$$

$$3U - 2a\omega = 3V \quad \times (-1) \quad -3U + 2a\omega = -3V \quad \times 4 \quad -12U + 8a\omega = -12V$$

$$\Rightarrow U = 7V$$

$\therefore V = \frac{1}{7}U$

● USING:  $V = U - \frac{1}{2}a\omega$

$$\frac{1}{7}U = U - \frac{1}{2}a\omega$$

$$2U = 14U - 7a\omega$$

$$7a\omega = 12U$$

$$\omega = \frac{12U}{7a}$$

● WE CAN ALSO MODEL THE MOTION AND CHECK WHETHER THERE IS ANOTHER COLLISION

LOCUS AT  $O$  &  $D$

$$\omega a = \frac{12U}{7} = \frac{1}{2} \omega$$

$$\theta = \frac{12U}{7a}$$

●  $CO = \frac{1}{2}a$   
 $CD = \frac{1}{2}a$

●  $BD = a$   
 $BD = a(r - \theta)$   
 $BD = \frac{12U}{7}$

● WHY WE NEED TO COMPARE TIMES

THE TIME  $T_1$  TO REACH  $D$  (ALONG THE ARC  $CO$ )

THE PERIOD  $T_2$  TO REACH  $D$  ALONG  $CD$

● PARTICLE TIME

$$T_1 = \frac{CO}{V} = \frac{\frac{1}{2}a}{\frac{1}{7}U} = \frac{7a}{2U}$$

● TIME  $T_2$

$$T_2 = \frac{BD}{\text{SPEED OF B}}$$

$$T_2 = \frac{a\omega}{U}$$

$$T_2 = \frac{a \times \frac{12U}{7a}}{U} = \frac{12a}{7U}$$

$$T_2 = \frac{12a}{7U}$$

$\therefore T_1 = \frac{7a}{2U}$  and  $T_2 = \frac{12a}{7U}$

IF THE PARTICLE'S TIME IS LONGER, SO IT DOES NOT HIT

$T_1 > T_2$