KINEMATICS & DYNAMICS

BY ADVANCED DIFFERENTIAL EQUATIONS

Question 1 (+)**

In this question take $g = 10 \text{ ms}^{-2}$.

A particle of mass M kg is released from rest from a height H m, and allowed to fall down through still air all the way to the ground.

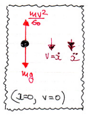
Let v ms^{-1} be the velocity of the particle t s after it was released.

The motion of the particle is subject to air resistance of magnitude $\dfrac{mv^2}{60}$.

Given that the particle reaches the ground with speed 14 ms^{-1}, find the value of H.

$$\boxed{H = 30\ln\left(\frac{150}{101}\right) \approx 11.865...}$$

FORMING A DIFFERENTIAL EQUATION FOR THE MOTION



$\Rightarrow m\ddot{x} = mg - \frac{mv^2}{60}$

$\Rightarrow \ddot{x} = g - \frac{1}{60}v^2$

$\Rightarrow 60\ddot{x} = 60g - v^2$

$\Rightarrow 60\frac{dv}{dt} = 600 - v^2 \quad (g=10)$

$\Rightarrow 60\frac{dv}{dx}\frac{dx}{dt} = 600 - v^2$

$\Rightarrow 60\frac{dv}{dx}v = 600 - v^2$

SOLVING BY SEPARATION OF VARIABLES SUBJECT TO CONDITIONS

$\Rightarrow \frac{60v}{600-v^2}\,dv = 1\,dx$

$\Rightarrow \int_{v=0}^{14} \frac{60v}{600-v^2}\,dv = \int_{x=0}^{x} 1\,dx$

$\Rightarrow -30\int_{v=0}^{14} \frac{-2v}{600-v^2}\,dv = \int_{x=0}^{x} 1\,dx$

$\Rightarrow -30\left[\ln|600-v^2|\right]_0^{14} = \left[x\right]_0^{x}$

$\Rightarrow -30\left[\ln 404 - \ln 600\right] = x$

$\Rightarrow x = 30\ln\frac{150}{101} \approx 11.8654...$ APPROX 11.90m

Question 2 (***)

An object is released from rest from a great height, and allowed to fall down through still air all the way to the ground.

Let v ms^{-1} be the velocity of the object t seconds after it was released.

The velocity of the object is increasing at the constant rate of 10 ms^{-1} every second, but at the same time due to the air resistance its velocity is decreasing at a rate proportional its velocity at that time.

The maximum velocity that the particle can achieve is 100 ms^{-1}.

By forming and solving a differential equation, show that

$$v = 100\left(1 - e^{-0.1t}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

FORMING A DIFFERENTIAL EQUATION

$$\frac{dv}{dt} = 10 - kv$$

DECREASE AT A RATE PROPORTIONAL TO ITS VELOCITY

CONSTANT RATE OF VELOCITY INCREASE ($g = 10$ HERE)

RATE OF CHANGE OF VELOCITY

NEXT WE ARE GIVEN THE TERMINAL VELOCITY AS 100

$\Rightarrow$ WHEN $v = 100$, $\frac{dv}{dt} = 0$

$\Rightarrow 0 = 10 - k \times 100$

$\Rightarrow 100k = 10$

$\Rightarrow k = \frac{1}{10}$

$$\therefore \frac{dv}{dt} = 10 - \frac{1}{10}v$$

SOLVING THE O.D.E BY SEPARATION OF VARIABLES

$\Rightarrow 10\frac{dv}{dt} = 100 - v$

$\Rightarrow \frac{10}{100 - v}\,dv = 1\,dt$

INTEGRATE SUBJECT TO THE CONDITION, $t = 0, v = 0$

$\Rightarrow \int_0^v \frac{10}{100 - v}\,dv = \int_0^t 1\,dt$

$\Rightarrow \left[-10\ln(100 - v)\right]_0^v = \left[t\right]_0^t$

$\Rightarrow \left[\ln(100 - v)\right]_0^v = \left[-\frac{1}{10}t\right]_0^t$

$\Rightarrow \ln(100 - v) - \ln 100 = -\frac{1}{10}t$

$\Rightarrow \ln\left(\frac{100 - v}{100}\right) = -\frac{1}{10}t$

$\Rightarrow \frac{100 - v}{100} = e^{-\frac{1}{10}t}$

$\Rightarrow 100 - v = 100e^{-\frac{1}{10}t}$

$\Rightarrow 100 - 100e^{-\frac{1}{10}t} = v$

$\Rightarrow v = 100\left(1 - e^{-\frac{1}{10}t}\right)$ //

Question 3 (***)

A small raindrop of mass m kg, is released from rest from a rain cloud and is falling through still air under the action of its own weight. The raindrop is subject to air resistance of magnitude kmv N, where v ms^{-1} is the speed of the raindrop t s after release, and k is a positive constant.

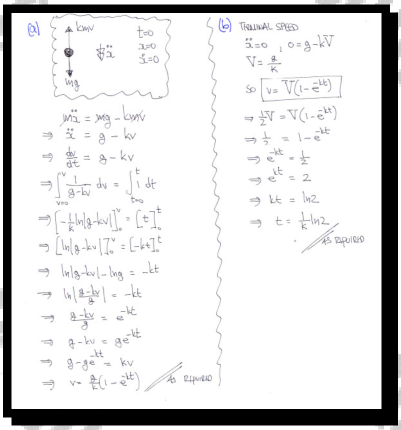
a) Show clearly that

$$v = \frac{g}{k}\left(1 - e^{-kt}\right).$$

The raindrop has terminal speed V.

b) Show that the raindrop reaches a speed of $\frac{1}{2}V$ in time $\frac{1}{k}\ln 2$ seconds.

$\boxed{\text{proof}}$



$m\ddot{x} = mg - kmv$

$\Rightarrow \ddot{x} = g - kv$

$\Rightarrow \frac{dv}{dt} = g - kv$

$\Rightarrow \int_{v=0}^{v} \frac{1}{g-kv}\,dv = \int_{t=0}^{t} 1\,dt$

$\Rightarrow \left[-\frac{1}{k}\ln|g-kv|\right]_0^v = \left[t\right]_0^t$

$\Rightarrow \left[\ln|g-kv|\right]_0^v = \left[-kt\right]_0^t$

$\Rightarrow \ln|g-kv| - \ln g = -kt$

$\Rightarrow \ln\left|\frac{g-kv}{g}\right| = -kt$

$\Rightarrow \frac{g-kv}{g} = e^{-kt}$

$\Rightarrow g - kv = ge^{-kt}$

$\Rightarrow g - ge^{-kt} = kv$

$\Rightarrow v = \frac{g}{k}\left(1 - e^{-kt}\right)$ // AS REQUIRED

(b) TERMINAL SPEED

$\ddot{x} = 0$, $0 = g - kV$

$V = \frac{g}{k}$

SO $\boxed{v = V\left(1 - e^{-kt}\right)}$

$\Rightarrow \frac{1}{2}V = V\left(1 - e^{-kt}\right)$

$\Rightarrow \frac{1}{2} = 1 - e^{-kt}$

$\Rightarrow e^{-kt} = \frac{1}{2}$

$\Rightarrow e^{kt} = 2$

$\Rightarrow kt = \ln 2$

$\Rightarrow t = \frac{1}{k}\ln 2$ // AS REQUIRED

Question 4 (***)

A particle of mass 2 kg is attached to one end of a light elastic spring of natural length 0.5 m and modulus of elasticity 5 N. The other end of the string is attached to a fixed point O on a smooth horizontal plane.

The particle is held at rest on the plane with the spring stretched to a length of 1 m and released at time $t = 0$ s.

During the subsequent motion, when the particle is moving with speed v ms^{-1} it experiences a resistance of magnitude $8v$ N. At time t s after the particle is released, the length of the spring is $(0.5 + x)$ m, where $-0.5 \le x \le 0.5$.

a) Show that x is a solution of the differential equation

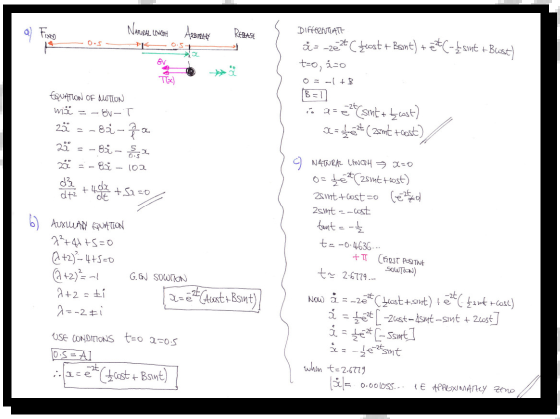
$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0,$$

where a and b are positive integers to be found.

b) Hence express x in terms of t.

c) Show further that the particle almost comes to rest, as the spring returns to its natural length.

$\boxed{a = 4}$, $\boxed{b = 5}$, $\boxed{x = \frac{1}{2}e^{-2t}\left[2\sin t + \cos t\right]}$


a) FIXED NATURAL LENGTH ARBITRARY RELEASE
0.5 0.5 x
$8v$ $T(x)$ $\ddot{x}$
EQUATION OF MOTION
$m\ddot{x} = -8v - T$
$2\ddot{x} = -8\dot{x} - \frac{\lambda}{\ell}x$
$2\ddot{x} = -8\dot{x} - \frac{5}{0.5}x$
$2\ddot{x} = -8\dot{x} - 10x$
$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0$
b) AUXILIARY EQUATION
$\lambda^2 + 4\lambda + 5 = 0$
$(\lambda+2)^2 - 4 + 5 = 0$
$(\lambda+2)^2 = -1$
$\lambda + 2 = \pm i$
$\lambda = -2 \pm i$
G.N SOLUTION $\boxed{x = e^{-2t}(A\cos t + B\sin t)}$
USE CONDITIONS $t=0$ $x=0.5$
$\boxed{0.5 = A}$
$\therefore \boxed{x = e^{-2t}(\frac{1}{2}\cos t + B\sin t)}$
DIFFERENTIATE
$\dot{x} = -2e^{-2t}(\frac{1}{2}\cos t + B\sin t) + e^{-2t}(-\frac{1}{2}\sin t + B\cos t)$
$t=0,\ \dot{x}=0$
$0 = -1 + B$
$\boxed{B=1}$
$\therefore x = e^{-2t}(\sin t + \frac{1}{2}\cos t)$
$x = \frac{1}{2}e^{-2t}(2\sin t + \cos t)$
c) NATURAL LENGTH $\Rightarrow x = 0$
$0 = \frac{1}{2}e^{-2t}(2\sin t + \cos t)$
$2\sin t + \cos t = 0 \quad (e^{-2t} \neq 0)$
$2\sin t = -\cos t$
$\tan t = -\frac{1}{2}$
$t = -0.4636\ldots$
$+\pi$ (FIRST POSITIVE SOLUTION)
$t \simeq 2.6779\ldots$
NOW $\dot{x} = -2e^{-2t}(\frac{1}{2}\cos t + \sin t) + e^{-2t}(-\frac{1}{2}\sin t + \cos t)$
$\dot{x} = \frac{1}{2}e^{-2t}[-2\cos t - 4\sin t - \sin t + 2\cos t]$
$\dot{x} = \frac{1}{2}e^{-2t}[-5\sin t]$
$\dot{x} = -\frac{5}{2}e^{-2t}\sin t$
when $t = 2.6779$
$|\dot{x}| = 0.001055\ldots$ I.E. APPROXIMATELY ZERO

Question 5 (***)

A particle P of mass M kg is attached to one end of a light elastic spring of natural length a m and modulus of elasticity $4Ma$ N. The other end of the string is attached to a fixed point O on a smooth horizontal plane.

The particle is held at rest on the plane with the spring at its natural length.

At time $t = 0$ s, P is projected with speed $\sqrt[3]{4}$ ms^{-1} in the direction PO.

During the subsequent motion, when the particle is moving with speed v ms^{-1} it experiences an additional resistive force of magnitude $5Mv$ N. At time t s after the particle is released, the length of the spring is $(a+x)$ m, where $-a \le x \le a$.

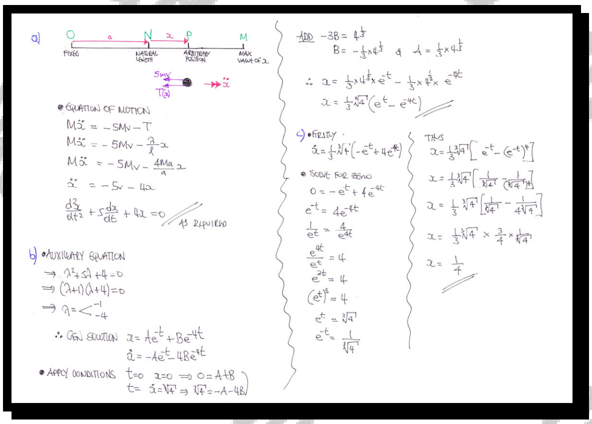
a) Show that x is a solution of the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 5\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0 .$$

b) Hence express x in terms of t.

c) Determine the greatest value of x.

$\boxed{x = \frac{1}{3}\sqrt[3]{4}\left[\mathrm{e}^{-t} + \mathrm{e}^{-4t}\right]}$, $\boxed{x_{\max} = 0.25 \text{ m}}$



Question 6 (***)

A small truck, of mass 1500 kg, travels along a straight horizontal road, with the engine working at the constant rate of 30 kW. The truck starts from rest and t s later its speed is v ms^{-1}.

The truck during its motion experiences air resistance proportional to its speed.

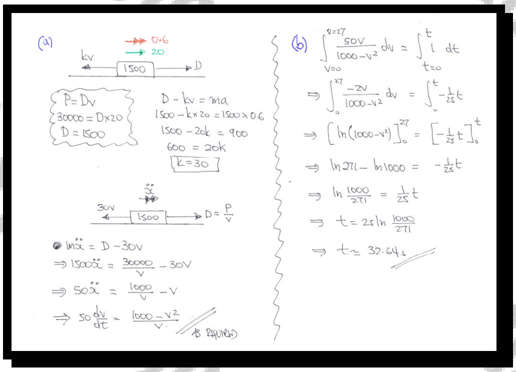
When the speed of the truck reaches 20 ms^{-1} its acceleration is 0.6 ms^{-2}.

a) Show clearly that

$$50\frac{dv}{dt} = \frac{1000 - v^2}{v}.$$

b) Calculate the time it takes the truck to reach a speed of 27 ms^{-1}.

$\boxed{t \approx 32.64 \text{ s}}$



Question 7 (***+)

A small truck, of mass 1800 kg, travels along a straight horizontal road, with the engine working at the constant rate of 45 kW.

The total resistance experienced by the truck during its motion is $25v$, where $v \text{ ms}^{-1}$ is the speed of the truck at time t s.

The track takes T s to accelerate from 18 ms^{-1} to 24 ms^{-1}, and in that time it coves a distance X m.

a) By forming and solving a differential equation, show clearly that

$$T = 36\ln\left(\frac{4}{3}\right).$$

b) Determine the value of X.

$$\boxed{X = -216 + 1080\ln\left(\frac{3}{2}\right) \approx 222 \text{ m}}$$

a) $\ddot{x}$, v; $50v$ ← $\boxed{1800}$ → $\frac{45000}{v}$

$\Rightarrow m\ddot{x} = F$

$\Rightarrow 1800\ddot{x} = \frac{45000}{v} - 50v$

$\Rightarrow 1800\ddot{x} = \frac{45000 - 50v^2}{v}$

$\Rightarrow 1800\ddot{x} = \frac{50(900 - v^2)}{v}$

$\Rightarrow \ddot{x} = \frac{50(900 - v^2)}{1800v}$

$\Rightarrow \boxed{\ddot{x} = \frac{900 - v^2}{36v}}$

$\Rightarrow \frac{dv}{dt} = \frac{900 - v^2}{36v}$

$\Rightarrow \int_{18}^{24} \frac{36v}{900 - v^2}\,dv = \int_{t_1}^{t_2} 1\,dt$

$\Rightarrow \left[-18\ln(900 - v^2)\right]_{18}^{24} = \left[t\right]_{t_1}^{t_2}$

$\Rightarrow 18\left[\ln(900 - v^2)\right]_{24}^{18} = t_2 - t_1$

$\Rightarrow T = 18\left[\ln 576 - \ln 324\right]$

$\Rightarrow T = 18\ln\frac{576}{324}$

$\Rightarrow T = 18\ln\frac{16}{9}$

$\Rightarrow T = 18\ln\left(\frac{4}{3}\right)^2$

$\Rightarrow T = 36\ln\frac{4}{3}$

b) $\frac{dv}{dt} = \frac{900 - v^2}{36v}$

$\Rightarrow v\frac{dv}{dx} = \frac{900 - v^2}{36v}$

$\Rightarrow \frac{36v^2}{900 - v^2}\frac{dv}{dx} = 1$

$\Rightarrow \frac{-v^2}{900 - v^2}\frac{dv}{dx} = -\frac{1}{36}$

$\Rightarrow \int_{18}^{24} \frac{(900 - v^2) - 900}{900 - v^2}\,dv = \int_{x_1}^{x_2} -\frac{1}{36}\,dx$

$\Rightarrow \int_{18}^{24} 1 - \frac{900}{(30 - v)(30 + v)}\,dv = \int_{x_1}^{x_2} -\frac{1}{36}\,dx$

PARTIAL FRACTIONS BY COVER UP

$\Rightarrow \int_{18}^{24} 1 - \left(\frac{15}{30 - v} - \frac{15}{30 + v}\right) dv = \left[-\frac{1}{36}x\right]_{x_1}^{x_2}$

$\Rightarrow \left[v + 15\ln(30 - v) + 15\ln(30 + v)\right]_{18}^{24} = \left[-\frac{1}{36}x\right]_{x_1}^{x_2}$

$\Rightarrow \left[v + 15\ln\left(\frac{30 - v}{30 + v}\right)\right]_{18}^{24} = -\frac{1}{36}(x_2 - x_1)$

$\Rightarrow \left(24 + 15\ln\frac{1}{9}\right) - \left(18 + 15\ln\frac{1}{4}\right) = -\frac{1}{36}(x_2 - x_1)$

$\Rightarrow 6 + 15\ln\frac{1}{9} - 15\ln\frac{1}{4} = -\frac{1}{36}(x_2 - x_1)$

$\Rightarrow 6 - 15\ln 9 + 15\ln 4 = -\frac{1}{36}(x_2 - x_1)$

$\Rightarrow 6 + 15\ln\frac{4}{9} = -\frac{1}{36}(x_2 - x_1)$

$\Rightarrow x_2 - x_1 = -36\left(6 + 15\ln\left(\frac{2}{3}\right)^2\right)$

$\Rightarrow X = -216 - 540\ln\left(\frac{2}{3}\right)^2$

$\Rightarrow X = -216 - 1080\ln\frac{2}{3}$

$\Rightarrow X = -216 + 1080\ln\frac{3}{2}$

$\simeq 222 \text{ m}$

Question 8 **(***+)**

An object is placed on the still water of a lake and allowed to fall down through the water to the bottom of the lake.

Let v ms^{-1} be the velocity of the object t seconds after it was released.

The velocity of the object is increasing at the constant rate of 9.8 ms^{-1} every second.

At the same time due to the resistance of the water its velocity is decreasing at a rate proportional to the square of its velocity at that time.

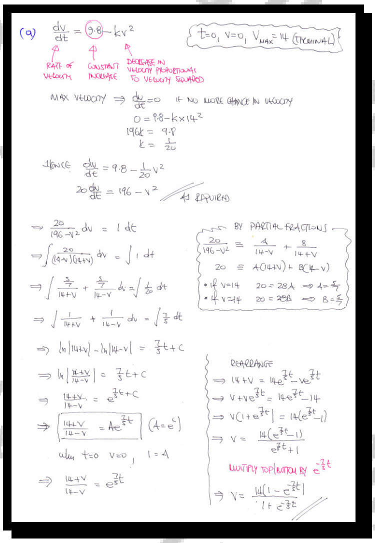
The maximum velocity that the particle can achieve is 14 ms^{-1}.

Show clearly that …

a) … $20\dfrac{dv}{dt} = 196 - v^2$.

b) … $v = 14\left(\dfrac{1-\mathrm{e}^{-1.4t}}{1+\mathrm{e}^{-1.4t}}\right)$

proof



(a) $\dfrac{dv}{dt} = 9.8 - kv^2$

RATE OF VELOCITY

CONSTANT INCREASE

DECREASE IN VELOCITY PROPORTIONAL TO VELOCITY SQUARED

$t=0,\ v=0,\ V_{MAX}=14$ (TERMINAL)

MAX VELOCITY $\Rightarrow \dfrac{dv}{dt}=0$ IF NO MORE CHANGE IN VELOCITY

$0 = 9.8 - k\times 14^2$

$196k = 9.8$

$k = \frac{1}{20}$

HENCE $\dfrac{dv}{dt} = 9.8 - \frac{1}{20}v^2$

$20\dfrac{dv}{dt} = 196 - v^2$ AS REQUIRED

$\Rightarrow \dfrac{20}{196-v^2}\,dv = 1\,dt$

$\Rightarrow \displaystyle\int \frac{20}{(14-v)(14+v)}\,dv = \int 1\,dt$

$\Rightarrow \displaystyle\int \frac{\frac{5}{7}}{14+v} + \frac{\frac{5}{7}}{14-v}\,dv = \int \frac{1}{20}\,dt$

$\Rightarrow \displaystyle\int \frac{1}{14+v} + \frac{1}{14-v}\,dv = \int \frac{7}{5}\,dt$

$\Rightarrow \ln|14+v| - \ln|14-v| = \frac{7}{5}t + C$

$\Rightarrow \ln\left|\dfrac{14+v}{14-v}\right| = \frac{7}{5}t + C$

$\Rightarrow \dfrac{14+v}{14-v} = e^{\frac{7}{5}t+C}$

$\Rightarrow \boxed{\dfrac{14+v}{14-v} = Ae^{\frac{7}{5}t}}$ $(A = e^C)$

when $t=0$ $v=0$, $1 = A$

$\Rightarrow \dfrac{14+v}{14-v} = e^{\frac{7}{5}t}$

BY PARTIAL FRACTIONS

$\dfrac{20}{196-v^2} \equiv \dfrac{A}{14-v} + \dfrac{B}{14+v}$

$20 \equiv A(14+v) + B(14-v)$

- if $v=14$ $20 = 28A \Rightarrow A = \frac{5}{7}$
- if $v=-14$ $20 = 28B \Rightarrow B = \frac{5}{7}$

REARRANGE

$\Rightarrow 14 + v = 14e^{\frac{7}{5}t} - ve^{\frac{7}{5}t}$

$\Rightarrow v + ve^{\frac{7}{5}t} = 14e^{\frac{7}{5}t} - 14$

$\Rightarrow v(1+e^{\frac{7}{5}t}) = 14(e^{\frac{7}{5}t} - 1)$

$\Rightarrow v = \dfrac{14(e^{\frac{7}{5}t}-1)}{e^{\frac{7}{5}t}+1}$

MULTIPLY TOP/BOTTOM BY $e^{-\frac{7}{5}t}$

$\Rightarrow v = \dfrac{14(1-e^{-\frac{7}{5}t})}{1+e^{-\frac{7}{5}t}}$

Question 9 (****)

A particle P of mass m is attached to the midpoint of a light elastic spring AB, of natural length l and modulus of elasticity λ. The end A of the spring is attached to a fixed point on a smooth horizontal floor. The end B is held at a point on the floor where $|AB| = 2a$, $a > l$.

At time $t = 0$, P is at rest on the floor at the point M, where $|MA| = a$.

The end B is now moved along the floor in such a way that AMB remains in a straight line and at time t s, $t \geq 0$

$$|AB| = 2a + A\sin 2t,$$

where A is a positive constant.

a) Show that, for $t \geq 0$

$$\frac{d^2x}{dt^2} + \frac{2\lambda}{ml}x = \frac{A\lambda}{ml}\sin 2t,$$

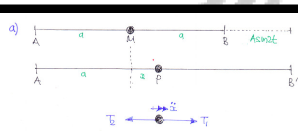
where $x = |MP|$ for $t \geq 0$.

It is now given that $m = 0.5$ kg, $l = 1$ m, $\lambda = 4$ N and $A = 1.5$ m.

b) Find the time at which P first comes to instantaneous rest.

$$\boxed{t = \frac{1}{3}\pi}$$

a)



$T_2 = \frac{\lambda}{l}(a + x - l)$

$T_1 = \frac{\lambda}{l}(2a + A\sin 2t - a - x - l) = \frac{\lambda}{l}(a - x - l + A\sin 2t)$

EQUATION OF MOTION

$m\ddot{x} = T_1 - T_2$

$m\ddot{x} = \frac{\lambda}{l}(\cancel{a} - x - l + A\sin 2t) - \frac{\lambda}{l}(\cancel{a} + x - l)$

$m\ddot{x} = \frac{\lambda}{l}\left[-2x + A\sin 2t\right]$

$m\ddot{x} + \frac{2\lambda}{l}x = \frac{A\lambda}{l}\sin 2t$

$\ddot{x} + \frac{2\lambda}{ml}x = \frac{A\lambda}{ml}\sin 2t$ // AS REQUIRED

b) O.D.E BECOMES $\ddot{x} + \frac{2\times 4}{0.5\times 1}x = \frac{1.5\times 4}{0.5\times 1}\sin 2t$

$\ddot{x} + 16x = 12\sin 2t$

COMPLIMENTARY FUNCTION

$x(t) = P\cos 4t + Q\sin 4t$

PARTICULAR INTEGRAL

$x = R\sin 2t$

$\dot{x} = 2R\cos 2t$

$\ddot{x} = -4R\sin 2t$

$-4R + 16R = 12$

$\boxed{R = 1}$

THUS GENERAL SOLUTION IS

$x(t) = P\cos 4t + Q\sin 4t + \sin 2t$

$\dot{x}(t) = -4P\sin 4t + 4Q\cos 4t + 2\cos 2t$

$t = 0,\ x = 0,\ \dot{x} = 0$

$\boxed{0 = P}$

$0 = 4Q + 2$

$\boxed{Q = -\frac{1}{2}}$

$\therefore \boxed{x(t) = \sin 2t - \frac{1}{2}\sin 4t}$

$\dot{x}(t) = 2\cos 2t - 2\cos 4t$

$0 = 2\cos 2t - 2\cos 4t$

$\cos 4t = \cos 2t$

$\begin{cases} 4t = 2t \pm 2n\pi \\ 4t = -2t \pm 2n\pi \end{cases}$ $n = 0, 1, 2, 3, \ldots$

$\begin{cases} 2t = 0 \pm 2n\pi \\ 6t = 0 \pm 2n\pi \end{cases}$

$\begin{cases} t = \pm n\pi \\ t = \pm \frac{n\pi}{3} \end{cases}$

$\therefore$ FIRST POSITIVE SOLUTION OCCURS $t = \frac{\pi}{3}$ //

Question 10 (****)

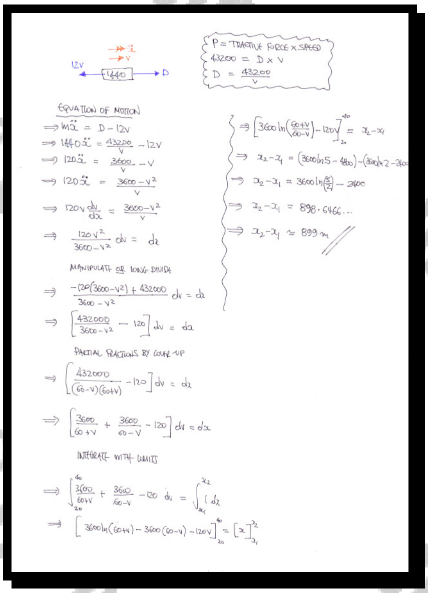
A car of mass 1440 kg is moving along a straight horizontal road.

The engine of the car is working at a constant rate of 43.2 kW.

When the speed of the car is $v \text{ ms}^{-1}$, the resistance to motion has magnitude $12v \text{ N}$.

Calculate the distance travelled by the car as it accelerates from a speed of $v \text{ ms}^{-1}$ to a speed of $v \text{ ms}^{-1}$.

$$\boxed{d = 3600\ln\left(\frac{5}{2}\right) - 2400 \approx 899 \text{ m}}$$



$P = \text{TRACTIVE FORCE} \times \text{SPEED}$
$43200 = D \times v$
$D = \frac{43200}{v}$

EQUATION OF MOTION

$\Rightarrow m\ddot{x} = D - 12v$

$\Rightarrow 1440\ddot{x} = \frac{43200}{v} - 12v$

$\Rightarrow 120\ddot{x} = \frac{3600}{v} - v$

$\Rightarrow 120\ddot{x} = \frac{3600 - v^2}{v}$

$\Rightarrow 120v\frac{dv}{dx} = \frac{3600 - v^2}{v}$

$\Rightarrow \frac{120v^2}{3600 - v^2}\,dv = dx$

MANIPULATE OR LONG DIVIDE

$\Rightarrow \frac{-120(3600 - v^2) + 432000}{3600 - v^2}\,dv = dx$

$\Rightarrow \left[\frac{432000}{3600 - v^2} - 120\right]dv = dx$

PARTIAL FRACTIONS BY COVER UP

$\Rightarrow \left[\frac{432000}{(60 - v)(60 + v)} - 120\right]dv = dx$

$\Rightarrow \left[\frac{3600}{60 + v} + \frac{3600}{60 - v} - 120\right]dv = dx$

INTEGRATE WITH LIMITS

$\Rightarrow \int_{20}^{40} \frac{3600}{60 + v} + \frac{3600}{60 - v} - 120\,dv = \int_{x_1}^{x_2} 1\,dx$

$\Rightarrow \left[3600\ln(60 + v) - 3600(60 - v) - 120v\right]_{20}^{40} = \left[x\right]_{x_1}^{x_2}$

$\Rightarrow \left[3600\ln\left(\frac{60 + v}{60 - v}\right) - 120v\right]_{20}^{40} = x_2 - x_1$

$\Rightarrow x_2 - x_1 = (3600\ln 5 - 4800) - (3600\ln 2 - 2400$

$\Rightarrow x_2 - x_1 = 3600\ln\left(\frac{5}{2}\right) - 2400$

$\Rightarrow x_2 - x_1 = 898.6466\ldots$

$\Rightarrow x_2 - x_1 \simeq 899 \text{ m}$

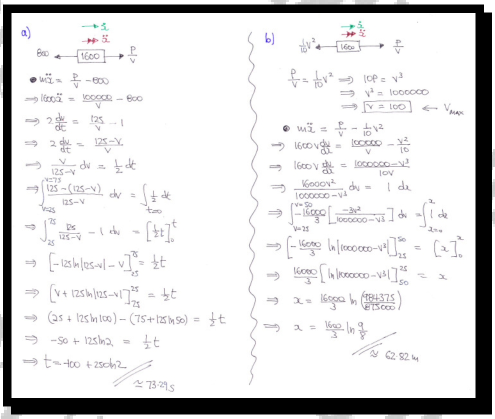
Question 11 (****)

The engine of a racing car, of mass 1600 kg, is working at constant power of 100 kW.

a) Given the total resistances to motion is 800 N, determine the time it takes the car to accelerate from 25 ms^{-1} to 75 ms^{-1}.

b) Given instead that the total resistances to motion have magnitude $\frac{1}{10}v$ N, where v is the speed car, determine the distance the car covers in accelerating from 25 ms^{-1} to half the maximum speed of the car.

$\boxed{t = -100 + 250\ln 2 \approx 73.29 \text{ s}}$, $\boxed{x = \frac{1600}{3}\ln\left(\frac{9}{8}\right) \approx 62.82 \text{ m}}$



a)

$$m\ddot{x} = \frac{P}{v} - 800$$
$$\Rightarrow 1600\ddot{x} = \frac{100000}{v} - 800$$
$$\Rightarrow 2\frac{dv}{dt} = \frac{125}{v} - 1$$
$$\Rightarrow 2\frac{dv}{dt} = \frac{125-v}{v}$$
$$\Rightarrow \frac{v}{125-v}\,dv = \frac{1}{2}\,dt$$
$$\Rightarrow \int_{v=25}^{v=75} \frac{125-(125-v)}{125-v}\,dv = \int_{t=0} \frac{1}{2}\,dt$$
$$\Rightarrow \int_{25}^{75} \frac{125}{125-v} - 1\,dv = \left[\frac{1}{2}t\right]_0^t$$
$$\Rightarrow \left[-125\ln|125-v| - v\right]_{25}^{75} = \frac{1}{2}t$$
$$\Rightarrow \left[v + 125\ln|125-v|\right]_{75}^{25} = \frac{1}{2}t$$
$$\Rightarrow (25 + 125\ln 100) - (75 + 125\ln 50) = \frac{1}{2}t$$
$$\Rightarrow -50 + 125\ln 2 = \frac{1}{2}t$$
$$\Rightarrow t = -100 + 250\ln 2$$
$$\simeq 73.29\text{ s}$$

b)

$$\frac{P}{v} = \frac{1}{10}v^2 \Rightarrow 10P = v^3 \Rightarrow v^3 = 1000000 \Rightarrow \boxed{v = 100} \leftarrow V_{MAX}$$
$$m\ddot{x} = \frac{P}{v} - \frac{1}{10}v^2$$
$$\Rightarrow 1600v\frac{dv}{dx} = \frac{100000}{v} - \frac{v^2}{10}$$
$$\Rightarrow 1600v\frac{dv}{dx} = \frac{1000000 - v^3}{10v}$$
$$\Rightarrow \frac{16000v^2}{1000000 - v^3}\,dv = 1\,dx$$
$$\Rightarrow \int_{v=25}^{v=50} -\frac{16000}{3}\left[\frac{-3v^2}{1000000 - v^3}\right]dv = \int_{x=0}^{x} 1\,dx$$
$$\Rightarrow \left[-\frac{16000}{3}\ln|1000000 - v^3|\right]_{25}^{50} = \left[x\right]_0^x$$
$$\Rightarrow \frac{16000}{3}\left[\ln|1000000 - v^3|\right]_{50}^{25} = x$$
$$\Rightarrow x = \frac{16000}{3}\ln\left(\frac{984375}{875000}\right)$$
$$\Rightarrow x = \frac{1600}{3}\ln\frac{9}{8}$$
$$\approx 62.82\text{ m}$$

Question 12 (****)

A small raindrop of mass m kg, is released from rest from a rain cloud and is falling through still air under the action of its own weight.

The raindrop is subject to air resistance of magnitude kmv^2 N, where v ms^{-1} is the speed of the raindrop x m below the point of release, and k is a positive constant.

a) Solve the differential equation to show that

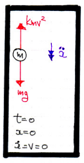
$$v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right).$$

The raindrop has a terminal velocity U.

b) Show further that the raindrop reaches a speed of $\frac{1}{2}U$, after falling through a distance of $\frac{U^2}{2g}\ln\left(\frac{4}{3}\right)$ metres.

$\boxed{}$, $\boxed{\text{proof}}$

FORMING THE EQUATION OF MOTION



$\Rightarrow m\ddot{x} = mg - kmv^2$

$\Rightarrow \ddot{x} = g - kv^2$

$\Rightarrow v\frac{dv}{dx} = g - kv^2$

SOLVING BY SEPARATION OF VARIABLES, SUBJECT TO THE INITIAL CONDITIONS GIVEN

$\Rightarrow v\,dv = (g - kv^2)\,dx$

$\Rightarrow \frac{v}{g - kv^2}\,dv = 1\,dx$

$\Rightarrow \int_{v=0}^{v} \frac{v}{g - kv^2}\,dv = \int_{x=0}^{x} 1\,dx$ $\quad \times(-2k)$

$\Rightarrow \int_{v=0}^{v} \frac{-2kv}{g - kv^2}\,dv = \int_{x=0}^{x} -2k\,dx$

$\Rightarrow \left[\ln|g - kv^2|\right]_{v=0}^{v=v} = \left[-2kx\right]_{x=0}^{x}$

$\Rightarrow \ln|g - kv^2| - \ln g = -2kx - 0$

$\Rightarrow \ln\left|\frac{g - kv^2}{g}\right| = -2kx$

$\Rightarrow \frac{g - kv^2}{g} = e^{-2kx}$

$\Rightarrow 1 - \frac{k}{g}v^2 = e^{-2kx}$

$\Rightarrow 1 - e^{-2kx} = \frac{k}{g}v^2$

$\Rightarrow v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$ // AS REQUIRED

b) FIRSTLY THE TERMINAL VELOCITY IS $\sqrt{\frac{g}{k}}$

[EITHER AS $x \to \infty$ $\quad v^2 \longrightarrow \frac{g}{k}$

OR $\frac{dv}{dx} = 0$ (RATE OF CHANGE OF VELOCITY W.R.T x IS ZERO)

$g - kv^2 = 0$

$v^2 = \frac{g}{k}$]

$\Rightarrow$ WHEN $v = \frac{1}{2}U = \frac{1}{2}\sqrt{\frac{g}{k}}$

$\Rightarrow \frac{1}{4}\frac{g}{k} = \frac{g}{k}\left(1 - e^{-2kx}\right)$

$\Rightarrow \frac{1}{4} = 1 - e^{-2kx}$

$\Rightarrow e^{-2kx} = \frac{3}{4}$

$\Rightarrow e^{2kx} = \frac{4}{3}$

$\Rightarrow 2kx = \ln\frac{4}{3}$

$\Rightarrow x = \frac{1}{2k}\ln\left(\frac{4}{3}\right)$ // AS REQUIRED

Question 13 (****)

A particle of mass m kg, is attached to one end A of a light elastic string AB, of natural length L m and modulus of elasticity $2mL$ N. Initially the particle and the string lie at rest on a smooth horizontal surface, with $|AB| = L$ m

At time $t = 0$, the end B of the string is set in motion with constant speed $2U$ ms^{-1}, in the direction AB, and at time t s, the extension of the string is x m and the displacement of the particle from its initial position is y m.

There is air resistance impeding the motion of the particle, of magnitude $3mv$, where v ms^{-1} is the speed of the particle at time t s.

a) Show that while the string is taut

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 6U .$$

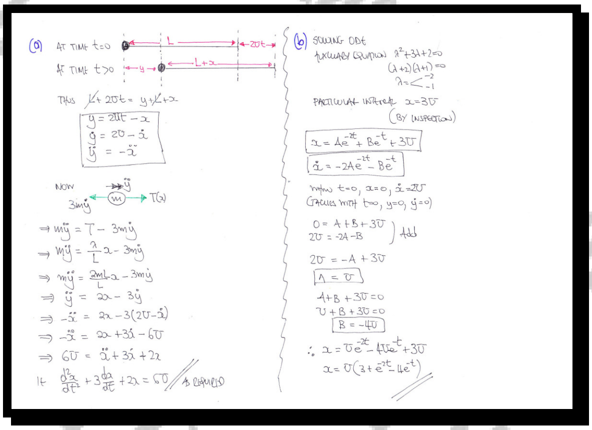
b) Express x in terms of U and t.

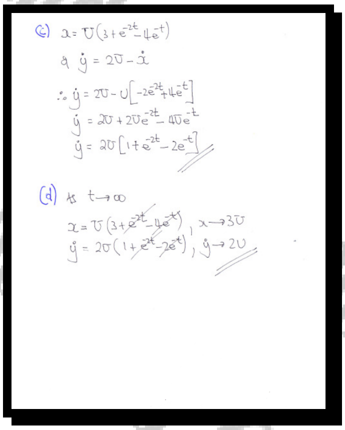
c) Hence find the speed of the particle at time t s.

d) State the extension of the string and the speed of the particle as t gets infinitely large

$\boxed{x = U\left(3 + e^{-2t} - 4e^{-t}\right)}$, $\boxed{v = 2U\left(1 + e^{-2t} - 2e^{-t}\right)}$,

$\boxed{\text{as } t \to \infty, \quad x \to 3U \quad \text{and} \quad v \to 2U}$





Question 14 (****)

A particle of mass 2 kg, is attached to one end A of a light elastic spring AB, of natural length 1.5 m and modulus of elasticity 12 N. Initially the end of the spring B is held at rest, with the particle hanging in equilibrium vertically below B.

At time $t = 0$, the end B of the spring is set in oscillatory motion so that the vertical displacement of B **below** its initial position is given by $5\sin 2t$ m, where t is measured in s.

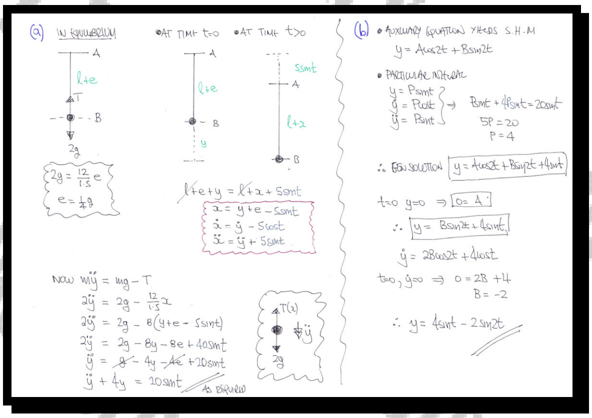
At time t s, the extension of the spring is x m and the displacement of the particle from below its initial position is y m. It is assumed that there is no air resistance impeding the motion of the particle.

a) Show clearly that

$$\frac{d^2y}{dt^2} + +4y = 20\sin t \,.$$

b) Express y in terms of t.

$$\boxed{y = 4\sin t - 2\sin 2t}$$



Question 15 (****)

A particle P, of mass 2 kg, is attached to one end of a light elastic spring of natural length 0.55 m and stiffness 8 Nm^{-1}.

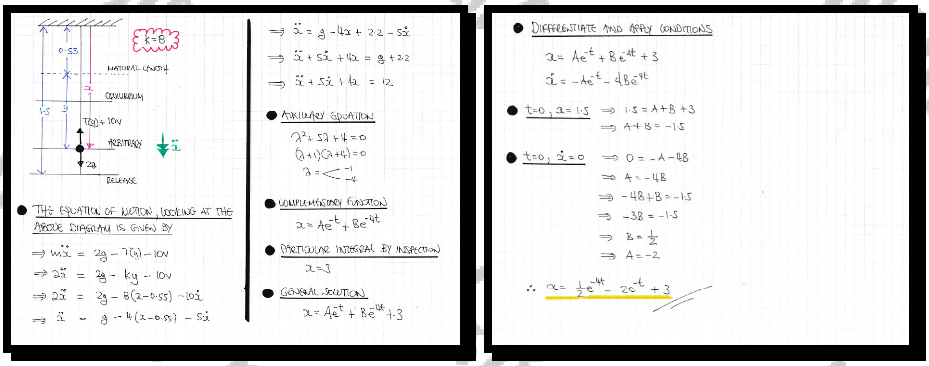
The other end of the spring is attached to a fixed point O, so that P is hanging in equilibrium vertically below O.

At time $t=0$, P is pulled vertically downwards, so that $OP = 1.5$ m ,and released from rest.

The motion of P takes place in a medium which provides resistance of magnitude $10|v|$ N, where $|v| \text{ ms}^{-1}$ is the speed of P at time t s.

If x denotes the distance of the particle from O at time t, express x in terms of t.

$\boxed{}$, $\boxed{x = \frac{1}{2}e^{-4t} - 2e^{-t} + 3}$



Question 16 (****+)

A small raindrop of mass m kg, is released from rest from a rain cloud and is falling through still air under the action of its own weight.

The raindrop is subject to air resistance of magnitude kmv N, where v ms^{-1} is the speed of the raindrop x m below the point of release, and k is a positive constant.

a) Show, by forming and solving a differential equation, that

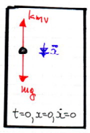
$$x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}.$$

The raindrop has a limiting speed V.

b) Show further that the raindrop reaches a speed of $\frac{1}{2}V$, after a falling through a distance of $\frac{V^2}{2g}(-1+\ln 4)$ metres.

$\boxed{}$, $\boxed{\text{proof}}$

a) SOLVING THE DIFFERENTIAL EQUATION



$$\Rightarrow m\ddot{x} = mg - kmv$$
$$\Rightarrow \ddot{x} = g - kv$$
$$\Rightarrow v\frac{dv}{dx} = g - kv$$
$$\Rightarrow \frac{v}{g-kv}\,dv = 1\,dx$$
$$\Rightarrow \frac{-kv}{g-kv}\,dv = -k\,dx$$
$$\Rightarrow \int_{v=0}^{v}\frac{-kv}{g-kv}\,dv = \int_{x=0}^{x}-k\,dx$$
$$\Rightarrow \int_0^v\frac{(g-kv)-g}{g-kv}\,dv = \int_0^x -k\,dx$$
$$\Rightarrow \int_0^v 1 - \frac{g}{g-kv}\,dv = \int_0^x -k\,dx$$
$$\Rightarrow \left[v + \frac{g}{k}\ln|g-kv|\right]_0^v = \left[-kx\right]_0^x$$
$$\Rightarrow \left[v + \frac{g}{k}\ln|g-kv|\right] - \left[\frac{g}{k}\ln g\right] = -kx - 0$$
$$\Rightarrow v + \frac{g}{k}\ln\left|\frac{g-kv}{g}\right| = -kx$$
$$\Rightarrow x = -\frac{1}{k}v - \frac{g}{k^2}\ln\left|\frac{g-kv}{g}\right|$$
$$\Rightarrow x = \frac{g}{k^2}\ln\left|\frac{g}{g-kv}\right| - \frac{v}{k}$$

AS REQUIRED

b) LOOKING AT THE ORIGINAL O.D.E

$$\ddot{x} = g - kv$$

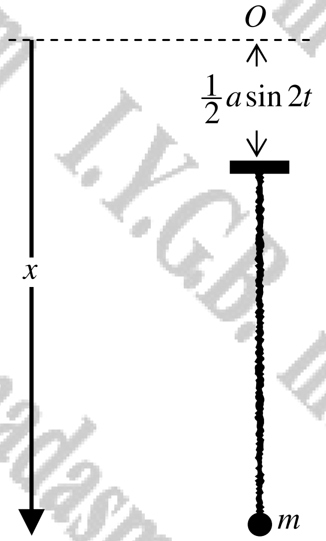
LIMITING SPEED $\Rightarrow \ddot{x} = 0$
$$\Rightarrow V = g/k$$

THUS WE HAVE IF $v = \frac{1}{2}V = g/2k$

$$x = \frac{g}{k^2}\ln\left|\frac{g}{g-kv}\right| - \frac{v}{k}$$
$$x = \frac{g}{k^2}\ln\left|\frac{g}{g-k\frac{g}{2k}}\right| - \frac{g/2k}{k}$$
$$x = \frac{g}{k^2}\ln 2 - \frac{g}{2k^2}$$
$$x = \frac{g}{2k^2}\left[2\ln 2 - 1\right]$$
$$x = \frac{1}{2g}\times\frac{g^2}{k^2}\left[\ln 4 - 1\right]$$
$$x = \frac{1}{2g}V^2\left[\ln 4 - 1\right]$$
$$x = \frac{V^2}{2g}\left[-1 + \ln 4\right]$$

AS REQUIRED

Question 17 (****+)



A particle, of mass m, is attached to one end of a light elastic spring of natural length a and modulus of elasticity $\frac{1}{2}mg$. The other end of the spring is initially stationary at the point O so that the particle is hanging in equilibrium vertically below O.

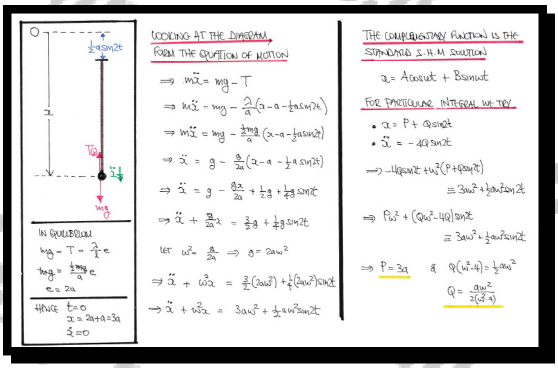
At time $t = 0$, the end of the spring which is at O begins to oscillate so that its **positive** displacement from O is given by $\frac{1}{2}a\sin 2t$.

If x denotes the distance of the particle from O at time t, show that

$$x = 3a + \frac{a\omega}{2\left(\omega^2 - 4\right)}\left[\omega\sin 2t - 2\sin\omega t\right],$$

where $\omega^2 = \frac{g}{2a}$.

$\boxed{}$, $\boxed{\text{proof}}$



$\frac{1}{2}a\sin 2t$

x

$T(x)$

$\ddot{x}$

mg

IN EQUILIBRIUM

$mg = T = \frac{\lambda}{l}e$

$mg = \frac{\frac{1}{2}mg}{a}e$

$e = 2a$

HENCE $t=0$

$x = 2a + a = 3a$

$\dot{x} = 0$

LOOKING AT THE DIAGRAM, FORM THE EQUATION OF MOTION

$\Rightarrow m\ddot{x} = mg - T$

$\Rightarrow m\ddot{x} = mg - \frac{\lambda}{a}\left(x - a - \frac{1}{2}a\sin 2t\right)$

$\Rightarrow m\ddot{x} = mg - \frac{\frac{1}{2}mg}{a}\left(x - a - \frac{1}{2}a\sin 2t\right)$

$\Rightarrow \ddot{x} = g - \frac{g}{2a}\left(x - a - \frac{1}{2}a\sin 2t\right)$

$\Rightarrow \ddot{x} = g - \frac{gx}{2a} + \frac{1}{2}g + \frac{1}{4}g\sin 2t$

$\Rightarrow \ddot{x} + \frac{g}{2a}x = \frac{3}{2}g + \frac{1}{4}g\sin 2t$

LET $\omega^2 = \frac{g}{2a} \Rightarrow g = 2a\omega^2$

$\Rightarrow \ddot{x} + \omega^2 x = \frac{3}{2}(2a\omega^2) + \frac{1}{4}(2a\omega^2)\sin 2t$

$\Rightarrow \ddot{x} + \omega^2 x = 3a\omega^2 + \frac{1}{2}a\omega^2\sin 2t$

THE COMPLEMENTARY FUNCTION IS THE STANDARD S.H.M SOLUTION

$x = A\cos\omega t + B\sin\omega t$

FOR PARTICULAR INTEGRAL WE TRY

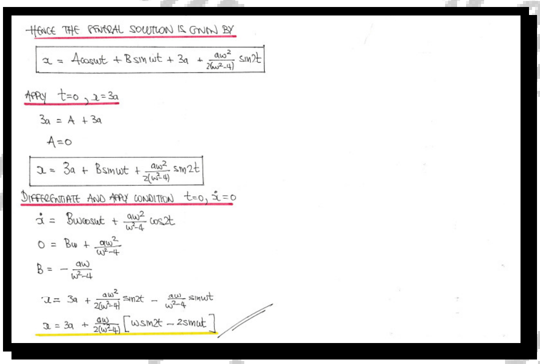
- $x = P + Q\sin 2t$
- $\ddot{x} = -4Q\sin 2t$

$\Rightarrow -4Q\sin 2t + \omega^2(P + Q\sin 2t) \equiv 3a\omega^2 + \frac{1}{2}a\omega^2\sin 2t$

$\Rightarrow P\omega^2 + (Q\omega^2 - 4Q)\sin 2t \equiv 3a\omega^2 + \frac{1}{2}a\omega^2\sin 2t$

$\Rightarrow P = 3a$ & $Q(\omega^2 - 4) = \frac{1}{2}a\omega^2$

$Q = \frac{a\omega^2}{2(\omega^2 - 4)}$



HENCE THE GENERAL SOLUTION IS GIVEN BY

$\boxed{x = A\cos\omega t + B\sin\omega t + 3a + \frac{a\omega^2}{2(\omega^2 - 4)}\sin 2t}$

APPLY $t = 0$, $x = 3a$

$3a = A + 3a$

$A = 0$

$\boxed{x = 3a + B\sin\omega t + \frac{a\omega^2}{2(\omega^2 - 4)}\sin 2t}$

DIFFERENTIATE AND APPLY CONDITION $t = 0$, $\dot{x} = 0$

$\dot{x} = B\omega\cos\omega t + \frac{a\omega^2}{\omega^2 - 4}\cos 2t$

$0 = B\omega + \frac{a\omega^2}{\omega^2 - 4}$

$B = -\frac{a\omega}{\omega^2 - 4}$

$x = 3a + \frac{a\omega^2}{2(\omega^2 - 4)}\sin 2t - \frac{a\omega}{\omega^2 - 4}\sin\omega t$

$x = 3a + \frac{a\omega}{2(\omega^2 - 4)}\left[\omega\sin 2t - 2\sin\omega t\right]$

Question 18 (****+)

A small raindrop of mass m kg, is released from rest from a rain cloud and is falling through still air under the action of its own weight. The raindrop is subject to air resistance of magnitude kv^2 N, where v ms^{-1} is the speed of the raindrop t s after release, and k is a positive constant.

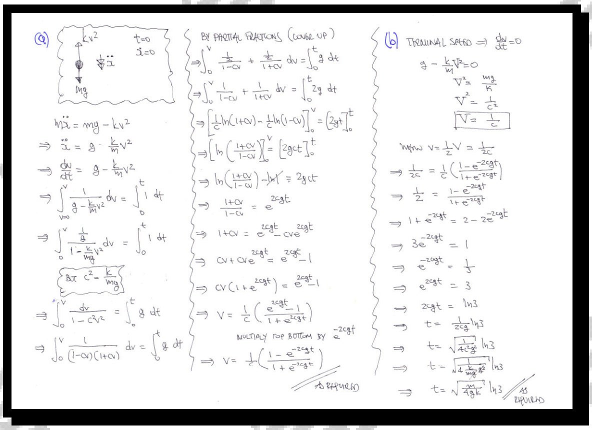
a) Show clearly that

$$v = \frac{1}{c}\left(\frac{1-e^{-2cgt}}{1+e^{-2cgt}}\right), \text{ where } c^2 = \frac{k}{mg}.$$

The raindrop has a terminal speed V.

b) Show that the raindrop reaches a speed of $\frac{1}{2}V$ in time $\sqrt{\frac{m}{4gk}}\ln 3$ seconds.

proof


(a)

kv^2 $\quad t=0 \quad \dot{x}=0$ $\quad \ddot{x}$ $\quad mg$

$m\ddot{x} = mg - kv^2$

$\Rightarrow \ddot{x} = g - \frac{k}{m}v^2$

$\Rightarrow \frac{dv}{dt} = g - \frac{k}{m}v^2$

$\Rightarrow \int_{v=0}^{v} \frac{1}{g - \frac{k}{m}v^2}\,dv = \int_0^t 1\,dt$

$\Rightarrow \int_0^v \frac{\frac{1}{g}}{1 - \frac{k}{mg}v^2}\,dv = \int_0^t 1\,dt$

$\boxed{\text{BUT } c^2 = \frac{k}{mg}}$

$\Rightarrow \int_0^v \frac{dv}{1 - c^2v^2} = \int_0^t g\,dt$

$\Rightarrow \int_0^v \frac{1}{(1-cv)(1+cv)}\,dv = \int_0^t g\,dt$

BY PARTIAL FRACTIONS (COVER UP)

$\Rightarrow \int_0^v \frac{\frac{1}{2}}{1-cv} + \frac{\frac{1}{2}}{1+cv}\,dv = \int_0^t g\,dt$

$\Rightarrow \int_0^v \frac{1}{1-cv} + \frac{1}{1+cv}\,dv = \int_0^t 2g\,dt$

$\Rightarrow \left[\frac{1}{c}\ln(1+cv) - \frac{1}{c}\ln(1-cv)\right]_0^v = \left[2gt\right]_0^t$

$\Rightarrow \left[\ln\left(\frac{1+cv}{1-cv}\right)\right]_0^v = \left[2gct\right]_0^t$

$\Rightarrow \ln\left(\frac{1+cv}{1-cv}\right) - \cancel{\ln 1} = 2gct$

$\Rightarrow \frac{1+cv}{1-cv} = e^{2cgt}$

$\Rightarrow 1+cv = e^{2cgt} - cve^{2cgt}$

$\Rightarrow cv + cve^{2cgt} = e^{2cgt} - 1$

$\Rightarrow cv(1+e^{2cgt}) = e^{2cgt} - 1$

$\Rightarrow v = \frac{1}{c}\left(\frac{e^{2cgt}-1}{1+e^{2cgt}}\right)$

MULTIPLY TOP BOTTOM BY e^{-2cgt}

$\Rightarrow v = \frac{1}{c}\left(\frac{1-e^{-2cgt}}{1+e^{-2cgt}}\right)$ // AS REQUIRED

(b) TERMINAL SPEED $\Rightarrow \frac{dv}{dt} = 0$

$g - \frac{k}{m}V^2 = 0$

$V^2 = \frac{mg}{k}$

$V^2 = \frac{1}{c^2}$

$\boxed{V = \frac{1}{c}}$

WHEN $v = \frac{1}{2}V = \frac{1}{2c}$

$\Rightarrow \frac{1}{2c} = \frac{1}{c}\left(\frac{1-e^{-2cgt}}{1+e^{-2cgt}}\right)$

$\Rightarrow \frac{1}{2} = \frac{1-e^{-2cgt}}{1+e^{-2cgt}}$

$\Rightarrow 1 + e^{-2cgt} = 2 - 2e^{-2cgt}$

$\Rightarrow 3e^{-2cgt} = 1$

$\Rightarrow e^{-2cgt} = \frac{1}{3}$

$\Rightarrow e^{2cgt} = 3$

$\Rightarrow 2cgt = \ln 3$

$\Rightarrow t = \frac{1}{2cg}\ln 3$

$\Rightarrow t = \sqrt{\frac{1}{4c^2g^2}}\ln 3$

$\Rightarrow t = \sqrt{\frac{1}{4\frac{k}{mg}g^2}}\ln 3$

$\Rightarrow t = \sqrt{\frac{m}{4gk}}\ln 3$ // AS REQUIRED

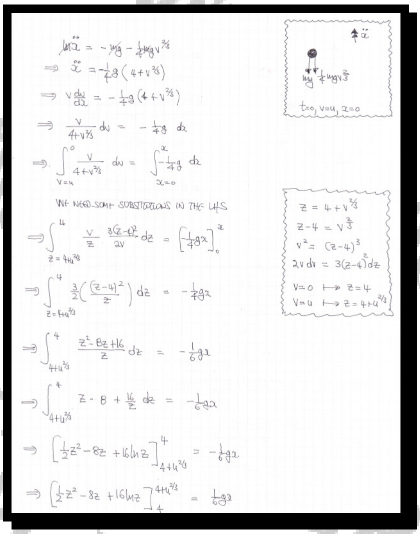
Question 19 (****+)

A particle, of mass m, is projected vertically upwards with speed u and moves under the action of its weight and air resistance of magnitude $\frac{1}{2}mgv^{\frac{2}{3}}$, where v is the speed of the particle at time t.

Show that the distance the particle covers until it comes to instantaneous rest is

$$\frac{3}{g}\left[u^{\frac{4}{3}} - 8u^{\frac{2}{3}} + 32\ln\left(1+\frac{1}{4}u^{\frac{2}{3}}\right)\right].$$

$\boxed{\text{proof}}$



$$\cancel{m}\ddot{x} = -\cancel{m}g - \tfrac{1}{4}\cancel{m}gv^{2/3}$$

$$\Rightarrow \ddot{x} = -\tfrac{1}{4}g\left(4+v^{2/3}\right)$$

$$\Rightarrow v\frac{dv}{dx} = -\tfrac{1}{4}g\left(4+v^{2/3}\right)$$

$$\Rightarrow \frac{v}{4+v^{2/3}}\,dv = -\tfrac{1}{4}g\,dx$$

$$\Rightarrow \int_{v=u}^{0}\frac{v}{4+v^{2/3}}\,dv = \int_{x=0}^{x}-\tfrac{1}{4}g\,dx$$

WE NEED SOME SUBSTITUTIONS IN THE LHS

$$z = 4+v^{2/3}$$
$$z-4 = v^{\frac{2}{3}}$$
$$v^2 = (z-4)^3$$
$$2v\,dv = 3(z-4)^2\,dz$$
$$v=0 \mapsto z=4$$
$$v=u \mapsto z=4+u^{2/3}$$

$$\Rightarrow \int_{z=4+u^{2/3}}^{4}\frac{v}{z}\,\frac{3(z-4)^2}{2v}\,dz = \left[-\tfrac{1}{4}gx\right]_0^x$$

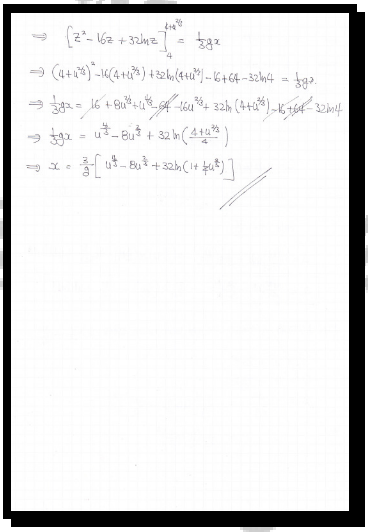
$$\Rightarrow \int_{z=4+u^{2/3}}^{4}\frac{3}{2}\left(\frac{(z-4)^2}{z}\right)dz = -\tfrac{1}{4}gx$$

$$\Rightarrow \int_{4+u^{2/3}}^{4}\frac{z^2-8z+16}{z}\,dz = -\tfrac{1}{6}gx$$

$$\Rightarrow \int_{4+u^{2/3}}^{4} z - 8 + \frac{16}{z}\,dz = -\tfrac{1}{6}gx$$

$$\Rightarrow \left[\tfrac{1}{2}z^2 - 8z + 16\ln z\right]_{4+u^{2/3}}^{4} = -\tfrac{1}{6}gx$$

$$\Rightarrow \left[\tfrac{1}{2}z^2 - 8z + 16\ln z\right]_{4}^{4+u^{2/3}} = \tfrac{1}{6}gx$$



$$\Rightarrow \left[z^2 - 16z + 32\ln z\right]_{4}^{4+u^{2/3}} = \tfrac{1}{3}gx$$

$$\Rightarrow \left(4+u^{2/3}\right)^2 - 16\left(4+u^{2/3}\right) + 32\ln\left(4+u^{2/3}\right) - 16 + 64 - 32\ln 4 = \tfrac{1}{3}gx$$

$$\Rightarrow \tfrac{1}{3}gx = \cancel{16} + 8u^{2/3} + u^{4/3} \cancel{-64} - 16u^{2/3} + 32\ln\left(4+u^{2/3}\right) \cancel{-16} \cancel{+64} - 32\ln 4$$

$$\Rightarrow \tfrac{1}{3}gx = u^{\frac{4}{3}} - 8u^{\frac{2}{3}} + 32\ln\left(\frac{4+u^{2/3}}{4}\right)$$

$$\Rightarrow x = \frac{3}{g}\left[u^{\frac{4}{3}} - 8u^{\frac{2}{3}} + 32\ln\left(1+\tfrac{1}{4}u^{\frac{2}{3}}\right)\right]$$

Question 20 (*****)

[In this question $g = 10 \text{ ms}^{-2}$]

Two particles A and B, or respective masses 8 kg and 2 kg, are attached to the ends of a light elastic string of natural length 2.5 m and modulus of elasticity 80 N.

The string passes through a small smooth hole on a rough horizontal table.

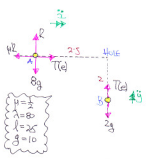
A is held at a distance of 2.5 m from the hole and B is held at a distance of 2 m vertically below the hole. The coefficient of friction between A and the table is 0.5.

Both particles are released simultaneously from rest.

a) Find an expression for the subsequent velocity of A and hence verify that A first comes to rest 0.47524 s after release.

b) Calculate the distance A covers until it first comes to rest.

$$\boxed{v = \frac{\sqrt{5}}{2}\sin\left(\sqrt{20t}\right) - 2t}, \quad \boxed{d \approx 0.15578\ldots \text{ m}}$$



$\mu = \frac{1}{2}$, $\lambda = 80$, $l = 2.5$, $g = 10$

- LET e BE THE EXTENSION IN THE STRING
  x THE DISPLACEMENT OF A
  y THE DISPLACEMENT OF B

THUS $e = 4.5 - l - x - y$

$\boxed{e = 2 - x - y} \Rightarrow \boxed{-x - y = e - 2}$

$\dot{e} = -\dot{x} - \dot{y}$

$\boxed{\ddot{e} = -\ddot{x} - \ddot{y}} \Rightarrow \boxed{\ddot{y} = -\ddot{x} - \ddot{e}}$

- EQUATION OF MOTION OF A

$\Rightarrow 8\ddot{x} = T - \mu R$
$\Rightarrow 8\ddot{x} = \frac{\lambda}{l}e - \frac{1}{2}(8g)$
$\Rightarrow 8\ddot{x} = \frac{80}{2.5}(2 - x - y) - \frac{1}{2}(8g)$
$\Rightarrow \ddot{x} = \frac{10}{2.5}(2 - x - y) - \frac{1}{2}g$
$\Rightarrow \ddot{x} = 4(2 - x - y) - \frac{1}{2}g$
$\Rightarrow \ddot{x} = 8 + 4(-x - y) - 5$
$\Rightarrow \ddot{x} = 3 + 4(e - 2)$
$\Rightarrow \ddot{x} = 4e - 5$

- EQUATION OF MOTION OF B

$\Rightarrow 2\ddot{y} = T - 2g$
$\Rightarrow 2\ddot{y} = \frac{\lambda}{l}e - 2g$
$\Rightarrow 2\ddot{y} = \frac{80}{2.5}(2 - x - y) - 2g$
$\Rightarrow \ddot{y} = \frac{40}{2.5}(2 - x - y) - g$
$\Rightarrow \ddot{y} = 16(2 - x - y) - 10$
$\Rightarrow \ddot{y} = 32 + 16(-x - y) - 10$
$\Rightarrow \ddot{y} = 32 + 16(e - 2) - 10$
$\Rightarrow \ddot{y} = 16e - 10$
$\Rightarrow -\ddot{x} - \ddot{e} = 16e - 10$

ADD EQUATIONS

$-\ddot{e} = 20e - 15$

$\ddot{e} + 20e = 15$ IE SHM WITH FORCING

GEN SOLUTION $e = A\cos\omega t + B\sin\omega t + \frac{3}{4}, \quad \omega = \sqrt{20}$

$t = 0, x = 0, y = 0 \Rightarrow e = 2$

$2 = A + \frac{3}{4}$

$\boxed{A = \frac{5}{4}}$

DIFFERENTIATE

$\dot{e} = B\omega\cos\omega t - A\omega\sin\omega t \qquad t = 0, \dot{x} = 0, \dot{y} = 0 \Rightarrow \dot{e} = 0$

$0 = B\omega$

$B = 0$

$\therefore e = \frac{5}{4}\cos\omega t + \frac{3}{4}$

NOW $-\ddot{x} - \ddot{e} = 16e - 10$

$-\ddot{x} - \ddot{e} = 16\left(\frac{5}{4}\cos\omega t + \frac{3}{4}\right) - 10$

$-\ddot{x} - \ddot{e} = 20\cos\omega t + 2$

INTEGRATE $-\dot{x} - \dot{e} = \frac{20}{\omega}\sin\omega t + 2t + C$

APPLY CONDITION $t = 0, \dot{x} = \dot{e} = 0 \Rightarrow C = 0$

$-\dot{x} - \dot{e} = \frac{20}{\omega}\sin\omega t + 2t$

$-\dot{x} - \left(-\frac{5}{4}\omega\sin\omega t\right) = \frac{20}{\omega}\sin\omega t + 2t$

$-\dot{x} = \frac{20}{\omega}\sin\omega t - \frac{5}{4}\omega\sin\omega t + 2t$

$\dot{x} = \left(\frac{5\omega}{4} - \frac{20}{\omega}\right)\sin\omega t - 2t$ OR $\dot{x} = \frac{\sqrt{5}}{2}\sin(\sqrt{20}t) - 2t$

$\dot{x}(0.47524) = \frac{\sqrt{5}}{2}\sin(\sqrt{20} \times 0.47524) - (2 \times 0.47524)$

$= 0.00000756\ldots$

AS REQUIRED

FINALLY $\dot{x} = \frac{\sqrt{5}}{2}\sin(\sqrt{20}t) - 2t$

INTEGRATE: $x = -\frac{1}{4}\cos(\sqrt{20}t) - t^2 + D$

$t = 0, x = 0$

$0 = -\frac{1}{4} + D$

$D = \frac{1}{4}$

$x = \frac{1}{4}\left(1 - \cos\sqrt{20}t\right) - t^2$

$x(0.47524) = \frac{1}{4}\left(1 - \cos\left(0.47524 \times \sqrt{20}\right)\right) - 0.47524^2 \approx 0.15578\ldots \text{m}$