POLAR CORDINATES
and
CENTRAL FORCES

Question 1 (**)

A particle P is moving on a cardioid with polar equation

$$r = a(1+\sin\theta),\ 0 \le \theta < 2\pi,$$

where a is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Find an expression for the speed of P in terms of a, ω and θ, and hence determine the maximum speed of the speed of P and the value of θ when this maximum speed occurs.

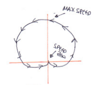
$\boxed{|\mathbf{v}| = a\omega\sqrt{2+2\sin\theta}}$, $\boxed{|\mathbf{v}|_{\text{max}} = 2a\omega}$, $\boxed{\theta = \frac{\pi}{2}}$

$r = a(1+\sin\theta)$ $\quad \dot{\theta} = \omega = \text{CONSTANT}$

IN POLARS

$\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = a(\cos\theta)\dot{\theta} = a\omega\cos\theta \leftarrow \dot{r}$



CARDIOID

THUS

$\Rightarrow \underline{v} = \underbrace{a\omega\cos\theta}_{\dot{r}}\hat{\underline{r}} + \underbrace{a(1+\sin\theta)}_{r}\underbrace{\omega}_{\dot{\theta}}\hat{\theta}$

$\Rightarrow |\underline{v}| = \sqrt{(a\omega\cos\theta) + [a\omega(1+\sin\theta)]^2}$

$\Rightarrow |\underline{v}| = a\omega\sqrt{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}$

$\Rightarrow |\underline{v}| = a\omega\sqrt{2+2\sin\theta}$

$\therefore V_{MAX} = 2a\omega$

IT OCCURS WHEN $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

Question 2 (**)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), by the parametric equations

$$r = 3\sqrt{5}t^2, \quad \theta = t^2 - 6t, \quad t \geq 0.$$

Determine the speed of P and the magnitude of its acceleration when $t = 2$.

$\boxed{|\mathbf{v}|_{t=2} = 60 \text{ ms}^{-1}}$, $\boxed{|\mathbf{a}|_{t=2} = \sqrt{20765} \approx 144 \text{ ms}^{-2}}$

$r = 3\sqrt{5}t^2 \qquad \theta = t^2 - 6t$

$\dot{r} = 6\sqrt{5}t \qquad \dot{\theta} = 2t - 6$

$\ddot{r} = 6\sqrt{5} \qquad \ddot{\theta} = 2$

$\Rightarrow \underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$\Rightarrow \underline{v} = (6\sqrt{5}t)\hat{\underline{r}} + (3\sqrt{5}t^2)(2t-6)\hat{\theta}$

$\Rightarrow \underline{v}_{t=2} = 12\sqrt{5}\,\hat{\underline{r}} + 24\sqrt{5}\,\hat{\underline{\theta}}$

$\Rightarrow |\underline{v}_{t=2}| = \sqrt{(12\sqrt{5})^2 + (24\sqrt{5})^2}$

$\Rightarrow |V_{t=2}| = 60 \text{ ms}^{-1}$

$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$

$\Rightarrow \underline{a} = \left[6\sqrt{5} - 3\sqrt{5}t^2(2t-6)^2\right]\hat{\underline{r}} + \left[2 \times 6\sqrt{5}t \times (2t-6) + 3\sqrt{5}t^2 \times 2\right]\hat{\underline{\theta}}$

$\Rightarrow \underline{a} = \left[6\sqrt{5} - 3\sqrt{5}(2t^2-6t)^2\right]\hat{\underline{r}} + \left[12\sqrt{5}(2t-6)t + 6\sqrt{5}t^2\right]\hat{\underline{\theta}}$

$\Rightarrow \underline{a}_{t=2} = \left[-43\sqrt{5}\right]\hat{\underline{r}} + \left[48\sqrt{5}\right]\hat{\underline{\theta}}$

$\Rightarrow |\underline{a}|_{t=2} = \sqrt{(-43\sqrt{5})^2 + (48\sqrt{5})^2}$

$\Rightarrow |\underline{a}|_{t=2} = \sqrt{20765} \approx 144 \text{ ms}^{-2}$

Question 3 (**)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

The path of P traces the spiral with polar equation

$$r = a\theta,$$

where a is a positive constant.

The radius vector OP rotates with constant angular speed ω.

Determine a simplified expression for the magnitude of the acceleration of P in terms of a, ω and r.

$$\boxed{|\mathbf{a}| = \omega^2\sqrt{4a^2 + r^2}}$$

$r = a\theta$ & $\dot{\theta} = \omega$ CONSTANT
$\ddot{\theta} = 0$

- ACCELERATION IN POLARS

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$$

- $\left\{\begin{array}{l}\dot{r} = a\dot{\theta} \\ \ddot{r} = a\ddot{\theta} = 0\end{array}\right\}$

- THUS

$$\Rightarrow \underline{a} = (0 - a\theta\omega^2)\hat{\underline{r}} + \frac{1}{a\theta}\frac{d}{dt}(a^2\theta^2\omega)\hat{\underline{\theta}}$$

$$\Rightarrow \underline{a} = -a\theta\omega^2\hat{\underline{r}} + \frac{a^2\omega}{a\theta}\frac{d}{dt}(\theta^2)\hat{\underline{\theta}}$$

$$\Rightarrow \underline{a} = -a\theta\omega^2\hat{\underline{r}} + \frac{a\omega}{\cancel{\theta}} \times 2\cancel{\theta}\dot{\theta}\,\hat{\underline{\theta}}$$

$$\Rightarrow \underline{a} = -a\theta\omega^2\hat{\underline{r}} + 2a\omega^2\hat{\underline{\theta}}$$

$$\Rightarrow |\underline{a}| = |-a\theta\omega^2, 2a\omega^2|$$

$$\Rightarrow |\underline{a}| = a\omega^2|-\theta, 2|$$

$$\Rightarrow |\underline{a}| = a\omega^2\sqrt{4 + \theta^2}$$

$$\Rightarrow |\underline{a}| = \omega^2\sqrt{4a^2 + a^2\theta^2}$$

$$\Rightarrow |\underline{a}| = \omega^2\sqrt{4a^2 + r^2} \;//$$

Question 4 **(**)**

A particle P is moving on a cardioid with polar equation

$$r = a(1 - \sin\theta),\ 0 \le \theta < 2\pi,$$

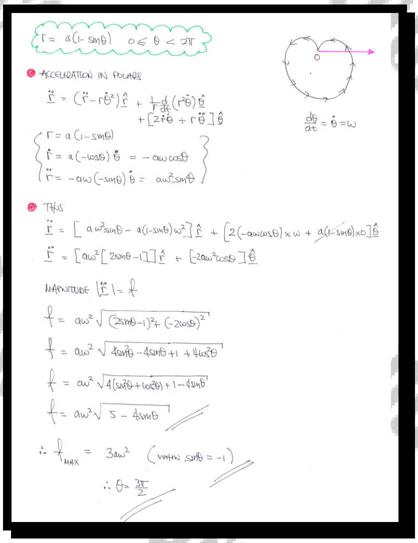
where a is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

The magnitude of the acceleration of P is denoted by f.

Find an expression for f in terms of a, ω and θ, and hence state the greatest value of f and the value of θ when this greatest value of f occurs.

$$\boxed{f = a\omega^2\sqrt{5 - 4\sin\theta}},\ \boxed{f_{\max} = 3a\omega^2},\ \boxed{\theta = \frac{3\pi}{2}}$$



$r = a(1 - \sin\theta) \quad 0 \le \theta < 2\pi$

ACCELERATION IN POLARS

$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$
$+ [2\dot{r}\dot{\theta} + r\ddot{\theta}]\hat{\underline{\theta}}$

$\frac{d\theta}{dt} = \dot{\theta} = \omega$

$r = a(1 - \sin\theta)$
$\dot{r} = a(-\cos\theta)\dot{\theta} = -a\omega\cos\theta$
$\ddot{r} = -a\omega(-\sin\theta)\dot{\theta} = a\omega^2\sin\theta$

THUS

$\ddot{\underline{r}} = [a\omega^2\sin\theta - a(1 - \sin\theta)\omega^2]\hat{\underline{r}} + [2(-a\omega\cos\theta) \times \omega + \cancel{a(1 - \sin\theta) \times 0}]\hat{\underline{\theta}}$

$\ddot{\underline{r}} = [a\omega^2[2\sin\theta - 1]]\hat{\underline{r}} + [-2a\omega^2\cos\theta]\hat{\underline{\theta}}$

MAGNITUDE $|\ddot{\underline{r}}| = f$

$f = a\omega^2\sqrt{(2\sin\theta - 1)^2 + (-2\cos\theta)^2}$

$f = a\omega^2\sqrt{4\sin^2\theta - 4\sin\theta + 1 + 4\cos^2\theta}$

$f = a\omega^2\sqrt{4(\sin^2\theta + \cos^2\theta) + 1 - 4\sin\theta}$

$f = a\omega^2\sqrt{5 - 4\sin\theta}$

$\therefore f_{MAX} = 3a\omega^2$ (WHEN $\sin\theta = -1$)

$\therefore \theta = \frac{3\pi}{2}$

Question 5 (**)

A particle P is moving on the curve with polar equation

$$r = k\,e^{\theta}\ ,\ 0 \le \theta < 2\pi\ ,$$

where k is a positive constant.

The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Find the magnitude and direction of the acceleration acting on P.

$\boxed{|\mathbf{a}| = 2mk\omega^2 e^{\theta} = 2mr\omega^2, \quad \text{transversly}}$

$r = ke^{\theta}$ & $\dot{\theta} = \omega = \text{CONSTANT}$

$\ddot{\theta} = 0$

$\dot{r} = ke^{\theta} \times \dot{\theta}$

$\dot{r} = \omega ke^{\theta}$

$\ddot{r} = \omega ke^{\theta} \times \dot{\theta}$

$\ddot{r} = \omega^2 ke^{\theta}$

RADIAL ACCELERATION $\hat{\underline{r}}$

$= \ddot{r} - r\dot{\theta}^2$

$= \omega^2 ke^{\theta} - (ke^{\theta})\omega^2 = 0$

$\therefore$ NO RADIAL ACCELERATION

NORMAL ACCELERATION $\hat{\underline{\theta}}$ (TRANSVERSE)

$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$= \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})$

$= 2\dot{r}\dot{\theta} + r\ddot{\theta}$

$= 2(\omega ke^{\theta})\omega + \cancel{ke^{\theta} \times 0}$

$= 2k\omega^2 e^{\theta}$

ZERO

$\therefore$ RESULTANT FORCE HAS MAGNITUDE

$2k\omega^2 e^{\theta} \times m$ OR $2\omega^2 r \times m$

IE $2mk\omega^2 e^{\theta}$ OR $2m\omega^2 r$

AND ACTS IN TRANSVERSE DIRECTION

Question 6 (+)**

In a plane polar coordinate system (r,θ), the base unit vectors are defined as $\hat{\mathbf{r}}$ in the direction of r increasing, and $\hat{\boldsymbol{\theta}}$ perpendicular to $\hat{\mathbf{r}}$, in the direction of θ increasing.

a) Given that the position vector $\mathbf{r}$ of a particle P is given by $\mathbf{r} = r\hat{\mathbf{r}}$, derive expressions for the velocity and acceleration of P in plane polar coordinates.

You may assume standard differentiation results for $\hat{\mathbf{r}}$ *and* $\hat{\boldsymbol{\theta}}$.

b) If $r^2 \dfrac{d\theta}{dt}$ is constant state what can be deduced about the force acting on P.

P is moving on the curve with polar equation

$$r = 2 + \cos\theta, \ 0 \le \theta < 2\pi,$$

with **constant** angular speed $\sqrt{5}$ rad s^{-1}.

c) Find the speed and the magnitude of the acceleration of P, when $\theta = \dfrac{\pi}{2}$.

$\boxed{}$, $\boxed{\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}}$, $\boxed{\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\boldsymbol{\theta}}}$, $\boxed{\text{no transverse force}}$,

$\boxed{|\mathbf{v}| = 5 \text{ ms}^{-1}}$, $\boxed{|\mathbf{a}| = 10\sqrt{2} \text{ ms}^{-2}}$

a) DIFFERENTIATE USING THE STANDARD RESULTS

$\frac{d}{d\theta}(\hat{r}) = \hat{\theta}$ & $\frac{d}{d\theta}(\hat{\theta}) = -\hat{r}$

$\Rightarrow \underline{r} = r\hat{r}$

$\Rightarrow \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$

$\Rightarrow \underline{v} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$ ← RELATED RATES/CHAIN RULE

$\Rightarrow \underline{v} = \dot{r}\hat{r} + r\hat{\theta}\dot{\theta}$

$\Rightarrow \underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$ //

DIFFERENTIATE AGAIN TO OBTAIN THE ACCELERATION VECTOR

$\Rightarrow \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{r}) + \frac{d}{dt}(r\dot{\theta}\hat{\theta})$

$\Rightarrow \underline{a} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$

$\Rightarrow \underline{a} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}$

$\Rightarrow \underline{a} = \ddot{r}\hat{r} + \dot{r}\hat{\theta}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\hat{r})\dot{\theta}$

$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$ //

WHICH CAN ALSO BE WRITTEN AS

$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$ //

b) IF $r^2\frac{d\theta}{dt}$ IS CONSTANT $\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow$ NO TRANSVERSE ACCELERATION

$\Rightarrow$ RESULTANT FORCE IS RADIAL (CENTRAL) //

c) $r = 2 + \cos\theta$ AND ANGULAR SPEED $= \dot{\theta} = \sqrt{5}$

$\Rightarrow r = 2 + \cos\theta$

$\Rightarrow \frac{dr}{dt} = -\sin\theta \times \dot{\theta}$

$\Rightarrow \dot{r} = -\sqrt{5}\sin\theta$ $\qquad \boxed{\dot{r}\big|_{\theta=\frac{\pi}{2}} = -\sqrt{5}}$

DIFFERENTIATE AGAIN

$\Rightarrow \ddot{r} = -\sqrt{5}\cos\theta \times \dot{\theta}$

$\Rightarrow \ddot{r} = -5\cos\theta$ $\qquad \boxed{\ddot{r}\big|_{\theta=\frac{\pi}{2}} = 0}$

- $\underline{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = -\sqrt{5}\hat{r} + 2\sqrt{5}\hat{\theta}$ $\qquad \left[r\big|_{\frac{\pi}{2}} = 2\right]$

  $|\underline{v}| = \sqrt{(-\sqrt{5})^2 + (2\sqrt{5})^2} = 5 \text{ ms}^{-1}$ //

- $\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

  $\underline{a} = [0 - 2(\sqrt{5})^2]\hat{r} + [2(-\sqrt{5})\sqrt{5} + 0]\hat{\theta}$ $\qquad [\ddot{\theta} = 0]$

  $\underline{a} = -10\hat{r} - 10\hat{\theta}$

  $|\underline{a}| = \sqrt{(-10)^2 + (-10)^2} = 10\sqrt{2} \text{ ms}^{-2}$

Question 7 (***)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

The radius vector OP rotates with constant angular speed ω.

The radial component of the acceleration of P has magnitude $r\omega^2$, and is directed towards O.

Initially, P is at the point with coordinates $(a,0)$, where a is a positive constant, and has radial velocity $2a\omega$.

Determine a polar equation for the path of P, in terms of a.

$$\boxed{r = a(2\theta+1)}$$

GIVEN

$\dot{\theta} = \omega$ $\quad (\ddot{r} - r\dot{\theta}^2) = -r\omega^2$ $\quad t=0,\ \theta=0,\ r=a,\ \dot{r}=2a\omega$

$\hat{r}$: $\ddot{r} - r\dot{\theta}^2 = -r\omega^2$

$\Rightarrow \ddot{r} - r\omega^2 = -r\omega^2$

$\Rightarrow \ddot{r} = 0$

$\Rightarrow \dot{r} = \text{constant}$

USING CONDITION

$\Rightarrow \boxed{\dot{r} = 2a\omega}$

$\left.\begin{aligned}\dot{r} &= \frac{dr}{dt} = 2a\omega\\ \dot{\theta} &= \frac{d\theta}{dt} = \omega\end{aligned}\right\} \Rightarrow$ DIVIDE

$\Rightarrow \frac{dr}{d\theta} = 2a$

$\Rightarrow 1\,dr = 2a\,d\theta$

$\Rightarrow \int_{r=a}^{r} 1\,dr = \int_{\theta=0}^{\theta} 2a\,d\theta$

$\Rightarrow [r]_a^r = [2a\theta]_0^\theta$

$\Rightarrow r - a = 2a\theta$

$\Rightarrow r = 2a\theta + a$

$\Rightarrow r = a(2\theta+1)$

Question 8 (***)

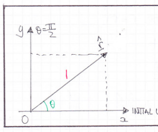
In a plane polar coordinate system (r,θ), the base unit vectors are defined as $\hat{\mathbf{r}}$ in the direction of r increasing, and $\hat{\boldsymbol{\theta}}$ perpendicular to $\hat{\mathbf{r}}$, in the direction of θ increasing.

a) Find expressions for $\dfrac{d}{d\theta}(\hat{\mathbf{r}})$ and $\dfrac{d}{d\theta}(\hat{\boldsymbol{\theta}})$

b) Given that the position vector $\mathbf{r}$ of a particle P is given by $\mathbf{r} = r\hat{\mathbf{r}}$, derive expressions for the velocity and acceleration of P in plane polar coordinates.

$$\boxed{\hat{\boldsymbol{\theta}}},\ \boxed{-\hat{\mathbf{r}}},\ \boxed{\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}},\ \boxed{\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\boldsymbol{\theta}}}$$

a)



LET A CARTESIAN SYSTEM SUPERIMPOSED BY A POLAR SYSTEM, SO THAT THE ORIGINS COINCIDE & THE INITIAL LINE COINCIDE WITH THE POSITIVE x AXIS

- LET A RADIAL VECTOR OF UNIT LENGTH BE $\hat{\underline{r}}$, SUBTENDING AN ANGLE θ WITH THE POSITIVE x AXIS
- THEN

$$\frac{d}{d\theta}(\hat{\underline{r}}) = \frac{d}{d\theta}(1\cos\theta\,\underline{i} + 1\sin\theta\,\underline{j})$$
$$= -\sin\theta\,\underline{i} + \cos\theta\,\underline{j} \quad \text{(AS } \underline{i} \text{ \& } \underline{j} \text{ ARE CONSTANT VECTORS)}$$
$$= \cos\left(\theta+\tfrac{\pi}{2}\right)\underline{i} + \sin\left(\theta+\tfrac{\pi}{2}\right)\underline{j}$$

$\cancel{\cos\theta\cos\tfrac{\pi}{2}} - \sin\theta\sin\tfrac{\pi}{2}$ $\cancel{\sin\theta\cos\tfrac{\pi}{2}} + \cos\theta\sin\tfrac{\pi}{2}$

$-\sin\theta$ $\cos\theta$

$$= \hat{\underline{\theta}}$$

IT ANOTHER UNIT VECTOR SINCE $\cos^2\left(\theta+\frac{\pi}{2}\right) + \sin^2\left(\theta+\frac{\pi}{2}\right) = 1$
ROTATION OF $\hat{\underline{r}}$ BY $\frac{\pi}{2}$ ANTICLOCKWISE

- SIMILARLY

$$\frac{d}{d\theta}\left[\hat{\underline{\theta}}\right] = \frac{d}{d\theta}\left[\cos\left(\theta+\tfrac{\pi}{2}\right)\underline{i} + \sin\left(\theta+\tfrac{\pi}{2}\right)\underline{j}\right]$$
$$= \frac{d}{d\theta}\left[-\sin\theta\,\underline{i} + \cos\theta\,\underline{j}\right]$$
$$= -\cos\theta\,\underline{i} - \sin\theta\,\underline{j}$$
$$= -\left[\cos\theta\,\underline{i} + \sin\theta\,\underline{j}\right]$$
$$= -\hat{\underline{r}}$$

b)

- STARTING WITH THE POLAR VECTOR $\underline{r} = r\hat{\underline{r}}$
- DIFFERENTIATE WITH RESPECT TO t

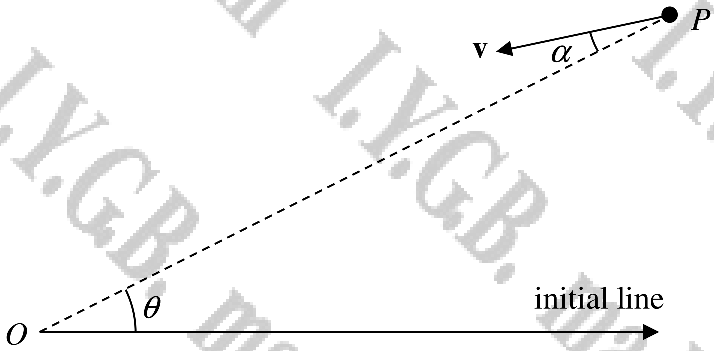
$$\Rightarrow \frac{d\underline{r}}{dt} = \frac{d}{dt}(r\hat{\underline{r}}) = \dot{r}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{dt}$$
$$\Rightarrow \underline{v} = \dot{r}\hat{\underline{r}} + r\frac{d\hat{\underline{r}}}{d\theta}\frac{d\theta}{dt} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$
$$\Rightarrow \underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$

$$\frac{d\hat{\underline{r}}}{d\theta} = \hat{\underline{\theta}} \qquad \frac{d\hat{\underline{\theta}}}{d\theta} = -\hat{\underline{r}}$$

- DIFFERENTIATE AGAIN W.R.T t, FOR ACCELERATION

$$\Rightarrow \frac{d\underline{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{\underline{r}}) + \frac{d}{dt}(r\dot{\theta}\hat{\underline{\theta}})$$
$$\Rightarrow \underline{a} = \ddot{r}\hat{\underline{r}} + \dot{r}\frac{d}{dt}\hat{\underline{r}} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}\frac{d}{dt}\hat{\underline{\theta}}$$
$$\Rightarrow \underline{a} = \ddot{r}\hat{\underline{r}} + \dot{r}\frac{d\hat{\underline{r}}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}\frac{d\hat{\underline{\theta}}}{d\theta}\frac{d\theta}{dt}$$
$$\Rightarrow \underline{a} = \ddot{r}\hat{\underline{r}} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + \dot{r}\dot{\theta}\hat{\underline{\theta}} + r\ddot{\theta}\hat{\underline{\theta}} + r\dot{\theta}^2(-\hat{\underline{r}})$$
$$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$$
$$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{\underline{\theta}}$$

Question 9 (***)



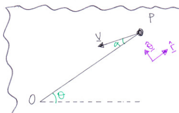
A particle P is moving on a polar plane (r,θ) so that its velocity vector $\mathbf{v}$ forms a constant angle α with OP, where O is the pole, as shown in the figure above.

Given further that P crosses the initial line at $r = 1$, show that the polar equation of the path of P is

$$r = e^{-\theta \cot \alpha}.$$

You may not use verification in this question.

proof



VELOCITY IN POLARS

$\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$

FROM DIAGRAM

$-|\underline{v}|\cos\alpha = \dot{r}$
$|v|\sin\alpha = r\dot{\theta}$ $\Big\} \Rightarrow$ DIVIDE

$\boxed{-\cot\alpha = \frac{\dot{r}}{r\dot{\theta}}}$

HENCE

$\Rightarrow \dfrac{\frac{dr}{dt}}{r\frac{d\theta}{dt}} = -\cot\alpha$

$\Rightarrow \dfrac{dr}{d\theta} = -r\cot\alpha$

CONSTANT

$\Rightarrow \frac{1}{r}\,dr = -\cot\alpha\,d\theta$

$\Rightarrow \int_{r=1}^{r} \frac{1}{r}\,dr = \int_{\theta=0}^{\theta} -\cot\alpha\,d\theta$

$\Rightarrow \left[\ln r\right]_1^r = \left[-\theta\cot\alpha\right]_0^\theta$

$\Rightarrow \ln r - \cancel{\ln 1} = -\theta\cot\alpha$

$\Rightarrow r = e^{-\theta\cot\alpha}$

Question 10 (***)

A particle P, of mass m, is moving on a path with polar equation

$$r = a\mathrm{e}^{k\theta}, \quad 0 \le \theta < 2\pi ,$$

where a and k are positive constants.

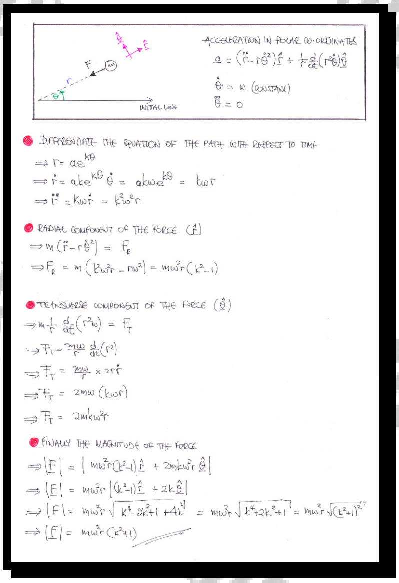
The radius vector OP, where O is the pole, rotates with constant angular speed ω.

Show that the magnitude of the resultant force acting on the plane of its polar path is

$$m\omega^2 r\left(k^2+1\right),$$

where r is the distance OP.

proof



ACCELERATION IN POLAR CO-ORDINATES

$\underline{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\underline{\theta}}$

$\dot{\theta} = \omega$ (CONSTANT)

$\ddot{\theta} = 0$

- DIFFERENTIATE THE EQUATION OF THE PATH WITH RESPECT TO TIME

$\Rightarrow r = ae^{k\theta}$

$\Rightarrow \dot{r} = ake^{k\theta}\dot{\theta} = ak\omega e^{k\theta} = k\omega r$

$\Rightarrow \ddot{r} = k\omega\dot{r} = k^2\omega^2 r$

- RADIAL COMPONENT OF THE FORCE $(\hat{\underline{r}})$

$\Rightarrow m\left(\ddot{r} - r\dot{\theta}^2\right) = F_R$

$\Rightarrow F_R = m\left(k^2\omega^2 r - r\omega^2\right) = m\omega^2 r\left(k^2-1\right)$

- TRANSVERSE COMPONENT OF THE FORCE $(\hat{\underline{\theta}})$

$\Rightarrow m\frac{1}{r}\frac{d}{dt}\left(r^2\omega\right) = F_T$

$\Rightarrow F_T = \frac{m\omega}{r}\frac{d}{dt}\left(r^2\right)$

$\Rightarrow F_T = \frac{m\omega}{r} \times 2r\dot{r}$

$\Rightarrow F_T = 2m\omega\left(k\omega r\right)$

$\Rightarrow F_T = 2mk\omega^2 r$

- FINALLY THE MAGNITUDE OF THE FORCE

$\Rightarrow |\underline{F}| = \left| m\omega^2 r\left(k^2-1\right)\hat{\underline{r}} + 2mk\omega^2 r\,\hat{\underline{\theta}}\right|$

$\Rightarrow |\underline{F}| = m\omega^2 r\left|\left(k^2-1\right)\hat{\underline{r}} + 2k\hat{\underline{\theta}}\right|$

$\Rightarrow |F| = m\omega^2 r\sqrt{k^4 - 2k^2 + 1 + 4k^2} = m\omega^2 r\sqrt{k^4+2k^2+1} = m\omega^2 r\sqrt{\left(k^2+1\right)^2}$

$\Rightarrow |\underline{F}| = m\omega^2 r\left(k^2+1\right)$

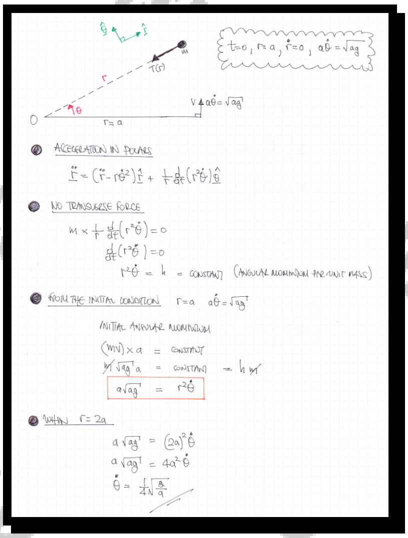
Question 11 (***)

A particle P rests on a smooth horizontal surface attached to a fixed point O on the surface by a light elastic string of natural length a.

When $|OP| = a$ the particle is projected with speed $\sqrt{ag}$ along the surface, in a direction perpendicular to OP.

Find the angular speed of P at the instant when $|OP| = 2a$.

$$\boxed{\sqrt{\frac{g}{16a}}}$$



Question 12 (***)

A particle P is moving on the curve with equation

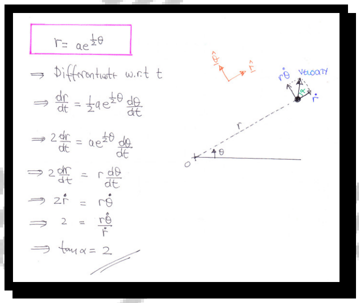
$$r = a\,e^{\frac{1}{2}\theta},$$

where (r,θ) are plane polar coordinates, and a is a positive constant.

The angle the velocity of P makes with OP, where O is the pole, is denoted by α.

Determine the value of $\tan\alpha$.

$\boxed{\tan\alpha = 2}$



Question 13 (***+)

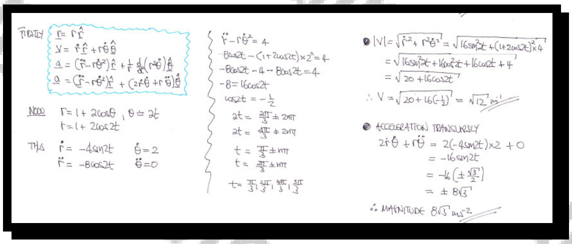
A particle is moving on path whose polar equation is

$$r = 1 + 2\cos\theta\,,\ 0 \le \theta < 2\pi\,.$$

The particle is moving in such a way so that $\theta = 2t$, where t represents the time in s, measured after a given instant. All distances are measure in m.

Determine the speed of the particle and the magnitude of its transverse acceleration when its radial acceleration is $4\ \text{ms}^{-1}$.

$\boxed{\sqrt{12}\ \text{ms}^{-1}}$, $\boxed{8\sqrt{3}\ \text{ms}^{-2}}$



FIRSTLY $\underline{r} = r\hat{\underline{r}}$

$\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\underline{\theta}}$

NOW $r = 1 + 2\cos\theta$, $\theta = 2t$

$r = 1 + 2\cos 2t$

THUS $\dot{r} = -4\sin 2t$ $\quad \dot{\theta} = 2$

$\ddot{r} = -8\cos 2t$ $\quad \ddot{\theta} = 0$

$\ddot{r} - r\dot{\theta}^2 = 4$

$-8\cos 2t - (1 + 2\cos 2t)\times 2^2 = 4$

$-8\cos 2t - 4 - 8\cos 2t = 4$

$-8 = 16\cos 2t$

$\cos 2t = -\frac{1}{2}$

$2t = \frac{2\pi}{3} \pm 2n\pi$

$2t = \frac{4\pi}{3} \pm 2n\pi$

$t = \frac{\pi}{3} \pm n\pi$

$t = \frac{2\pi}{3} \pm n\pi$

$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

• $|v| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = \sqrt{16\sin^2 2t + (1 + 2\cos 2t)^2 \times 4}$

$= \sqrt{16\sin^2 2t + 16\cos^2 2t + 16\cos 2t + 4}$

$= \sqrt{20 + 16\cos 2t}$

$\therefore V = \sqrt{20 + 16(-\frac{1}{2})} = \sqrt{12}\ \text{ms}^{-1}$

• ACCELERATION TRANSVERSLY

$2\dot{r}\dot{\theta} + r\ddot{\theta} = 2(-4\sin 2t)\times 2 + 0$

$= -16\sin 2t$

$= -16\left(\pm\frac{\sqrt{3}}{2}\right)$

$= \pm 8\sqrt{3}$

$\therefore$ MAGNITUDE $8\sqrt{3}\ \text{ms}^{-2}$

Question 14 **(***+)**

At time $t = 0$, a particle is on the initial line of a standard polar coordinate system (r, θ), and moving on a path with polar equation

$$r = \frac{1}{4}e^{k\theta}, \theta \geq 0,$$

where k is a constant.

Relative to the pole O, the particle has a constant angular velocity of 2 rad s^{-1}, throughout the motion.

Given that the initial magnitude of the acceleration of the particle is 1.04 ms^{-2}, determine the possible values of k.

$\boxed{\text{I456-D}}$, $\boxed{\pm\frac{1}{5}}$

$r = \frac{1}{4}e^{k\theta} \quad t=0, \ \theta=0 \quad \& \quad \dot{\theta} = \omega = 2$

- $\dot{\theta} = 2 \Longrightarrow \ddot{\theta} = 0$

$\Downarrow$

$\theta = 2t + C$
$\theta = 2t$ $\quad t=0, \theta=0$

- REWRITE THE EQUATION AS

$\Longrightarrow r = \frac{1}{4}e^{2kt}$

$\Longrightarrow \frac{dr}{dt} = \dot{r} = \frac{k}{2}e^{2kt}$

$\Longrightarrow \frac{d^2r}{dt^2} = \ddot{r} = k^2 e^{2kt}$

- RADIAL ACCELERATION ($\hat{r}$)

$\ddot{r} - r\dot{\theta}^2 = k^2e^{2kt} - \frac{1}{4}e^{2kt} \times 2^2 = k^2e^{2kt} - e^{2kt} = e^{2kt}(k^2-1)$

- TRANSVERSE ACCELERATION $\hat{\theta}$

$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{\frac{1}{4}e^{2kt}}\frac{d}{dt}\left[\frac{1}{16}e^{4kt} \times 2\right] = 4e^{-2kt}\frac{d}{dt}\left[\frac{1}{8}e^{4kt}\right]$

$= 4e^{-2kt} \times \frac{k}{2}e^{4kt} = 2ke^{2kt}$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

- WHEN $t=0$ $\quad |\underline{a}| = 1.04$

$\underline{a} = (k^2-1)\hat{\underline{r}} + 2k\hat{\underline{\theta}}$

$|\underline{a}| = \sqrt{(k^2-1)^2 + (2k)^2}$

$1.04 = \sqrt{k^4 - 2k + 1 + 4k^2}$

$1.04 = \sqrt{k^4 + 2k + 1}$

$1.04 = \sqrt{(k^2+1)^2}$

$1.04 = |k^2+1|$

$k^2 + 1 = \begin{cases} 1.04 \\ -1.04 \end{cases}$

$k^2 = \begin{cases} 0.04 \\ \cancel{-2.04} \end{cases}$

$k = \pm 0.2$

Question 15 (***+)

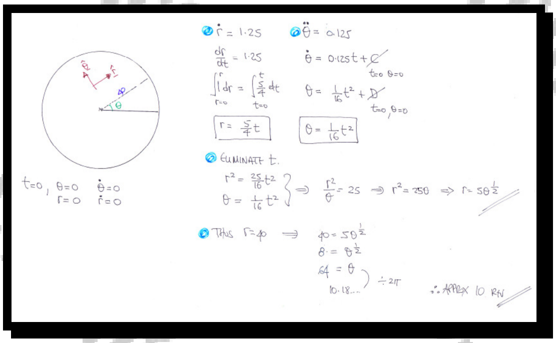
A man is standing at the centre at O of a circular platform, whose radius is $40\ \text{m}$, which is initially at rest.

At time $t = 0$ the platform begins to rotate about O with constant angular acceleration of $0.125\ \text{rad s}^{-1}$, and at the same time the man begins to walk with constant speed $1.25\ \text{ms}^{-1}$, radially outwards relative to the platform.

Let r be the radial distance of the man from O and θ the angle by which the platform has turned.

Determine a polar equation for the path of the man, relative to the ground, in the form $r = f(\theta)$ and hence show that the platform has completed 10 revolutions by the time the man reaches the edge of the platform.

$$\boxed{r = 5\theta^{\frac{1}{2}}}$$



Question 16 (****)

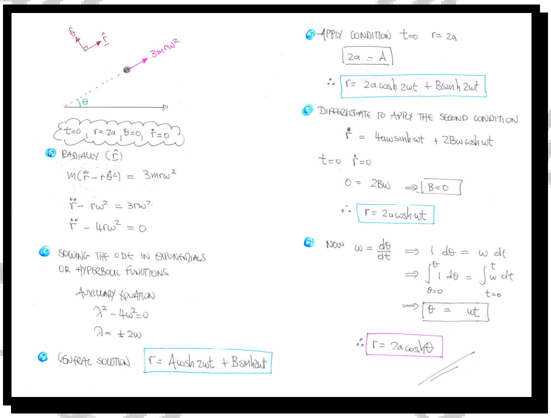
A particle of mass m is moving with constant angular velocity ω on a polar plane (r,θ), with pole at O. The only force acting on the particle has magnitude $3mr\omega^2$, which acts radially outwards.

When $t=0$, the particle is at the point $(2a,0)$, where a is a positive constant, and has no radial speed.

By forming and solving a suitable differential equation, show that the equation of the path of the particle is

$$r = 2a\cosh\theta .$$

$\boxed{\text{proof}}$



$t=0,\ r=2a,\ \theta=0,\ \dot{r}=0$

RADIALLY $(\hat{r})$

$m(\ddot{r}-r\dot{\theta}^2) = 3mr\omega^2$

$\ddot{r} - r\omega^2 = 3r\omega^2$

$\ddot{r} - 4r\omega^2 = 0$

SOLVING THE ODE IN EXPONENTIALS OR HYPERBOLIC FUNCTIONS

AUXILIARY EQUATION

$\lambda^2 - 4\omega^2 = 0$

$\lambda = \pm 2\omega$

GENERAL SOLUTION $\boxed{r = A\cosh 2\omega t + B\sinh 2\omega t}$

APPLY CONDITION $t=0$ $r=2a$

$\boxed{2a = A}$

$\therefore \boxed{r = 2a\cosh 2\omega t + B\sinh 2\omega t}$

DIFFERENTIATE TO APPLY THE SECOND CONDITION

$\dot{r} = 4a\omega\sinh \omega t + 2B\omega\cosh \omega t$

$t=0$ $\dot{r}=0$

$0 = 2B\omega \implies \boxed{B=0}$

$\therefore \boxed{r = 2a\cosh \omega t}$

NOW $\omega = \frac{d\theta}{dt} \implies 1\, d\theta = \omega\, dt$

$\implies \int_{\theta=0}^{\theta} 1\, d\theta = \int_{t=0}^{t} \omega\, dt$

$\implies \boxed{\theta = \omega t}$

$\therefore \boxed{r = 2a\cosh\theta}$

Question 17 (****)

Relative to a fixed origin O, a particle P is moving with constant angular velocity ω on the curve with polar equation

$$r = k\,\mathrm{e}^{\theta \cot \alpha},$$

where k and α are positive constants with $0 < \alpha < \frac{1}{4}\pi$.

Show that the magnitude of the acceleration of the particle is $\dfrac{v^2}{r}$, where v is the speed of the particle and r is the distance OP.

M4, proof

$$\boxed{r = k e^{\theta \cot\alpha} \quad (k, \alpha \text{ CONSTANTS}, 0 < \alpha \leqslant \tfrac{\pi}{4}) \qquad \text{CONSTANT ANGULAR VELOCITY} \quad \dot{\theta} = \omega, \ \ddot{\theta} = 0}$$

DIFFERENTIATE THE EQUATION OF THE PATH TO OBTAIN $\dot{r}$ & $\ddot{r}$

$$\Rightarrow r = k e^{\theta \cot\alpha}$$
$$\Rightarrow \dot{r} = k e^{\theta\cot\alpha} \times \dot{\theta}\cot\alpha$$
$$\Rightarrow \boxed{\dot{r} = r\omega\cot\alpha}$$
$$\Rightarrow \ddot{r} = \dot{r}\,\omega\cot\alpha = (r\omega\cot\alpha)\,\omega\cot\alpha$$
$$\Rightarrow \boxed{\ddot{r} = r\omega^2\cot^2\alpha}$$

NOW THE ACCELERATION IN POLARS IS GIVEN BY

$$\Rightarrow \ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{\underline{\theta}}$$
$$\Rightarrow \ddot{\underline{r}} = (\ddot{r} - r\omega^2)\,\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\omega)\,\hat{\underline{\theta}}$$
$$\Rightarrow \ddot{\underline{r}} = (\ddot{r} - r\omega^2)\,\hat{\underline{r}} + 2\dot{r}\omega\,\hat{\underline{\theta}}$$
$$\Rightarrow \ddot{\underline{r}} = (r\omega^2\cot^2\alpha - \omega^2 r)\,\hat{\underline{r}} + (2\omega^2 r\cot\alpha)\,\hat{\underline{\theta}}$$
$$\Rightarrow \ddot{\underline{r}} = r\omega^2\Big[\big[\cot^2\alpha - 1\big]\hat{\underline{r}} + \big[2\cot\alpha\big]\hat{\underline{\theta}}\Big]$$

NEXT THE MODULUS (MAGNITUDE OF ACCELERATION)

$$\Rightarrow |\ddot{\underline{r}}| = \left| r\omega^2\left[(\cot^2\alpha - 1)\hat{\underline{r}} + (2\cot\alpha)\hat{\underline{\theta}}\right]\right|$$
$$\Rightarrow |\ddot{\underline{r}}| = r\omega^2\left[(\cot^2\alpha - 1)^2 + (2\cot\alpha)^2\right]^{\frac{1}{2}}$$
$$\Rightarrow |\ddot{\underline{r}}| = r\omega^2\left[\cot^4\alpha - 2\cot^2\alpha + 1 + 4\cot^2\alpha\right]^{\frac{1}{2}}$$
$$\Rightarrow |\ddot{\underline{r}}| = r\omega^2\left[\cot^4\alpha + 2\cot^2\alpha + 1\right]^{\frac{1}{2}}$$
$$\Rightarrow |\ddot{\underline{r}}| = r\omega^2\sqrt{(\cot^2\alpha + 1)^2}$$
$$\Rightarrow \boxed{|\ddot{\underline{r}}| = r\omega^2\operatorname{cosec}^2\alpha}$$

NEXT WE REQUIRE A VELOCITY EXPRESSION, TO GET THE SPEED

$$\Rightarrow \underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$$
$$\Rightarrow \underline{v} = (r\omega\cot\alpha)\,\hat{\underline{r}} + r\omega\,\hat{\underline{\theta}}$$
$$\Rightarrow |\underline{v}| = \left| r\omega(\cot\alpha\,\hat{\underline{r}} + \hat{\theta})\right|$$
$$\Rightarrow |\underline{v}| = r\omega\sqrt{\cot^2\alpha + 1}$$
$$\Rightarrow |\underline{v}| = r\omega\sqrt{\operatorname{cosec}^2\alpha}$$
$$\Rightarrow \boxed{|\underline{v}| = r\omega\operatorname{cosec}\alpha}$$

FINALLY WE OBTAIN

$$|\ddot{\underline{r}}| = r\omega^2\operatorname{cosec}^2\alpha = \frac{1}{r}(r^2\omega^2\operatorname{cosec}^2\alpha) = \frac{|\underline{v}|^2}{r}$$

$$\therefore\ |\ddot{\underline{r}}| = \frac{|\underline{v}|^2}{r}$$

AS REQUIRED

Question 18 (****)

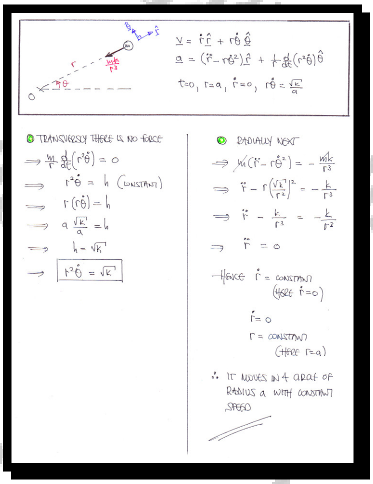
A particle P, of mass m, moves in a plane under the action of a force F which is directed towards a fixed origin O.

The magnitude of F is $\frac{mk}{r^3}$, where $r = |OP|$ and k is a positive constant.

Initially $r = a$ and the particle has speed $\frac{\sqrt{k}}{a}$ in a direction perpendicular to OP.

Use polar coordinates to describe the motion and path of P

$$\boxed{\text{moving on a circle of radius } a \text{ with constant speed}}$$



Question 19 (****)

A particle P of mass m is moving on a polar plane (r,θ), with pole at O.

The path of P traces the spiral with polar equation

$$r = a\mathrm{e}^{k\theta},$$

where a and k are positive constants.

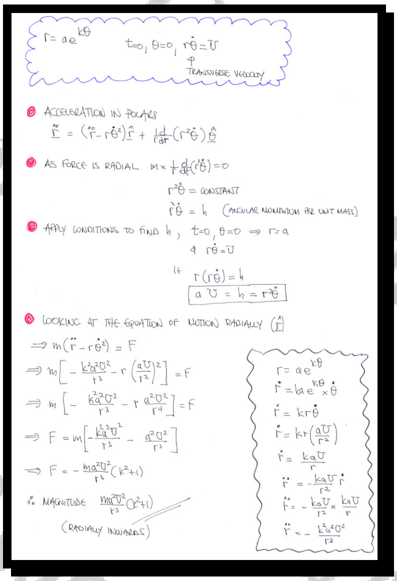
A variable force acts on P, acting in the radial direction with magnitude F.

Initially $\theta = 0$, and at that instant the transverse speed of P is U.

Show that

$$F = \frac{ma^2U^2}{r^3}\left(k^2+1\right).$$

$\boxed{\text{proof}}$



Question 20 (****)

A particle of mass 0.1 kg is attached to one end of a light elastic string and the other end is attached to a fixed point O on a smooth horizontal surface. The string has natural length 0.8 m and modulus of elasticity 61.74 N.

The string is then extended to 3.2 m and the particle is projected with speed u ms^{-1} at right angles to the string. In the subsequent motion, the polar coordinates of the particle relative to O are (r,θ).

a) Express $r^2\dot{\theta}$ in terms of u.

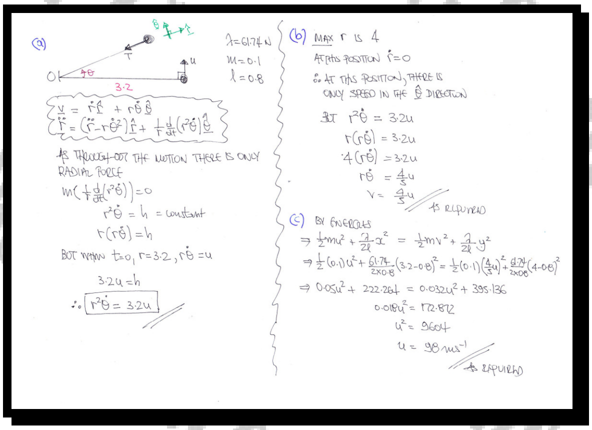
During the motion the maximum value of r is 4 m and at that position the particle has speed v ms^{-1}.

b) Show clearly that

$$v = \frac{4}{5}u \, .$$

c) By considering energies in two suitable positions, show that $v = 98$

$$\boxed{r^2\dot{\theta} = \frac{16}{5}u = 3.2u}$$



(a)

$\lambda = 61.74$ N
$m = 0.1$
$l = 0.8$

$\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}}$
$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

AS THROUGH-OUT THE MOTION THERE IS ONLY RADIAL FORCE

$m\left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right) = 0$

$r^2\dot{\theta} = h = \text{constant}$

$r(r\dot{\theta}) = h$

BUT WHEN $t=0$, $r = 3.2$, $r\dot{\theta} = u$

$3.2u = h$

$\therefore \boxed{r^2\dot{\theta} = 3.2u}$

(b) MAX r IS 4

AT THIS POSITION $\dot{r} = 0$

$\therefore$ AT THIS POSITION, THERE IS ONLY SPEED IN THE $\hat{\underline{\theta}}$ DIRECTION

BUT $r^2\dot{\theta} = 3.2u$

$r(r\dot{\theta}) = 3.2u$

$4(r\dot{\theta}) = 3.2u$

$r\dot{\theta} = \frac{4}{5}u$

$v = \frac{4}{5}u$

AS REQUIRED

(c) BY ENERGIES

$\Rightarrow \frac{1}{2}mu^2 + \frac{\lambda}{2l}x^2 = \frac{1}{2}mv^2 + \frac{\lambda}{2l}y^2$

$\Rightarrow \frac{1}{2}(0.1)u^2 + \frac{61.74}{2\times0.8}(3.2-0.8)^2 = \frac{1}{2}(0.1)\left(\frac{4}{5}u\right)^2 + \frac{61.74}{2\times0.8}(4-0.8)^2$

$\Rightarrow 0.05u^2 + 222.264 = 0.032u^2 + 395.136$

$0.018u^2 = 172.872$

$u^2 = 9604$

$u = 98$ ms^{-1}

AS REQUIRED

Question 21 (****)

A particle P is moving on a plane, and its position in time t s is described in plane polar coordinates (r,θ), where O is the pole.

The radius vector OP rotates with constant angular speed ω.

The radial component of the acceleration of P has magnitude $2r\omega^2$, and is directed towards O.

Initially, P is at the point with coordinates $(a,0)$, where a is a positive constant, and has radial velocity $\sqrt{3}\,a\omega$.

Determine, in terms of a, a polar equation for the path of P.

$\boxed{}$, $\boxed{r = 2a\sin\left(\theta + \frac{\pi}{6}\right)}$

GIVEN IN THE PROBLEM

$\dot{\theta} = \omega = \text{CONSTANT}$ $t=0$, $\theta=0$, $r=a$, $\dot{r}=\sqrt{3}a\omega$

$\hat{r}: \ (\ddot{r} - r\dot{\theta}^2) = -2\omega^2 r$

LOOKING AT THE ACCELERATION RADIALLY ($\hat{r}$)

$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -2\omega^2 r$

$\Rightarrow \ddot{r} - r\omega^2 = -2\omega^2 r$

$\Rightarrow \ddot{r} = -\omega^2 r$

SOLVING THE O.D.E, WHICH IS A STANDARD S.H.M SOLUTION

$\Rightarrow \frac{d^2r}{dt^2} = -\omega^2 r$

$\Rightarrow r(t) = A\cos\omega t + B\sin\omega t$

APPLY THE CONDITION $t=0$, $r=a$ YIELDS $a = A$

$\Rightarrow r(t) = a\cos\omega t + B\sin\omega t$

DIFFERENTIATE TO APPLY THE OTHER CONDITION

$\Rightarrow \dot{r}(t) = -a\omega\sin\omega t + B\omega\cos\omega t$ $t=0$, $\dot{r}=\sqrt{3}a\omega$

$\sqrt{3}a\omega = B\omega$

$B = \sqrt{3}a$

MANIPULATE AS FOLLOWS

$\Rightarrow r = a\cos\omega t + \sqrt{3}a\sin\omega t$

$\Rightarrow r = 2a\left[\frac{1}{2}\cos\omega t + \frac{\sqrt{3}}{2}\sin\omega t\right]$

$\Rightarrow r = 2a\left[\sin\frac{\pi}{6}\cos\omega t + \cos\frac{\pi}{6}\sin\omega t\right]$

$\Rightarrow r = 2a\sin\left(\omega t + \frac{\pi}{6}\right)$

FINALLY TO "LOSE" t

$\dot{\theta} = \omega$

$\frac{d\theta}{dt} = \omega$

$1\,d\theta = \omega\,dt$

$\int_{\theta=0}^{\theta} 1\,d\theta = \int_{t=0}^{t} \omega\,dt$

$[\theta]_0^{\theta} = [\omega t]_0^t$

$\theta = \omega t$

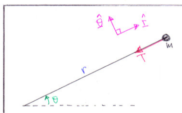
$\therefore r(\theta) = 2a\sin\left(\theta + \frac{\pi}{6}\right)$

Question 22 (****)

A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity mg. The other end of the string is attached to a fixed point O on a smooth horizontal surface. The particle is held in contact with the horizontal table so that $|OP| = 2a$ and projected with horizontal speed u in a direction perpendicular to OP.

Show that when $r = a$ and the radial speed of P is $\sqrt{3u^2 + 2ag}$.

proof



ACCELERATION IN POLARS

$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

MASS m
MODULUS $\lambda = mg$
NATURAL LENGTH a

INITIALLY $t=0$ $r=2a$
$\dot{r}=0$
$\omega r = u$

- TRANSVERSELY THERE IS NO FORCE

• $m \times \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$r^2\dot{\theta} = h$ (CONSTANT)

• INITIALLY $\boxed{\begin{aligned} &r = 2a \\ &\omega r = u \Rightarrow \omega(2a) = u \\ &\qquad\quad \Rightarrow \omega = \frac{u}{2a} \end{aligned}}$

$\Rightarrow h = (2a)^2\left(\frac{u}{2a}\right)$

$\Rightarrow h = 2au$

$\therefore \boxed{r^2\dot{\theta} = 2au}$

- RADIALLY WE HAVE THE TENSION OF THE STRING

$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -T$

$\Rightarrow m\left(\ddot{r} - r\left(\frac{2au}{r^2}\right)^2\right) = \frac{\lambda}{l}x$

$\Rightarrow m\left(\ddot{r} - \frac{4a^2u^2}{r^3}\right) = \frac{mg}{a}(r-a)$

$\Rightarrow \ddot{r} - \frac{4a^2u^2}{r^3} = \frac{g}{a}(r-a)$

$\Rightarrow \ddot{r} - \frac{4a^2u^2}{r^3} = \frac{gr}{a} - g$

- MULTIPLY THE O.D.E BY $2\dot{r}$ & INTEGRATE

$\Rightarrow 2\dot{r}\ddot{r} - \frac{8a^2u^2}{r^3}\dot{r} = \frac{2gr}{a}\dot{r} - g\dot{r}$

$\Rightarrow \frac{d}{dt}(\dot{r}^2) + \frac{d}{dt}\left(\frac{4a^2u^2}{r^2}\right) = \frac{d}{dt}\left(\frac{gr^2}{a}\right) - \frac{d}{dt}(gr) + C$

$\Rightarrow \dot{r}^2 + \frac{4a^2u^2}{r^2} = \frac{gr^2}{a} - gr + C$

- $t=0$, $\dot{r}=0$, $r=2a$ $\Rightarrow$ $\boxed{\begin{aligned} &u^2 = 4ga - 2ga + C \\ &C = u^2 - 2ag \end{aligned}}$

$\Rightarrow \dot{r}^2 + \frac{4a^2u^2}{r^2} = \frac{gr}{a}(r-a) + u^2 - 2ag$

- WHEN $r=a$

$\Rightarrow \dot{r}^2 + 4u^2 = u^2 - 2ag$

$\Rightarrow \dot{r}^2 = 3u^2 + 2ag$

$\Rightarrow |\dot{r}| = \sqrt{3u^2 + 2ag}$

Question 23 (****)

A particle P of mass 0.45 kg is attached to another particle Q of mass 2 kg by a light inextensible string of length 1.2 m.

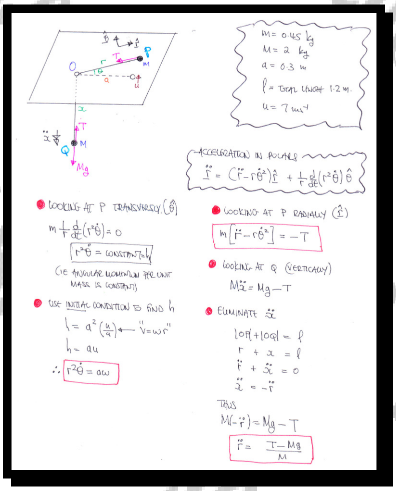
The string passes through a small smooth hole O on a smooth large table, so and P lies on the table and Q is hanging vertically below O.

When $|OP| = 0.3$ m, P is projected with horizontal speed 7 ms^{-1} at right angles to the taut string.

Show that when $|OP| = r$ m, the tension in the string T satisfies

$$T = \frac{9}{50}\left[20 - \frac{9}{r^3}\right].$$

proof



$m = 0.45$ kg
$M = 2$ kg
$a = 0.3$ m
$\ell =$ TOTAL LENGTH 1.2 m.
$u = 7 \text{ ms}^{-1}$

ACCELERATION IN POLARS
$\ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$

- LOOKING AT P TRANSVERSLY $(\hat{\theta})$

$m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\boxed{r^2\dot{\theta} = \text{CONSTANT} = h}$

(IE ANGULAR MOMENTUM PER UNIT MASS IS CONSTANT)

- USE INITIAL CONDITION TO FIND h

$h = a^2\left(\frac{u}{a}\right) \leftarrow$ "$v = \omega r$"

$h = au$

$\therefore \boxed{r^2\dot{\theta} = a\omega}$

- LOOKING AT P RADIALLY $(\hat{\underline{r}})$

$\boxed{m\left[\ddot{r} - r\dot{\theta}^2\right] = -T}$

- LOOKING AT Q (VERTICALLY)

$M\ddot{x} = Mg - T$

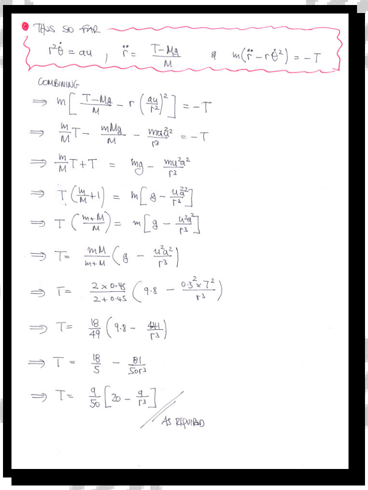
- ELIMINATE $\ddot{x}$

$|OP| + |OQ| = \ell$
$r + x = \ell$
$\ddot{r} + \ddot{x} = 0$
$\ddot{x} = -\ddot{r}$

THUS

$M(-\ddot{r}) = Mg - T$

$\boxed{\ddot{r} = \frac{T - Mg}{M}}$



- THUS SO FAR

$r^2\dot{\theta} = au$, $\ddot{r} = \frac{T - Mg}{M}$ & $m(\ddot{r} - r\dot{\theta}^2) = -T$

COMBINING

$\Rightarrow m\left[\frac{T - Mg}{M} - r\left(\frac{au}{r^2}\right)^2\right] = -T$

$\Rightarrow \frac{m}{M}T - \frac{mMg}{M} - \frac{ma^2u^2}{r^3} = -T$

$\Rightarrow \frac{m}{M}T + T = mg - \frac{mu^2a^2}{r^3}$

$\Rightarrow T\left(\frac{m}{M} + 1\right) = m\left[g - \frac{u^2a^2}{r^3}\right]$

$\Rightarrow T\left(\frac{m+M}{M}\right) = m\left[g - \frac{u^2a^2}{r^3}\right]$

$\Rightarrow T = \frac{mM}{m+M}\left(g - \frac{u^2a^2}{r^3}\right)$

$\Rightarrow T = \frac{2 \times 0.45}{2 + 0.45}\left(9.8 - \frac{0.3^2 \times 7^2}{r^3}\right)$

$\Rightarrow T = \frac{18}{49}\left(9.8 - \frac{4.41}{r^3}\right)$

$\Rightarrow T = \frac{18}{5} - \frac{81}{50r^3}$

$\Rightarrow T = \frac{9}{50}\left[20 - \frac{9}{r^3}\right]$

AS REQUIRED

Question 24 (****+)

A particle P of mass 0.5 kg is moving on the circle with equation

$$(x-1)^2 + y^2 = 1 .$$

The particle is subject to a force of magnitude F, which always acts in the direction PO, where O is the origin.

The particle is observed passing through the point $(2,0)$ with speed 0.125 ms^{-1}, tangential to the circle and parallel to the y axis.

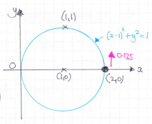
Show that if $|OP| = r$ m, then

$$F = \frac{1}{4r^5} .$$

proof

$\boxed{(x-1)^2+y^2=1}$

$t=0$ $\quad x=2$, $y=0$, $V=0.125$ (PARALLEL TO y AXIS)



WORK IN POLAR CO-ORDINATES

$$\Rightarrow x^2-2x+\cancel{1}+y^2=\cancel{1}$$
$$\Rightarrow x^2+y^2=2x$$
$$\Rightarrow r^2=2(r\cos\theta)$$
$$\Rightarrow r=2\cos\theta$$

ACCELERATION IN POLARS

$$\underline{a}=(\ddot{r}-r\dot{\theta}^2)\hat{\underline{r}}+\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$$

NO TRANSVERSE FORCE ($\hat{\underline{\theta}}$)

$$\Rightarrow \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta})=0$$
$$\Rightarrow r^2\dot{\theta}=\text{CONSTANT}$$
$$\Rightarrow r(r\dot{\theta})=\text{CONSTANT}$$
$$\Rightarrow 2(0.125)=\text{CONSTANT}$$
$$\therefore r^2\dot{\theta}=\frac{1}{4}$$

$t=0$ $\quad r=2$, $\theta=0$, $r\dot{\theta}=0.125$

RADIALLY THE EQUATION OF MOTION NOW YIELDS

$$\Rightarrow m(\ddot{r}-r\dot{\theta}^2)=-F \quad \text{(MINUS SIMPLY IMPLIES INWARDS)}$$
$$\Rightarrow \frac{1}{2}\left[\ddot{r}-r\left(\frac{1}{4r^2}\right)^2\right]=-F$$
$$\Rightarrow -F=\frac{1}{2}\left[\ddot{r}-\frac{1}{16r^3}\right]$$

NOW WE NEED AN EXPRESSION FOR $\ddot{r}$

$$\Rightarrow r=2\cos\theta$$
$$\Rightarrow \frac{dr}{dt}=-2\sin\theta\times\dot{\theta}$$
$$\Rightarrow \dot{r}=-2\sin\theta\times\frac{1}{4r^2}=-\frac{\sin\theta}{2(2\cos\theta)^2}=-\frac{\sin\theta}{8\cos^2\theta}$$
$$\Rightarrow \dot{r}=-\frac{1}{8}\tan\theta\sec\theta$$
$$\Rightarrow \ddot{r}=-\frac{1}{8}\left[\sec^3\theta+\tan^2\theta\sec\theta\right]\dot{\theta}$$
$$\Rightarrow \ddot{r}=-\frac{1}{8}\sec\theta\left[\sec^2\theta+\tan^2\theta\right]\times\frac{1}{4r^2}$$
$$\Rightarrow \ddot{r}=-\frac{1}{8}\sec\theta\left[2\sec^2\theta-1\right]\times\frac{1}{4r^2}$$

BUT $\quad \frac{r}{2}=\cos\theta$, $\quad \sec\theta=\frac{2}{r}$

$$\Rightarrow \ddot{r}=-\frac{1}{32r^2}\times\frac{2}{r}\times\left[2\times\frac{4}{r^2}-1\right]$$
$$\Rightarrow \ddot{r}=-\frac{1}{16r^3}\left(\frac{8}{r^2}-1\right)$$

RETURNING TO THE MAIN EXPRESSION

$$\Rightarrow -F=\frac{1}{2}\left[\ddot{r}-\frac{1}{16r^3}\right]$$
$$\Rightarrow -F=\frac{1}{2}\left[-\frac{1}{16r^3}\left(\frac{8}{r^2}-1\right)-\frac{1}{16r^3}\right]$$
$$\Rightarrow -F=\frac{1}{2}\left[-\frac{1}{2r^5}+\cancel{\frac{1}{16r^3}}-\cancel{\frac{1}{16r^3}}\right]$$
$$\Rightarrow -F=-\frac{1}{4r^5}$$
$$\Rightarrow F=\frac{1}{4r^5}$$

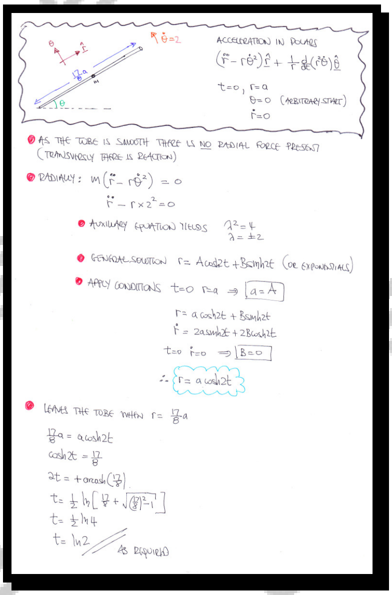
Question 25 (****+)

A particle of mass m is placed inside a smooth tube OA of length $\frac{17}{8}a$. Initially the particle is at rest at a distance a from O

The tube is made to rotate with constant angular velocity ω, in a horizontal plane through a vertical axis passing through O. The particle reaches A in time T.

Show that $T = \ln 2$.

$\boxed{\text{proof}}$



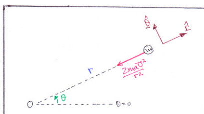
Question 26 (****+)

A particle P, of mass m, is moving on a plane passing though a fixed origin O under the action of a force F, which acts radially in the direction PO.

The distance PO at time t s is denoted by r. At time $t = 0$, $r = a$ and the speed of P is U, pointing in a direction perpendicular to PO.

Given that $F = \dfrac{2maU^2}{r^2}$ determine the least value of r in the subsequent motion.

$$\boxed{r_{\text{min}} = \frac{1}{3}a}$$



IN POLARS

$\underline{v} = \dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}$

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{\hat{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{\hat{\theta}}$

CONDITIONS

$t=0,\ r=a,\ \theta=0$ (ARBITRARY)

$a\dot{\theta} = U,\ \dot{r} = 0$

↑ TRANSVERSE VELOCITY ↑ RADIAL VELOCITY

- NO TRANSVERSE COMPONENT OF FORCE

$\underline{F} = m\underline{a}$

$\Rightarrow 0 = m \times \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow r^2\dot{\theta} = \text{CONSTANT} = h$ ← ANGULAR MOMENTUM PER UNIT MASS IS CONSTANT

- FROM THE INITIAL CONDITIONS WE CAN CALCULATE h

$h = r(r\dot{\theta}) = a \times U$

$\boxed{h = aU}$

- NEXT LOOKING AT THE EQUATION OF MOTION RADIALLY

$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -\frac{2maU^2}{r^2}$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -\frac{2aU^2}{r^2}$

$\Rightarrow \ddot{r} - r\left(\frac{aU}{r^2}\right)^2 = -\frac{2aU^2}{r^2}$ [SINCE $r^2\dot{\theta} = h = aU$]

$\Rightarrow \ddot{r} - \frac{a^2U^2}{r^3} + \frac{2aU^2}{r^2} = 0$

- TO SOLVE THE O.D.E, MULTIPLY THROUGH BY $2\dot{r}$

$\Rightarrow 2\dot{r}\ddot{r} - \frac{2a^2U^2\dot{r}}{r^3} + \frac{4aU^2\dot{r}}{r^2} = 0$

$\Rightarrow \frac{d}{dt}(\dot{r}^2) + \frac{d}{dt}\left(\frac{a^2U^2}{r^2}\right) - \frac{d}{dt}\left[\frac{4aU^2}{r}\right] = 0$

- INTEGRATE WITH RESPECT TO t

$\Rightarrow \dot{r}^2 + \frac{a^2U^2}{r^2} - \frac{4aU^2}{r} = C$

- APPLY CONDITION $t=0,\ \dot{r}=0$ (NO RADIAL VELOCITY), $r=a$

$0 + \frac{a^2U^2}{a^2} - \frac{4aU^2}{a} = C$

$C = -3U^2$

$\Rightarrow \dot{r}^2 + \frac{a^2U^2}{r^2} - \frac{4aU^2}{r} = -3U^2$

$\Rightarrow \dot{r}^2 = -3U^2 + \frac{4aU^2}{r} - \frac{a^2U^2}{r^2}$

$\Rightarrow \dot{r}^2 = \frac{-3r^2U^2 + 4arU^2 - a^2U^2}{r^2}$

$\Rightarrow \dot{r}^2 = -\frac{U^2}{r^2}(3r^2 - 4ar + a^2)$

$\Rightarrow \dot{r}^2 = -\frac{U^2}{r}(3r-a)(r-a)$

$\dot{r} = 0 \Rightarrow r = \cancel{a}$ ← INITIALLY, $\frac{a}{3}$

Question 27 (****+)

A particle P, of mass m, is moving on a plane passing though a fixed origin O under the action of a force F, which acts radially in the direction PO. The distance PO at time t s is denoted by r. The path of P has polar equation

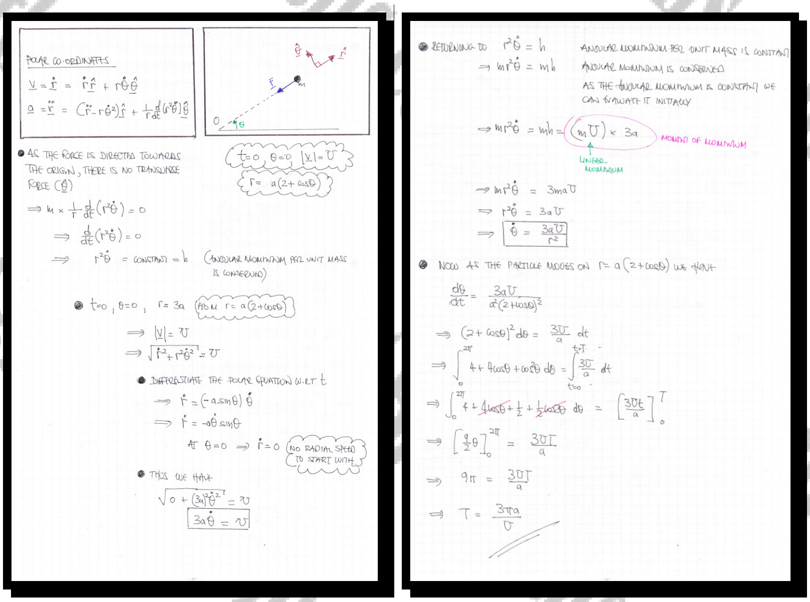
$$r = a(2 + \cos\theta),$$

where a is a positive constant.

At time $t = 0$, $\theta = 0$ and the speed of P is U.

Find, in terms of π, a and U, the time it takes P to return to its starting position.

$$\boxed{t = \frac{3\pi a}{U}}$$



POLAR CO-ORDINATES

$\underline{v} = \underline{\dot{r}} = \dot{r}\underline{\hat{r}} + r\dot{\theta}\underline{\hat{\theta}}$

$\underline{a} = \underline{\ddot{r}} = (\ddot{r} - r\dot{\theta}^2)\underline{\hat{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\underline{\hat{\theta}}$

- AS THE FORCE IS DIRECTED TOWARDS THE ORIGIN, THERE IS NO TRANSVERSE FORCE ($\underline{\hat{\theta}}$)

$t=0$, $\theta=0$, $|\underline{v}| = U$

$r = a(2+\cos\theta)$

$\Rightarrow m \times \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow r^2\dot{\theta} = \text{CONSTANT} = h$ (ANGULAR MOMENTUM PER UNIT MASS IS CONSERVED)

- $t=0$, $\theta=0$, $r = 3a$ (FROM $r = a(2+\cos\theta)$)

$\Rightarrow |\underline{v}| = U$

$\Rightarrow \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = U$

- DIFFERENTIATE THE POLAR EQUATION W.R.T t

$\Rightarrow \dot{r} = (-a\sin\theta)\dot{\theta}$

$\Rightarrow \dot{r} = -a\dot{\theta}\sin\theta$

AT $\theta = 0 \Rightarrow \dot{r} = 0$ (NO RADIAL SPEED TO START WITH)

- THUS WE HAVE

$\sqrt{0 + (3a)^2\dot{\theta}^2} = U$

$\boxed{3a\dot{\theta} = U}$

- RETURNING TO $r^2\dot{\theta} = h$ ANGULAR MOMENTUM PER UNIT MASS IS CONSTANT

$\Rightarrow mr^2\dot{\theta} = mh$ ANGULAR MOMENTUM IS CONSERVED

AS THE ANGULAR MOMENTUM IS CONSTANT WE CAN EVALUATE IT INITIALLY

$\Rightarrow mr^2\dot{\theta} = mh = (mU) \times 3a$ MOMENT OF MOMENTUM

LINEAR MOMENTUM

$\Rightarrow mr^2\dot{\theta} = 3maU$

$\Rightarrow r^2\dot{\theta} = 3aU$

$\Rightarrow \boxed{\dot{\theta} = \frac{3aU}{r^2}}$

- NOW AS THE PARTICLE MOVES ON $r = a(2+\cos\theta)$ WE HAVE

$\frac{d\theta}{dt} = \frac{3aU}{a^2(2+\cos\theta)^2}$

$\Rightarrow (2+\cos\theta)^2\,d\theta = \frac{3U}{a}\,dt$

$\Rightarrow \int_0^{2\pi} 4 + 4\cos\theta + \cos^2\theta\,d\theta = \int_{t=0}^{t=T} \frac{3U}{a}\,dt$

$\Rightarrow \int_0^{2\pi} 4 + \cancel{4\cos\theta} + \frac{1}{2} + \cancel{\frac{1}{2}\cos 2\theta}\,d\theta = \left[\frac{3Ut}{a}\right]_0^T$

$\Rightarrow \left[\frac{9}{2}\theta\right]_0^{2\pi} = \frac{3UT}{a}$

$\Rightarrow 9\pi = \frac{3UT}{a}$

$\Rightarrow T = \frac{3\pi a}{U}$

Question 28 (****+)

When a particle is on the initial line of a standard polar coordinate system (r,θ), it has transverse velocity a, where a is a positive constant.

The particle is moving on a path with polar equation

$$r = \frac{a}{1+\sin\theta}, \quad -\pi < \theta < \pi .$$

If the particle experiences a force, which directed towards the pole at all times, show that the radial acceleration of the particle is $-\dfrac{4a^3}{r^2}$.

M45, proof

AS THERE IS NO TRANSVERSE FORCE ACTING WE HAVE NO ACCELERATION IN THE $\hat{\theta}$, DIRECTION

$r^2\dot{\theta} = h$ (CONSTANT)

EVALUATE THE CONSTANT USING THE INITIAL CONDITIONS $\theta=0$, $r=a$, $r\dot{\theta}=a$

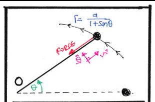
$\Rightarrow r(r\dot{\theta}) = h$

$\Rightarrow a(2a) = h$

$\Rightarrow h = 2a^2$

HENCE WE HAVE

$r^2\dot{\theta} = 2a^2$



- $t=0$ (ARBITRARY)
- $\theta=0$
- $r=a$ (FROM EQUATION)
- $r\dot{\theta} = 2a$ (GIVEN)

ACCELERATION IN POLARS

$\underline{a} = (\ddot{r}-r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

LOOKING AT THE RADIAL ACCELERATION, WE REQUIRE $\ddot{r}$

$r = a(1+\sin\theta)^{-1}$

$\Rightarrow \frac{dr}{dt} = \dot{r} = -a(1+\sin\theta)^{-2}(\cos\theta)\dot{\theta} = -\frac{(a\cos\theta)\dot{\theta}}{(1+\sin\theta)^2}$

$\Rightarrow \dot{r} = -a\dot{\theta}\cos\theta \times \frac{1}{(1+\sin\theta)^2} = -a\cos\theta \times \left(\frac{r}{a}\right)^2 \times \left(\frac{2a^2}{r^2}\right)$

$r=\frac{a}{1+\sin\theta}$ $\quad \dot{\theta}$

$\Rightarrow \dot{r} = -2a\cos\theta$

$\Rightarrow \frac{d}{dt}(\dot{r}) = \frac{d}{dt}(-2a\cos\theta) = (2a\sin\theta)\times\dot{\theta}$

$\Rightarrow \ddot{r} = 2a\sin\theta \times \left(\frac{2a^2}{r^2}\right)$

$\dot{\theta}$ $\quad r=\frac{a}{1+\sin\theta}$

$\Rightarrow \ddot{r} = \frac{4a^3}{r^2}\times\sin\theta = \frac{2a^3}{r^2}\left(\frac{a}{r}-1\right)$

FINALLY WE HAVE

$\ddot{r} - r\dot{\theta}^2 = \frac{4a^3}{r^2}\left(\frac{a}{r}-1\right) - r\left(\frac{2a^2}{r^2}\right)^2$

$= \cancel{\frac{4a^4}{r^3}} - \frac{4a^3}{r^2} - \cancel{\frac{4a^4}{r^3}}$

$= -\frac{4a^3}{r^2}$

Question 29 (****+)

A circular rough wire of radius a and centre O is fixed with the plane of the wire in a horizontal position. A particle of mass m is threaded on the wire. The system lies in a field which exerts a vertical force on the particle in such a way so that the particle is weightless whilst inside the field. The coefficient of friction between the particle and the wire is μ.

When $t=0$, the particle is at the point with polar coordinates $(r,\theta)=(a,0)$, and is given an initial angular speed ω.

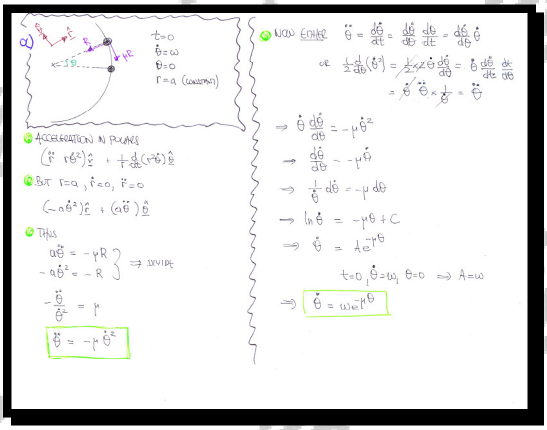
a) By forming and solving a suitable differential equation, show that the angular speed of the particle $\frac{d\theta}{dt}$, in time t satisfies

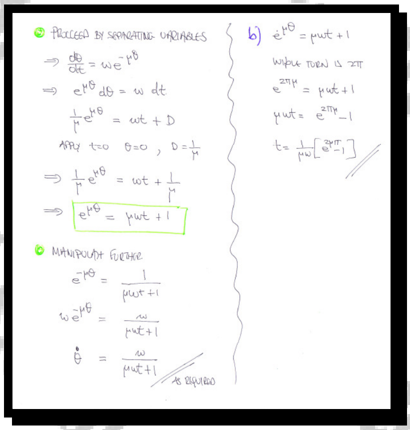
$$\frac{d\theta}{dt}=\frac{\omega}{\mu\omega t+1}.$$

b) Show further that the time T it takes the particle to complete its first revolution is given by

$$T=\frac{e^{2\mu\pi}-1}{\mu\omega}.$$

proof





Question 30 (****+)

A particle P, of mass m, is moving around a fixed origin O under the action of a single force of magnitude $\frac{mk}{r^2}$, where k is a positive constant.

This force is always directed along PO towards O.

At time t the length OP is r and the angular velocity of P around O is $\frac{d\theta}{dt}$.

a) Show that if h is a positive constant

$$\frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

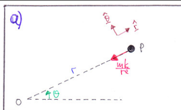
b) By using the substitution $u = \frac{1}{r}$ show further that

$$r = \frac{1}{A\cos\theta + B\sin\theta + C},$$

where A, B and C are constants.

proof

a)



ACCELERATION IN POLARS

$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\underline{\theta}}$

● TRANSVERSELY ($\hat{\underline{\theta}}$)

$\Rightarrow m\left(\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right) = 0$

$\Rightarrow \frac{d}{dt}(r^2\dot{\theta}) = 0$

$\Rightarrow \boxed{r^2\dot{\theta} = h}$ ← CONSTANT (ANGULAR MOMENTUM PER UNIT MASS)

● RADIALLY ($\hat{\underline{r}}$)

$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -\frac{mk}{r^2}$

$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -\frac{k}{r^2}$

$\Rightarrow \ddot{r} - r\left(\frac{h}{r^2}\right)^2 = -\frac{k}{r^2}$

$\Rightarrow \ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}$

b)

$u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$

(differentiate w.r.t t)

$\Rightarrow \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right)$

$\Rightarrow \dot{r} = -\frac{1}{u^2}\frac{du}{dt}$

$\Rightarrow \dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

$\Rightarrow \dot{r} = -r^2\frac{du}{d\theta}\dot{\theta}$ $\quad (h = r^2\dot{\theta})$

$\Rightarrow \dot{r} = -h\frac{du}{d\theta}$

(differentiate w.r.t t again)

$\Rightarrow \ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$

$\Rightarrow \ddot{r} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$

$\Rightarrow \ddot{r} = -h\dot{\theta}\frac{d^2u}{d\theta^2}$

$\Rightarrow \ddot{r} = -h\left(\frac{h}{r^2}\right)\frac{d^2u}{d\theta^2}$

$\Rightarrow \ddot{r} = -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2}$

$\Rightarrow -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2} - \frac{h^2}{r^3} = -\frac{k}{r^2}$ $\quad \left(\text{PART (a)} \quad \ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}\right)$

$\Rightarrow -h^2\frac{d^2u}{d\theta^2} - \frac{h^2}{r} = -k$

$\Rightarrow \frac{d^2u}{d\theta^2} + \frac{1}{r} = \frac{k}{h^2}$

$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}}$

THIS IS S.H.M WITH

C.F. $u = A\cos\theta + B\sin\theta$

P.I. $u = \frac{k}{h^2}$

$\Rightarrow u(\theta) = A\cos\theta + B\sin\theta + \frac{k}{h^2}$

$\Rightarrow r(\theta) = \frac{1}{A\cos\theta + B\sin\theta + \frac{k}{h^2}}$

$\Rightarrow r(\theta) = \frac{1}{A\cos\theta + B\sin\theta + C}$

Question 31 (*****)

A particle of mass m is free to move on a smooth horizontal surface. The particle is attached to one end of a light elastic spring of natural length l and modulus of elasticity λ. The other end of the spring is attached to a fixed point O, on the surface.

The particle is held on the surface with the spring at its natural length and then is projected with speed U at right angles to the spring.

Ignoring air resistance and assuming that in standard S.I. units $m=1$, $l=1$, $9\lambda=8$ and $U=2$, determine the range of values of the length of the spring in the subsequent motion.

$\boxed{1 \le \text{length} \le 3}$