ROTATIONAL MOTION

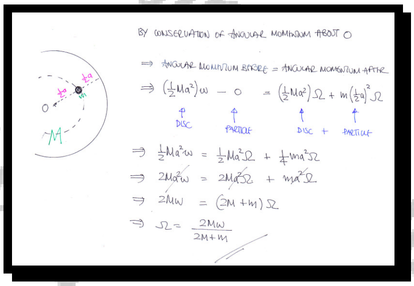
Question 1 (**)

A uniform disc, of mass M and radius a, is rotating with constant angular velocity ω, in a horizontal plane, about a fixed smooth vertical axis L, which is perpendicular to the disc and passes through its centre O.

A particle of mass m is gently lowered on to the disc at a distance $\frac{1}{2}a$ from O, and as soon as it touches the disc it adheres to the disc.

Determine the new angular speed of the disc in terms of m, M and ω.

$$\boxed{\Omega = \frac{2M\omega}{2M+m}}$$



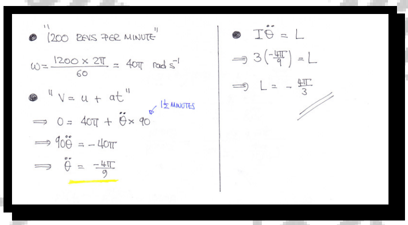
Question 2 (**)

A flywheel, of moment of inertia M and radius a, is rotating at 1200 revolutions per minute, when the source which was maintaining this rotation is switched off.

The flywheel comes to rest after $1\frac{1}{2}$ minutes due to a resistive couple L.

Determine the exact value of L.

$$\boxed{L = -\frac{4}{3}\pi}$$



Question 3 (**)

The centre of mass of a rigid body B, of mass m kg, lies at the origin O.

The point A with coordinates $(3,-4,1)$ lies on B.

When a force $\mathbf{F} = (27\mathbf{i} + 16\mathbf{j} - 17\mathbf{k})$ N acts on A, it causes B an angular acceleration of 22.75 s^{-2}, about O.

Determine the moment of inertia of B about O.

$$\boxed{I_O = 8 \text{ kg m}^2}$$

$\underline{F} = (27, 16, -17)$

$\underline{r} = (3, -4, 1)$

$$\underline{G}_0 = \underline{r} \wedge \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -4 & 1 \\ 27 & 16 & -17 \end{vmatrix} = (52, 78, 156)$$

$$|G_0| = |52, 78, 156|$$
$$= 26|2, 3, 6| = 26\sqrt{4+9+36}$$
$$= 26 \times 7 = 182$$

USING $L = I\ddot{\theta}$

$182 = I \times 22.75$

$I = 8 \text{ kgm}^2$

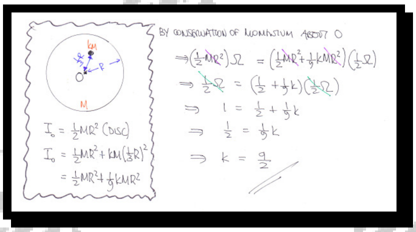
Question 4 (**)

A uniform circular disc, of mass M and radius R, is rotating with constant angular velocity in a horizontal plane about a vertical axis through its centre O.

A particle of mass kM, where k is a positive constant is gently placed on the disc at a distance $\frac{1}{3}R$ from O. The particle becomes instantly attached to the disc.

Given that the disc now rotates with half its original angular velocity. Determine the value of k.

$$\boxed{k = \frac{9}{2}}$$



Question 5 (**)

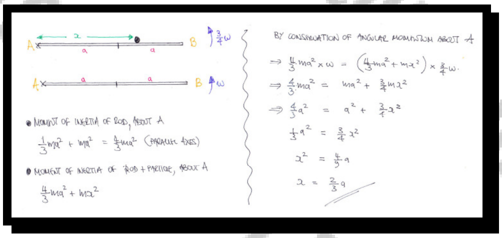
A uniform rod AB, of mass m and length $2a$, is free to rotate in a horizontal plane about a fixed smooth vertical axis L, which is perpendicular to the rod and passes through A.

The rod has angular speed ω when it strikes a stationary particle P of mass m, which adheres to the rod.

Just before P adheres to the rod, P is at a distance x from A.

Given that after P adheres to the rod, the angular speed of the rod reduces to $\frac{3}{4}\omega$, express x in terms of a.

$$\boxed{x = \frac{2}{3}a}$$



• MOMENT OF INERTIA OF ROD, ABOUT A

$\frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$ (PARALLEL AXES)

• MOMENT OF INERTIA OF ROD + PARTICLE, ABOUT A

$\frac{4}{3}ma^2 + mx^2$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT A

$\Rightarrow \frac{4}{3}ma^2 \times \omega = \left(\frac{4}{3}ma^2 + mx^2\right) \times \frac{3}{4}\omega$

$\Rightarrow \frac{4}{3}ma^2 = ma^2 + \frac{3}{4}mx^2$

$\Rightarrow \frac{4}{3}a^2 = a^2 + \frac{3}{4}x^2$

$\frac{1}{3}a^2 = \frac{3}{4}x^2$

$x^2 = \frac{4}{9}a$

$x = \frac{2}{3}a$

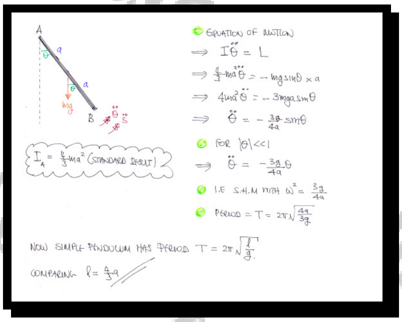
Question 6 (**)

A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to the rod and passes through A.

When the rod is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period τ.

Find the length, in terms of a, of a simple pendulum whose period of small amplitude oscillations is also τ.

$$\boxed{l = \frac{4}{3}a}$$



Question 7 (**)

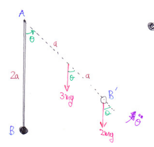
A uniform rod AB, of mass $3m$ and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to the rod and passes through A. A particle of mass $2m$ is attached to B.

When the rod is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period T.

Show that

$$T \approx 2\pi\sqrt{\frac{12a}{7g}}.$$

$\boxed{\text{proof}}$



FIRSTLY, MOMENT OF INERTIA ABOUT A

$$\underset{(\text{ROD})}{\tfrac{4}{3}(3m)a^2} + \underset{\text{PARTICLE}}{2m(2a)^2} = 12ma^2$$

EQUATION OF MOTION

$\Rightarrow I\ddot{\theta} = L$

$\Rightarrow (12ma^2)\ddot{\theta} = -2mg\sin\theta \times 2a + -3mg\sin\theta \times a$

$\Rightarrow 12ma^2\ddot{\theta} = -4mga\sin\theta - 3mga\sin\theta$

$\Rightarrow 12ma^2\ddot{\theta} = -7mga\sin\theta$

$\Rightarrow 12a\ddot{\theta} = -7g\sin\theta$

$\Rightarrow \ddot{\theta} = -\frac{7g}{12a}\sin\theta$

FOR SMALL AMPLITUDE OSCILLATIONS $\sin\theta \simeq \theta$

$\Rightarrow \ddot{\theta} = -\frac{7g}{12a}\theta$

I.E SHM WITH PERIOD $T = \frac{2\pi}{\omega}$

IE $T = 2\pi\sqrt{\frac{12a}{7g}}$ //

Question 8 (**+)

A pulley is in the shape of a disc of radius a and mass M.

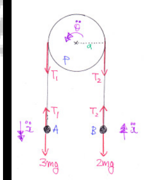
It is free to rotate in a vertical plane about a fixed smooth horizontal axis through its centre O. A light inextensible string passes over the pulley and has a particle of mass $3m$ attached at one of its ends and a particle of mass $2m$ attached at the other end.

The particles are held initially at rest, at the same horizontal level and at the same vertical plane as the pulley, with the string taut.

The system is released from rest and the particles begin to move with the string not slipping on the pulley.

Given that each the particles experience an acceleration of $\frac{1}{10}g$, express M in terms of m.

$\boxed{M = 10m}$



[PULLEY]: $I\ddot{\theta} = T_1 a - T_2 a$

$\left(\frac{1}{2}Ma^2\right)\ddot{\theta} = (T_1 - T_2)a$

$\boxed{\frac{1}{2}Ma\ddot{\theta} = T_1 - T_2}$

[A]: $3m\ddot{x} = 3mg - T_1$

$3m\left(\frac{1}{10}g\right) = 3mg - T_1$

$T_1 = 3mg - \frac{3}{10}mg$

$\boxed{T_1 = \frac{27}{10}mg}$

[B]: $2m\ddot{x} = T_2 - 2mg$

$2m\left(\frac{1}{10}g\right) = T_2 - 2mg$

$\frac{1}{5}mg = T_2 - 2mg$

$\boxed{T_2 = \frac{11}{5}mg}$

HENCE

$\Rightarrow \left(\frac{1}{2}Ma\right)\ddot{\theta} = T_1 - T_2$

$\Rightarrow \frac{1}{2}Ma\ddot{\theta} = \frac{27}{10}mg - \frac{11}{5}mg$

$\Rightarrow \frac{1}{2}Ma\ddot{\theta} = \frac{1}{2}mg$

$\Rightarrow Ma\ddot{\theta} = mg$

$\Rightarrow M(\ddot{x}) = mg$

$\Rightarrow M\left(\frac{1}{10}g\right) = mg$ (AS THERE IS NO SLIPPING $\ddot{x} = a\ddot{\theta}$)

$\Rightarrow \frac{1}{10}M = m$

$\Rightarrow M = 10m$

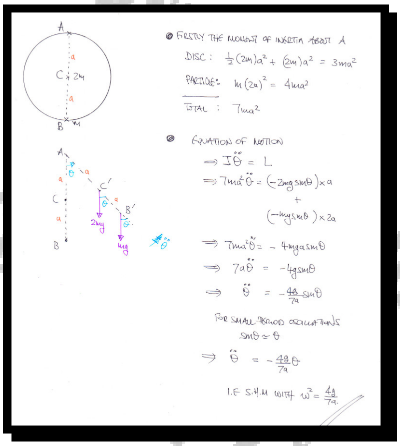
Question 9 (+)**

A uniform circular disc, with centre C, has mass $2m$ and radius a.

A particle of mass m is attached at the point B on the circumference of the disc. The disc is free to rotate about a smooth fixed horizontal axis L, which is perpendicular to the plane of the disc and passes through the point A on the circumference of the disc.

Given that the straight line ACB is a diameter of the disc, show that if the disc is slightly disturbed from its position of stable equilibrium, its subsequent motion will be approximately simple harmonic.

$\boxed{\text{proof}}$



- FIRSTLY THE MOMENT OF INERTIA ABOUT A

DISC : $\frac{1}{2}(2m)a^2 + (2m)a^2 = 3ma^2$

PARTICLE : $m(2a)^2 = 4ma^2$

TOTAL : $7ma^2$

- EQUATION OF MOTION

$\Rightarrow I\ddot{\theta} = L$

$\Rightarrow 7ma^2\ddot{\theta} = (-2mg\sin\theta)\times a + (-mg\sin\theta)\times 2a$

$\Rightarrow 7ma^2\ddot{\theta} = -4mga\sin\theta$

$\Rightarrow 7a\ddot{\theta} = -4g\sin\theta$

$\Rightarrow \ddot{\theta} = -\frac{4g}{7a}\sin\theta$

FOR SMALL PERIOD OSCILLATIONS $\sin\theta \simeq \theta$

$\Rightarrow \ddot{\theta} = -\frac{4g}{7a}\theta$

I.E S.H.M WITH $\omega^2 = \frac{4g}{7a}$

Question 10 (**+)

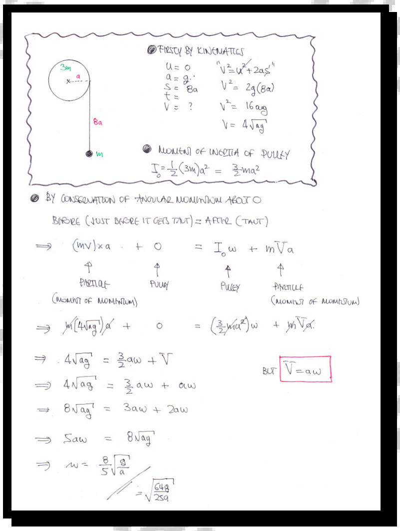
A pulley is in the shape of a disc of radius a and mass $3m$.

It is free to rotate in a vertical plane about a fixed smooth horizontal axis through its centre O. A light inextensible string has one end attached to a point on the rim of the pulley and is wound several times around the rim of the pulley. The portion of the string not wound on the pulley has length $8a$ and has a particle of mass m attached to its free end.

The particle is held at the same level as O, close to the rim of the still pulley and is released from rest.

Determine, in terms of a and g, the angular velocity of the pulley immediately after the string becomes taut.

$$\boxed{\omega = \frac{8}{5}\sqrt{\frac{g}{a}} = \sqrt{\frac{64g}{25a}}}$$



Question 11 (**+)

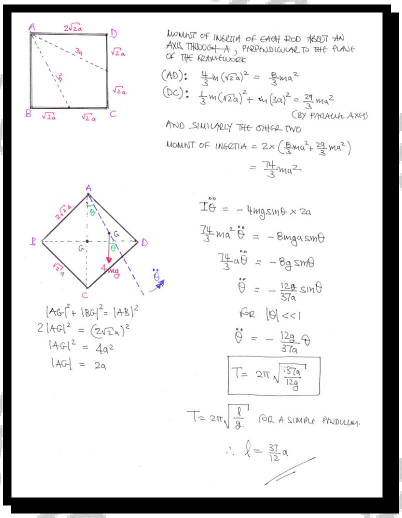
Four uniform rods, each of mass m and length $2\sqrt{2}a$, are rigidly joined together to form a square framework $ABCD$.

The framework is free to rotate in a vertical plane about a fixed smooth horizontal axis, which is perpendicular to plane of the framework and passes through A.

When the framework is slightly displaced from its position of stable equilibrium it performs small amplitude oscillations, with period τ.

Find the length, in terms of a, of a simple pendulum whose period of small amplitude oscillations is also τ.

$$\boxed{l = \frac{37}{12}a}$$



MOMENT OF INERTIA OF EACH ROD ABOUT AN AXIS THROUGH A, PERPENDICULAR TO THE PLANE OF THE FRAMEWORK

(AD): $\frac{4}{3}m(\sqrt{2}a)^2 = \frac{8}{3}ma^2$

(DC): $\frac{1}{3}m(\sqrt{2}a)^2 + m(3a)^2 = \frac{29}{3}ma^2$

(BY PARALLEL AXIS)

AND SIMILARLY THE OTHER TWO

MOMENT OF INERTIA $= 2\times\left(\frac{8}{3}ma^2 + \frac{29}{3}ma^2\right)$

$= \frac{74}{3}ma^2$

$I\ddot{\theta} = -4mg\sin\theta \times 2a$

$\frac{74}{3}ma^2\ddot{\theta} = -8mga\sin\theta$

$\frac{74}{3}a\ddot{\theta} = -8g\sin\theta$

$\ddot{\theta} = -\frac{12g}{37a}\sin\theta$

FOR $|\theta| \ll 1$

$\ddot{\theta} = -\frac{12g}{37a}\theta$

$\boxed{T = 2\pi\sqrt{\frac{37a}{12g}}}$

$|AG|^2 + |BG|^2 = |AB|^2$

$2|AG|^2 = (2\sqrt{2}a)^2$

$|AG|^2 = 4a^2$

$|AG| = 2a$

$T = 2\pi\sqrt{\frac{l}{g}}$ FOR A SIMPLE PENDULUM.

$\therefore\ l = \frac{37}{12}a$

Question 12 **(**+)**

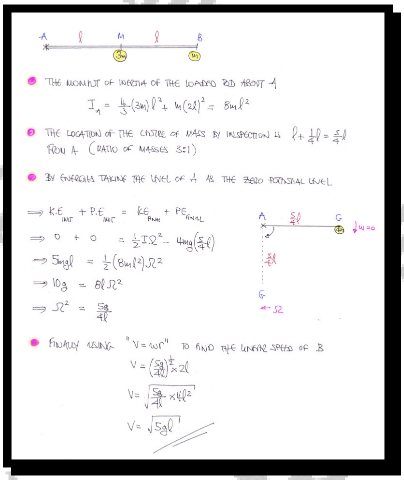
A uniform rod AB, of mass $3m$ and length $2l$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the rod and passes through A.

A particle of mass m is attached to the rod at B.

The loaded rod is held in a horizontal position and is released from rest.

Find, in terms of g and l, the speed of the particle when the rod is first vertical.

$$\boxed{v = \sqrt{5gl}}$$



- THE MOMENT OF INERTIA OF THE LOADED ROD ABOUT A

$$I_A = \frac{4}{3}(3m)l^2 + m(2l)^2 = 8ml^2$$

- THE LOCATION OF THE CENTRE OF MASS BY INSPECTION IS $l + \frac{1}{4}l = \frac{5}{4}l$ FROM A (RATIO OF MASSES 3:1)

- BY ENERGIES TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL

$$\Longrightarrow K.E_{INIT} + P.E_{INIT} = KE_{FINAL} + PE_{FINAL}$$
$$\Longrightarrow 0 + 0 = \frac{1}{2}I\Omega^2 - 4mg\left(\frac{5}{4}l\right)$$
$$\Longrightarrow 5mgl = \frac{1}{2}(8ml^2)\Omega^2$$
$$\Longrightarrow 10g = 8l\Omega^2$$
$$\Longrightarrow \Omega^2 = \frac{5g}{4l}$$

- FINALLY USING "$V = \omega r$" TO FIND THE LINEAR SPEED OF B

$$V = \left(\frac{5g}{4l}\right)^{\frac{1}{2}} \times 2l$$
$$V = \sqrt{\frac{5g}{4l} \times 4l^2}$$
$$V = \sqrt{5gl}$$

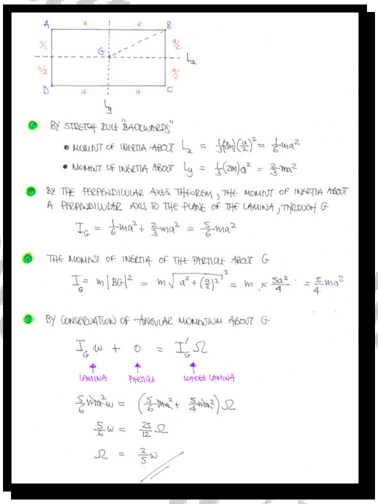
Question 13 (**+)

A uniform rectangular lamina $ABCD$, where $|AB| = 2a$ and $|BC| = a$, has mass $2m$.

The lamina is rotating with angular speed ω, in a horizontal plane about a smooth fixed vertical axis which passes through the centre of the lamina. A particle of mass m is held at rest just above the surface of the lamina when it adheres to the corner B.

Find, in terms of ω, the new angular speed of the now loaded lamina.

$$\boxed{\Omega = \frac{2}{5}\omega}$$



- BY STRETCH RULE "BACKWARDS"
  - MOMENT OF INERTIA ABOUT $L_x = \frac{1}{3}(2m)\left(\frac{a}{2}\right)^2 = \frac{1}{6}ma^2$
  - MOMENT OF INERTIA ABOUT $L_y = \frac{1}{3}(2m)a^2 = \frac{2}{3}ma^2$
- BY THE PERPENDICULAR AXES THEOREM, THE MOMENT OF INERTIA ABOUT A PERPENDICULAR AXIS TO THE PLANE OF THE LAMINA, THROUGH G

$$I_G = \frac{1}{6}ma^2 + \frac{2}{3}ma^2 = \frac{5}{6}ma^2$$

- THE MOMENT OF INERTIA OF THE PARTICLE ABOUT G

$$I_G = m|BG|^2 = m\left[\sqrt{a^2 + \left(\frac{a}{2}\right)^2}\right]^2 = m \times \frac{5a^2}{4} = \frac{5}{4}ma^2$$

- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT G

$$\underbrace{I_G\omega}_{\text{LAMINA}} + \underbrace{0}_{\text{PARTICLE}} = \underbrace{I_G'\Omega}_{\text{LOADED LAMINA}}$$

$$\frac{5}{6}ma^2\omega = \left(\frac{5}{6}ma^2 + \frac{5}{4}ma^2\right)\Omega$$

$$\frac{5}{6}\omega = \frac{25}{12}\Omega$$

$$\Omega = \frac{2}{5}\omega$$

Question 14 (***)

A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the rod and passes through the point O of the rod, where $OA = \frac{1}{3}a$.

a) Find the moment of inertia of the rod about L.

The rod is held at rest with B vertically above O and is slightly displaced.

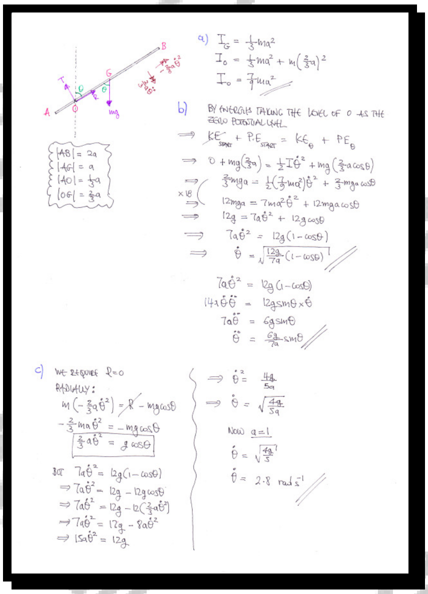
b) Determine, when OB makes an angle θ with the upward vertical, ...

i. … the angular speed of the rod.

ii. … the magnitude of the angular acceleration of the rod.

c) Given that the length of the rod is 2 m, calculate the angular speed of the rod when the force acting on the rod at O is perpendicular to the rod.

$$\boxed{I_O = \frac{7}{9}ma^2}, \quad \boxed{\dot{\theta} = \sqrt{\frac{12g}{7a}(1-\cos\theta)}}, \quad \boxed{\ddot{\theta} = \frac{6g}{7a}\sin\theta}, \quad \boxed{\omega = 2.8 \text{ rad s}^{-1}}$$



Question 15 (***)

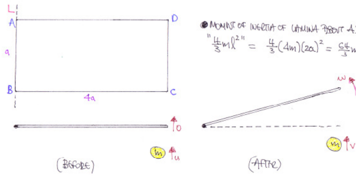
A uniform rectangular lamina $ABCD$, where $AB = a$ and $BC = 4a$, has mass $4m$. The lamina is free to rotate about the edge AB, which is fixed and vertical.

A particle, of mass m, is moving horizontally with speed u in a direction which is perpendicular to the lamina. The lamina is at rest when it is struck by the particle at C.

The coefficient of restitution between the particle and the lamina is 0.75.

Find the angular speed of the lamina immediately after the impact.

$$\boxed{\omega = \frac{3u}{16a}}$$



- MOMENT OF INERTIA OF LAMINA ABOUT AB (STRETCH RULE BACKWARDS)

$$\text{"}\tfrac{4}{3}ml^2\text{"} = \tfrac{4}{3}(4m)(2a)^2 = \tfrac{64}{3}ma^2$$

- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT L

$$(mu)\times 4a + 0 = (mV)\times 4a + I\omega$$
$$4mua = 4mVa + \tfrac{64}{3}ma^2\omega$$
$$\boxed{4u = 4V + \tfrac{64}{3}a\omega}$$

- BY RESTITUTION

$$e = \frac{SEP}{APP}$$
$$\frac{3}{4} = \frac{(\omega \times 4a) - V}{u}$$
$$\boxed{3u = 16\omega a - 4V}$$

ADD EQUATIONS

$$7u = \tfrac{112}{3}\omega a$$
$$\omega = \frac{3u}{16a}$$

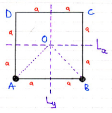
Question 16 (***)

A uniform square lamina $ABCD$ has side length a and mass m. The lamina is free to rotate in a vertical plane about a fixed horizontal axis, which is perpendicular to the plane of the lamina and passes through O, the centre of the lamina. Two particles, each of mass m, are attached to the vertices A and B. The system is released from rest with AB vertical.

Find, in terms of a and g, the angular velocity of the system when AB is horizontal.

$\boxed{}$, $\boxed{\omega = \sqrt{\frac{6g}{7a}}}$

START BY DETERMINING THE MOMENT OF INERTIA OF THE LOADED LAMINA, ABOUT O



- BY THE "STRETCH RULE" BACKWARDS

$$I_{L_x} = I_{L_y} = \frac{1}{3}ma^2 ;$$

AS THE LAMINA COMPRESSES TO A ROD OF LENGTH $2a$ & MASS m, WITH ITS CENTRE AT O

- BY THE PERPENDICULAR AXES THEOREM, I_O, PERPENDICULAR TO THE PLANE OF THE LAMINA IS GIVEN BY

$$I_O = \frac{1}{3}ma^2 + \frac{1}{3}ma^2 = \frac{2}{3}ma^2$$

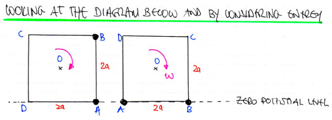
- $|OA| = |OB| = \sqrt{2}a$ (BY PYTHAGORAS)
- CONTRIBUTION OF EACH PARTICLE, TO THE MOMENT OF INERTIA ABOUT O, IS GIVEN BY

$$m(\sqrt{2}a)^2 = 2ma^2$$

- TOTAL MOMENT OF INERTIA ABOUT O

$$I_{TOT} = \frac{2}{3}ma^2 + 2(2ma^2) = \frac{14}{3}ma^2$$

LOOKING AT THE DIAGRAM BELOW AND BY CONSIDERING ENERGY



NOTE THAT THE CENTRE OF MASS OF THE LAMINA (MASS m) DOES NOT MOVE AND HENCE WE MAY IGNORE IT

$$\Rightarrow KE_{BEFORE} + PE_{BEFORE} = KE_{AFTER} + PE_{AFTER}$$

$$\Rightarrow 0 + mg(2a) = \frac{1}{2}I_{TOT}\omega^2 + 0$$

$$\Rightarrow 2mga = \frac{1}{2}\left(\frac{14}{3}ma^2\right)\omega^2$$

$$\Rightarrow 2\cancel{m}g\cancel{a} = \frac{7}{3}\cancel{m}a^{\cancel{2}}\omega^2$$

$$\Rightarrow 2g = \frac{7}{3}a\omega^2$$

$$\Rightarrow \frac{6g}{7a} = \omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{6g}{7a}}$$

Question 17 (***)

A uniform circular disc, with centre C, has mass $5m$ and radius a.

The straight line AB is a diameter of the disc.

A particle of mass m is attached to the disc at the point M, where M is the midpoint of AC. The disc is free to rotate about a smooth horizontal axis L, which lies in the plane of the disc and is a tangent to the disc at B.

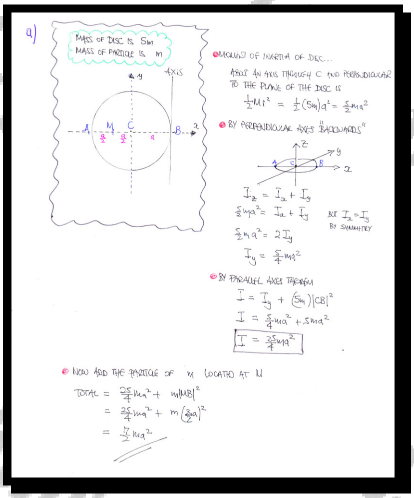
a) Find the moment of inertia of the loaded disc about L.

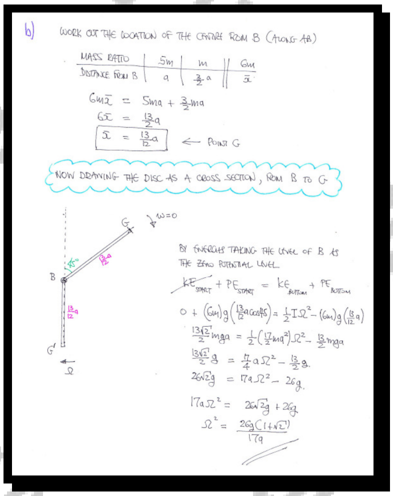
The loaded disc is released from rest with AB at an angle of 45° to the upward vertical. When A is vertically below B, the loaded disc has angular speed Ω.

b) Show that

$$\Omega^2 = \frac{26g\left(1+\sqrt{2}\right)}{17a}.$$

$$\boxed{I = \frac{17}{2}ma^2}$$





Question 18 (***)

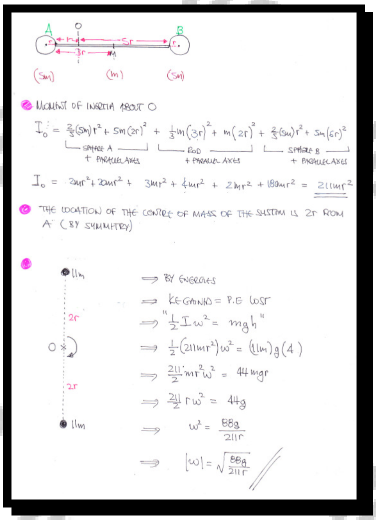
Two uniform spheres, each of mass $5m$ and radius r, are attached to each of the ends of a thin uniform rod AB, of mass m and length $6r$. The centres of the spheres are collinear with AB, and are located $8r$ apart.

The above described system is free to rotate about a fixed smooth horizontal axis, perpendicular to AB, and passing through a point on the rod O, where $|AO| = r$.

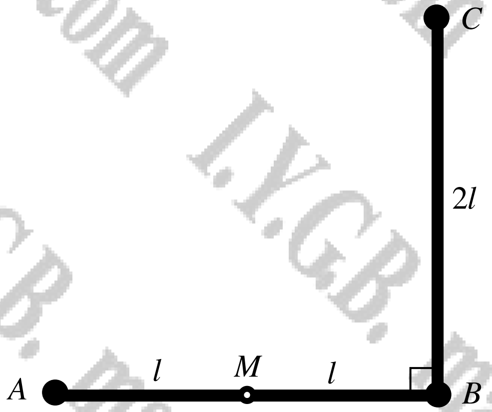
The system is slightly disturbed from rest with B vertically above A.

Determine the angular velocity of the system when A vertically above B.

$$\boxed{|\omega| = \sqrt{\frac{88g}{211r}}}$$



Question 19 **(***)**



Two identical uniform rods AB and BC, each of mass m and length l are rigidly joined at B, so that $\measuredangle ABC = 90°$. Three particles of masses m, $2m$ and $3m$ are fixed at A, B and C, respectively. The system of the two rods and the three particles can rotate freely in a vertical plane about a horizontal axis through M, where M is the midpoint of AB.

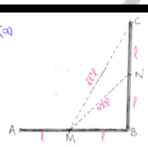
a) Show clearly that the moment of inertia of the system about an axis through M and perpendicular to the plane ABC is $\frac{62}{3}ml^2$.

The system is released from rest with AB horizontal and C vertically above B.

b) Determine, in terms of g and l, the angular velocity of the system when BC is horizontal and B is vertically below A.

$$\boxed{\omega = \sqrt{\frac{31g}{32l}}}$$

(a)



• $|MN| = \sqrt{2}\,l$ (BY PYTHAGORAS)
• $|MC| = \sqrt{5}\,l$ (BY PYTHAGORAS)

I_{AB} ABOUT $M = \frac{1}{3}ml^2$

I_{BC} ABOUT $M = \frac{1}{3}ml^2 + m|MN|^2$ (PARALLEL AXES)
$= \frac{1}{3}ml^2 + m(\sqrt{2}l)^2$
$= \frac{7}{3}ml^2$

I_m ABOUT $M = ml^2$

I_{2m} ABOUT $M = 2ml^2$

I_{3m} ABOUT $M = 3m|MC|^2 = 3m(\sqrt{5}l)^2 = 15ml^2$

ADDING: $\frac{1}{3}ml^2 + \frac{7}{3}ml^2 + ml^2 + 2ml^2 + 15ml^2 = \frac{62}{3}ml^2$ // AS REQUIRED

(b) BY ENERGIES, TAKING THE LEVEL OF M AS THE ZERO POTENTIAL LEVEL AND TREATING EACH OBJECT SEPARATELY

$P.E_{\text{LOSS}} = KE_{\text{GAIN}}$

$\Rightarrow \underset{\text{ROD AB}}{0} + \underset{\text{ROD BC}}{\cancel{mgl}} + \underset{\text{"2m"}}{2mgl} + \underset{\text{"3m"}}{3mg \times 3l} - \underset{\text{"m"}}{\cancel{mgl}} = \frac{1}{2}I\omega^2$

MINUS BECAUSE IT GAINED POTENTIAL ENERGY

$\Rightarrow 11mgl = \frac{1}{2} \times \frac{62}{3}ml^2\omega^2$

$\Rightarrow 11g = \frac{31}{3}l\omega^2$

$\Rightarrow 33g = 31l\omega^2$

$\Rightarrow \omega = \sqrt{\frac{31g}{33l}}$ //

Question 20 (***)

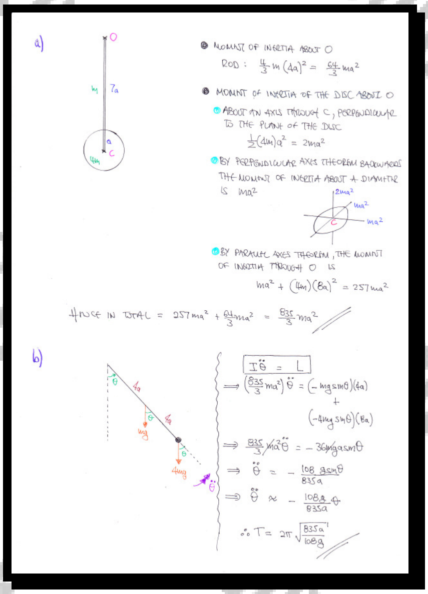
A compound pendulum consists of a thin uniform rod OC of length $8a$ and mass m is rigidly attached at C to the centre of a thin uniform circular disc of radius a and mass $4m$. The rod is in the same vertical plane as the disc.

The pendulum is free to rotate in this vertical plane, through a smooth horizontal axis through O, perpendicular to the plane of the disc.

a) Show that the moment of inertia of the pendulum about the above described axis is $\frac{835}{3}ma^2$.

b) Show further that the period of small amplitude oscillations of the pendulum, about the position of the stable equilibrium is $2\pi\sqrt{\frac{835a}{108g}}$.

$\boxed{\text{proof}}$



a)

- MOMENT OF INERTIA ABOUT O

ROD : $\frac{4}{3}m(4a)^2 = \frac{64}{3}ma^2$

- MOMENT OF INERTIA OF THE DISC ABOUT O
  - ABOUT AN AXIS THROUGH C, PERPENDICULAR TO THE PLANE OF THE DISC
    $\frac{1}{2}(4m)a^2 = 2ma^2$
  - BY PERPENDICULAR AXES THEOREM BACKWARDS THE MOMENT OF INERTIA ABOUT A DIAMETER IS ma^2
  - BY PARALLEL AXES THEOREM, THE MOMENT OF INERTIA THROUGH O IS
    $ma^2 + (4m)(8a)^2 = 257ma^2$

HENCE IN TOTAL $= 257ma^2 + \frac{64}{3}ma^2 = \frac{835}{3}ma^2$

b)

$\boxed{I\ddot{\theta} = L}$

$\Longrightarrow \left(\frac{835}{3}ma^2\right)\ddot{\theta} = (-mg\sin\theta)(4a) + (-4mg\sin\theta)(8a)$

$\Longrightarrow \frac{835}{3}\cancel{m}a^2\ddot{\theta} = -36\cancel{m}ga\sin\theta$

$\Longrightarrow \ddot{\theta} = -\frac{108g\sin\theta}{835a}$

$\Longrightarrow \ddot{\theta} \approx -\frac{108g}{835a}\theta$

$\therefore T = 2\pi\sqrt{\frac{835a}{108g}}$

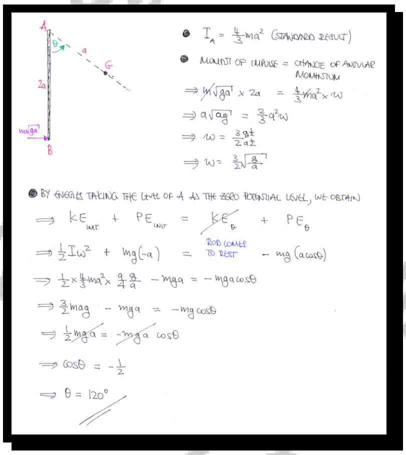
Question 21 (***)

A thin uniform rod AB, of length $2a$ and mass m, is free to rotate in a vertical plane, about a fixed smooth horizontal axis through A.

The rod is hanging in equilibrium, with B vertically below A, when it receives a horizontal impulse of magnitude $m\sqrt{ag}$, in a direction perpendicular to the axis through A.

Find the angle by which the rod turns before coming to instantaneous rest.

$\boxed{120°}$



- $I_A = \frac{4}{3}ma^2$ (STANDARD RESULT)
- MOMENT OF IMPULSE = CHANGE OF ANGULAR MOMENTUM

$\Rightarrow \cancel{m}\sqrt{ga} \times 2a = \frac{4}{3}\cancel{m}a^2 \times \omega$

$\Rightarrow a\sqrt{ag} = \frac{2}{3}a^2\omega$

$\Rightarrow \omega = \frac{3g^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$

$\Rightarrow \omega = \frac{3}{2}\sqrt{\frac{g}{a}}$

- BY ENERGIES TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL, WE OBTAIN

$\Rightarrow KE_{INIT} + PE_{INIT} = \cancel{KE_\theta} + PE_\theta$

$\Rightarrow \frac{1}{2}I\omega^2 + mg(-a) =$ ROD COMES TO REST $- mg(a\cos\theta)$

$\Rightarrow \frac{1}{2} \times \frac{4}{3}ma^2 \times \frac{9}{4}\frac{g}{a} - mga = -mga\cos\theta$

$\Rightarrow \frac{3}{2}mag - mga = -mg\cos\theta$

$\Rightarrow \frac{1}{2}\cancel{mga} = -\cancel{mga}\cos\theta$

$\Rightarrow \cos\theta = -\frac{1}{2}$

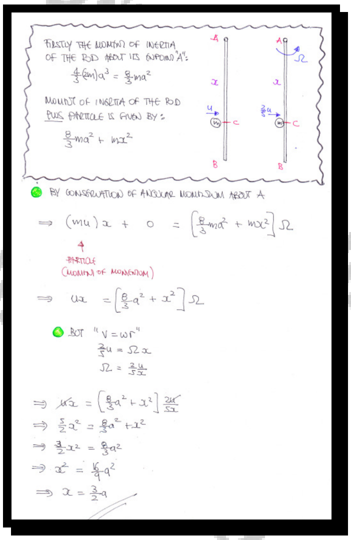
$\Rightarrow \theta = 120°$

Question 22 (***)

A thin uniform rod AB of length $2a$ and mass $2m$ is free to rotate in a vertical plane, through a fixed smooth horizontal axis through A. The rod is hanging in equilibrium with B vertically below A. A particle of mass m, moving horizontally with speed u in a vertical plane perpendicular to the axis through A, strikes the rod at the point C and adheres to it.

Given that the speed of the particle immediately after it adheres to the rod is $\frac{2}{5}u$, determine the distance AC

$$\boxed{|AC| = \frac{3}{2}a}$$



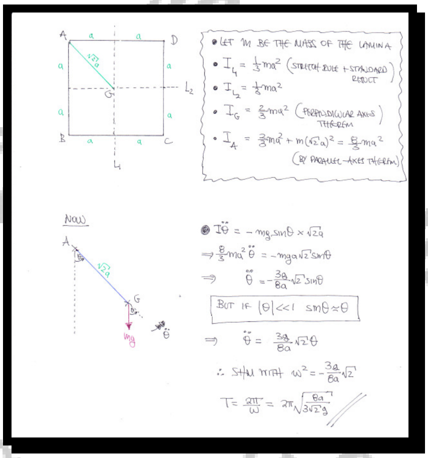
Question 23 (***)

A uniform square lamina has side length $2a$. The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the lamina and passes through one of the vertices of the lamina.

The lamina is suspended through L, and hanging in stable equilibrium when it is slightly displaced from that position.

Determine the period of small oscillation about this position, in terms of π, a and g.

$$\boxed{T = 2\pi\sqrt{\frac{8a}{3\sqrt{2}g}}}$$

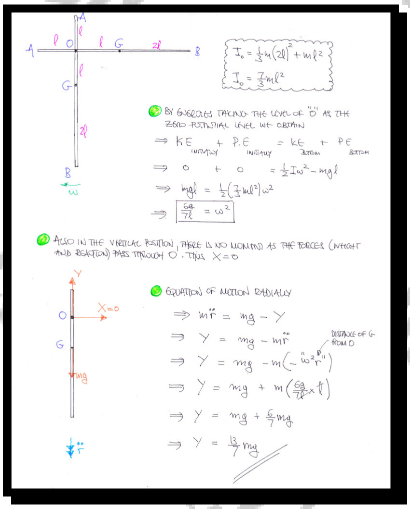


Question 24 (***)

A thin uniform rod AB of length $4l$ and mass m is free to rotate in a vertical plane, through a fixed smooth horizontal axis through a point O on the rod, which is at a distance l from A. The rod is released from rest in a horizontal position and when the rod is vertical for the first time its angular velocity is ω.

Show that when the rod is first vertical, the magnitude of the force acting on the rod at O is $\frac{13}{7}mg$.

$\boxed{\text{proof}}$



Question 25 (***)

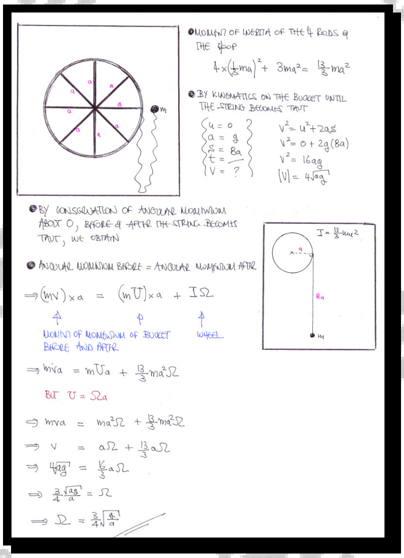
A cartwheel, consisting of 8 spokes and a circular rim, is placed over a well. Each of the 8 spokes is modelled as a uniform rod of mass $\frac{1}{2}m$ and length a. The rim is modelled as a uniform circular hoop of mass $3m$.

The 8 spokes are equally spaced on the rim and the meet at the centre of the hoop O. The cartwheel is modelled as a two dimensional rigid structure. A bucket of mass m, which is modelled as a particle, is attached to one end of a light inextensible string of length $8a$. The other end of the string is attached to a point P on the rim of the cartwheel, so that OP is horizontal.

The bucket is held next to P and released from rest with string slack.

Determine, in terms of a and g, the angular velocity of the cartwheel just after the instant the string becomes taut.

$$\boxed{\omega = \sqrt{\frac{9a}{16g}}}$$


● MOMENT OF INERTIA OF THE 4 RODS & THE HOOP

$4\times\left(\frac{1}{3}ma\right)^2 + 3ma^2 = \frac{13}{3}ma^2$

● BY KINEMATICS ON THE BUCKET UNTIL THE STRING BECOMES TAUT

$u = 0$, $a = g$, $s = 8a$, $t = $ —, $V = ?$

$V^2 = u^2 + 2as$

$V^2 = 0 + 2g(8a)$

$V^2 = 16ag$

$|V| = 4\sqrt{ag}$

● BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O, BEFORE & AFTER THE STRING BECOMES TAUT, WE OBTAIN

$I = \frac{13}{3}ma^2$

● ANGULAR MOMENTUM BEFORE = ANGULAR MOMENTUM AFTER

$\Rightarrow (mV)\times a = (mU)\times a + I\Omega$

MOMENT OF MOMENTUM OF BUCKET BEFORE AND AFTER — WHEEL

$\Rightarrow mVa = mUa + \frac{13}{3}ma^2\Omega$

BUT $U = \Omega a$

$\Rightarrow mVa = ma^2\Omega + \frac{13}{3}ma^2\Omega$

$\Rightarrow V = a\Omega + \frac{13}{3}a\Omega$

$\Rightarrow 4\sqrt{ag} = \frac{16}{3}a\Omega$

$\Rightarrow \frac{3}{4}\frac{\sqrt{ag}}{a} = \Omega$

$\Rightarrow \Omega = \frac{3}{4}\sqrt{\frac{g}{a}}$

Question 26 (***)

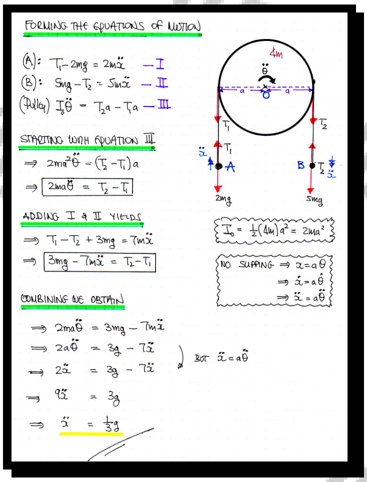
A pulley is modelled as a uniform circular disc of mass $4m$ and radius a.

The pulley is free to rotate about a fixed smooth horizontal axis through its centre and perpendicular to its plane. A light inextensible string passes over the pulley and two particles A and B, of respective mass $2m$ and $5m$ are attached to the two ends of the string.

The particles are released from rest with the string taut.

Assuming further that there is no slipping between the string and the pulley find, in terms of g, the acceleration of the particles.

$\boxed{\ddot{x} = \frac{1}{3}g}$



FORMING THE EQUATIONS OF MOTION

(A): $T_1 - 2mg = 2m\ddot{x}$ — I

(B): $5mg - T_2 = 5m\ddot{x}$ — II

(Pulley) $I_0\ddot{\theta} = T_2 a - T_1 a$ — III

STARTING WITH EQUATION III

$\Rightarrow 2ma^2\ddot{\theta} = (T_2 - T_1)a$

$\Rightarrow \boxed{2ma\ddot{\theta} = T_2 - T_1}$

ADDING I & II YIELDS

$\Rightarrow T_1 - T_2 + 3mg = 7m\ddot{x}$

$\Rightarrow \boxed{3mg - 7m\ddot{x} = T_2 - T_1}$

COMBINING WE OBTAIN

$\Rightarrow 2ma\ddot{\theta} = 3mg - 7m\ddot{x}$

$\Rightarrow 2a\ddot{\theta} = 3g - 7\ddot{x}$

$\Rightarrow 2\ddot{x} = 3g - 7\ddot{x}$

BUT $\ddot{x} = a\ddot{\theta}$

$\Rightarrow 9\ddot{x} = 3g$

$\Rightarrow \ddot{x} = \frac{1}{3}g$

Question 27 (***+)

A thin uniform rod of mass $2m$ and length $2a$ is freely pivoted at A and is hanging at rest in a vertical position with B below A.

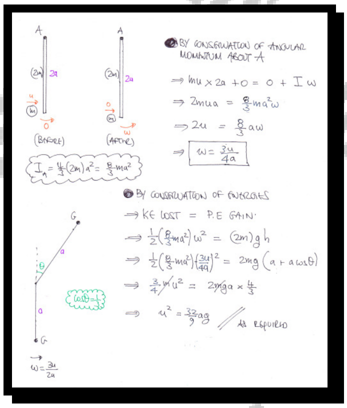
A particle of mass m, travelling horizontally with speed u, strikes the rod at B.

After the impact the particle remains at rest and the rod begins to rotate coming to instantaneous rest when AB is at $\arccos\frac{1}{3}$ to the upward vertical through A.

Using a clear method, show that

$$u^2 = \frac{32}{9}ag .$$

proof



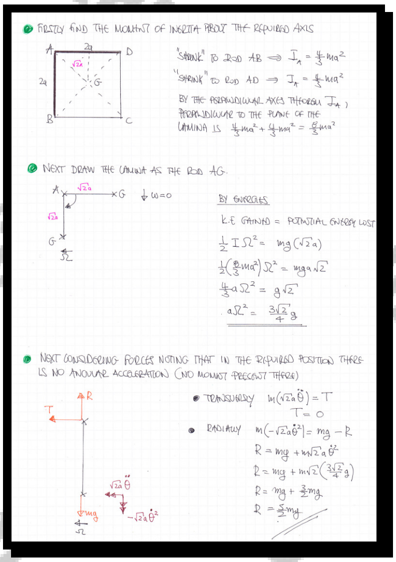
Question 28 (***+)

A uniform square lamina $ABCD$, of mass m and side $2a$, is free to rotate in a vertical plane about a fixed, smooth, horizontal axis L, which passes through A and is perpendicular to the plane of the lamina.

The lamina is released from rest with AC horizontal.

Determine, in terms of mg, the magnitude of the component of the force exerted by the lamina on L, along AC, when AC is vertical with C below A.

$$\boxed{\frac{5}{2}mg}$$



- FIRSTLY FIND THE MOMENT OF INERTIA ABOUT THE REQUIRED AXIS

"STRINK" TO ROD AB $\Rightarrow I_A = \frac{4}{3}ma^2$

"STRINK" TO ROD AD $\Rightarrow I_A = \frac{4}{3}ma^2$

BY THE PERPENDICULAR AXES THEOREM I_A, PERPENDICULAR TO THE PLANE OF THE LAMINA IS $\frac{4}{3}ma^2 + \frac{4}{3}ma^2 = \frac{8}{3}ma^2$

- NEXT DRAW THE LAMINA AS THE ROD AG.

BY ENERGIES

K.E GAINED = POTENTIAL ENERGY LOST

$\frac{1}{2}I\Omega^2 = mg(\sqrt{2}a)$

$\frac{1}{2}\left(\frac{8}{3}ma^2\right)\Omega^2 = mga\sqrt{2}$

$\frac{4}{3}a\Omega^2 = g\sqrt{2}$

$a\Omega^2 = \frac{3\sqrt{2}}{4}g$

- NEXT CONSIDERING FORCES NOTING THAT IN THE REQUIRED POSITION THERE IS NO ANGULAR ACCELERATION (NO MOMENT PRESENT THERE)

- TRANSVERSLY $m(\sqrt{2}a\ddot{\theta}) = T$

$T = 0$

- RADIALLY $m(-\sqrt{2}a\dot{\theta}^2) = mg - R$

$R = mg + m\sqrt{2}a\dot{\theta}^2$

$R = mg + m\sqrt{2}\left(\frac{3\sqrt{2}}{4}g\right)$

$R = mg + \frac{3}{2}mg$

$R = \frac{5}{2}mg$

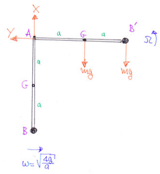
Question 29 (***+)

A thin uniform rod of mass m and length $2a$ has a particle of mass m attached at B.

The rod is freely pivoted at A and is hanging at rest in a vertical position, with B below A, when it is given an angular velocity about A of magnitude $\sqrt{\dfrac{4g}{a}}$.

Determine, in terms of m and g, the horizontal and vertical components of the force exerted on the axis of rotation when AB is in a horizontal position.

$$\boxed{F_{hor} = \frac{69}{8}mg}, \quad \boxed{F_{ver} = \frac{5}{16}mg}$$



● FIRSTLY, MOMENT OF INERTIA ABOUT A

$$I = \underbrace{\tfrac{4}{3}ma^2}_{\text{ROD}} + \underbrace{m(2a)^2}_{\text{PARTICLE}} = \tfrac{16}{3}ma^2$$

● BY ENERGIES (TAKE INITIAL POSITION AS THE ZERO POTENTIAL LEVEL)

$$\Rightarrow KE_{\text{START}} + \cancel{PE_{\text{START}}} = KE_{\text{HORIZONTAL}} + PE_{\text{HORIZONTAL}}$$

$$\Rightarrow \tfrac{1}{2}I\omega^2 = mga + mg(2a) + \tfrac{1}{2}I\Omega^2$$

$$\Rightarrow \tfrac{1}{2}\left(\tfrac{16}{3}ma^2\right)\left(\tfrac{4g}{a}\right) = 3mga + \tfrac{1}{2}\left(\tfrac{16}{3}ma^2\right)\Omega^2$$

$$\Rightarrow \tfrac{32}{3}mag = 3mga + \tfrac{8}{3}ma^2\Omega^2$$

$$\Rightarrow \tfrac{23}{3}mga = \tfrac{8}{3}ma^2\Omega^2$$

$$\Rightarrow 23g = 8\Omega^2 a$$

$$\Rightarrow \boxed{\Omega^2 = \frac{23g}{8a}}$$

● RADIALLY (ACC: $-r\dot{\theta}^2$)

$$m(-a\Omega^2) + m(-2a\Omega^2) = -Y$$

$$Y = 3ma\Omega^2$$

$$Y = 3ma\left(\frac{23g}{8a}\right)$$

$$Y = \frac{69}{8}mg$$

● TRANSVERSELY (ACC: $r\ddot{\theta}$)

$$I\ddot{\theta} = L$$

$$\tfrac{16}{3}ma^2\ddot{\theta} = -mga - mg(2a)$$

$$\ddot{\theta} = -\frac{9}{16}\frac{g}{a}$$

THUS

$$m(a\ddot{\theta}) + m(2a\ddot{\theta}) = X - 2mg$$

$$3ma\left(-\frac{9}{16}\frac{g}{a}\right) = X - 2mg$$

$$-\frac{27}{16}mg = X - 2mg$$

$$X = \frac{5}{16}mg$$

Question 30 (***+)

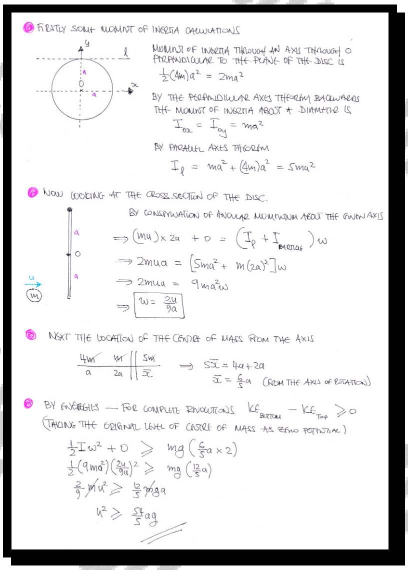
A uniform disc, of mass $4m$ and radius a, is free to rotate about a smooth, fixed horizontal axis which is tangential to a point on the rim of the disc and lies on the plane of the disc.

The disc is hanging in stable equilibrium when it is struck by a particle of mass m moving with speed u in a direction perpendicular to the plane of the disc. The particle hits the disc at the lowest point of the disc and immediately adheres to it.

Given that in the subsequent motion the disc performs full revolutions about its axis, show that

$$u^2 \geq \frac{54}{5}ag .$$

proof



FIRSTLY SOME MOMENT OF INERTIA CALCULATIONS

MOMENT OF INERTIA THROUGH AN AXIS THROUGH O PERPENDICULAR TO THE PLANE OF THE DISC IS

$\frac{1}{2}(4m)a^2 = 2ma^2$

BY THE PERPENDICULAR AXES THEOREM BACKWARDS THE MOMENT OF INERTIA ABOUT A DIAMETER IS

$I_{ox} = I_{oy} = ma^2$

BY PARALLEL AXES THEOREM

$I_\ell = ma^2 + (4m)a^2 = 5ma^2$

NOW LOOKING AT THE CROSS SECTION OF THE DISC.

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT THE GIVEN AXIS

$\Rightarrow (mu) \times 2a + 0 = (I_\ell + I_{\text{PARTICLE}})\omega$

$\Rightarrow 2mua = [5ma^2 + m(2a)^2]\omega$

$\Rightarrow 2mua = 9ma^2\omega$

$\Rightarrow \boxed{\omega = \frac{2u}{9a}}$

NEXT THE LOCATION OF THE CENTRE OF MASS FROM THE AXIS

4m	m	5m
a	2a	$\bar{x}$

$\Rightarrow 5\bar{x} = 4a + 2a$

$\bar{x} = \frac{6}{5}a$ (FROM THE AXIS OF ROTATION)

BY ENERGIES — FOR COMPLETE REVOLUTIONS $KE_{\text{BOTTOM}} - KE_{\text{TOP}} \geq 0$
(TAKING THE ORIGINAL LEVEL OF CENTRE OF MASS AS ZERO POTENTIAL)

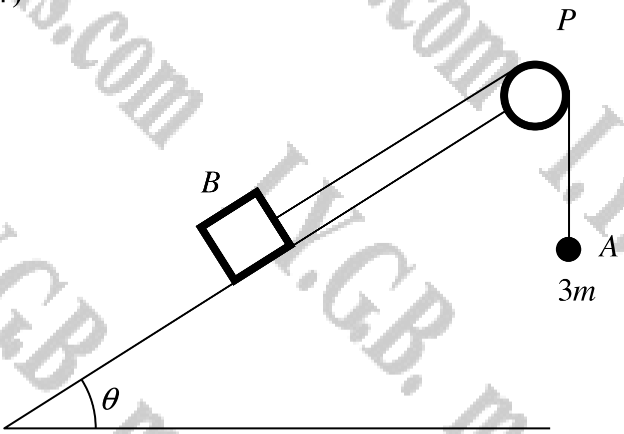
$\frac{1}{2}I\omega^2 + 0 \geq mg\left(\frac{6}{5}a \times 2\right)$

$\frac{1}{2}(9ma^2)\left(\frac{2u}{9a}\right)^2 \geq mg\left(\frac{12}{5}a\right)$

$\frac{2}{9}\cancel{m}u^2 \geq \frac{12}{5}\cancel{m}ga$

$u^2 \geq \frac{54}{5}ag$

Question 31 (***+)



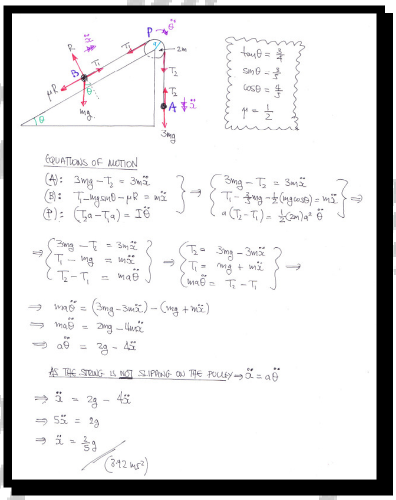
A small box B and a particle A, of mass m and $3m$ respectively, are attached to each of the ends of a light inextensible string.

The string passes over a pulley P, at the top of a fixed rough plane, inclined at an angle θ to the horizontal, where $\tan\theta = \frac{3}{4}$. The pulley is modelled as a uniform disc of mass $2m$ and radius a, rotating about a smooth fixed horizontal axis.

The small box B is placed at rest on the incline plane while A is hanging freely at the end of the incline plane vertically below P, as shown in the figure above. It is further given that A, B, P and the string lie in a vertical plane parallel to the line of greatest slope of the incline plane.

The system is released from rest with the string taut. The box B begins to move up the incline plane, where the coefficient between B and the plane is 0.5. Ignoring air resistance, find the acceleration of A, immediately after the system is set in motion.

$$\boxed{\ddot{x} = \frac{2}{5}g = 3.92\,\text{ms}^{-2}}$$



Question 32 (***+)

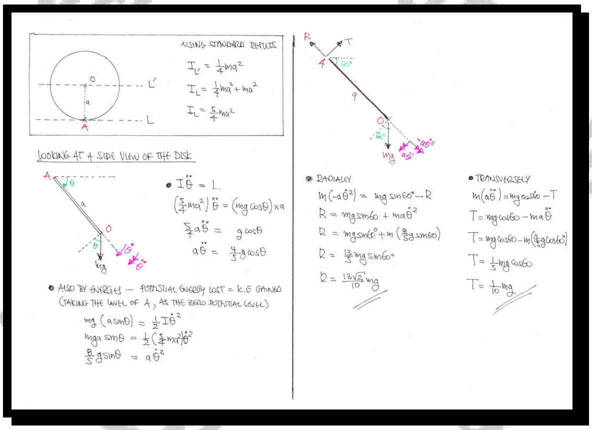
A uniform circular disc, of mass m and radius a, is free to rotate about a fixed smooth horizontal axis L, tangential to a point A on the circumference of the disc.

The centre O of the disc moves in a vertical plane that is perpendicular to L.

The disc is held with its plane horizontal and released from rest.

Determine the magnitude of each of the components, in the radial and transverse directions to the motion of the disc, of the force on L, when the disc has turned through an angle of $60°$.

$$\boxed{F_R = \frac{13\sqrt{2}}{10}mg}, \quad \boxed{F_T = \frac{1}{10}mg}$$



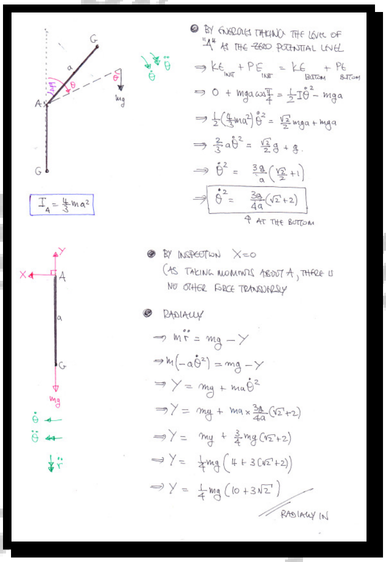
Question 33 (***+)

A thin uniform rod AB of length $2a$ and mass m is free to rotate in a vertical plane, about a smooth horizontal axis through A.

The rod is held at $\frac{\pi}{4}$ to the upward vertical through A, and released from rest.

Determine, in terms of m and g, the magnitude and direction of the force exerted on the axis at A, when B is vertically below A.

$$\boxed{\frac{1}{4}mg\left[10+3\sqrt{2}\right], \text{ radially inwards}}$$



$\boxed{I_A = \frac{4}{3}ma^2}$

- BY ENERGIES TAKING THE LEVEL OF "A" AS THE ZERO POTENTIAL LEVEL

$\Rightarrow KE_{INIT} + PE_{INIT} = KE_{BOTTOM} + PE_{BOTTOM}$

$\Rightarrow 0 + mga\cos\frac{\pi}{4} = \frac{1}{2}I\dot{\theta}^2 - mga$

$\Rightarrow \frac{1}{2}\left(\frac{4}{3}ma^2\right)\dot{\theta}^2 = \frac{\sqrt{2}}{2}mga + mga$

$\Rightarrow \frac{2}{3}a\dot{\theta}^2 = \frac{\sqrt{2}}{2}g + g$

$\Rightarrow \dot{\theta}^2 = \frac{3g}{a}\left(\frac{\sqrt{2}}{2}+1\right)$

$\Rightarrow \boxed{\dot{\theta}^2 = \frac{3g}{4a}(\sqrt{2}+2)}$

AT THE BOTTOM

- BY INSPECTION $X=0$
(AS TAKING MOMENTS ABOUT A, THERE IS NO OTHER FORCE TRANSVERSLY

- RADIALLY

$\Rightarrow m\ddot{r} = mg - Y$

$\Rightarrow m(-a\dot{\theta}^2) = mg - Y$

$\Rightarrow Y = mg + ma\dot{\theta}^2$

$\Rightarrow Y = mg + ma \times \frac{3g}{4a}(\sqrt{2}+2)$

$\Rightarrow Y = mg + \frac{3}{4}mg(\sqrt{2}+2)$

$\Rightarrow Y = \frac{1}{4}mg\left(4 + 3(\sqrt{2}+2)\right)$

$\Rightarrow Y = \frac{1}{4}mg\left(10 + 3\sqrt{2}\right)$

RADIALLY IN

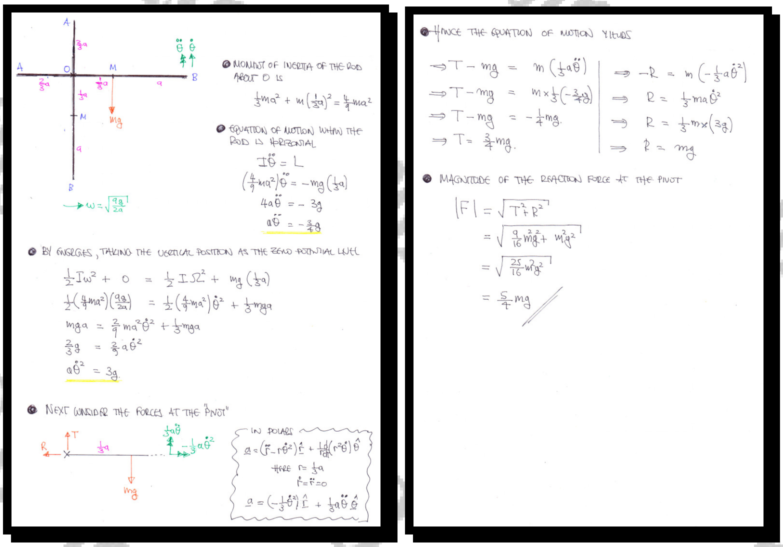
Question 34 (***+)

A thin uniform rod AB of length $2a$ and mass m is free to rotate in a vertical plane, about a smooth horizontal axis through O, where $|AO| = \frac{2}{3}a$.

When the rod is vertical with B below O, the rod has angular velocity $\sqrt{\frac{9g}{2a}}$.

Determine, in terms of m and g, the magnitude and direction of the force exerted on the axis at O, when AB is horizontal

$\boxed{\frac{5}{4}mg}$


MOMENT OF INERTIA OF THE ROD ABOUT O IS

$\frac{1}{3}ma^2 + m\left(\frac{1}{3}a\right)^2 = \frac{4}{9}ma^2$

EQUATION OF MOTION WHEN THE ROD IS HORIZONTAL

$I\ddot{\theta} = L$

$\left(\frac{4}{9}ma^2\right)\ddot{\theta} = -mg\left(\frac{1}{3}a\right)$

$4a\ddot{\theta} = -3g$

$a\ddot{\theta} = -\frac{3}{4}g$

BY ENERGIES, TAKING THE VERTICAL POSITION AS THE ZERO POTENTIAL LEVEL

$\frac{1}{2}I\omega^2 + 0 = \frac{1}{2}I\Omega^2 + mg\left(\frac{1}{3}a\right)$

$\frac{1}{2}\left(\frac{4}{9}ma^2\right)\left(\frac{9g}{2a}\right) = \frac{1}{2}\left(\frac{4}{9}ma^2\right)\dot{\theta}^2 + \frac{1}{3}mga$

$mga = \frac{2}{9}ma^2\dot{\theta}^2 + \frac{1}{3}mga$

$\frac{2}{3}g = \frac{2}{9}a\dot{\theta}^2$

$a\dot{\theta}^2 = 3g$

NEXT CONSIDER THE FORCES AT THE "PIVOT"

IN POLARS

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

HERE $r = \frac{1}{3}a$

$\dot{r} = \ddot{r} = 0$

$\underline{a} = \left(-\frac{1}{3}\dot{\theta}^2\right)\hat{\underline{r}} + \frac{1}{3}a\ddot{\theta}\hat{\underline{\theta}}$

HENCE THE EQUATION OF MOTION YIELDS

$\Rightarrow T - mg = m\left(\frac{1}{3}a\ddot{\theta}\right)$

$\Rightarrow T - mg = m \times \frac{1}{3}\left(-\frac{3}{4}g\right)$

$\Rightarrow T - mg = -\frac{1}{4}mg$

$\Rightarrow T = \frac{3}{4}mg$

$\Rightarrow -R = m\left(-\frac{1}{3}a\dot{\theta}^2\right)$

$\Rightarrow R = \frac{1}{3}ma\dot{\theta}^2$

$\Rightarrow R = \frac{1}{3}m \times (3g)$

$\Rightarrow R = mg$

MAGNITUDE OF THE REACTION FORCE AT THE PIVOT

$|F| = \sqrt{T^2 + R^2}$

$= \sqrt{\frac{9}{16}m^2g^2 + m^2g^2}$

$= \sqrt{\frac{25}{16}m^2g^2}$

$= \frac{5}{4}mg$

Question 35 (***+)

A uniform rod of mass 5 kg and length 3 m is free to rotate in a vertical plane about a fixed horizontal axis through one of the two ends of the rod.

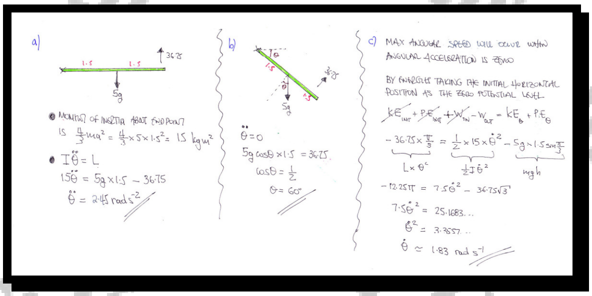
The rod is released from rest in a horizontal position. A constant frictional couple of magnitude 36.75 Nm opposes the motion.

a) Find the initial angular acceleration of the rod.

b) Determine the angle that the rod makes with the horizontal when its angular acceleration is zero.

c) Calculate the greatest angular speed of the rod.

$\boxed{\ddot{\theta} = 2.45 \text{ rad s}^{-2}}$, $\boxed{\theta = 60°}$, $\boxed{\dot{\theta} \approx 1.83 \text{ rad s}^{-1}}$



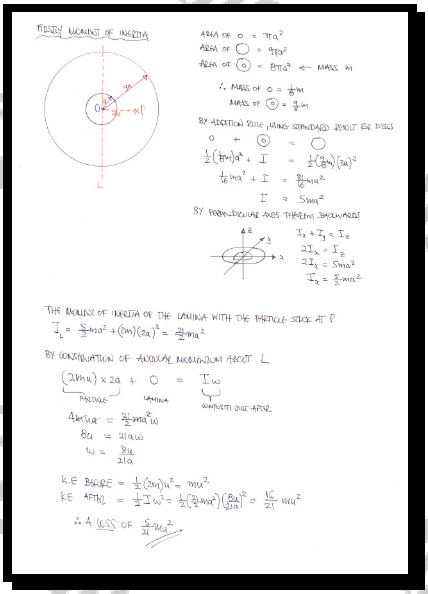
Question 36 (***+)

A plane shape S of mass m is formed by removing a circular disc with centre O and radius a from a uniform circular disc with centre O and radius $3a$.

S is free to rotate about a fixed smooth horizontal axis L, which passes through O and lies in the plane of S. Initially S is at rest in a horizontal plane when a particle of mass $2m$ falls vertically and strikes S at the point P, where $OP = 2a$ and OP is perpendicular to L. Immediately before the particle strikes P the speed of the particle is u. The particle adheres to S at P.

Find, in terms of m and u, the loss in kinetic energy due to the impact.

$$\boxed{\frac{5}{21}mu^2}$$



FIRSTLY MOMENT OF INERTIA

AREA OF ○ $= \pi a^2$
AREA OF ◯ $= 9\pi a^2$
AREA OF ◎ $= 8\pi a^2 \leftarrow$ MASS m

$\therefore$ MASS OF ○ $= \frac{1}{8}m$
MASS OF ◎ $= \frac{9}{8}m$

BY ADDITION RULE, USING STANDARD RESULT FOR DISCS

○ + ◎ = ◯

$\frac{1}{2}\left(\frac{1}{8}m\right)a^2 + I = \frac{1}{2}\left(\frac{9}{8}m\right)(3a)^2$

$\frac{1}{16}ma^2 + I = \frac{81}{16}ma^2$

$I = 5ma^2$

BY PERPENDICULAR AXES THEOREM BACKWARDS

$I_x + I_y = I_z$
$2I_x = I_z$
$2I_x = 5ma^2$
$I_x = \frac{5}{2}ma^2$

THE MOMENT OF INERTIA OF THE LAMINA WITH THE PARTICLE STUCK AT P

$I_L = \frac{5}{2}ma^2 + (2m)(2a)^2 = \frac{21}{2}ma^2$

BY CONSERVATION OF ANGULAR MOMENTUM ABOUT L

$\underbrace{(2mu)\times 2a}_{\text{PARTICLE}} + \underbrace{0}_{\text{LAMINA}} = \underbrace{I\omega}_{\text{COMPOSITE JUST AFTER}}$

$4mua = \frac{21}{2}ma^2\omega$

$8u = 21a\omega$

$\omega = \frac{8u}{21a}$

K.E BEFORE $= \frac{1}{2}(2m)u^2 = mu^2$
KE AFTER $= \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{21}{2}ma^2\right)\left(\frac{8u}{21a}\right)^2 = \frac{16}{21}mu^2$

$\therefore$ A LOSS OF $\frac{5}{21}mu^2$

Question 37 (***+)

The points A, B, C and D lie on the circumference of a circular hoop of mass m and radius a, so that AC and BD are two perpendicular diameters of the hoop.

Two particles, each of mass M, are attached to A and B.

The loaded hoop is free to rotate in a vertical plane, about a fixed smooth horizontal axis through D.

The system is released from rest, with AC in a vertical position, A uppermost.

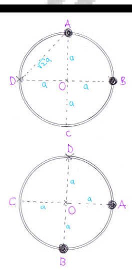
When AC is in a horizontal position the angular velocity of the system is ω.

Show that

$$\omega^2 = \frac{(m+4M)g}{(m+3M)a},$$

and hence deduce ω if the mass of the hoop is insignificant compared to that of the two particles.

$$\boxed{|\omega| = \sqrt{\frac{4g}{3a}}}$$



MOMENT OF INERTIA ABOUT D

$\underbrace{ma^2 + ma^2}_{\text{hoop}} + \underbrace{M(\sqrt{2}a)^2}_{A} + \underbrace{M(2a)^2}_{B} = 2ma^2 + 2Ma^2 + 4Ma^2 = 6Ma^2 + 2ma^2$

BY ENERGIES — WHEN DB IS VERTICAL THE SYSTEM WOULD HAVE LOST POTENTIAL ENERGY EQUAL TO

$g[\underset{\text{hoop}}{ma} + \underset{A}{M(2a)} + \underset{B}{M(2a)}] = g(ma + 4Ma) = (m+4M)ag$

HENCE KINETIC ENERGY GAIN = POTENTIAL ENERGY LOSS

$\frac{1}{2}I\omega^2 = (m+4M)ag$

$\frac{1}{2}[6Ma^2 + 2ma^2]\omega^2 = (m+4M)ag$

$(3M+m)a\omega^2 = (m+4M)g$

$\omega^2 = \frac{m+4M}{3M+m}\frac{g}{a}$

AS REQUIRED

IF $m \ll M$

$\omega^2 = \frac{4M}{3M}\frac{g}{a} = \frac{4g}{3a}$

$|\omega| = \sqrt{\frac{4g}{3a}}$

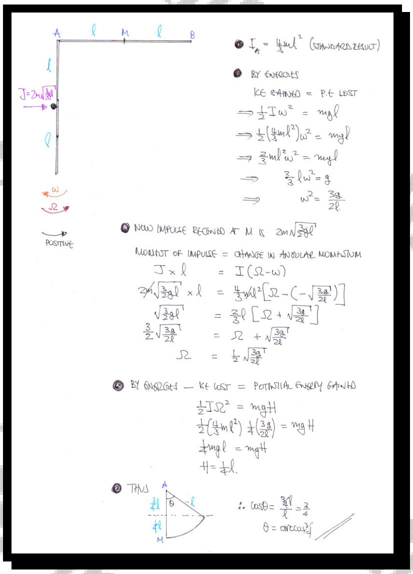
Question 38 (***+)

A uniform rod AB, of mass m and length $2l$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which is perpendicular to the rod and passes through A. The rod is released from rest in a horizontal position and when the rod first becomes vertical it hits a smooth peg at a distance l vertically below A.

The peg exerts an impulse J on the rod and the rod next comes to instantaneous rest at $\arccos x$ to the downward vertical through A.

Determine the value of x given that $J = 2m\sqrt{\frac{3}{2}gl}$.

$$\boxed{k = \frac{3}{4}}$$



- $I_A = \frac{4}{3}ml^2$ (STANDARD RESULT)
- BY ENERGIES
  KE GAINED = P.E LOST
  $\Rightarrow \frac{1}{2}I\omega^2 = mgl$
  $\Rightarrow \frac{1}{2}\left(\frac{4}{3}ml^2\right)\omega^2 = mgl$
  $\Rightarrow \frac{2}{3}ml^2\omega^2 = mgl$
  $\Rightarrow \frac{2}{3}l\omega^2 = g$
  $\Rightarrow \omega^2 = \frac{3g}{2l}$
- NOW IMPULSE RECEIVED AT M IS $2m\sqrt{\frac{3}{2}gl}$
  MOMENT OF IMPULSE = CHANGE IN ANGULAR MOMENTUM
  $J \times l = I(\Omega - \omega)$
  $2\cancel{m}\sqrt{\frac{3}{2}gl} \times l = \frac{4}{3}\cancel{m}l^2\left[\Omega - \left(-\sqrt{\frac{3g}{2l}}\right)\right]$
  $\sqrt{\frac{3}{2}gl} = \frac{2}{3}l\left[\Omega + \sqrt{\frac{3g}{2l}}\right]$
  $\frac{3}{2}\sqrt{\frac{3g}{2l}} = \Omega + \sqrt{\frac{3g}{2l}}$
  $\Omega = \frac{1}{2}\sqrt{\frac{3g}{2l}}$
- BY ENERGIES — KE LOST = POTENTIAL ENERGY GAINED
  $\frac{1}{2}I\Omega^2 = mgH$
  $\frac{1}{2}\left(\frac{4}{3}ml^2\right)\frac{1}{4}\left(\frac{3g}{2l}\right) = mgH$
  $\frac{1}{4}mgl = mgH$
  $H = \frac{1}{4}l$.
- THUS

$\therefore \cos\theta = \frac{\frac{3}{4}l}{l} = \frac{3}{4}$
$\theta = \arccos\frac{3}{4}$

Question 39 (***+)

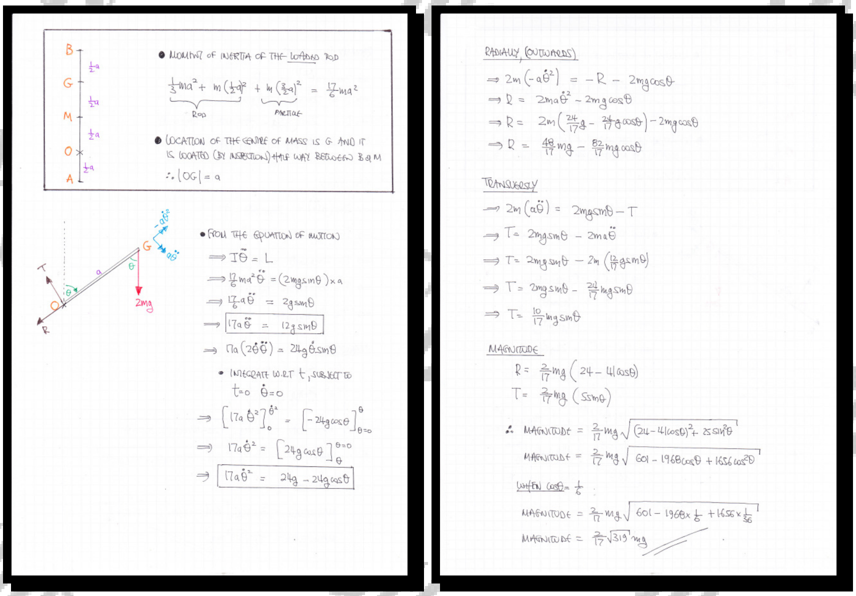
A particle of mass m is attached to the point B of a uniform rod AB, of mass m and length $2a$.

The loaded rod is free to rotate about a smooth, horizontal axis through the point O on the rod, where $|OA| = \frac{1}{2}a$.

The rod is held in a vertical position with B above O and is slightly disturbed. When the rod has turned by an angle θ from the upward vertical the magnitude of the force exerted by the rod on the axis is F.

Determine an expression for F, in terms of m, g and θ, and hence determine in terms of m and g, an expression for F when $\cos\theta = \frac{1}{6}$

$$\boxed{F = \frac{2}{17}mg\sqrt{601 - 1968\cos\theta + 1656\cos^2\theta}}, \quad \boxed{F = \frac{2}{17}mg\sqrt{319}}$$



Question 40 (***+)

A bucket of mass $3m$ is attached to one end of rope and moves in a vertical line.

The rope passes vertically up from the bucket and is wrapped several times around a cylindrical drum of mass $2m$ and radius a.

The drum is free to rotate about its axis of symmetry which remains in a fixed horizontal position.

The bucket is released from rest, with the rope taut, and begins to move vertically downwards.

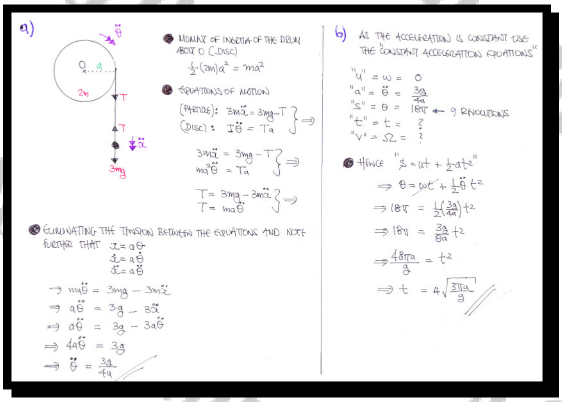
The bucket is modelled as a particle, the drum as a uniform cylinder rotating about its fixed smooth axis, the rope as a light inextensible string.

Ignoring that air resistance show that

a) … the angular acceleration of the drum is $\dfrac{3g}{4a}$.

b) … the time it takes the drum to complete 9 full revolutions is $4\sqrt{\dfrac{3\pi a}{g}}$.

$\boxed{\text{proof}}$



a)

MOMENT OF INERTIA OF THE DRUM ABOUT O (DISC)

$\frac{1}{2}(2m)a^2 = ma^2$

EQUATIONS OF MOTION

(PARTICLE): $3m\ddot{x} = 3mg - T$
(DISC): $I\ddot{\theta} = Ta$

$3m\ddot{x} = 3mg - T$
$ma^2\ddot{\theta} = Ta$

$T = 3mg - 3m\ddot{x}$
$T = ma\ddot{\theta}$

ELIMINATING THE TENSION BETWEEN THE EQUATIONS AND NOTE FURTHER THAT $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

$\Rightarrow ma\ddot{\theta} = 3mg - 3m\ddot{x}$
$\Rightarrow a\ddot{\theta} = 3g - 3\ddot{x}$
$\Rightarrow a\ddot{\theta} = 3g - 3a\ddot{\theta}$
$\Rightarrow 4a\ddot{\theta} = 3g$
$\Rightarrow \ddot{\theta} = \frac{3g}{4a}$

b) AS THE ACCELERATION IS CONSTANT USE THE "CONSTANT ACCELERATION EQUATIONS"

"u" $= \omega = 0$
"a" $= \ddot{\theta} = \frac{3g}{4a}$
"s" $= \theta = 18\pi$ ← 9 REVOLUTIONS
"t" $= t = ?$
"v" $= \Omega = ?$

HENCE "$s = ut + \frac{1}{2}at^2$"

$\Rightarrow \theta = \cancel{\omega t} + \frac{1}{2}\ddot{\theta}t^2$
$\Rightarrow 18\pi = \frac{1}{2}\left(\frac{3g}{4a}\right)t^2$
$\Rightarrow 18\pi = \frac{3g}{8a}t^2$
$\Rightarrow \frac{48\pi a}{g} = t^2$
$\Rightarrow t = 4\sqrt{\frac{3\pi a}{g}}$

Question 41 (***+)

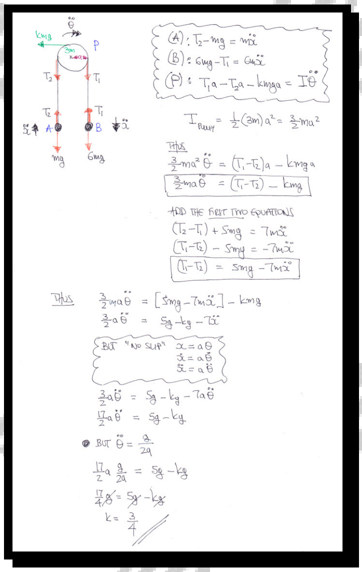
A light inextensible string has a particle of mass m attached to one end and a particle of mass $6m$ attached to the other end. The string passes over a rough pulley, which is modelled as a uniform disc of mass $3m$ and radius a.

The pulley rotates in a vertical plane through a fixed smooth horizontal axis which passes through the centre of the pulley and is perpendicular to the plane of the pulley.

The system is released from rest with the string taut and the parts of the string not in contact with the pulley vertical. The string does not slip on the pulley during the consequent motion. The pulley is experiencing a constant frictional couple of magnitude $kmga$, where k is a positive constant.

Given that the angular acceleration of the pulley is $\frac{g}{2a}$, determine the value of k.

$$\boxed{k = \frac{3}{4}}$$



(A): $T_2 - mg = m\ddot{x}$
(B): $6mg - T_1 = 6m\ddot{x}$
(P): $T_1 a - T_2 a - kmga = I\ddot{\theta}$

$I_{\text{PULLEY}} = \frac{1}{2}(3m)a^2 = \frac{3}{2}ma^2$

THUS

$\frac{3}{2}ma^2\ddot{\theta} = (T_1 - T_2)a - kmga$

$\boxed{\frac{3}{2}ma\ddot{\theta} = (T_1 - T_2) - kmg}$

ADD THE FIRST TWO EQUATIONS

$(T_2 - T_1) + 5mg = 7m\ddot{x}$

$(T_1 - T_2) - 5mg = -7m\ddot{x}$

$\boxed{(T_1 - T_2) = 5mg - 7m\ddot{x}}$

THUS $\frac{3}{2}ma\ddot{\theta} = [5mg - 7m\ddot{x}] - kmg$

$\frac{3}{2}a\ddot{\theta} = 5g - kg - 7\ddot{x}$

BUT "NO SLIP" $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

$\frac{3}{2}a\ddot{\theta} = 5g - kg - 7a\ddot{\theta}$

$\frac{17}{2}a\ddot{\theta} = 5g - kg$

BUT $\ddot{\theta} = \frac{g}{2a}$

$\frac{17}{2}a\frac{g}{2a} = 5g - kg$

$\frac{17}{4}\cancel{g} = 5\cancel{g} - k\cancel{g}$

$k = \frac{3}{4}$

Question 42 (***+)

A particle A of mass m is connected to small box B of mass $2m$ by a light inextensible string. The string passes over a pulley P, which is located at the end of a smooth horizontal table. The box is held on the table with the particle hanging vertically at the end of the table, as shown in the figure above. The pulley is modelled as a disc of mass $4m$ and radius a, rotating about a smooth horizontal axis through its centre. The system is released from rest with the string taut.

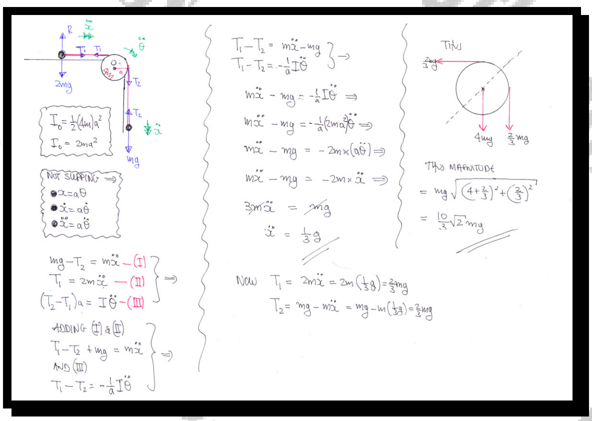
In the subsequent motion …

- … the string does not slip on the pulley.
- … the section of the string PB not in contact with the pulley remains horizontal at all times and the section of the string AP not in contact with the pulley remains vertical at all times.

Find the acceleration of the system and hence show that while the system is in motion the force exerted on the pulley has magnitude

$$\frac{10}{3}\sqrt{2}mg$$

$$\boxed{\ddot{x} = \frac{1}{3}g}$$



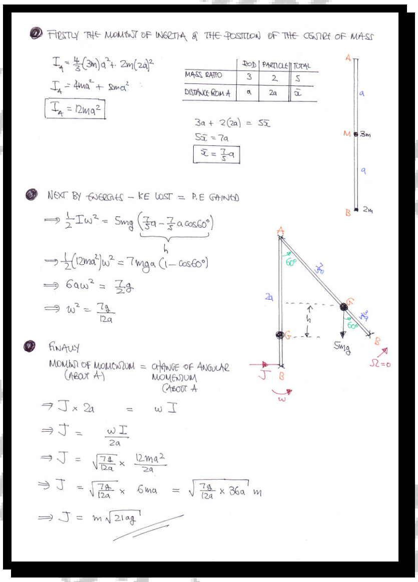
Question 43 (***+)

A pendulum is modelled as a uniform rod AB, of mass $3m$ and length $2a$, which has a particle of mass $2m$ attached at B. The pendulum is free to rotate in a vertical plane about a fixed smooth horizontal axis L which passes through A. The vertical plane is perpendicular to L.

The pendulum is hanging at rest in a vertical position, with B below A, when it receives a horizontal impulse of magnitude J. The impulse acts at B in a vertical plane which is perpendicular to L.

Given that the pendulum turns through an angle of $60°$ before first coming to instantaneous rest show that $J = m\sqrt{21ag}$.

proof



FIRSTLY THE MOMENT OF INERTIA & THE POSITION OF THE CENTRE OF MASS

$I_A = \frac{4}{3}(3m)a^2 + 2m(2a)^2$

$I_A = 4ma^2 + 8ma^2$

$\boxed{I_A = 12ma^2}$

	ROD	PARTICLE	TOTAL
MASS RATIO	3	2	5
DISTANCE FROM A	a	2a	$\bar{x}$

$3a + 2(2a) = 5\bar{x}$

$5\bar{x} = 7a$

$\boxed{\bar{x} = \frac{7}{5}a}$

NEXT BY ENERGIES – KE LOST = P.E GAINED

$\Longrightarrow \frac{1}{2}I\omega^2 = 5mg\underbrace{\left(\frac{7}{5}a - \frac{7}{5}a\cos60°\right)}_{h}$

$\Longrightarrow \frac{1}{2}(12ma^2)\omega^2 = 7mga(1 - \cos60°)$

$\Longrightarrow 6a\omega^2 = \frac{7}{2}g$

$\Longrightarrow \omega^2 = \frac{7g}{12a}$

FINALLY

MOMENT OF MOMENTUM (ABOUT A) = CHANGE OF ANGULAR MOMENTUM (ABOUT A)

$\Rightarrow J \times 2a = \omega I$

$\Rightarrow J = \frac{\omega I}{2a}$

$\Rightarrow J = \sqrt{\frac{7g}{12a}} \times \frac{12ma^2}{2a}$

$\Rightarrow J = \sqrt{\frac{7g}{12a}} \times 6ma = \sqrt{\frac{7g}{12a} \times 36a^2}\, m$

$\Rightarrow J = m\sqrt{21ag}$

Question 44 (***+)

A pulley is modelled as a uniform circular disc of mass $16m$ and radius a. The pulley is free to rotate about a fixed horizontal axis through its centre and perpendicular to its plane. A light inextensible string passes over the pulley and two particles A and B, of respective mass $2m$ and $5m$ are attached to the two ends of the string.

The particles are released from rest with the string taut.

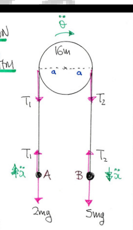
A constant couple of magnitude of mga resists the rotation of the pulley about its axis.

In the consequent motion there is no slipping between the string and the pulley.

Determine, in terms of mg, the tension in each of the two sections of the string to which the two particles are attached.

$\boxed{T_A = \frac{34}{15}mg}$, $\boxed{T_B = \frac{13}{3}mg}$

- FORMING THE EQUATION OF MOTION FOR EACH COMPONENT OF THE SYSTEM



LOOKING AT A

$T_1 - 2mg = 2m\ddot{x}$

LOOKING AT B

$5mg - T_2 = 5m\ddot{x}$

LOOKING AT PULLEY

$T_2a - T_1a - mga = I\ddot{\theta}$

$(T_2 - T_1)a - mga = 8ma^2\ddot{\theta}$

$T_2 - T_1 - mg = 8ma\ddot{\theta}$

- MOMENT OF INERTIA OF PULLEY IS $\frac{1}{2}(16m)a^2$ $8ma^2$
- FRICTIONAL COUPLE ON PULLEY mga
- NO SLIP $\Rightarrow$ $x = a\theta$, $\dot{x} = a\dot{\theta}$, $\ddot{x} = a\ddot{\theta}$

- "SUBTRACTING" THE FIRST TWO EQUATIONS GIVES

$\Rightarrow T_1 - T_2 + 3mg = 7m\ddot{x}$

$\Rightarrow 3mg - 7m\ddot{x} = T_2 - T_1$

- SUBSTITUTING INTO THE THIRD EQUATION

$\Rightarrow (T_2 - T_1) - mg = 8ma\ddot{\theta}$

$\Rightarrow (3mg - 7m\ddot{x}) - mg = 8ma\ddot{\theta}$

$\Rightarrow 3g - 7\ddot{x} - g = 8a\ddot{\theta}$

$\Rightarrow 2g - 7\ddot{x} = 8\ddot{x}$

$\Rightarrow 15\ddot{x} = 2g$

$\Rightarrow \ddot{x} = \frac{2g}{15}$

- HENCE WE OBTAIN

$T_1 - 2mg = 2m\ddot{x}$ & $5mg - T_2 = 5m\ddot{x}$

$T_1 - 2mg = 2m\left(\frac{2g}{15}\right)$ $\quad$ $5mg - T_2 = 5m\left(\frac{2}{15}g\right)$

$T_1 - 2mg = \frac{4}{15}mg$ $\quad$ $5mg - T_2 = \frac{2}{3}mg$

$T_1 = \frac{34}{15}mg$ $\quad$ $T_2 = \frac{13}{3}mg$

Question 45 (***+)

A uniform rod AB, of mass m, is free to rotate about a smooth fixed horizontal axis L, which passes through A.

The rotation of the rod takes place in a vertical plane.

The rod is held so that AB makes an angle of $60°$ with the upward vertical and released from rest.

a) Given that the moment of inertia of the rod about L is $12ma^2$, show that in the subsequent motion

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{8a}(1-2\cos\theta),$$

where θ is the angle that AB makes with the upward vertical.

b) Determine, in terms of m, g and θ, the magnitude and direction of the radial force exerted on L by the rod.

$\boxed{}$, $\boxed{\frac{1}{2}mg(26\cos\theta - 9), \text{ radially outwards}}$

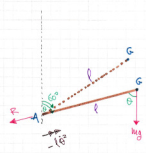
a) STARTING WITH A DIAGRAM, WHERE G DENOTES THE MIDPOINT OF THE ROD

$I_A = \frac{4}{3}ml^2$

$12\cancel{m}a^2 = \frac{4}{3}\cancel{m}l^2$

$l^2 = 9a^2$

$l = 3a$



BY ENERGIES TAKING THE LEVEL OF "A", AS THE ZERO POTENTIAL LEVEL

$\Rightarrow P.E_{60°} + \cancel{K.E_{60°}} = KE_\theta + P.E_\theta$

$\Rightarrow mgl\cos60° = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta$

$\Rightarrow mg(3a)\times\frac{1}{2} = \frac{1}{2}(12ma^2)\dot{\theta}^2 + mg(3a)\cos\theta$

$\Rightarrow \frac{3}{2}\cancel{mga} = 4\cancel{m}a^{\cancel{2}}\dot{\theta}^2 + 3\cancel{mga}\cos\theta$

$\Rightarrow 3g = 8a\dot{\theta}^2 + 6g\cos\theta$

$\Rightarrow 8a\dot{\theta}^2 = 3g - 6g\cos\theta$

$\Rightarrow 8a\dot{\theta}^2 = 3g(1-2\cos\theta)$

$\Rightarrow \dot{\theta}^2 = \frac{3g}{8a}(1-2\cos\theta)$ //

b) LOOKING IN THE RADIAL DIRECTION

$\Rightarrow$ "$m\ddot{r}$ = RESULTANT FORCE"

$\Rightarrow m(-3a\dot{\theta}^2) = -R - mg\cos\theta$

$\Rightarrow R = 3ma\dot{\theta}^2 - mg\cos\theta$

$\Rightarrow R = 3ma\left(\frac{3g}{8a}(1-2\cos\theta)\right) - mg\cos\theta$

$\Rightarrow R = \frac{9}{8}mg(1-2\cos\theta) - mg\cos\theta$

$\Rightarrow R = \frac{1}{8}mg(9 - 18\cos\theta - 8\cos\theta)$

$\Rightarrow R = \frac{1}{8}mg(9 - 26\cos\theta)$

HENCE THE REQUIRED COMPONENT IS

$\frac{1}{8}mg(9-26\cos\theta)$, RADIALLY INWARDS

OR

$\frac{1}{8}mg(26\cos\theta - 9)$, RADIALLY OUTWARDS //

Question 46 (****)

The point O lies on a uniform rod AB so that the ratio $|AO|:|OB|$ is $3:5$.

The rod is held in a horizontal position on a rough horizontal table so that AB is perpendicular to the straight edge of the table.

The part of the rod AO is in contact with the table and the part OB overhangs the edge of the table.

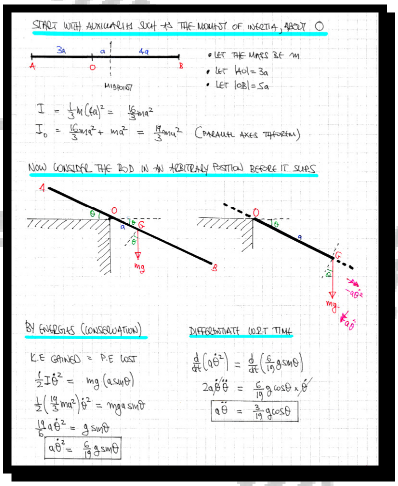
The rod is released from rest and begins to rotate about O.

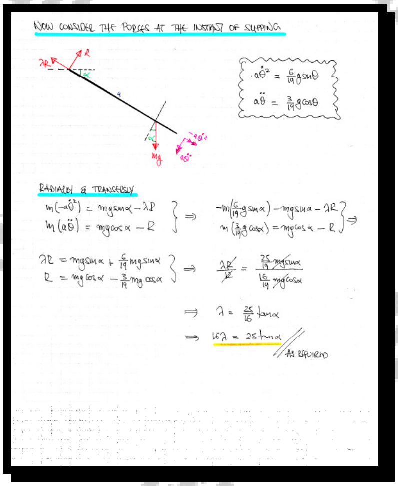
When the rod has turned by an angle α it begins to slip.

If the coefficient of friction between the rod and the table is λ, show that

$$16\lambda = 25\tan\alpha \; .$$

$\boxed{}$, $\boxed{\text{proof}}$





Question 47 (****)

Four identical rods, each of mass m and length $2a$ are joined together to form a square rigid framework $ABCD$.

A fifth rod AC, of mass $3m$, is added to the framework for extra support.

The 5 rod framework is free to rotate about a smooth fixed horizontal axis L, which passes through A, so that the rotation of the framework takes place in a vertical plane.

The framework is held so that D is vertically above A and released from rest.

On the subsequent rotation, when B is vertically below A, a stationary particle of mass M adheres to B.

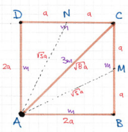
Given that the angular speed of the framework, after the particle has adhered to it, is

$$\frac{2}{9}\sqrt{\frac{21g}{a}},$$

determine M in terms of m.

$\boxed{M = \frac{2}{3}m}$

START BY A DIAGRAM



- LENGTH OF AC IS $2\sqrt{2}a$
- LENGTH OF AN OR AM IS $\sqrt{5}a$

MOMENT OF INERTIA OF THE ROD AB OR ROD AD ABOUT A

$I = \frac{1}{3}ma^2 + ma^2 = \frac{4}{3}ma^2$

MOMENT OF INERTIA OF THE ROD BC (OR DC) ABOUT A

$I = \frac{1}{3}ma^2 + m(\sqrt{5}a)^2 = \frac{1}{3}ma^2 + 5ma^2 = \frac{16}{3}ma^2$

MOMENT OF INERTIA OF THE ROD AC, ABOUT A

$I = \frac{1}{3}(3m)(\sqrt{2}a)^2 + 3m(\sqrt{2}a)^2 = 2ma^2 + 6ma^2 = 8ma^2$

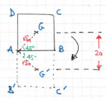
ADDING TOGETHER THE MOMENT OF INERTIA OF ALL THE RODS GIVES

$I_{TOT} = \underset{(AB)}{\frac{4}{3}ma^2} + \underset{(AD)}{\frac{4}{3}ma^2} + \underset{(BC)}{\frac{16}{3}ma^2} + \underset{(DC)}{\frac{16}{3}ma^2} + \underset{(AC)}{8ma^2}$

$I_{TOTAL} = \frac{64}{3}ma^2$

WHEN "B" IS VERTICALLY BELOW "A", THE CENTRE OF MASS "G" OF THE SYSTEM WOULD HAVE "DROPPED" FROM a ABOVE THE LEVEL OF "A" TO $\sqrt{2}a\cos 45 = a$ BELOW THE LEVEL OF A

BY ENERGIES



$\Rightarrow \frac{1}{2}I\omega^2 = 7mg(2a)$

$\Rightarrow \frac{1}{2}\left(\frac{64}{3}ma^2\right)\omega^2 = 14mga$

$\Rightarrow \frac{32}{3}\cancel{m}a^{\cancel{2}}\omega^2 = 14\cancel{m}g\cancel{a}$

$\Rightarrow \frac{32}{3}a\omega^2 = 14g$

$\Rightarrow \frac{16}{3}a\omega^2 = 7g.$

$\Rightarrow \omega^2 = \frac{21g}{16a}$

$\Rightarrow \omega = \frac{1}{4}\sqrt{\frac{21g}{a}}$

NOW BY CONSERVATION OF ANGULAR MOMENTUM ABOUT A

- $I_{NEW\ TOTAL} = \frac{64}{3}ma^2 + M(2a)^2$

  $= \frac{64}{3}ma^2 + 4Ma^2$

- $I_{TOT} \times \omega = I_{NEW\ TOT} \times \Omega$

$\left(\frac{64}{3}ma^2\right) \times \frac{1}{4}\sqrt{\frac{21g}{a}} = \left(\frac{64}{3}ma^2 + 4Ma^2\right)\left(\frac{2}{9}\sqrt{\frac{21g}{a}}\right)$

$\Rightarrow \frac{16}{3}\cancel{\sqrt{\frac{21g}{a}}}m\cancel{a^2} = \frac{2}{9}\cancel{\sqrt{\frac{21g}{a}}}\left(\frac{64}{3}m + 4M\right)\cancel{a^2}$

$\Rightarrow \frac{16}{3}m = \frac{2}{9}\left(\frac{64}{3}m + 4M\right)$

$\Rightarrow \frac{16}{3}m = \frac{128}{27}m + \frac{8}{9}M$ $\quad \div 8$

$\Rightarrow \frac{2}{3}m = \frac{16}{27}m + \frac{1}{9}M$ $\quad \times 27$

$\Rightarrow 18m = 16m + 3M$

$\Rightarrow 2m = 3M$

$\Rightarrow M = \frac{2}{3}m$

Question 48 (****)

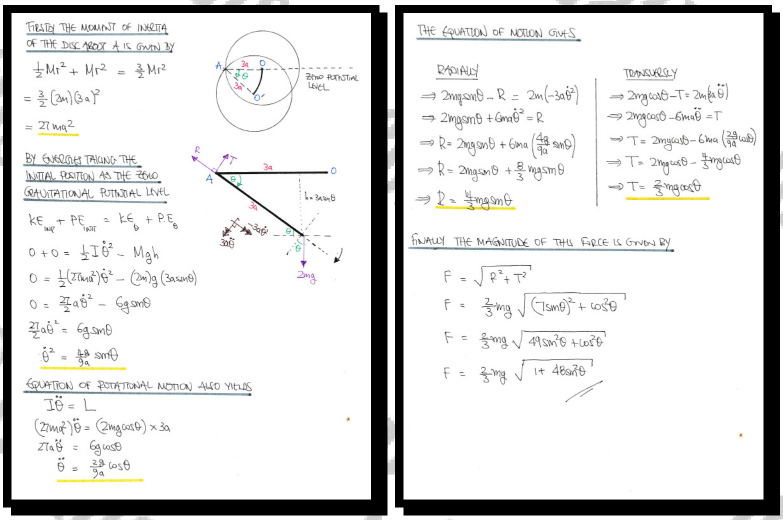
A uniform circular disc, of radius $3a$ and mass $2m$, is free to rotate about a smooth fixed horizontal axis L, which is perpendicular to the plane of the disc and is passing through the point A, which lies at the circumference of the disc. The disc is held with its centre O at the same horizontal level as A, and released from rest.

Show that the horizontal component of the force exerted on L has magnitude

$$\frac{2}{3}mg\sqrt{1+48\sin^2\theta},$$

where θ is the angle that AO makes with the horizontal.

$\boxed{}$, $\boxed{\text{proof}}$



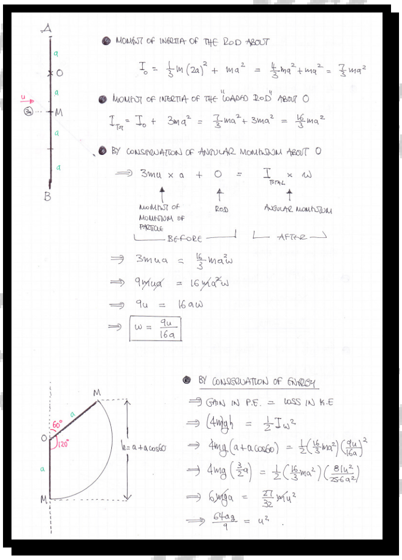
Question 49 (****)

A uniform rod AB, of mass m and length $4a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L passing through the point O, where $|AO| = a$. The vertical plane is perpendicular to L.

The rod is hanging at rest in a vertical position, with B below A, when it is struck at its midpoint by a particle P of mass $3m$, travelling horizontally with speed u. The path of P on impact with the rod is in a vertical plane which is perpendicular to L.

Given that P attaches itself at the midpoint of the rod on impact, determine, in terms of m and g, the magnitude of the force acting on the rod at L, when the rod first comes to instantaneous rest.

$$\boxed{\frac{1}{2}mg\sqrt{19}}$$



- MOMENT OF INERTIA OF THE ROD ABOUT

$$I_O = \frac{1}{3}m(2a)^2 + ma^2 = \frac{4}{3}ma^2 + ma^2 = \frac{7}{3}ma^2$$

- MOMENT OF INERTIA OF THE "LOADED ROD" ABOUT O

$$I_{TOT} = I_O + 3ma^2 = \frac{7}{3}ma^2 + 3ma^2 = \frac{16}{3}ma^2$$

- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O

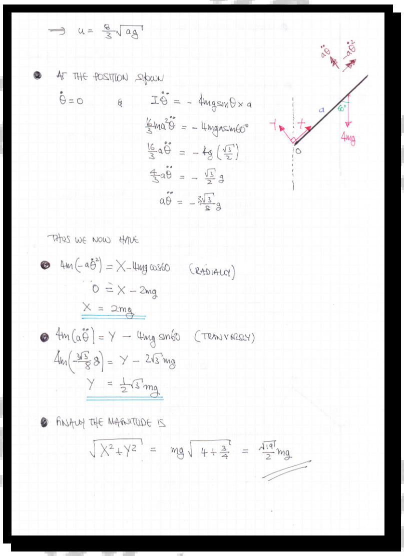
$$\Longrightarrow \underbrace{\underbrace{3mu \times a}_{\text{MOMENT OF MOMENTUM OF PARTICLE}} + \underbrace{0}_{\text{ROD}}}_{\text{BEFORE}} = \underbrace{\underbrace{I_{TOTAL} \times \omega}_{\text{ANGULAR MOMENTUM}}}_{\text{AFTER}}$$

$$\Longrightarrow 3mua = \frac{16}{3}ma^2\omega$$
$$\Longrightarrow 9\cancel{m}u\cancel{a} = 16\cancel{m}a^{\cancel{2}}\omega$$
$$\Longrightarrow 9u = 16a\omega$$
$$\Longrightarrow \boxed{\omega = \frac{9u}{16a}}$$

- BY CONSERVATION OF ENERGY

$\Longrightarrow$ GAIN IN P.E. = LOSS IN K.E

$$\Longrightarrow (4m)gh = \frac{1}{2}I\omega^2$$
$$\Longrightarrow 4mg(a + a\cos 60) = \frac{1}{2}\left(\frac{16}{3}ma^2\right)\left(\frac{9u}{16a}\right)^2$$
$$\Longrightarrow 4mg\left(\frac{3}{2}a\right) = \frac{1}{2}\left(\frac{16}{3}ma^2\right)\left(\frac{81u^2}{256a^2}\right)$$
$$\Longrightarrow 6\cancel{m}ga = \frac{27}{32}\cancel{m}u^2$$
$$\Longrightarrow \frac{64ag}{9} = u^2$$



$$\Longrightarrow u = \frac{8}{3}\sqrt{ag}$$

- AT THE POSITION SHOWN

$\dot{\theta} = 0$ & $I\ddot{\theta} = -4mg\sin\theta \times a$

$$\frac{16}{3}ma^2\ddot{\theta} = -4mga\sin 60°$$
$$\frac{16}{3}a\ddot{\theta} = -4g\left(\frac{\sqrt{3}}{2}\right)$$
$$\frac{4}{3}a\ddot{\theta} = -\frac{\sqrt{3}}{2}g$$
$$a\ddot{\theta} = -\frac{3\sqrt{3}}{8}g$$

THUS WE NOW HAVE

- $4m(-a\dot{\theta}^2) = X - 4mg\cos 60$ (RADIALLY)

$$0 = X - 2mg$$
$$X = 2mg$$

- $4m(a\ddot{\theta}) = Y - 4mg\sin 60$ (TRANVERSLY)

$$4m\left(-\frac{3\sqrt{3}}{8}g\right) = Y - 2\sqrt{3}mg$$
$$Y = \frac{1}{2}\sqrt{3}mg$$

- FINALLY THE MAGNITUDE IS

$$\sqrt{X^2 + Y^2} = mg\sqrt{4 + \frac{3}{4}} = \frac{\sqrt{19}}{2}mg$$

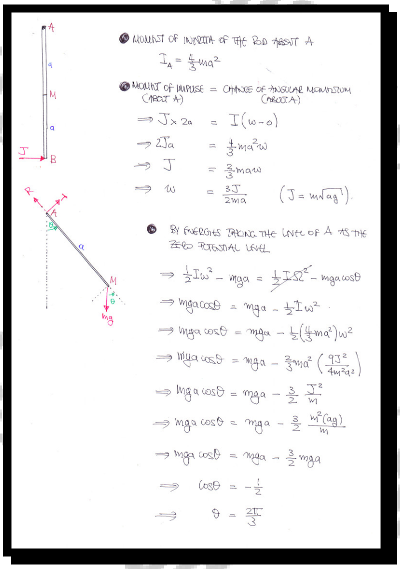
Question 50 (****)

A uniform rod AB, of mass m and length $2a$, is free to rotate in a vertical plane about a fixed smooth horizontal axis L passing through A. The vertical plane is perpendicular to L.

The rod is hanging at rest in a vertical position, with B below A, when it receives a horizontal impulse of magnitude $m\sqrt{ag}$. The impulse acts at B in a vertical plane which is perpendicular to L.

Determine, in terms of m and g, the magnitude of the force acting on the rod at L, when the rod first comes to instantaneous rest.

$$\boxed{\frac{1}{2}mg\sqrt{163}}$$



MOMENT OF INERTIA OF THE ROD ABOUT A

$I_A = \frac{4}{3}ma^2$

MOMENT OF IMPULSE = CHANGE OF ANGULAR MOMENTUM (ABOUT A)

$\Rightarrow J \times 2a = I(\omega - 0)$

$\Rightarrow 2Ja = \frac{4}{3}ma^2\omega$

$\Rightarrow J = \frac{2}{3}ma\omega$

$\Rightarrow \omega = \frac{3J}{2ma}$ $\quad (J = m\sqrt{ag})$

BY ENERGIES TAKING THE LEVEL OF A AS THE ZERO POTENTIAL LEVEL

$\Rightarrow \frac{1}{2}I\omega^2 - mga = \frac{1}{2}I\Omega^2 - mga\cos\theta$

$\Rightarrow mga\cos\theta = mga - \frac{1}{2}I\omega^2$

$\Rightarrow mga\cos\theta = mga - \frac{1}{2}\left(\frac{4}{3}ma^2\right)\omega^2$

$\Rightarrow mga\cos\theta = mga - \frac{2}{3}ma^2\left(\frac{9J^2}{4m^2a^2}\right)$

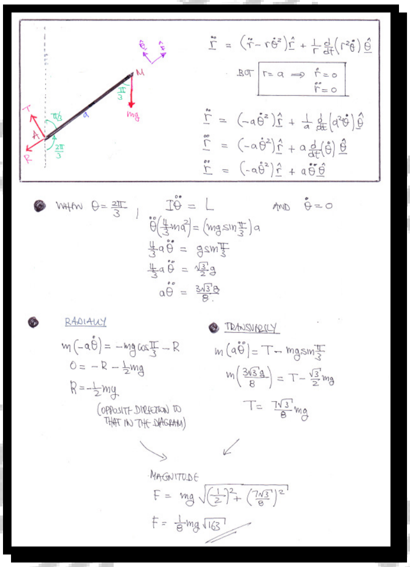
$\Rightarrow mga\cos\theta = mga - \frac{3}{2}\frac{J^2}{m}$

$\Rightarrow mga\cos\theta = mga - \frac{3}{2}\frac{m^2(ag)}{m}$

$\Rightarrow mga\cos\theta = mga - \frac{3}{2}mga$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$



$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

BUT $r = a \Rightarrow \dot{r} = 0$, $\ddot{r} = 0$

$\ddot{\underline{r}} = (-a\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{a}\frac{d}{dt}(a^2\dot{\theta})\hat{\underline{\theta}}$

$\ddot{\underline{r}} = (-a\dot{\theta}^2)\hat{\underline{r}} + a\frac{d}{dt}(\dot{\theta})\hat{\underline{\theta}}$

$\ddot{\underline{r}} = (-a\dot{\theta}^2)\hat{\underline{r}} + a\ddot{\theta}\hat{\underline{\theta}}$

WHEN $\theta = \frac{2\pi}{3}$, $I\ddot{\theta} = L$ AND $\dot{\theta} = 0$

$\ddot{\theta}\left(\frac{4}{3}ma^2\right) = \left(mg\sin\frac{\pi}{3}\right)a$

$\frac{4}{3}a\ddot{\theta} = g\sin\frac{\pi}{3}$

$\frac{4}{3}a\ddot{\theta} = \frac{\sqrt{3}}{2}g$

$a\ddot{\theta} = \frac{3\sqrt{3}g}{8}$

RADIALLY

$m(-a\dot{\theta}^2) = -mg\cos\frac{\pi}{3} - R$

$0 = -R - \frac{1}{2}mg$

$R = -\frac{1}{2}mg$

(OPPOSITE DIRECTION TO THAT IN THE DIAGRAM)

TRANSVERSELY

$m(a\ddot{\theta}) = T - mg\sin\frac{\pi}{3}$

$m\left(\frac{3\sqrt{3}g}{8}\right) = T - \frac{\sqrt{3}}{2}mg$

$T = \frac{7\sqrt{3}}{8}mg$

MAGNITUDE

$F = mg\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{7\sqrt{3}}{8}\right)^2}$

$F = \frac{1}{8}mg\sqrt{163}$

Question 51 (****)

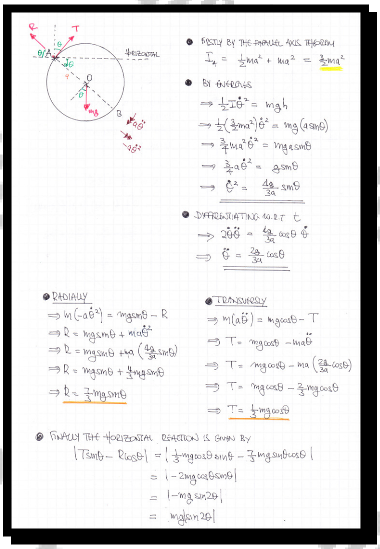
A uniform circular disc, of radius a and mass m, is free to rotate about a smooth fixed horizontal axis L, which is perpendicular to the plane of the disc and is passing through the point A, which lies at the circumference of the disc. The disc is held with its centre O at the same horizontal level as A, and released from rest.

Show that the horizontal component of the force exerted on L has magnitude

$$mg\left|\sin 2\theta\right|,$$

where θ is the angle that AO makes with the horizontal.

$\boxed{\text{proof}}$



- FIRSTLY BY THE PARALLEL AXIS THEOREM
  $I_A = \frac{1}{2}ma^2 + ma^2 = \frac{3}{2}ma^2$
- BY ENERGIES
  $\Rightarrow \frac{1}{2}I\dot{\theta}^2 = mgh$
  $\Rightarrow \frac{1}{2}\left(\frac{3}{2}ma^2\right)\dot{\theta}^2 = mg(a\sin\theta)$
  $\Rightarrow \frac{3}{4}ma^2\dot{\theta}^2 = mga\sin\theta$
  $\Rightarrow \frac{3}{4}a\dot{\theta}^2 = g\sin\theta$
  $\Rightarrow \dot{\theta}^2 = \frac{4g}{3a}\sin\theta$
- DIFFERENTIATING W.R.T t
  $\Rightarrow 2\dot{\theta}\ddot{\theta} = \frac{4g}{3a}\cos\theta\,\dot{\theta}$
  $\Rightarrow \ddot{\theta} = \frac{2g}{3a}\cos\theta$
- RADIALLY
  $\Rightarrow m(-a\dot{\theta}^2) = mg\sin\theta - R$
  $\Rightarrow R = mg\sin\theta + ma\dot{\theta}^2$
  $\Rightarrow R = mg\sin\theta + ma\left(\frac{4g}{3a}\sin\theta\right)$
  $\Rightarrow R = mg\sin\theta + \frac{4}{3}mg\sin\theta$
  $\Rightarrow R = \frac{7}{3}mg\sin\theta$
- TRANSVERSLY
  $\Rightarrow m(a\ddot{\theta}) = mg\cos\theta - T$
  $\Rightarrow T = mg\cos\theta - ma\ddot{\theta}$
  $\Rightarrow T = mg\cos\theta - ma\left(\frac{2g}{3a}\cos\theta\right)$
  $\Rightarrow T = mg\cos\theta - \frac{2}{3}mg\cos\theta$
  $\Rightarrow T = \frac{1}{3}mg\cos\theta$
- FINALLY THE HORIZONTAL REACTION IS GIVEN BY
  $|T\sin\theta - R\cos\theta| = \left|\frac{1}{3}mg\cos\theta\sin\theta - \frac{7}{3}mg\sin\theta\cos\theta\right|$
  $= |-2mg\cos\theta\sin\theta|$
  $= |-mg\sin 2\theta|$
  $= mg|\sin 2\theta|$

Question 52 (****)

A circular flywheel F, of radius 0.2 m, is free to rotate about an axis L, which is perpendicular to the plane of the flywheel and through its centre. The motion of F is smooth.

The flywheel receives a tangential impulse, in the plane of F, of 180 Ns.

a) Given that the moment of inertia of F about L is 6 kg m^2, determine the angular speed of F after it receives the impulse.

A **resistive** couple C Nm is then applied to F whose magnitude is given by

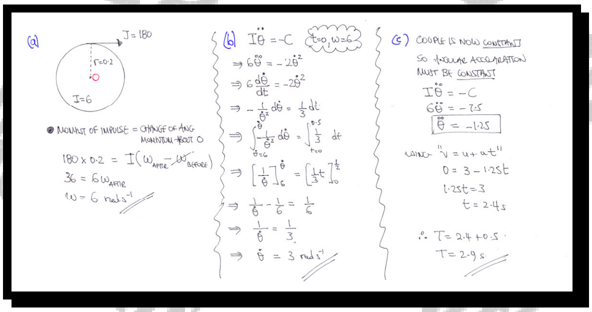
$$\begin{cases} 2\dot{\theta}^2 & 0 \le t \le 0.5 \\ 7.5 & 0.5 < t \le T \end{cases},$$

where $\dot{\theta}$ is the angular speed of F at time t s, bringing F to rest in time T s.

b) Form and solve a differential equation, in $\dot{\theta}$, to find the angular speed of F when $t = 0.5$ s.

c) Calculate the value of T.

$\boxed{\omega = 6 \text{ rad s}^{-1}}$, $\boxed{\dot{\theta}\big|_{t=0.5} = 3 \text{ rad s}^{-1}}$, $\boxed{T = 2.9 \text{ s}}$



Question 53 (****)

A uniform circular disc with centre O has mass m and radius a.

The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis through a point A on the disc, where $OA = \frac{1}{2}a$.

The disc is held at rest in a position with O vertically above A. The disc is then released and begins to rotate about O. The angle between OA and the upward vertical is denoted by θ.

a) Find, in terms of a, g and θ, …

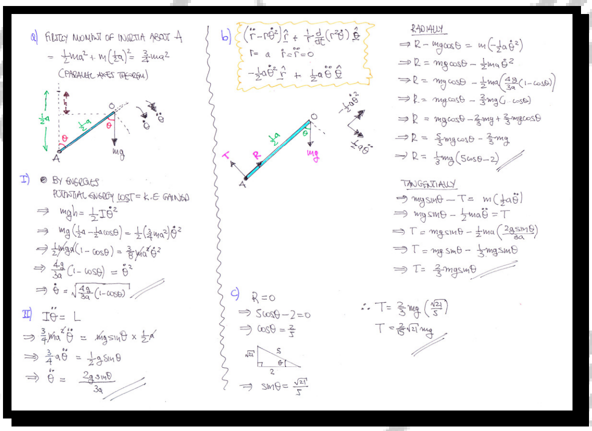
i. … the angular speed of the disc.

ii. … the angular acceleration of the disc.

b) Determine, in terms of m, g and θ, the radial and tangential component of the force acting at A.

c) Calculate, in terms of mg, the magnitude of the force acting at A, when the radial component of the force is zero.

$\boxed{\dot{\theta} = \sqrt{\frac{4g(1-\cos\theta)}{3a}}}$, $\boxed{\ddot{\theta} = \frac{2g\sin\theta}{3a}}$, $\boxed{R = \frac{1}{3}mg(5\cos\theta - 2)}$, $\boxed{T = \frac{2}{3}mg\sin\theta}$, $\boxed{F = \frac{2\sqrt{21}}{15}mg}$



Question 54 (****)

A uniform equilateral triangular lamina ABC has mass m and side length of $\sqrt{3}a$.

a) Show, by integration, that the moment of inertia of the lamina about an axis through one of its vertices and perpendicular to the plane of the lamina is

$$\frac{5}{4}ma^2.$$

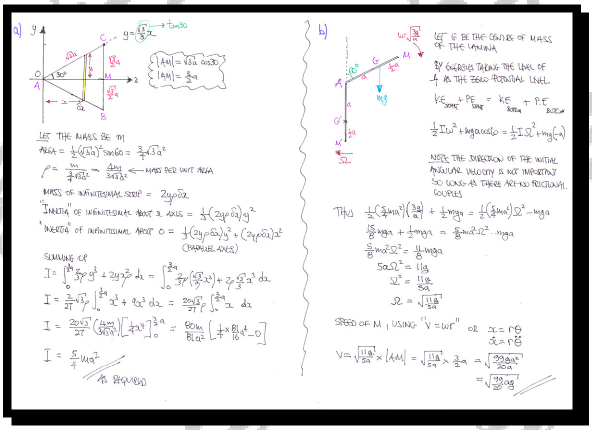
[*In this proof, you may assume standard results for the moment of inertia of uniform rods.*]

The lamina is free to rotate in a vertical plane about a fixed smooth horizontal axis L, which passes through A and is perpendicular to the lamina. The midpoint of BC is the point M.

The lamina is held with AM making an angle of $60°$ with the upward vertical through A and is projected with angular speed $\sqrt{\frac{3g}{a}}$.

b) Find, in terms of a and g, the speed of M when M is vertically below A.

$$\boxed{v = \sqrt{\frac{99}{20}ag}}$$



Question 55 (****)

A uniform circular disc with centre at O, has radius a and mass m. The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis. This axis is perpendicular to the plane of the disc and passes through a point P, at the circumference of the disc.

The disc is held at rest with PQ horizontal, where PQ is a diameter of the disc, and released from rest. At time t after release, the diameter PQ makes an angle θ below the horizontal, where θ is acute.

a) Find expressions, in terms of m, g and θ, for …

i. … the radial component of the force exerted on the disc by the axis.

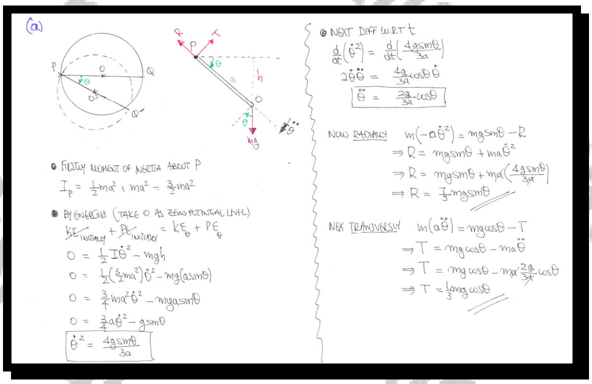
ii. … the transverse component of the force exerted on the disc by the axis.

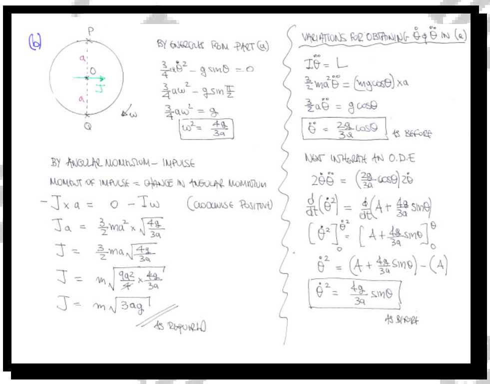
When PQ is vertical the disc is brought to instantaneous rest by a horizontal impulse J, acting through O.

b) Show clearly that

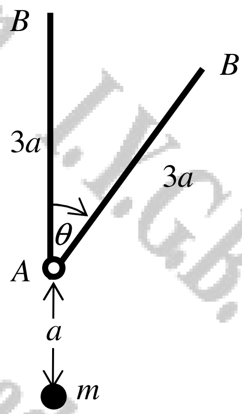
$$J = m\sqrt{3ag}\,.$$

$$\boxed{R_{\text{radial}} = \frac{7}{3}mg\sin\theta}\,,\ \boxed{R_{\text{transverse}} = \frac{1}{3}mg\cos\theta}$$





Question 56 (****)



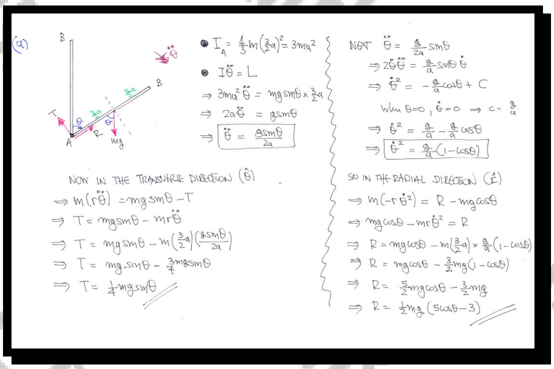
A uniform rod AB of mass m and length $3a$ is free to rotate in a vertical plane about a horizontal axis through A. The rod is held at rest with B is vertically above A and is released from rest. At time t after release the rod makes an angle θ with the upward vertical.

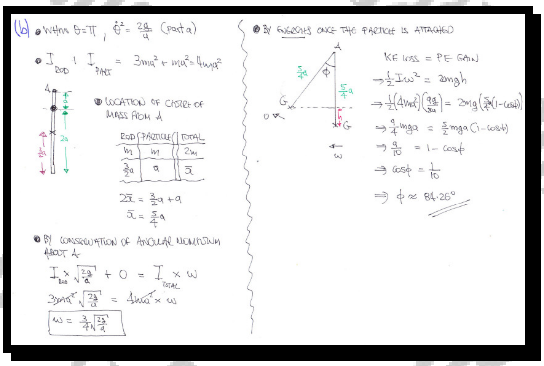
a) Determine expressions, in terms of m, g and θ, for the magnitudes of the components of the reaction at A, parallel to AB and perpendicular to AB.

When B gets vertically below A, the rod collides with a particle of mass m which was at rest at a distance a vertically below A. The particle attaches to the rod and the resulting system continues to move until the rod AB makes an angle φ with the downward vertical.

b) Calculate the value of φ.

$\boxed{R_{\perp AB} = \left|\frac{1}{2}mg\sin\theta\right|}$, $\boxed{R_{\parallel AB} = \left|\frac{1}{2}mg(5\cos\theta - 3)\right|}$, $\boxed{\varphi = \arccos\frac{1}{10} \approx 84.26°}$





Question 57 (****)

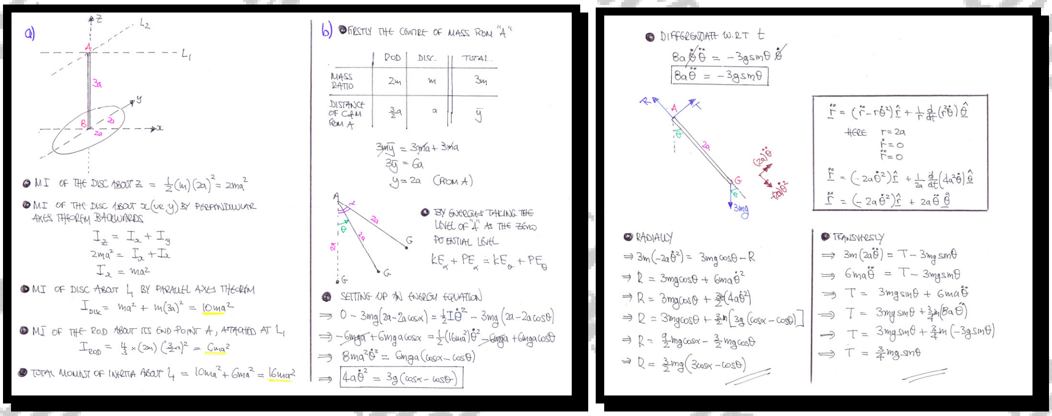
A composite body consists of a thin uniform rod AB, of mass m and length $3a$, with the end B rigidly attached to the centre O of a uniform circular lamina, of radius $2a$ and mass m. The rod is perpendicular to the plane of the lamina. The body is free to rotate in a vertical plane about a fixed smooth horizontal axis through A, and perpendicular to AB.

a) Find the moment of inertia of the body about the above described axis.

The body is released from rest with AB making an angle α with the downward vertical through A.

b) Determine simplified expression for the transverse and radial components of the force acting on the axis, when AB is making an angle θ with the downward vertical through A, where $\theta < \alpha$.

$\boxed{I = 16ma^2}$, $\boxed{R_{\text{radial}} = \frac{3}{2}mg(3\cos\alpha - \cos\theta)}$, $\boxed{R_{\text{transverse}} = \frac{3}{4}mg\sin\theta}$



Question 58 (****+)

A uniform circular disc with centre at O, has radius r and mass m.

The disc is free to rotate in a vertical plane about a fixed smooth horizontal axis. This axis is perpendicular to the plane of the disc and passes through a point P, which is $\frac{3}{4}r$ from O.

The disc is initially at rest with O vertically below P.

A horizontal impulse of magnitude $\frac{2m}{35}\sqrt{255gr}$ is applied at the lowest point on the circumference of the disc and in the plane of the disc.

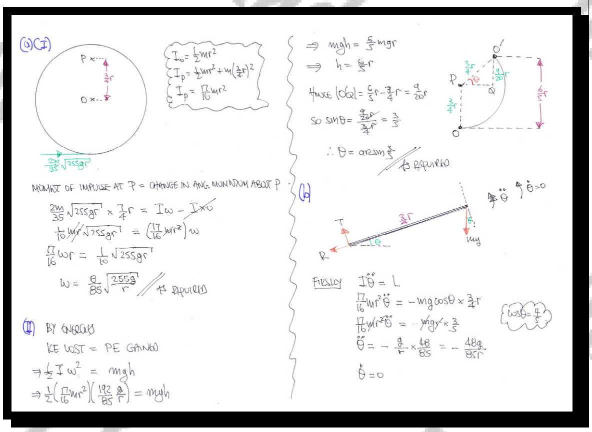
a) Show clearly that the disc …

i. … begins to move with angular velocity $\frac{8}{85}\sqrt{\frac{255g}{r}}$.

ii. … first comes to rest when PO is inclined at $\arcsin\frac{3}{5}$ above the horizontal.

b) Determine the magnitude of the force exerted on the disc by the axis, when the disc first comes to rest.

$$\boxed{F = \frac{\sqrt{145}}{17}mg}$$



(a)(I)

$I_O = \frac{1}{2}mr^2$

$I_P = \frac{1}{2}mr^2 + m\left(\frac{3}{4}r\right)^2$

$I_P = \frac{17}{16}mr^2$

MOMENT OF IMPULSE AT P = CHANGE IN ANG. MOMENTUM ABOUT P

$\frac{2m}{35}\sqrt{255gr} \times \frac{7}{4}r = I\omega - \cancel{I \times 0}$

$\frac{1}{10}mr\sqrt{255gr} = \left(\frac{17}{16}mr^2\right)\omega$

$\frac{17}{16}\omega r = \frac{1}{10}\sqrt{255gr}$

$\omega = \frac{8}{85}\sqrt{\frac{255g}{r}}$ AS REQUIRED

(II) BY ENERGY

KE LOST = PE GAINED

$\Rightarrow \frac{1}{2}I\omega^2 = mgh$

$\Rightarrow \frac{1}{2}\left(\frac{17}{16}mr^2\right)\left(\frac{192}{85}\frac{g}{r}\right) = mgh$

$\Rightarrow mgh = \frac{6}{5}mgr$

$\Rightarrow h = \frac{6}{5}r$

HENCE $|O'Q| = \frac{6}{5}r - \frac{3}{4}r = \frac{9}{20}r$

SO $\sin\theta = \frac{\frac{9}{20}r}{\frac{3}{4}r} = \frac{3}{5}$

$\therefore \theta = \arcsin\frac{3}{5}$ AS REQUIRED

(b)

$\ddot{\theta}$, $\dot{\theta} = 0$

FIRSTLY $I\ddot{\theta} = L$

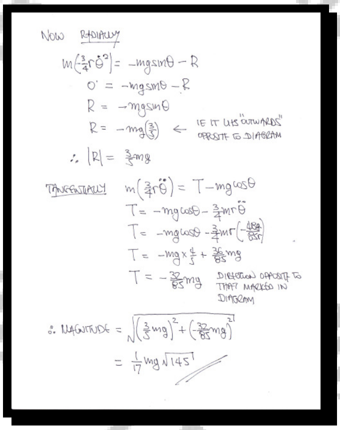
$\frac{17}{16}mr^2\ddot{\theta} = -mg\cos\theta \times \frac{3}{4}r$

$\cos\theta = \frac{4}{5}$

$\frac{17}{16}mr^2\ddot{\theta} = -mgr \times \frac{3}{5}$

$\ddot{\theta} = -\frac{g}{r} \times \frac{48}{85} = -\frac{48g}{85r}$

$\dot{\theta} = 0$



NOW RADIALLY

$m\left(-\frac{3}{4}r\dot{\theta}^2\right) = -mg\sin\theta - R$

$0 = -mg\sin\theta - R$

$R = -mg\sin\theta$

$R = -mg\left(\frac{3}{5}\right)$ ← IE IT ACTS "OUTWARDS" OPPOSITE TO DIAGRAM

$\therefore |R| = \frac{3}{5}mg$

TANGENTIALLY $m\left(\frac{3}{4}r\ddot{\theta}\right) = T - mg\cos\theta$

$T = -mg\cos\theta - \frac{3}{4}mr\ddot{\theta}$

$T = -mg\cos\theta - \frac{3}{4}mr\left(-\frac{48g}{85r}\right)$

$T = -mg \times \frac{4}{5} + \frac{36}{85}mg$

$T = -\frac{32}{85}mg$ DIRECTION OPPOSITE TO THAT MARKED IN DIAGRAM

$\therefore$ MAGNITUDE $= \sqrt{\left(\frac{3}{5}mg\right)^2 + \left(-\frac{32}{85}mg\right)^2}$

$= \frac{1}{17}mg\sqrt{145}$

Question 59 (****+)

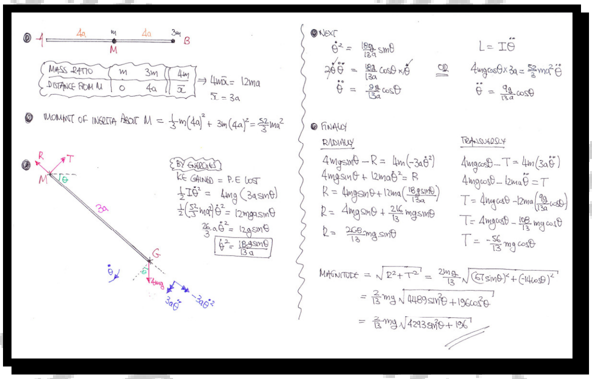
A system consists of a rod AB of length $8a$ and mass m and a particle of mass $3m$ attached at B. The system is freely hinged at the midpoint of the rod and can rotate in a vertical plane.

The system is held in a horizontal position and released from rest.

When the system has turned by an angle θ, the magnitude of the reaction force at the axis of rotation is F, where $F = f(m, g, \theta)$.

Determine an expression for F.

$$\boxed{F = \frac{2}{13} mg\sqrt{196 + 4293\sin^2\theta}}$$



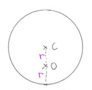
Question 60 (****+)

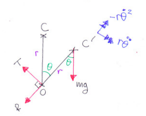
A uniform circular disc, of radius $2r$ and mass m, is free to rotate about a smooth fixed horizontal axis L, which is perpendicular to the plane of the disc and is at a distance r from the centre of the disc C.

The disc is held at rest with C vertically above L. The disc is slightly disturbed and from its position of rest and begins to rotate about C.

Determine, in terms of g and r, the angular velocity of the disc at the two positions where the magnitude of the force exerted on the axis has magnitude $\frac{2}{3}mg$.

$$\boxed{\dot{\theta} = \sqrt{\frac{2g}{9r}}, \quad \dot{\theta} = \sqrt{\frac{10g}{21r}}}$$





$I_0 = \frac{1}{2}m(2r)^2 + mr^2$

$I_0 = 3mr^2$

ACCELERATION $\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$

HERE $r = r =$ CONSTANT $\Rightarrow \ddot{r} = \dot{r} = 0$

SO $\ddot{\underline{r}} = -r\dot{\theta}^2\hat{\underline{r}} + r\ddot{\theta}\hat{\underline{\theta}}$

- BY ENERGIES TAKING THE LEVEL OF O AS THE ZERO POTENTIAL LEVEL

$$KE_{INIT} + P.E_{INIT} = KE_\theta + PE_\theta$$
$$0 + mgr = \frac{1}{2}I\dot{\theta}^2 + mg(r\cos\theta)$$
$$mgr = \frac{1}{2}(3mr^2)\dot{\theta}^2 + mgr\cos\theta$$
$$gr = \frac{3}{2}r^2\dot{\theta}^2 + gr\cos\theta$$
$$gr - gr\cos\theta = \frac{3}{2}r^2\dot{\theta}^2$$
$$\boxed{r\dot{\theta}^2 = \frac{2}{3}g(1-\cos\theta)}$$

- NEXT USING $L = I\ddot{\theta}$

$$(mg\sin\theta)\times r = 3mr^2\ddot{\theta}$$
$$mgr\sin\theta = 3mr^2\ddot{\theta}$$
$$g\sin\theta = 3r\ddot{\theta}$$
$$\boxed{r\ddot{\theta} = \frac{1}{3}g\sin\theta}$$

ALTERNATIVE

$$\frac{d}{dt}(r\dot{\theta}^2) = \frac{d}{dt}\left[\frac{2}{3}g(1-\cos\theta)\right]$$
$$2r\cancel{\dot{\theta}}\ddot{\theta} = \frac{2}{3}g\sin\theta \times \cancel{\dot{\theta}}$$
$$r\ddot{\theta} = \frac{1}{3}g\sin\theta$$

- NOW THE EQUATIONS OF MOTION

RADIALLY

$$\Rightarrow m(-r\dot{\theta}^2) = -R - mg\cos\theta$$
$$\Rightarrow mr\dot{\theta}^2 = R + mg\cos\theta$$
$$\Rightarrow m\left(\frac{2}{3}g(1-\cos\theta)\right) = R + mg\cos\theta$$
$$\Rightarrow \frac{2}{3}mg(1-\cos\theta) = R + mg\cos\theta$$
$$\Rightarrow R = \frac{2}{3}mg(1-\cos\theta) - mg\cos\theta$$
$$\Rightarrow R = \frac{1}{3}mg\left[2 - 2\cos\theta - 3\cos\theta\right]$$
$$\Rightarrow R = \frac{1}{3}mg(2 - 5\cos\theta)$$

TRANSVERSELY

$$\Rightarrow mr\ddot{\theta} = mg\sin\theta - T$$
$$\Rightarrow T = mg\sin\theta - mr\ddot{\theta}$$
$$\Rightarrow T = mg\sin\theta - m\left(\frac{1}{3}g\sin\theta\right)$$
$$\Rightarrow T = \frac{2}{3}mg\sin\theta$$

$$\text{MAGNITUDE} = \sqrt{T^2 + R^2}$$
$$= mg\sqrt{\frac{1}{9}(2-5\cos\theta)^2 + \frac{4}{9}\sin^2\theta}$$
$$= mg \times \frac{1}{3}\sqrt{(2-5\cos\theta)^2 + 4\sin^2\theta}$$
$$= mg \times \frac{1}{3}\sqrt{4 - 20\cos\theta + 25\cos^2\theta + 4\sin^2\theta}$$
$$= \frac{1}{3}mg\sqrt{4 - 20\cos\theta + 21\cos^2\theta + (4\cos^2\theta + 4\sin^2\theta)}$$
$$= \frac{1}{3}mg\sqrt{8 - 20\cos\theta + 21\cos^2\theta}$$

NOW $\frac{2}{3}mg = \frac{1}{3}mg\sqrt{8 - 20\cos\theta + 21\cos^2\theta}$

$$2 = \sqrt{8 - 20\cos\theta + 21\cos^2\theta}$$
$$4 = 8 - 20\cos\theta + 21\cos^2\theta$$
$$\Rightarrow 21\cos^2\theta - 20\cos\theta + 4 = 0$$
$$\Rightarrow (3\cos\theta - 2)(7\cos\theta - 2) = 0$$
$$\Rightarrow \cos\theta = \begin{cases} 2/7 \\ 2/3 \end{cases}$$

- NOW $r\dot{\theta}^2 = \frac{2}{3}g(1-\cos\theta)$

$$r\dot{\theta}^2 = \frac{2}{3}g \times \frac{1}{3}$$
OR
$$r\dot{\theta}^2 = \frac{2}{3}g \times \frac{5}{7}$$

$$r\dot{\theta}^2 = \frac{2}{9}g$$
$$r\dot{\theta}^2 = \frac{10}{21}g$$

$$\dot{\theta} = \begin{cases} \sqrt{\frac{2g}{9r}} \\ \sqrt{\frac{10g}{21r}} \end{cases}$$

Question 61 (**+)**

A pulley is in the shape of a disc of radius a and mass $4m$.

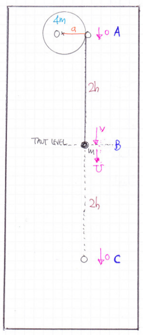
The pulley is free to rotate in a vertical plane about a rough horizontal axis through its centre O. The rotation of the pulley is opposed by a couple of magnitude C.

A light inextensible string has one end attached to a point on the rim of the pulley and is wound several times around the rim of the pulley. The portion of the string not wound on the pulley has length $2h$ and has a particle of mass m attached to its free end. The particle is held at the same level as O, close to the rim of the still pulley and is released from rest.

The particle comes to rest at a vertical distance $4h$ below the level of its release.

Determine, in terms of m, g and a, the magnitude of C.

$$\boxed{C = \frac{4}{3}mga}$$



- FIRSTLY $I_0 = \frac{1}{2}(4m)a^2 = 2ma^2$
- KINEMATICS OR ENERGIES

$\Rightarrow$ "$v^2 = \cancel{u^2} + 2as$"

$\Rightarrow v^2 = 2g(2h)$

$\Rightarrow v^2 = 4gh$

$\Rightarrow v = 2\sqrt{gh}$

- BY CONSERVATION OF ANGULAR MOMENTUM ABOUT O

$$\Rightarrow \underbrace{\underbrace{(mv)\times a}_{\text{PARTICLE}} + \underbrace{0}_{\text{PULLEY}}}_{\text{BEFORE}} = \underbrace{\underbrace{I\omega}_{\text{PULLEY}} + \underbrace{(mU)a}_{\text{PARTICLE}}}_{\text{AFTER}}$$

$\Rightarrow \cancel{m}v\cancel{a} = 2\cancel{m}a^{\cancel{2}}\omega + \cancel{m}U\cancel{a}$

$v = 2a\omega + U$

$\Rightarrow 2\sqrt{gh} = 2a\omega + U$

BUT $U = a\omega$

$\Rightarrow 2\sqrt{gh} = 2a\omega + a\omega$

$\Rightarrow 2\sqrt{gh} = 3a\omega$

- FINALLY BY ENERGIES TAKING THE LEVEL OF C AS THE ZERO POTENTIAL LEVEL

$\Rightarrow KE_B + P.E_B + W_{IN} - W_{OUT} = \cancel{KE_C} + \cancel{P.E_C}$

$\Rightarrow \left(\frac{1}{2}mU^2 + \frac{1}{2}I\omega^2\right) + mg(2h) - C\times\theta = 0$

$\Rightarrow \frac{1}{2}mU^2 + \frac{1}{2}(2ma^2)\omega^2 + 2mgh - C\theta = 0$

$\Rightarrow mU^2 + 2ma^2\omega^2 + 4mgh - 2C\theta = 0.$

$\Rightarrow 2C\theta = mU^2 + 2ma^2\omega^2 + 4mgh$

BUT AS THE PULLEY UNWINDS

"$x = a\theta$"

$2h = a\theta$

$\theta = \frac{2h}{a}$

- SWITCHING ALSO U^2 IN THE ABOVE EQUATION

$2C\left(\frac{2h}{a}\right) = m(a\omega)^2 + 2ma^2\omega^2 + 4mgh$

$\frac{4Ch}{a} = ma^2\omega^2 + 2ma^2\omega^2 + 4mgh$

$\frac{4Ch}{a} = 3ma^2\omega^2 + 4mgh$

$\frac{4Ch}{a} = 3m\left(\frac{2}{3}\sqrt{gh}\right)^2 + 4mgh$

$\frac{4Ch}{a} = 3m \times \frac{4}{9}gh + 4mgh$

$\frac{\cancel{4}C\cancel{h}}{a} = \frac{\cancel{4}}{3}mg\cancel{h} + \cancel{4}mg\cancel{h}$

$\frac{C}{a} = \frac{1}{3}mg + mg$

$\frac{C}{a} = \frac{4}{3}mg$

$C = \frac{4}{3}mga$

Question 62 (****+)

A **heavy** pulley is modelled as a uniform circular disc of radius a, free to rotate through a horizontal axis passing though the centre of the disc and perpendicular to the plane of the disc.

A light inextensible string passes over the **rough** rim of the pulley.

Two particles of mass m and $2m$ are attached to each of the two ends of the string and hang vertically with the string taut until the moment that are gently released from rest, from the same level above horizontal ground.

After the two particles are released, the string does not slip in the pulley, both particles are moving in vertical directions and neither particle reaches the ground or the pulley.

In the subsequent motion the ratio of the tension in the section of the string that the particle of mass m is attached, to that of the tension in the section of the string that the particle of mass $2m$ is attached, is $2:3$.

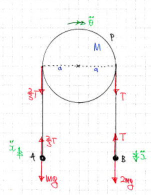
When the angular velocity of the pulley reaches ω the string suddenly breaks.

A couple of constant magnitude brings the pulley to rest.

If the pulley covers an angle π since the couple was applied, show that the magnitude of this couple is $\frac{2}{\pi}m\omega^2 a^2$

$\boxed{}$, $\boxed{\text{proof}}$

START WITH THE STANDARD DIAGRAM – LET THE MASS OF THE PULLEY BE "M" AND THE TENSION ON THE STRING OF THE HEAVIEST PARTICLE BE "T"



$\frac{T_A}{T_B} = \frac{2}{3}$ (GIVEN) $\Rightarrow T_A = \frac{2}{3}T_B$

EQUATIONS OF MOTION FOR A & B

$2mg - T = 2m\ddot{x}$
$\frac{2}{3}T - mg = m\ddot{x}$) $\times 2$

$2mg - T = 2m\ddot{x}$
$\frac{4}{3}T - 2mg = 2m\ddot{x}$

$\frac{4}{3}T - 2mg = 2mg - T$
$\frac{7}{3}T = 4mg$
$T = \frac{12}{7}mg$

USING ONE OF THE EQUATIONS TO FIND $\ddot{x}$

$\Rightarrow \frac{2}{3}T - mg = m\ddot{x}$
$\Rightarrow \frac{8}{7}mg - mg = m\ddot{x}$
$\Rightarrow \ddot{x} = \frac{1}{7}g$

NEXT THE EQUATION OF MOTION OF THE PULLEY

$\rightarrow Ta - \frac{2}{3}Ta = I\ddot{\theta}$
$\rightarrow \frac{1}{3}Ta = \frac{1}{2}Ma^2\ddot{\theta}$

BUT THERE IS NO SLIPPING, IE $\ddot{x} = a\ddot{\theta}$

$\Rightarrow \frac{1}{3}T = \frac{1}{2}Ma\ddot{\theta}$
$\Rightarrow \frac{1}{3}\left(\frac{12}{7}mg\right) = \frac{1}{2}M\ddot{x}$
$\Rightarrow \frac{4}{7}mg = \frac{1}{2}M\left(\frac{1}{7}g\right)$
$\Rightarrow \frac{4}{7}m\cancel{g} = \frac{1}{14}M\cancel{g}$
$\Rightarrow M = 8m$

NOW "KINEMATICS"

"$v^2 = u^2 + 2as$"
$\Omega^2 = \omega^2 + 2\ddot{\theta}\theta$ (CONSTANT ANGULAR DECELERATION)
$0 = \omega^2 + 2\ddot{\theta}_{\text{CONST}} \times \pi$
$\ddot{\theta}_{\text{CONST}} = -\frac{\omega^2}{2\pi}$

FINALLY $L = I\ddot{\theta}$

$L = \frac{1}{2}Ma^2 \times \left(-\frac{\omega^2}{2\pi}\right)$
$L = -\frac{1}{4}\frac{Ma^2\omega^2}{\pi}$
$L = -\frac{8ma^2\omega^2}{4\pi}$
$L = -\frac{2ma^2\omega^2}{\pi}$

$\therefore$ MAGNITUDE $\frac{2}{\pi}(ma^2\omega^2)$

Question 63 (****+)

A uniform circular disc, of radius a and mass m, is free to rotate about a smooth fixed horizontal axis L, which is coplanar to the disc and tangential to a point A at its circumference.

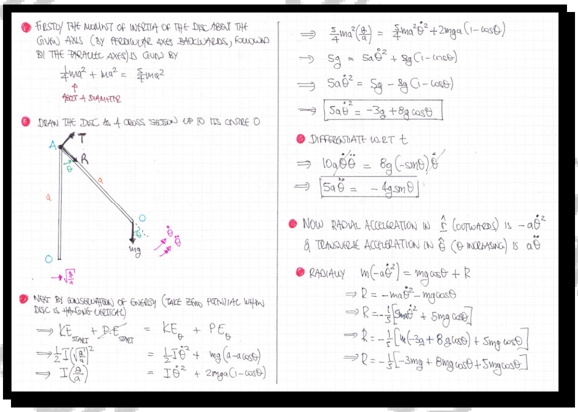
When the centre of the disc, O, is vertically below A, the angular velocity of the disc is $\sqrt{\dfrac{a}{g}}$.

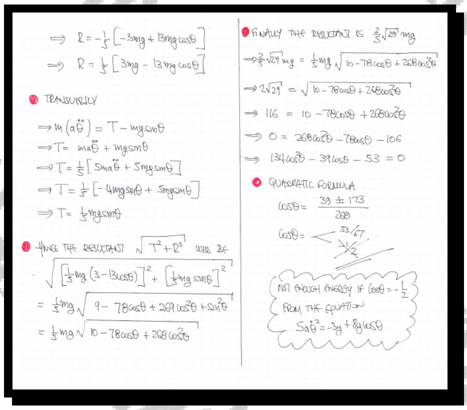
The angle OA makes with the downward vertical through A is denoted by θ.

When $\cos\theta = k$ the magnitude of the resultant force on the axis is $\dfrac{2\sqrt{29}}{5}mg$.

Determine the exact value of k.

$$\boxed{k = \frac{53}{67}}$$





Question 64 (*****)

A uniform rod, of mass m and length $2a$, lies at rest on a smooth horizontal surface and its free to rotate about a smooth vertical axis through its centre O.

A particle of mass m, moving on the surface with speed U, strikes the rod at right angles at the point C on the rod, so that $|OC| = \frac{1}{2}a$.

Given that the collision is perfectly elastic, determine whether there is another collision between the particle and the rod.

$\boxed{\text{there is an other colission}}$