RECURRENCE RELATIONS

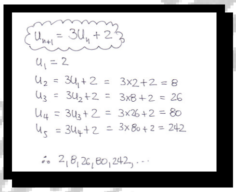
Question 1 (**)

A sequence $u_1, u_2, u_3, u_4, \dots$ is given by the recurrence relation

$$u_{n+1} = 3u_n + 2, \quad u_1 = 2.$$

Find the value of u_2, u_3, u_4 and u_5.

$\boxed{u_2 = 8}$, $\boxed{u_3 = 26}$, $\boxed{u_4 = 80}$, $\boxed{u_5 = 242}$



Question 2 (**)

A sequence $y_1, y_2, y_3, y_4, \dots$ is given by

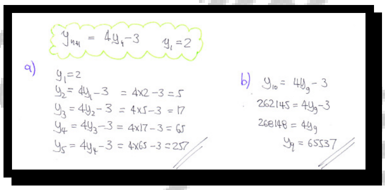
$$y_{n+1} = 4y_n - 3, \quad y_1 = 2.$$

a) Find the value of y_2, y_3, y_4 and y_5.

It is further given that $y_{10} = 262145$.

b) Calculate the value of y_9.

$\boxed{y_2 = 5}$, $\boxed{y_3 = 17}$, $\boxed{y_4 = 65}$, $\boxed{y_5 = 257}$, $\boxed{y_9 = 65537}$



Question 3 (**)

A sequence u_1, u_2, u_3, u_4, … is given by the recurrence relation

$$u_{n+1} = 2u_n - n^2 + 3, \quad u_1 = 2.$$

Find the value of u_2, u_3, u_4 and u_5.

$\boxed{u_2 = 6}$, $\boxed{u_3 = 11}$, $\boxed{u_4 = 16}$, $\boxed{u_5 = 19}$

$u_{n+1} = 2u_n - n^2 + 3$

$u_1 = 2$ (Given)

$u_2 = 2u_1 - 1^2 + 3 = 2\times2 - 1 + 3 = 6$

$u_3 = 2u_2 - 2^2 + 3 = 2\times6 - 4 + 3 = 11$

$u_4 = 2u_3 - 3^2 + 3 = 2\times11 - 9 + 3 = 16$

$u_5 = 2u_4 - 4^2 + 3 = 2\times16 - 16 + 3 = 19$

$\therefore 2, 6, 11, 16, 19,$

Question 4 (**)

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_{n+1} = (3 - u_n)^2, \quad u_1 = 4.$$

a) Find the value of u_2, u_3 and u_4.

b) State the value of u_{10}.

$\boxed{u_2 = 1}$, $\boxed{u_3 = 4}$, $\boxed{u_4 = 1}$, $\boxed{u_{10} = 1}$

(a) $u_{n+1} = (3 - u_n)^2$

$u_1 = 4$

$u_2 = (3 - u_1)^2 = (3-4)^2 = 1$

$u_3 = (3 - u_2)^2 = (3-1)^2 = 4$

$u_4 = (3 - u_3)^2 = (3-4)^2 = 1$

(b) $u_{10} = 1$

(since all "evens" are 1 and all "odds" are 4)

Question 5 (**)

A sequence b_1, b_2, b_3, b_4, … is given by

$$b_{n+1} = 5b_n - 3, \quad b_1 = k,$$

where k is a non zero constant.

a) Find the value of b_4 in terms of k.

b) Given that $b_4 = 7$, determine the value of k.

$\boxed{}$, $\boxed{b_4 = 125k - 93}$, $\boxed{k = \frac{4}{5}}$

a) $b_{n+1} = 5b_n - 3$

- $b_1 = k$
- $b_2 = 5b_1 - 3 = 5k - 3$
- $b_3 = 5b_2 - 3 = 5(5k-3) - 3 = 25k - 18$
- $b_4 = 5b_3 - 3 = 5(25k-18) - 3 = 125k - 93$ //

b) $b_4 = 7$

$125k - 93 = 7$

$125k = 100$

$k = \frac{100}{125}$

$k = \frac{4}{5}$ //

Question 6 (**)

A recurrence relation is defined for $n \geq 1$ by

$$a_{n+1} = 3a_n + 4\,, \quad a_1 = k\,,$$

where k is a non zero constant.

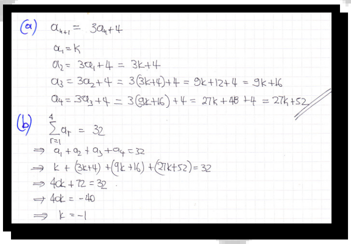
a) Find the value of a_4 in terms of k.

It is further given that

$$\sum_{r=1}^{4} a_r = 32\,.$$

b) Determine the value of k.

$\boxed{a_4 = 27k + 52}$, $\boxed{k = -1}$



(a) $a_{n+1} = 3a_n + 4$

$a_1 = k$

$a_2 = 3a_1 + 4 = 3k + 4$

$a_3 = 3a_2 + 4 = 3(3k+4) + 4 = 9k + 12 + 4 = 9k + 16$

$a_4 = 3a_3 + 4 = 3(9k+16) + 4 = 27k + 48 + 4 = 27k + 52$

(b) $\sum_{r=1}^{4} a_r = 32$

$\Rightarrow a_1 + a_2 + a_3 + a_4 = 32$

$\Rightarrow k + (3k+4) + (9k+16) + (27k+52) = 32$

$\Rightarrow 40k + 72 = 32$

$\Rightarrow 40k = -40$

$\Rightarrow k = -1$

Question 7 (**)

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_{n+1} = 3u_n - 9, \quad u_1 = k,$$

where k is a non zero constant.

a) Find the value of u_3 in terms of k.

It is further given that

$$\sum_{r=1}^{4} u_r = 38.$$

b) Find the value of k.

$\boxed{u_3 = 9k - 36}$, $\boxed{k = 5}$

(a) $u_{n+1} = 3u_n - 9$

$\Rightarrow u_1 = k$

$\Rightarrow u_2 = 3u_1 - 9 = 3k - 9$

$\Rightarrow u_3 = 3u_2 - 9 = 3(3k-9) - 9 = 9k - 36$

$\Rightarrow u_4 = 3u_3 - 9 = 3(9k-36) - 9 = 27k - 117$

$\therefore u_3 = 9k - 36$

(b) $\sum_{r=1}^{4} u_r = 38$

$\Rightarrow u_1 + u_2 + u_3 + u_4 = 38$

$\Rightarrow k + (3k-9) + (9k-36) + (27k-117) = 38$

$\Rightarrow 40k - 162 = 38$

$\Rightarrow 40k = 200$

$\Rightarrow k = 5$

Question 8 (**)

A recurrence relation is defined for $n \geq 1$ by

$$t_{n+1} = kt_n - 1, \quad t_1 = 2,$$

where k is a non zero constant.

a) Find the value of t_3 in terms of k.

b) Given that $t_3 = 14$ find the possible values of k.

$\boxed{t_3 = 2k^2 - k - 1}$, $\boxed{k = -\frac{5}{2}, 3}$

(a) $t_{n+1} = kt_n - 1$

$t_1 = 2$

$t_2 = kt_1 - 1 = k \times 2 - 1 = 2k - 1$

$t_3 = kt_2 - 1 = k(2k-1) - 1 = 2k^2 - k - 1$

(b) $2k^2 - k - 1 = 14$

$2k^2 - k - 15 = 0$

$(2k+5)(k-3) = 0$

$k = 3, -\frac{5}{2}$

Question 9 (**)

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_{n+1} = ku_n + 5, \quad u_1 = 2,$$

where k is a non zero constant.

a) Find the value of u_3 in terms of k.

It is further given that

$$\sum_{r=1}^{3} u_r = 7.$$

b) Find the possible values of k.

$\boxed{u_3 = 2k^2 + 5k + 5}$, $\boxed{k = -1, -\frac{5}{2}}$

(a) $u_{n+1} = ku_n + 5$

$\Rightarrow u_1 = 2$
$\Rightarrow u_2 = k \times u_1 + 5 = 2k + 5$
$\Rightarrow u_3 = ku_2 + 5 = k(2k+5) + 5$

$\therefore u_3 = 2k^2 + 5k + 5$

(b) $\sum_{r=1}^{3} u_r = 7$

$\Rightarrow u_1 + u_2 + u_3 = 7$
$\Rightarrow 2 + (2k+5) + (2k^2+5k+5) = 7$
$\Rightarrow 2k^2 + 7k + 12 = 7$
$\Rightarrow 2k^2 + 7k + 5 = 0$
$\Rightarrow (2k+5)(k+1) = 0$
$\Rightarrow k = -1, -\frac{5}{2}$

Question 10 (**)

A sequence x_1, x_2, x_3, x_4, … is given by the recurrence formula

$$x_{n+1} = 5(x_n + 1) - 2n^2, \quad x_1 = \frac{4}{5}.$$

Calculate the value of x_2, x_3, x_4, and x_5.

$\boxed{x_2 = 7}$, $\boxed{x_3 = 32}$, $\boxed{x_4 = 147}$, $\boxed{x_5 = 708}$

$x_{n+1} = 5(x_n + 1) - 2n^2$

$x_1 = \frac{4}{5}$

$x_2 = 5(x_1+1) - 2\times1^2 = 5(\frac{4}{5}+1) - 2 = 4 + 5 - 2 = 7$

$x_3 = 5(x_2+1) - 2\times2^2 = 5(7+1) - 8 = 40 - 8 = 32$

$x_4 = 5(x_3+1) - 2\times3^2 = 5(32+1) - 18 = 5\times33 - 18 = 147$

$x_5 = 5(x_4+1) - 2\times4^2 = 5(147+1) - 32 = 5\times148 - 32 = 708$

Question 11 (**)

A sequence t_1, t_2, t_3, t_4, t_5, t_6 … is given by the recurrence formula

$$t_{n+1} = nt_n - (t_n)^2 + 4, \quad t_1 = 2.$$

Find the value of t_2, t_3, t_4, t_5 and t_6.

$\boxed{t_2 = 2}$, $\boxed{t_3 = 4}$, $\boxed{t_4 = 0}$, $\boxed{t_5 = 4}$, $\boxed{t_6 = 8}$

$t_{n+1} = nt_n - t_n^2 + 4$

- $t_1 = 2$
- $t_2 = 1\times t_1 - t_1^2 + 4 = 1\times2 - 2^2 + 4 = 2 - 4 + 4 = 2$
- $t_3 = 2\times t_2 - t_2^2 + 4 = 2\times2 - 2^2 + 4 = 4 - 4 + 4 = 4$
- $t_4 = 3\times t_3 - t_3^2 + 4 = 3\times4 - 4^2 + 4 = 12 - 16 + 4 = 0$
- $t_5 = 4t_4 - t_4^2 + 4 = 4\times0 - 0^2 + 4 = 0 - 0 + 4 = 4$
- $t_6 = 5t_5 - t_5^2 + 4 = 5\times4 - 4^2 + 4 = 20 - 16 + 4 = 8$

Question 12 (**+)

A recurrence relation is defined for $n \geq 1$ by

$$a_{n+1} = 7a_n - n^3 - 3, \quad a_1 = 1.$$

a) Find the value of a_4.

b) Evaluate the sum

$$\sum_{r=1}^{5} a_r .$$

$\boxed{\quad}$, $\boxed{a_4 = 40}$, $\boxed{267}$

a) $a_{n+1} = 7a_n - n^3 - 3$

$a_1 = 1$
$a_2 = 7a_1 - 1^3 - 3 = 7 \times 1 - 1 - 3 = 3$
$a_3 = 7a_2 - 2^3 - 3 = 7 \times 3 - 8 - 3 = 10$
$a_4 = 7a_3 - 3^3 - 3 = 7 \times 10 - 27 - 3 = 40$
$\underbrace{\left[\, a_5 = 7a_4 - 4^3 - 3 \,\right]}_{\text{NEEDED FOR PART (b)}} = 7 \times 40 - 64 - 3 = 213$

b) ADDING THE FIRST 5 TERMS

$\sum_{r=1}^{5} a_r = a_1 + a_2 + a_3 + a_4 + a_5$
$= 1 + 3 + 10 + 40 + 213$
$= 267$

Question 13 (**+)

A sequence $u_1, u_2, u_3, u_4, u_5, u_6, \ldots$ is given by

$$u_{n+2} = u_{n+1} + 2u_n, \quad u_2 = 4, \quad u_3 = 8.$$

a) Find the value of u_4, u_5 and u_6.

b) Determine the value of u_1.

$\boxed{u_4 = 16}$, $\boxed{u_5 = 32}$, $\boxed{u_6 = 64}$, $\boxed{u_1 = 2}$

$u_{n+2} = u_{n+1} + 2u_n \qquad u_2 = 4, \; u_3 = 8$

(a) $u_4 = u_3 + 2u_2 = 8 + 2\times 4 = 16$
$u_5 = u_4 + 2u_3 = 16 + 2\times 8 = 32$
$u_6 = u_5 + 2u_4 = 32 + 2\times 16 = 64$

(b) $u_3 = u_2 + 2u_1$
$8 = 4 + 2u_1$
$4 = 2u_1$
$u_1 = 2$

OR

BY SPOTTING THAT nth TERM IS

$u_n = 2^n$

$u_1 = 2$

Question 14 (+)**

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{1}{1-u_n}, \quad n \geq 1, \quad u_1 = 2.$$

a) Find the value of u_2, u_3 and u_4.

b) State the value of u_{12}.

c) Show clearly that

$$\sum_{r=1}^{12} u_r = 6.$$

$\boxed{}$, $\boxed{u_2 = -1, \ u_3 = \frac{1}{2}, \ u_4 = 2}$, $\boxed{u_{12} = \frac{1}{2}}$

(a) $u_{n+1} = \frac{1}{1-u_n}$

- $u_1 = 2$ (GIVEN)
- $u_2 = \frac{1}{1-u_1} = \frac{1}{1-2} = -1$
- $u_3 = \frac{1}{1-u_2} = \frac{1}{1-(-1)} = \frac{1}{2}$
- $u_4 = \frac{1}{1-u_3} = \frac{1}{1-\frac{1}{2}} = 2$

(b) $u_{12} = \frac{1}{2}$

2	−1	$\frac{1}{2}$
u_1	u_2	u_3
u_4	u_5	u_6
u_7	u_8	u_9
u_{10}	u_{11}	u_{12}

(c)
$$\sum_{r=1}^{12} u_r = u_1 + u_2 + u_3 + \dots + u_{12}$$
$$= \left(2 - 1 + \frac{1}{2}\right) \times 4$$
$$= 1.5 \times 4$$
$$= 6$$
AS REQUIRED

Question 15 (**+)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = ku_n + 4, \quad n \geq 1, \quad u_1 = 16,$$

where k is a non zero constant.

a) If $u_3 = 10$, find the possible values of k .

b) Determine the value of u_4, given that $k > 0$.

$\boxed{k = \frac{1}{2}, -\frac{3}{4}}$, $\boxed{u_4 = 9}$

(a) $u_{n+1} = ku_n + 4$

$u_1 = 16$

$u_2 = ku_1 + 4 = k \times 16 + 4 = 16k + 4$

$u_3 = ku_2 + 4 = k(16k+4) + 4 = 16k^2 + 4k + 4$

NOW $u_3 = 10$

$16k^2 + 4k + 4 = 10$

$16k^2 + 4k - 6 = 0$

$8k^2 + 2k - 3 = 0$

$(2k-1)(4k+3) = 0$

$k = \frac{1}{2}, -\frac{3}{4}$

(b) IF $k > 0$, $u_{n+1} = \frac{1}{2}u_n + 4$ $\quad u_3 = 10$

$\therefore u_4 = \frac{1}{2}u_3 + 4 = \frac{1}{2} \times 10 + 4 = 5 + 4 = 9$

Question 16 (**+)

A sequence t_1, t_2, t_3, t_4, t_5, … is given by

$$t_{n+1} = 2t_n + 1, \quad t_5 = 103.$$

Find the value of t_1 .

$\boxed{t_1 = \frac{11}{2}}$

$t_{n+1} = 2t_n + 1$ $\quad t_5 = 103$

$t_{n+1} - 1 = 2t_n$

$\boxed{t_n = \frac{t_{n+1} - 1}{2}}$

$t_5 = 103$

$t_4 = \frac{t_5 - 1}{2} = \frac{103-1}{2} = \frac{102}{2} = 51$

$t_3 = \frac{t_4 - 1}{2} = \frac{51-1}{2} = \frac{50}{2} = 25$

$t_2 = \frac{t_3 - 1}{2} = \frac{25-1}{2} = \frac{24}{2} = 12$

$t_1 = \frac{t_2 - 1}{2} = \frac{12-1}{2} = \frac{11}{2}$ $(= 5.5)$

Question 17 (**+)

A sequence y_1, y_2, y_3, y_4, ... is given by the recurrence formula

$$y_{n+1} = 2y_n - 4n + 4, \quad y_5 = 52.$$

a) Make y_n the subject of the above recurrence formula.

b) Hence determine the value of y_4, y_3, y_2 and y_1.

$$\boxed{y_n = \frac{y_{n+1} + 4n - 4}{2}}, \ \boxed{y_4 = 32}, \ \boxed{y_3 = 20}, \ \boxed{y_2 = 12}, \ \boxed{y_1 = 6}$$

a) $y_{n+1} = 2y_n - 4n + 4$

$\Rightarrow y_{n+1} + 4n - 4 = 2y_n$

$\Rightarrow y_n = \frac{y_{n+1} + 4n - 4}{2}$

(b) Thus

$y_5 = 52$

$y_4 = \frac{y_5 + 4\times4 - 4}{2} = \frac{52 + 16 - 4}{2} = 32$

$y_3 = \frac{y_4 + 4\times3 - 4}{2} = \frac{32 + 12 - 4}{2} = 20$

$y_2 = \frac{y_3 + 4\times2 - 4}{2} = \frac{20 + 8 - 4}{2} = 12$

$y_1 = \frac{y_2 + 4\times1 - 4}{2} = \frac{12 + 4 - 4}{2} = 6$

Question 18 (***)

A sequence is defined for $n \geq 1$ by the recurrence relation

$$u_{n+1} = 2u_n + 1, \quad u_1 = 3.$$

a) Find the first five terms of the sequence.

b) By considering the first few powers of 2, write down an expression for the n^{th} term of the sequence.

$$\boxed{3, 7, 15, 31, 63, \ldots}, \ \boxed{u_n = 2^{n+1} - 1}$$

(a) $u_{n+1} = 2u_n + 1$

$u_1 = 3$

$\Rightarrow u_2 = 2u_1 + 1 = 2\times3+1 = 7$

$\Rightarrow u_3 = 2u_2 + 1 = 2\times7+1 = 15$

$\Rightarrow u_4 = 2u_3 + 1 = 2\times15+1 = 31$

$\Rightarrow u_5 = 2u_4 + 1 = 2\times31+1 = 63$

$\therefore 3, 7, 15, 31, 63, \ldots$

(b) Powers of 2

$2^1, 2^2, 2^3, 2^4, \ldots$

$2, 4, 8, 16, 32, 64$

$3 \quad 7 \quad 15 \quad 31 \quad 63$

$\therefore u_n = 2^{n+1} - 1$

Question 19 (***)

A sequence of numbers is given by the recurrence relation

$$a_{n+1} = 5 - \frac{18}{4 + a_n}, \quad n \geq 1, \quad a_2 = 0.$$

a) Find the value of a_3, a_4 and a_5.

b) Determine the value of a_1.

c) Calculate the value of

$$\sum_{r=1}^{5} a_r .$$

$\boxed{}$, $\boxed{u_3 = \frac{1}{2}, \quad u_3 = 1, \quad u_5 = \frac{7}{5}}$, $\boxed{u_1 = -\frac{2}{5}}$, $\boxed{\sum_{r=1}^{5} u_r = 2.5}$

$$\boxed{a_{n+1} = 5 - \frac{18}{4 + a_n}}$$

(a) $a_2 = 0$ (Given)

$a_3 = 5 - \frac{18}{4 + a_2} = 5 - \frac{18}{4} = 5 - \frac{9}{2} = \frac{1}{2}$

$a_4 = 5 - \frac{18}{4 + a_3} = 5 - \frac{18}{4 + \frac{1}{2}} = 5 - \frac{18}{4.5} = 5 - 4 = 1$

$a_5 = 5 - \frac{18}{4 + a_4} = 5 - \frac{18}{4 + 1} = 5 - \frac{18}{5} = \frac{25}{5} - \frac{18}{5} = \frac{7}{5}$

(b) $a_2 = 5 - \frac{18}{4 + a_1}$

$\Rightarrow 0 = 5 - \frac{18}{4 + a_1}$

$\Rightarrow \frac{18}{4 + a_1} = 5$

$\Rightarrow 18 = 5(4 + a_1)$

$\Rightarrow 18 = 20 + 5a_1$

$\Rightarrow -2 = 5a_1$

$\Rightarrow a_1 = -\frac{2}{5}$

(c) $\sum_{r=1}^{5} a_r = a_1 + a_2 + a_3 + a_4 + a_5$

$= -\frac{2}{5} + 0 + \frac{1}{2} + 1 + \frac{7}{5}$

$= 2.5$

Question 20 (***)

A sequence x_1, x_2, x_3, x_4, ... is given by

$$x_{n+1} = \frac{k-5x_n}{x_n}, \quad x_1 = 1, \quad k > 5,$$

where k is a non zero constant.

a) Determine the value of x_3 in terms of k, giving the final answer as a single simplified fraction.

It is further given that $x_3 > 6$.

b) Find the range of the values of k.

$\boxed{}$, $\boxed{x_3 = \dfrac{25-4k}{k-5}}$, $\boxed{5 < k < \dfrac{11}{2}}$

$$\boxed{x_{n+1} = \frac{k-5x_n}{x_n}}$$

- $x_1 = 1$
- $x_2 = \dfrac{k-5x_1}{x_1} = \dfrac{k-5\times1}{1} = k-5$
- $x_3 = \dfrac{k-5x_2}{x_2} = \dfrac{k-5(k-5)}{k-5} = \dfrac{k-5k+25}{k-5} = \dfrac{25-4k}{k-5}$

NOW WE ARE GIVEN THAT $x_3 > 6$ & $k > 5$ SO $k-5>0$

$\Rightarrow x_3 > 6$

$\Rightarrow \dfrac{25-4k}{k-5} > 6$

$\Rightarrow 25-4k > 6(k-5)$

$\Rightarrow 25-4k > 6k-30$

$\Rightarrow -10k > -55$

$\Rightarrow k < \dfrac{55}{10} = \dfrac{11}{2}$

$\therefore 5 < k < \dfrac{11}{2}$

Question 21 (***)

A sequence u_1, u_2, u_3, u_4, … is given by the recurrence relation

$$u_{n+1} = 2u_n + (-1)^n \left(n^2 + 2\right), \quad u_1 = 10.$$

Find the value of u_2, u_3, u_4 and u_5.

$\boxed{u_2 = 17}$, $\boxed{u_3 = 40}$, $\boxed{u_4 = 69}$, $\boxed{u_5 = 156}$

$u_{n+1} = 2u_n + (-1)^n(n^2+2)$, $u_1 = 10$

$u_1 = 10$ (Given)

$u_2 = 2u_1 + (-1)^1(1^2+2) = 2\times10 - 3 = 17$

$u_3 = 2u_2 + (-1)^2(2^2+2) = 2\times17 + 6 = 40$

$u_4 = 2u_3 + (-1)^3(3^2+2) = 2\times40 - 11 = 69$

$u_5 = 2u_4 + (-1)^4(4^2+2) = 2\times69 + 18 = 156$

10, 17, 40, 69, 156, …

Question 22 (***+)

A sequence u_1, u_2, u_3, u_4, … is defined by the recursive relation

$$u_{n+1} = 2u_n + 3, \quad u_1 = k,$$

where k is a non zero constant.

a) Given that $u_6 = 189$, find the value of u_5.

b) Determine the value of k.

$\boxed{u_5 = 93}$, $\boxed{k = 3}$

$u_{n+1} = 2u_n + 3$, $u_1 = k$

(a) $u_6 = 2u_5 + 3$

$189 = 2u_5 + 3$

$186 = 2u_5$

$u_5 = 93$

(b) $u_{n+1} - 3 = 2u_n$

$u_n = \frac{1}{2}(u_{n+1} - 3)$

$u_4 = \frac{1}{2}(u_5 - 3) = \frac{1}{2}(93-3) = 45$

$u_3 = \frac{1}{2}(u_4 - 3) = \frac{1}{2}(45-3) = 21$

$u_2 = \frac{1}{2}(u_3 - 3) = \frac{1}{2}(21-3) = 9$

$u_1 = \frac{1}{2}(u_2 - 3) = \frac{1}{2}(9-3) = 3$

$\therefore k = 3$

Question 23 (***+)

A sequence a_1, a_2, a_3, a_4, ... is given by

$$a_{n+1} = p + qa_n,$$

where p and q are non zero constants.

It is given that $a_1 = 250$, $a_2 = 220$ and $a_3 = 196$.

a) Determine the value of p and the value of q.

b) Show clearly that the sequence converges to 100.

$\boxed{}$, $\boxed{p = 20}$, $\boxed{q = \frac{4}{5}}$

a) SUBSTITUTING INTO THE RECURRENCE RELATION

$\Rightarrow a_{n+1} = p + qa_n$

$\Rightarrow a_2 = p + qa_1 \quad \therefore 220 = p + q \times 250$

$\Rightarrow a_3 = p + qa_2 \quad \therefore 196 = p + q \times 220$

SOLVING SIMULTANEOUSLY

$\left. \begin{array}{l} p + 250q = 220 \\ p + 220q = 196 \end{array} \right) \Rightarrow 30q = 24$

$q = \frac{4}{5}$ //

$\Rightarrow p + 250 \times \frac{4}{5} = 220$

$p + 200 = 220$

$p = 20$ //

b) LET THE REQUIRED LIMIT BE L

AS $n \to \infty \qquad a_n \approx a_{n+1} \longrightarrow L$

$\Rightarrow a_{n+1} = 20 + \frac{4}{5}a_n$

$\Rightarrow L = 20 + \frac{4}{5}L$

$\Rightarrow 5L = 100 + 4L$

$\Rightarrow L = 100$ //

INDEED IT CONVERGES TO 100

Question 24 (***+)

A sequence x_1, x_2, x_3, x_4,... is given by

$$x_{n+1} = \frac{a+2x_n}{x_n}, \quad x_1 = 2,$$

where a is a non zero constant.

a) Find a simplified expression for x_3 in terms of a.

It is given that $x_3 = 12$.

b) Determine the value of a.

$\boxed{}$, $\boxed{x_3 = \dfrac{4a+8}{a+4}}$, $\boxed{a=-5}$

(a) $x_{n+1} = \dfrac{a+2x_n}{x_n}$

$x_1 = 2$

$x_2 = \dfrac{a+2x_1}{x_1} = \dfrac{a+4}{2}$

$x_3 = \dfrac{a+2x_2}{x_2} = \dfrac{a+2\left(\frac{a+4}{2}\right)}{\frac{a+4}{2}} = \dfrac{a+a+4}{\frac{a+4}{2}} = \dfrac{2a+4}{\frac{a+4}{2}} = \dfrac{4a+8}{a+4}$

(b) $x_3 = 12$

$\dfrac{4a+8}{a+4} = 12$

$4a+8 = 12a+48$

$-40 = 8a$

$a = -5$

Question 25 (***+)

A recurrence relation is defined for $n \geq 1$ by

$$u_{n+1} = a + \frac{1}{2}u_n, \quad u_1 = 520,$$

where a is a non zero constant.

a) Given that $u_4 = 72$, find the value of a.

b) Given further that $u_{10} = 9$, find the value of u_9.

$\boxed{\text{C1}}$, $\boxed{a = 4}$, $\boxed{u_9 = 10}$

(a) $u_{n+1} = a + \frac{1}{2}u_n$

- $u_1 = 520$
- $u_2 = a + \frac{1}{2}u_1 = a + \frac{1}{2} \times 520 = a + 260$
- $u_3 = a + \frac{1}{2}u_2 = a + \frac{1}{2}(a + 260) = \frac{3}{2}a + 130$
- $u_4 = a + \frac{1}{2}u_3 = a + \frac{1}{2}\left(\frac{3}{2}a + 130\right) = a + \frac{3}{4}a + 65 = \frac{7}{4}a + 65$

$\therefore \frac{7}{4}a + 65 = 72$

$\frac{7}{4}a = 7$

$7a = 28$

$\therefore a = 4$

(b) $u_{n+1} = 4 + \frac{1}{2}u_n$

$\Rightarrow u_{10} = 4 + \frac{1}{2}u_9$

$\Rightarrow 9 = 4 + \frac{1}{2}u_9$

$\Rightarrow 5 = \frac{1}{2}u_9$

$\Rightarrow u_9 = 10$

Question 26 (***+)

A sequence of positive numbers is given by the recurrence relation for $n \geq 1$ by

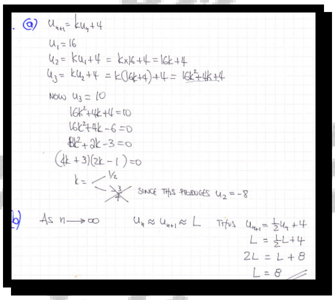
$$u_{n+1} = ku_n + 4, \quad u_1 = 16,$$

where k is a non zero constant.

a) Given that $u_3 = 10$, find the value of k .

b) Given further that the sequence converges to a limit L , use an algebraic method to determine the value of L .

$\boxed{k = \frac{1}{2}}$, $\boxed{L = 8}$



(a) $u_{n+1} = ku_n + 4$

$u_1 = 16$

$u_2 = ku_1 + 4 = k \times 16 + 4 = 16k + 4$

$u_3 = ku_2 + 4 = k(16k+4) + 4 = 16k^2 + 4k + 4$

NOW $u_3 = 10$

$16k^2 + 4k + 4 = 10$

$16k^2 + 4k - 6 = 0$

$8k^2 + 2k - 3 = 0$

$(4k+3)(2k-1) = 0$

$k = \frac{1}{2}$, $\cancel{-\frac{3}{4}}$ SINCE THIS PRODUCES $u_2 = -8$

(b) As $n \to \infty$ $u_n \approx u_{n+1} \approx L$ THUS $u_{n+1} = \frac{1}{2}u_n + 4$

$L = \frac{1}{2}L + 4$

$2L = L + 8$

$L = 8$

Question 27 (*+)**

A sequence u_1, u_2, u_3, u_4, … is given by

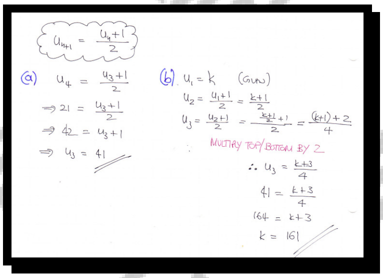
$$u_{n+1} = \frac{u_n + 1}{2}, \quad u_1 = k\,,$$

where k is a non zero constant.

a) Given that $u_4 = 21$ find the value of u_3.

b) Determine the value of k.

$\boxed{}$, $\boxed{u_3 = 41}$, $\boxed{k = 161}$



Question 28 (***+)

A sequence u_1, u_2, u_3, u_4, … is given by

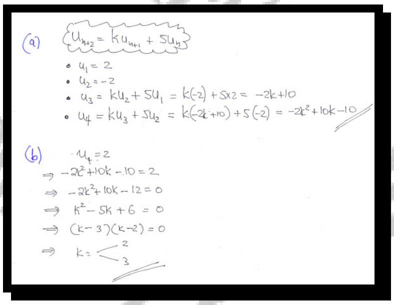
$$u_{n+2} = ku_{n+1} + 5u_n, \quad u_1 = 2, \quad u_2 = -2,$$

where k is a non zero constant.

a) Find the value of u_4 in terms of k.

b) Given that $u_4 = 2$, find the possible values of k.

$\boxed{u_4 = -2k^2 + 10k - 10}$, $\boxed{k = 2, 3}$


(a) $u_{n+2} = ku_{n+1} + 5u_n$

- $u_1 = 2$
- $u_2 = -2$
- $u_3 = ku_2 + 5u_1 = k(-2) + 5\times2 = -2k + 10$
- $u_4 = ku_3 + 5u_2 = k(-2k+10) + 5(-2) = -2k^2 + 10k - 10$

(b) $u_4 = 2$

$\Rightarrow -2k^2 + 10k - 10 = 2$

$\Rightarrow -2k^2 + 10k - 12 = 0$

$\Rightarrow k^2 - 5k + 6 = 0$

$\Rightarrow (k-3)(k-2) = 0$

$\Rightarrow k = 2, 3$

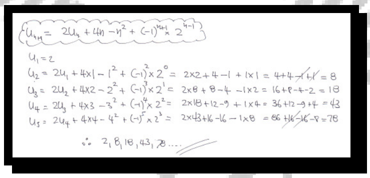
Question 29 (***+)

A sequence is defined for $n \geq 1$ by the recurrence relation

$$u_{n+1} = 2u_n + 4n - n^2 + (-1)^{n+1} 2^{n-1}, \quad u_1 = 2.$$

Find the first five terms of the sequence.

$\boxed{2, 8, 18, 43, 78, ...}$


$u_{n+1} = 2u_n + 4n - n^2 + (-1)^{n+1} \times 2^{n-1}$

$u_1 = 2$

$u_2 = 2u_1 + 4\times1 - 1^2 + (-1)^2\times2^0 = 2\times2 + 4 - 1 + 1\times1 = 4 + 4 \cancel{-1} \cancel{+1} = 8$

$u_3 = 2u_2 + 4\times2 - 2^2 + (-1)^3\times2^1 = 2\times8 + 8 - 4 - 1\times2 = 16 + 8 - 4 - 2 = 18$

$u_4 = 2u_3 + 4\times3 - 3^2 + (-1)^4\times2^2 = 2\times18 + 12 - 9 + 1\times4 = 36 + 12 - 9 + 4 = 43$

$u_5 = 2u_4 + 4\times4 - 4^2 + (-1)^5\times2^3 = 2\times43 + 16 - 16 - 1\times8 = 86 \cancel{+16} \cancel{-16} - 8 = 78$

$\therefore$ 2, 8, 18, 43, 78 ….

Question 30 (***+)

A sequence a_1, a_2, a_3, a_4, … is given by the recurrence formula

$$a_{n+1} = \frac{a_n}{1+a_n}, \quad a_1 = 1$$

a) Determine the value of a_2, a_3, a_4 and a_5.

b) State an expression for the n^{th} term of the sequence and verify that it satisfies the above recurrence formula.

$\boxed{a_2 = \frac{1}{2}}$, $\boxed{a_3 = \frac{1}{3}}$, $\boxed{a_4 = \frac{1}{4}}$, $\boxed{a_5 = \frac{1}{5}}$, $\boxed{a_n = \frac{1}{n}}$

(a) $\boxed{a_{n+1} = \frac{a_n}{1+a_n}}$

$a_1 = 1$

$a_2 = \frac{a_1}{1+a_1} = \frac{1}{1+1} = \frac{1}{2}$

$a_3 = \frac{a_2}{1+a_2} = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{2}{6} = \frac{1}{3}$

$a_4 = \frac{a_3}{1+a_3} = \frac{\frac{1}{3}}{1+\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{3}{12} = \frac{1}{4}$

$a_5 = \frac{a_4}{1+a_4} = \frac{\frac{1}{4}}{1+\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{5}{4}} = \frac{4}{20} = \frac{1}{5}$

(b) $a_n = \frac{1}{n}$

THUS

• $\frac{a_n}{1+a_n} = \frac{\frac{1}{n}}{1+\frac{1}{n}} = \frac{\frac{1}{n}}{\frac{n+1}{n}}$

$= \frac{n}{n(n+1)} = \frac{1}{n+1} = a_{n+1}$

AS REQUIRED

Question 31 **(***+)**

A sequence $u_1, u_2, u_3, u_4, \ldots$ is defined by the recursive relation

$$u_{n+2} = 3u_{n+1} - 2u_n, \quad u_1 = k,$$

where k is a non zero constant.

Given that $u_6 = 33$ and $u_7 = 65$, determine the value of k.

$\boxed{k = 2}$

$u_{n+2} = 3u_{n+1} - 2u_n$

$2u_n = 3u_{n+1} - u_{n+2}$

$u_n = \frac{3}{2}u_{n+1} - \frac{1}{2}u_{n+2}$ $\quad u_6 = 33$ & $u_7 = 65$

$u_5 = \frac{3}{2}u_6 - \frac{1}{2}u_7 = \frac{3}{2}\times 33 - \frac{1}{2}\times 65 = \frac{99}{2} - \frac{65}{2} = 17$

$u_4 = \frac{3}{2}u_5 - \frac{1}{2}u_6 = \frac{3}{2}\times 17 - \frac{1}{2}\times 33 = \frac{51}{2} - \frac{33}{2} = 9$

$u_3 = \frac{3}{2}u_4 - \frac{1}{2}u_5 = \frac{3}{2}\times 9 - \frac{1}{2}\times 17 = \frac{27}{2} - \frac{17}{2} = 5$

$u_2 = \frac{3}{2}u_3 - \frac{1}{2}u_4 = \frac{3}{2}\times 5 - \frac{1}{2}\times 9 = \frac{15}{2} - \frac{9}{2} = 3$

$u_1 = \frac{3}{2}u_2 - \frac{1}{2}u_3 = \frac{3}{2}\times 3 - \frac{1}{2}\times 5 = \frac{9}{2} - \frac{5}{2} = 2$

Question 32 **(***+)**

A sequence $u_1, u_2, u_3, u_4, \ldots$ is given by

$$u_{n+1} = 4u_n + ku_{n-1},$$

where k is a non zero constant.

It is further given that $u_2 = 4$, $u_3 = 12$ and $u_5 = 178$.

Determine the value of k.

$\boxed{k = -\frac{1}{2}}$

$u_{n+1} = 4u_n + ku_{n-1}$ $\quad u_2 = 4,\ u_3 = 12,\ u_5 = 178$

$\left.\begin{array}{l} u_5 = 4u_4 + ku_3 \\ u_4 = 4u_3 + ku_2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 178 = 4u_4 + k\times 12 \\ u_4 = 4\times 12 + k\times 4 \end{array}\right\} \to \left.\begin{array}{l} 4u_4 + 12k = 178 \\ u_4 = 4k + 48 \end{array}\right\}$

$\Rightarrow 4(4k+48) + 12k = 178$

$\Rightarrow 16k + 192 + 12k = 178$

$\Rightarrow 28k = -14$

$\Rightarrow k = -\frac{1}{2}$

Question 33 (*+)**

A recurrence relation is defined for $n \geq 1$ by

$$t_{n+1} = at_n + b, \quad t_1 = 2,$$

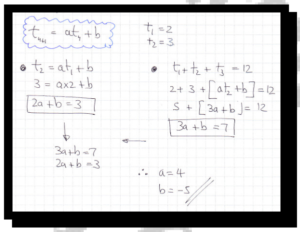
where a and b are non zero constants.

Given further that

$$t_2 = 3 \quad \text{and} \quad \sum_{r=1}^{3} t_r = 12,$$

find the possible value of a and the possible value of b.

$\boxed{}$, $\boxed{a = 4}$, $\boxed{b = -5}$



Question 34 (***+)

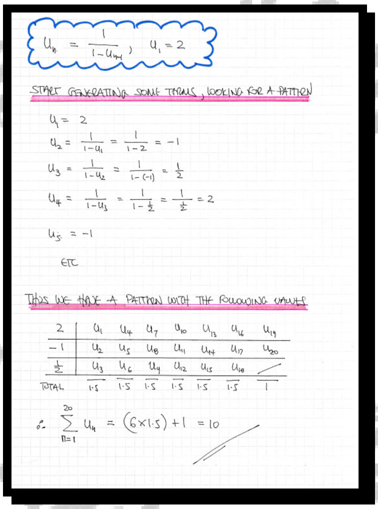
A sequence of numbers, u_1, u_2, u_3, u_4, ... , is defined by

$$u_n = \frac{1}{1-u_{n-1}}, \quad u_1 = 2.$$

Determine the value of

$$\sum_{n=1}^{20} u_n .$$

$$\boxed{\sum_{n=1}^{20} u_n = 10}$$



$$u_n = \frac{1}{1-u_{n-1}}, \quad u_1 = 2$$

START GENERATING SOME TERMS, LOOKING FOR A PATTERN

$u_1 = 2$

$u_2 = \frac{1}{1-u_1} = \frac{1}{1-2} = -1$

$u_3 = \frac{1}{1-u_2} = \frac{1}{1-(-1)} = \frac{1}{2}$

$u_4 = \frac{1}{1-u_3} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

$u_5 = -1$

ETC

THUS WE HAVE A PATTERN WITH THE FOLLOWING VALUES

2	u_1	u_4	u_7	u_{10}	u_{13}	u_{16}	u_{19}
-1	u_2	u_5	u_8	u_{11}	u_{14}	u_{17}	u_{20}
$\frac{1}{2}$	u_3	u_6	u_9	u_{12}	u_{15}	u_{18}	
TOTAL	1.5	1.5	1.5	1.5	1.5	1.5	1

$$\therefore \sum_{n=1}^{20} u_n = (6 \times 1.5) + 1 = 10$$

Question 35 (****)

A recurrence relation is defined for $n \geq 1$ by

$$u_{n+1} = k + (-1)^n u_n, \quad u_1 = 4,$$

where k is a non zero constant.

a) Show clearly that $u_5 = 4$.

b) State, in terms of k, the value of u_{26}.

c) Given further that

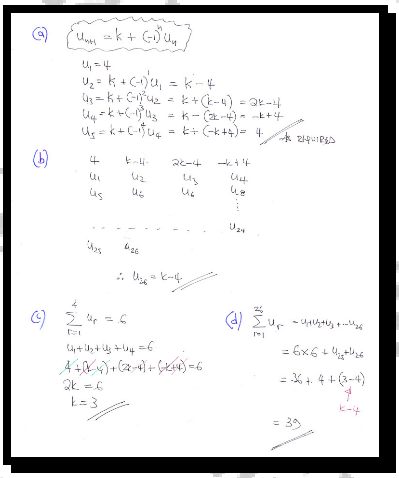
$$\sum_{r=1}^{4} u_r = 6,$$

find the value of k.

d) Evaluate the sum

$$\sum_{r=1}^{26} u_r.$$

$\boxed{}$, $\boxed{u_{26} = k-4}$, $\boxed{k=3}$, $\boxed{S_{26} = 39}$



(a) $u_{n+1} = k + (-1)^n u_n$

$u_1 = 4$
$u_2 = k + (-1)^1 u_1 = k - 4$
$u_3 = k + (-1)^2 u_2 = k + (k-4) = 2k - 4$
$u_4 = k + (-1)^3 u_3 = k - (2k-4) = -k + 4$
$u_5 = k + (-1)^4 u_4 = k + (-k+4) = 4$ AS REQUIRED

(b)
4	$k-4$	$2k-4$	$-k+4$
u_1	u_2	u_3	u_4
u_5	u_6	u_6	u_8
			$\vdots$
			u_{24}
u_{25}	u_{26}		

$\therefore u_{26} = k - 4$

(c) $\sum_{r=1}^{4} u_r = 6$
$u_1 + u_2 + u_3 + u_4 = 6$
$\cancel{4} + (\cancel{k}-\cancel{4}) + (2k - \cancel{4}) + (\cancel{-k}+\cancel{4}) = 6$
$2k = 6$
$k = 3$

(d) $\sum_{r=1}^{26} u_r = u_1 + u_2 + u_3 + \dots u_{26}$
$= 6 \times 6 + u_{25} + u_{26}$
$= 36 + 4 + (3-4)$
$k-4$
$= 39$

Question 36 (****)

A sequence u_1, u_2, u_3, u_4, u_5, … is given by

$$u_{n+2} = u_{n+1} + 6u_n, \quad u_1 = 1, \quad u_2 = 13.$$

a) Find the value of u_3, the value of u_4 and the value of u_5.

b) Find a simplified expression for the n^{th} term of the above sequence by considering the first few terms of the sequence shown below

$$3-2,\ 9+4,\ 27-8,\ 81+16,\ 243-32,\ \ldots$$

$\boxed{}$, $\boxed{u_3 = 19}$, $\boxed{u_4 = 97}$, $\boxed{u_5 = 211}$, $\boxed{u_n = 3^n + (-2)^n}$

(a) $u_{n+2} = u_{n+1} + 6u_n$

$u_1 = 1$

$u_2 = 13$

$u_3 = u_2 + 6u_1 = 13 + 6\times1 = 19$

$u_4 = u_3 + 6u_2 = 19 + 6\times13 = 19 + 78 = 97$

$u_5 = u_4 + 6u_3 = 97 + 6\times19 = 97 + 114 = 211$

(b)

$u_1 = 1 = 3 - 2 = 3^1 - 2^1 = 3^1 + (-2)^1$

$u_2 = 13 = 9 + 4 = 3^2 + 2^2 = 3^2 + (-2)^2$

$u_3 = 19 = 27 - 8 = 3^3 - 2^3 = 3^3 + (-2)^3$

$u_4 = 97 = 81 + 16 = 3^4 + 2^4 = 3^4 + (-2)^4$

$u_5 = 211 = 243 - 32 = 3^5 - 2^5 = 3^5 + (-2)^5$

$\therefore u_n = 3^n + (-2)^n$

Question 37 (****)

A sequence t_1, t_2, t_3, t_4, … is given by

$$t_{n+1} = a + bt_n ,$$

where a and b are non zero constants.

It is given that $t_3 = 320$, $t_4 = 240$ and $t_5 = 200$.

a) Determine the value of a and the value of b.

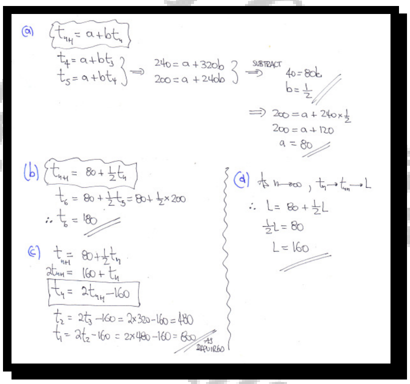
b) Find the value of t_6.

c) Show clearly that $t_1 = 800$.

The sequence converges to a limit L.

d) Determine the value of L.

$\boxed{a = 80}$, $\boxed{b = \frac{1}{2}}$, $\boxed{t_6 = 180}$, $\boxed{L = 160}$


(a) $\boxed{t_{n+1} = a + bt_n}$

$\left.\begin{array}{l} t_4 = a + bt_3 \\ t_5 = a + bt_4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 240 = a + 320b \\ 200 = a + 240b \end{array}\right\} \xrightarrow{\text{SUBTRACT}} 40 = 80b$

$b = \frac{1}{2}$

$\Rightarrow 200 = a + 240 \times \frac{1}{2}$

$200 = a + 120$

$a = 80$

(b) $\boxed{t_{n+1} = 80 + \frac{1}{2}t_n}$

$t_6 = 80 + \frac{1}{2}t_5 = 80 + \frac{1}{2} \times 200$

$\therefore t_6 = 180$

(c) $t_{n+1} = 80 + \frac{1}{2}t_n$

$2t_{n+1} = 160 + t_n$

$\boxed{t_n = 2t_{n+1} - 160}$

$t_2 = 2t_3 - 160 = 2 \times 320 - 160 = 480$

$t_1 = 2t_2 - 160 = 2 \times 480 - 160 = 800$ AS REQUIRED

(d) As $n \to \infty$, $t_n \to t_{n+1} \to L$

$\therefore L = 80 + \frac{1}{2}L$

$\frac{1}{2}L = 80$

$L = 160$

Question 38 (****)

A recurrence relation is defined for $n \geq 1$ by

$$a_{n+1} = (a_n)^2 - 4, \quad a_1 = k,$$

where k is a non zero constant.

a) Find the value of a_3 in terms of k.

It is given that $a_2 + a_3 = 26$.

b) Find the possible values of k.

$\boxed{}$, $\boxed{a_3 = k^4 - 8k^2 + 12}$, $\boxed{k = \pm 3}$

(a) $\boxed{a_{n+1} = (a_n)^2 - 4}$

- $a_1 = k$
- $a_2 = (a_1)^2 - 4 = k^2 - 4$
- $a_3 = (a_2)^2 - 4 = (k^2 - 4)^2 - 4 = k^4 - 8k^2 + 16 - 4 = k^4 - 8k^2 + 12$

(b) $a_2 + a_3 = 26$

$\Rightarrow (k^2 - 4) + (k^4 - 8k^2 + 12) = 26$

$\Rightarrow k^4 - 7k^2 - 18 = 0$

$\Rightarrow (k^2 - 9)(k^2 + 2) = 0$

$k^2 = 9$ or $\cancel{-2}$

$k = 3$ or -3

Question 39 (****)

A sequence y_1, y_2, y_3, y_4, y_5, y_6 … is given by

$$y_{n+2} = y_{n+1} + 2y_n, \quad y_1 = 1, \quad y_2 = 5.$$

a) Find the value of y_3, y_4, y_5 and y_6.

b) Find a simplified expression for the n^{th} term of the sequence, by considering the first few powers of 2.

$\boxed{}$, $\boxed{y_3 = 7}$, $\boxed{y_4 = 17}$, $\boxed{y_5 = 31}$, $\boxed{y_6 = 65}$ $\boxed{y_n = 2^n + (-1)^n}$

(a) $\boxed{y_{n+2} = y_{n+1} + 2y_n.}$

$y_1 = 1$
$y_2 = 5$
$y_3 = y_2 + 2y_1 = 5 + 2\times1 = 7$
$y_4 = y_3 + 2y_2 = 7 + 2\times5 = 17$
$y_5 = y_4 + 2y_3 = 17 + 2\times7 = 31$
$y_6 = y_5 + 2y_4 = 31 + 2\times17 = 65$

(b) $x_n = 2^n$: 2 4 8 16 32 64

y_n : 1 5 7 17 31 65

$-1 \quad +1 \quad -1 \quad +1 \quad -1 \quad +1$

$\therefore y_n = 2^n + (-1)^n$

Question 40 (****)

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_{n+1} = k - \frac{12}{u_n}, \quad u_1 = 1,$$

where k is a non zero constant.

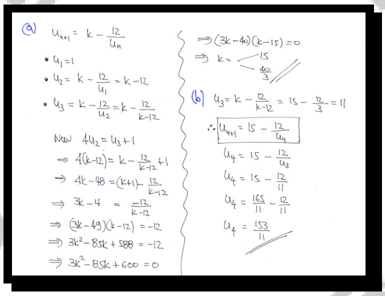
It is further given that

$$4u_2 = u_3 + 1.$$

a) Show that one of the possible values of k is 15 and find the other.

b) If $k = 15$ find the exact value of u_4.

$\boxed{k = \frac{40}{3}}$, $\boxed{u_4 = \frac{153}{11}}$


(a) $u_{n+1} = k - \frac{12}{u_n}$

- $u_1 = 1$
- $u_2 = k - \frac{12}{u_1} = k - 12$
- $u_3 = k - \frac{12}{u_2} = k - \frac{12}{k-12}$

Now $4u_2 = u_3 + 1$

$\Rightarrow 4(k-12) = k - \frac{12}{k-12} + 1$

$\Rightarrow 4k - 48 = (k+1) - \frac{12}{k-12}$

$\Rightarrow 3k - 4 = \frac{-12}{k-12}$

$\Rightarrow (3k-49)(k-12) = -12$

$\Rightarrow 3k^2 - 85k + 588 = -12$

$\Rightarrow 3k^2 - 85k + 600 = 0$

$\Rightarrow (3k-40)(k-15) = 0$

$\Rightarrow k = 15, \frac{40}{3}$

(b) $u_3 = k - \frac{12}{k-12} = 15 - \frac{12}{3} = 11$

$\therefore \boxed{u_{n+1} = 15 - \frac{12}{u_n}}$

$u_4 = 15 - \frac{12}{u_3}$

$u_4 = 15 - \frac{12}{11}$

$u_4 = \frac{165}{11} - \frac{12}{11}$

$u_4 = \frac{153}{11}$

Question 41 (****)

A sequence of numbers is defined by the recurrence relation for $n \geq 1$

$$u_{n+1} = ku_n + 6, \quad u_1 = 4,$$

where k is a non zero constant.

a) Find, in terms of k, the value of u_2 and the value of u_3

b) Given that $u_3 = 10$ find the possible values of k.

The sequence tends to a limit L.

c) Find the value of u_4.

d) Determine the value of L.

$\boxed{u_2 = 4k + 6}$, $\boxed{u_3 = 4k^2 + 6k + 6}$, $\boxed{k = -2, \frac{1}{2}}$, $\boxed{u_4 = 11}$, $\boxed{L = 12}$

(a) $u_{n+1} = ku_n + 6$
$u_1 = 4$
$u_2 = ku_1 + 6 = 4k + 6$
$u_3 = ku_2 + 6 = k(4k+6) + 6$
$= 4k^2 + 6k + 6$

(b) $u_3 = 10$
$\Rightarrow 4k^2 + 6k + 6 = 10$
$\Rightarrow 4k^2 + 6k - 4 = 0$
$\Rightarrow 2k^2 + 3k - 2 = 0$
$\Rightarrow (2k -)(k + 2) = 0$
$\Rightarrow k = -2, \frac{1}{2}$

(c) IF SEQUENCE CONVERGES $k = \frac{1}{2}$
($-1 < k < 1$ TO CONVERGE)
$\therefore u_{n+1} = \frac{1}{2}u_n + 6$
$u_4 = \frac{1}{2}u_3 + 6 = \frac{1}{2} \times 10 + 6 = 11$

(d) AS $n \to \infty$, $u_n \to u_{n+1} \to L$
$\therefore L = \frac{1}{2}L + 6$
$2L = L + 12$
$L = 12$

Question 42 (****)

A recurrence relation is defined for $n \geq 1$ by

$$U_{n+1} = aU_n + b, \quad U_1 = k,$$

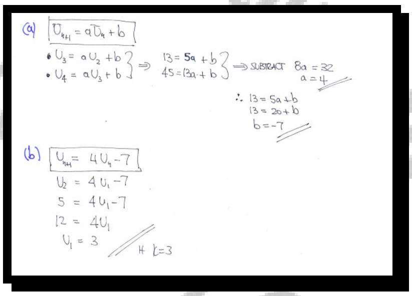
where a, b and k are non zero constants.

It is given that $U_2 = 5$, $U_3 = 13$ and $U_4 = 45$.

a) Find the value of a and the value of b.

b) Determine the value of k.

$\boxed{a = 4}$, $\boxed{b = -7}$, $\boxed{U_1 = k = 3}$



(a) $\boxed{U_{n+1} = aU_n + b}$

- $U_3 = aU_2 + b$
- $U_4 = aU_3 + b$

$\Rightarrow$ $13 = 5a + b$, $45 = 13a + b$ $\Rightarrow$ SUBTRACT $8a = 32$, $a = 4$

$\therefore 13 = 5a + b$
$13 = 20 + b$
$b = -7$

(b) $\boxed{U_{n+1} = 4U_n - 7}$

$U_2 = 4U_1 - 7$
$5 = 4U_1 - 7$
$12 = 4U_1$
$U_1 = 3$ // $k = 3$

Question 43 (****)

A sequence of numbers is given by the recurrence relation

$$t_{n+1} = At_n + B, \quad n \geq 1,$$

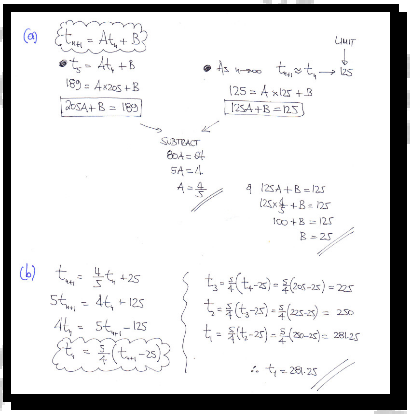
where A and B are non zero constants.

It is given that $t_4 = 205$ and $t_5 = 189$, and the sequence converges to 125.

a) By forming and solving two equations show that $A = \frac{4}{5}$ and $B = 25$.

b) Find the value of t_1.

$$\boxed{t_1 = 281.25}$$



(a) $t_{n+1} = At_n + B$

- $t_5 = At_4 + B$

$189 = A \times 205 + B$

$\boxed{205A + B = 189}$

- As $n \to \infty$ $t_{n+1} \approx t_n \to 125$ LIMIT

$125 = A \times 125 + B$

$\boxed{125A + B = 125}$

SUBTRACT

$80A = 64$

$5A = 4$

$A = \frac{4}{5}$

$125A + B = 125$

$125 \times \frac{4}{5} + B = 125$

$100 + B = 125$

$B = 25$

(b) $t_{n+1} = \frac{4}{5}t_n + 25$

$5t_{n+1} = 4t_n + 125$

$4t_n = 5t_{n+1} - 125$

$t_n = \frac{5}{4}(t_{n+1} - 25)$

$t_3 = \frac{5}{4}(t_4 - 25) = \frac{5}{4}(205 - 25) = 225$

$t_2 = \frac{5}{4}(t_3 - 25) = \frac{5}{4}(225 - 25) = 250$

$t_1 = \frac{5}{4}(t_2 - 25) = \frac{5}{4}(250 - 25) = 281.25$

$\therefore t_1 = 281.25$

Question 44 (****)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = pu_n + q, \quad n \geq 1,$$

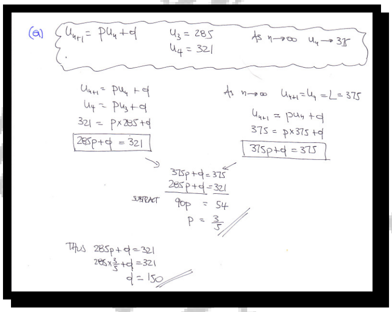
where p and q are non zero constants.

It is given that $u_3 = 285$ and $u_4 = 321$, and the sequence converges to 375.

a) Find the value of p and the value of q

b) Determine the value of u_1.

$$\boxed{p = \frac{3}{5}}, \quad \boxed{q = 150}, \quad \boxed{u_1 = 125}$$



(a) $u_{n+1} = pu_n + q$ $\quad u_3 = 285$, $u_4 = 321$ $\quad$ As $n \to \infty$ $u_n \to 375$

$u_{n+1} = pu_n + q$

$u_4 = pu_3 + q$

$321 = p \times 285 + q$

$\boxed{285p + q = 321}$

As $n \to \infty$ $u_{n+1} = u_n = L = 375$

$u_{n+1} = pu_n + q$

$375 = p \times 375 + q$

$\boxed{375p + q = 375}$

$375p + q = 375$

$285p + q = 321$

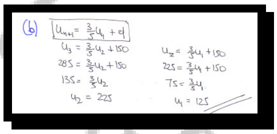
SUBTRACT $\quad 90p = 54$

$p = \frac{3}{5}$

Thus $285p + q = 321$

$285 \times \frac{3}{5} + q = 321$

$q = 150$



(b) $\boxed{u_{n+1} = \frac{3}{5}u_n + q}$

$u_3 = \frac{3}{5}u_2 + 150$

$285 = \frac{3}{5}u_2 + 150$

$135 = \frac{3}{5}u_2$

$u_2 = 225$

$u_2 = \frac{3}{5}u_1 + 150$

$225 = \frac{3}{5}u_1 + 150$

$75 = \frac{3}{5}u_1$

$u_1 = 125$

Question 45 (****)

$$P_{n+1} = A + BP_n, \ t > 1.$$

The relationship above gives the amount of money Adrian pays into a pension scheme each year P_n, where n is the pension contribution in the n^{th} year.

Adrian's annual contributions in the 2^{nd}, 3^{rd} and 4^{th} years were £1625, £2425 and £3065, respectively.

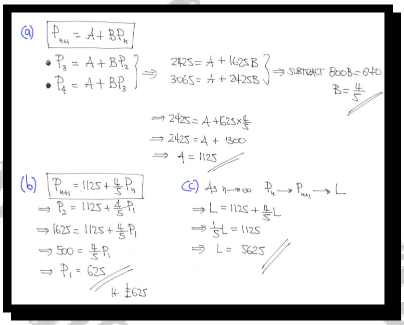
a) Find the value of A and the value of B.

b) Determine Adrian's annual contributions in the first year.

Adrian's annual contributions cannot exceed a certain amount L.

c) Find the value of L.

$\boxed{A = 1125}$, $\boxed{B = 0.8}$, $\boxed{P_1 = 625}$, $\boxed{L = 5625}$



Question 46 (****)

In a clinical trial the concentration C, of a certain blood agent, is measured at one hour intervals since a trial drug was first administered to a patient.

The following readings were obtained

$$C_3 = 88, \; C_4 = 76 \text{ and } C_5 = 70,$$

where C_t denotes the reading t hours after the drug was first administered.

It is thought that C satisfies the relationship

$$C_{t+1} = a + bC_t, \; t \geq 0.$$

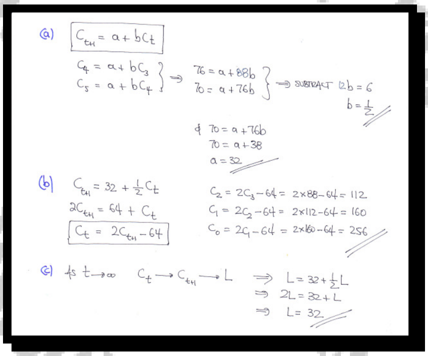
a) Find the value of a and the value of b.

b) Determine the **initial** concentration of the blood agent, when the drug was first administered.

The value of C converges to a limit L.

c) Find the value of L.

$\boxed{}$, $\boxed{a = 32}$, $\boxed{b = \frac{1}{2}}$, $\boxed{C_0 = 256}$, $\boxed{L = 64}$



(a) $\boxed{C_{t+1} = a + bC_t}$

$\left.\begin{array}{l} C_4 = a + bC_3 \\ C_5 = a + bC_4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 76 = a + 88b \\ 70 = a + 76b \end{array}\right\} \Rightarrow$ SUBTRACT $12b = 6$

$b = \frac{1}{2}$

& $70 = a + 76b$

$70 = a + 38$

$a = 32$

(b) $C_{t+1} = 32 + \frac{1}{2}C_t$

$2C_{t+1} = 64 + C_t$

$\boxed{C_t = 2C_{t+1} - 64}$

$C_2 = 2C_3 - 64 = 2 \times 88 - 64 = 112$

$C_1 = 2C_2 - 64 = 2 \times 112 - 64 = 160$

$C_0 = 2C_1 - 64 = 2 \times 160 - 64 = 256$

(c) As $t \to \infty$ $\quad C_t \to C_{t+1} \to L$ $\quad \Rightarrow L = 32 + \frac{1}{2}L$

$\Rightarrow 2L = 32 + L$

$\Rightarrow L = 32$

Question 47 (****)

A sequence u_1, u_2, u_3, u_4, … is given by

$$(u_{n+1})(u_n)^2 = \frac{n^4}{n+2}, \quad u_1 = \frac{1}{2}.$$

Calculate the value of u_2, u_3, u_4 and u_5, and hence write an expression for the n^{th} term of the sequence.

$$\boxed{\frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6}, \ldots}, \quad \boxed{u_n = \frac{n^2}{n+1}}$$

$$\boxed{u_{n+1}u_n^2 = \frac{n^4}{n+2}} \quad \text{OR} \quad \boxed{u_{n+1} = \frac{n^4}{n+2} \times \frac{1}{(u_n)^2}}$$

- $u_1 = \frac{1}{2}$
- $u_2 = \frac{1^4}{1+2} \times \frac{1}{(u_1)^2} = \frac{1}{3} \times \frac{1}{(\frac{1}{2})^2} = \frac{1}{3} \times \frac{1}{\frac{1}{4}} = \frac{1}{3} \times 4 = \frac{4}{3}$
- $u_3 = \frac{2^4}{2+2} \times \frac{1}{(u_2)^2} = \frac{16}{4} \times \frac{1}{(\frac{4}{3})^2} = 4 \times \frac{1}{\frac{16}{9}} = \cancel{4} \times \frac{9}{\cancel{16}_4} = \frac{9}{4}$
- $u_4 = \frac{3^4}{3+2} \times \frac{1}{(u_3)^2} = \frac{81}{5} \times \frac{1}{(\frac{9}{4})^2} = \frac{81}{5} \times \frac{1}{\frac{81}{16}} = \frac{\cancel{81}}{5} \times \frac{16}{\cancel{81}} = \frac{16}{5}$
- $u_5 = \frac{4^4}{4+2} \times \frac{1}{(u_4)^2} = \frac{4^2 \times 4^2}{6} \times \frac{1}{(\frac{16}{5})^2} = \frac{16 \times 16}{6} \times \frac{1}{\frac{16 \times 16}{25}} = \frac{\cancel{16 \times 16}}{6} \times \frac{25}{\cancel{16 \times 16}} = \frac{25}{6}$

$\therefore \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \frac{25}{6}, \ldots, \frac{n^2}{n+1},$

$\therefore u_n = \frac{n^2}{n+1}$

Question 48 (****)

A sequence x_1, x_2, x_3, x_4, … is given by the recurrence formula

$$x_{n+1} = \frac{x_n}{2nx_n + x_n + 1}, \quad x_1 = 1$$

a) Determine the value of x_2, x_3, x_4 and x_5.

b) State an expression for the n^{th} term of the sequence and verify that it satisfies the above recurrence formula.

$$\boxed{x_2 = \frac{1}{4}}, \ \boxed{x_3 = \frac{1}{9}}, \ \boxed{x_4 = \frac{1}{16}}, \ \boxed{x_5 = \frac{1}{25}}, \ \boxed{x_n = \frac{1}{n^2}}$$

(a) $\boxed{x_{n+1} = \frac{x_n}{2nx_n + x_n + 1}}$

$x_1 = 1$

$x_2 = \frac{x_1}{2\times1\times x_1 + x_1 + 1} = \frac{1}{2\times1\times1+1+1} = \frac{1}{4}$

$x_3 = \frac{x_2}{2\times2\times x_2 + x_2 + 1} = \frac{\frac{1}{4}}{2\times2\times\frac{1}{4}+\frac{1}{4}+1} = \frac{\frac{1}{4}}{1+\frac{1}{4}+1} = \frac{\frac{1}{4}}{\frac{9}{4}} = \frac{4}{36} = \frac{1}{9}$

$x_4 = \frac{x_3}{2\times3\times x_3 + x_3 + 1} = \frac{\frac{1}{9}}{2\times3\times\frac{1}{9}+\frac{1}{9}+1} = \frac{\frac{1}{9}}{\frac{6}{9}+\frac{1}{9}+1} = \frac{\frac{1}{9}}{\frac{16}{9}} = \frac{9}{9\times16} = \frac{1}{16}$

$x_5 = \frac{x_4}{2\times4\times x_4 + x_4 + 1} = \frac{\frac{1}{16}}{2\times4\times\frac{1}{16}+\frac{1}{16}+1} = \frac{\frac{1}{16}}{\frac{8}{16}+\frac{1}{16}+1} = \frac{\frac{1}{16}}{\frac{25}{16}} = \frac{16}{16\times25} = \frac{1}{25}$

(b) $x_n = \frac{1}{n^2}$

Hence

$\frac{x_n}{2nx_n + x_n + 1} = \frac{\frac{1}{n^2}}{2n\left(\frac{1}{n^2}\right)+\frac{1}{n^2}+1} = \frac{\frac{1}{n^2}}{\frac{2}{n}+\frac{1}{n^2}+1} = \ldots$ MULTIPLY TOP/BOTTOM BY n^2

$= \frac{1}{2n+1+n^2} = \frac{1}{n^2+2n+1} = \frac{1}{(n+1)^2} = x_{n+1}$

AS REQUIRED

Question 49 (****)

A recurrence relation obeys the relationship

$$x_{n+1} = \sqrt{x_n + 12}\ ,\ \ x_1 = k\ ,$$

where k is a non zero constant.

This recurrence relation converges to a limit L, for a suitable range of values of k.

a) Find the value of L.

b) Determine the **range** of values of k, so L exists.

$\boxed{L = 4}$, $\boxed{k \geq -12}$

(a) $x_{n+1} = \sqrt{x_n + 12}$

• As $n \to \infty$ $x_n \to x_{n+1} \to L$

$\Rightarrow L = \sqrt{L+12}$

$\Rightarrow L^2 = L + 12$

$\Rightarrow L^2 - L - 12 = 0$

$\Rightarrow (L-4)(L+3) = 0$

$\Rightarrow L = 4$, $\cancel{-3}$

(b) L ONLY EXISTS IF $x_1 + 12 \geq 0$

$\therefore x_1 \geq -12$

$k \geq -12$

Question 50 (****)

A sequence u_1, u_2, u_3, u_4, ... is given by

$$u_{n+1} = \frac{1}{1-u_n}, \quad u_1 = k,$$

where k is a non zero constant.

a) Show clearly that $u_4 = k$.

b) Given that $u_2 \times u_3 = -\frac{1}{2}$, determine the value of k .

c) State the value of u_{110} .

$\boxed{k=2}$, $\boxed{u_{110}=-1}$

(a) $\boxed{u_{n+1} = \frac{1}{1-u_n}}$

- $u_1 = k$
- $u_2 = \frac{1}{1-u_1} = \frac{1}{1-k}$
- $u_3 = \frac{1}{1-u_2} = \frac{1}{1-\frac{1}{1-k}} = \frac{1}{\frac{\cancel{1}-k-\cancel{1}}{1-k}} = \frac{1}{\frac{-k}{1-k}} = \frac{1-k}{-k} = \frac{k-1}{k}$
- $u_4 = \frac{1}{1-u_3} = \frac{1}{1-\frac{k-1}{k}} = \frac{1}{\frac{k-(k-1)}{k}} = \frac{1}{\frac{1}{k}} = \frac{\frac{1}{1}}{\frac{1}{k}} = k$ // AS REQUIRED

(b) $u_2 \times u_3 = -\frac{1}{2}$

$\Rightarrow \frac{1}{1-k} \times \frac{k-1}{k} = -\frac{1}{2}$

$\Rightarrow \frac{k-1}{k(1-k)} = -\frac{1}{2}$ $\quad \times(-1)$

$\Rightarrow \frac{\cancel{k-1}}{k\cancel{(k-1)}} = \frac{1}{2}$ $\quad (k \neq 1)$

$\Rightarrow \frac{1}{k} = \frac{1}{2}$

$\Rightarrow k = 2$ //

(c)

k	$\frac{1}{1-k}$	$\frac{k-1}{k}$
u_1	u_2	u_3
u_4	u_5	u_6
$\vdots$	$\vdots$	$\vdots$
...	...	u_{99}
...	...	u_{102}
...	...	u_{105}
...	...	u_{108}
u_{109}	u_{110}	u_{111}

$\therefore u_{110} = \frac{1}{1-k} = \frac{1}{1-2} = -1$ //

Question 51 (****)

A sequence u_1, u_2, u_3, u_4, u_5 ... satisfies

$$u_{n+1} = Au_n + B,$$

where A and B are non zero constants.

The second and third term of this sequence are 464 and 428, respectively.

Given further that the sequence converges to 320, find the value of the fourth term of this sequence.

$\boxed{}$, $\boxed{u_4 = 401}$

FORM SOME EQUATIONS BASED ON THE RECURRENCE FORMULA

- $u_{n+1} = Au_n + B$
  $u_3 = Au_2 + B$
  $428 = A \times 464 + B$
  $464A + B = 428$

- As $n \to \infty$ $u_n \to u_{n+1} \to 320$
  $u_{n+1} = Au_n + B$
  $320 = A \times 320 + B$
  $320A + B = 320$

SUBTRACTING THE EQUATIONS

$144A = 108$
$A = \frac{3}{4}$

AND B CAN NOW BE FOUND

$320A + B = 320$
$320 \times \frac{3}{4} + B = 320$
$240 + B = 320$
$B = 80$

THUS WE NOW HAVE

$u_{n+1} = \frac{3}{4}u_n + 80$
$u_4 = \frac{3}{4}u_3 + 80$
$u_4 = \frac{3}{4} \times 428 + 80$
$u_4 = 401$

Question 52 (****+)

A sequence of numbers is given by the recurrence relation

$$u_{n+1} = \frac{Au_n + 2}{4 + Bu_n}, \quad n \geq 1, \quad u_1 = \frac{1}{2},$$

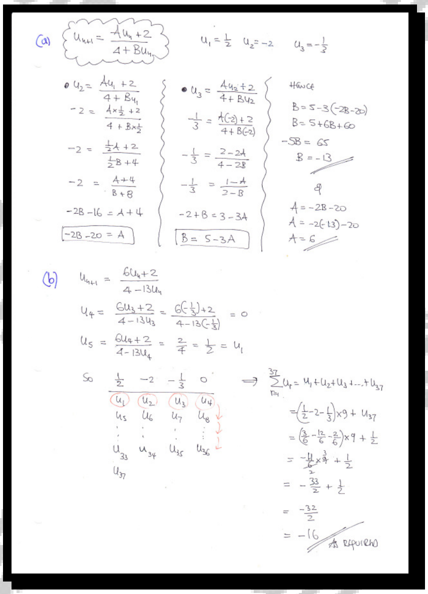
where A and B are non zero constants.

a) If $u_2 = -2$ and $u_3 = -\frac{1}{3}$, find the value of A and the value of B.

b) Show clearly that

$$\sum_{r=1}^{37} u_r = -16.$$

$\boxed{}$, $\boxed{A = 6}$, $\boxed{B = -13}$



Question 53 (****+)

A sequence u_1, u_2, u_3, u_4, ... is given by

$$u_{n+1} = \frac{u_n}{a} + \frac{a}{u_n}, \quad u_1 = 2,$$

where a is a non zero constant.

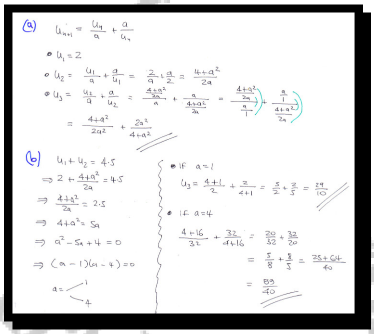
a) Find the value of u_3 in terms of a.

It is further given that

$$u_1 + u_2 = 4.5.$$

b) Find the possible values of u_3.

$$\boxed{u_3 = \frac{4+a^2}{2a^2} + \frac{2a^2}{4+a^2}}, \quad \boxed{u_3 = \frac{29}{10}, \frac{89}{40}}$$



(a) $u_{n+1} = \frac{u_n}{a} + \frac{a}{u_n}$

- $u_1 = 2$
- $u_2 = \frac{u_1}{a} + \frac{a}{u_1} = \frac{2}{a} + \frac{a}{2} = \frac{4+a^2}{2a}$
- $u_3 = \frac{u_2}{a} + \frac{a}{u_2} = \frac{\frac{4+a^2}{2a}}{a} + \frac{a}{\frac{4+a^2}{2a}} = \frac{\frac{4+a^2}{2a}}{\frac{a}{1}} + \frac{\frac{a}{1}}{\frac{4+a^2}{2a}}$

$= \frac{4+a^2}{2a^2} + \frac{2a^2}{4+a^2}$

(b) $u_1 + u_2 = 4.5$

$\Rightarrow 2 + \frac{4+a^2}{2a} = 4.5$

$\Rightarrow \frac{4+a^2}{2a} = 2.5$

$\Rightarrow 4 + a^2 = 5a$

$\Rightarrow a^2 - 5a + 4 = 0$

$\Rightarrow (a-1)(a-4) = 0$

$a = 1, 4$

- If $a = 1$

$u_3 = \frac{4+1}{2} + \frac{2}{4+1} = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$

- If $a = 4$

$\frac{4+16}{32} + \frac{32}{4+16} = \frac{20}{32} + \frac{32}{20}$

$= \frac{5}{8} + \frac{8}{5} = \frac{25+64}{40}$

$= \frac{89}{40}$

Question 54 (****+)

A sequence t_1, t_2, t_3, t_4, t_5, … is given by

$$t_{n+1} = a t_n + 3n + 2, \quad t \in \mathbb{N}, \quad t_1 = -2,$$

where a is a non zero constant.

a) Given that $\displaystyle\sum_{r=1}^{3}\left(r^3 + t_r\right) = 12$, determine the possible values of a.

b) Evaluate $\displaystyle\sum_{r=8}^{31}\left(t_{r+1} - a t_r\right)$.

$\boxed{}$, $\boxed{a = 5, \ a = -\frac{7}{2}}$, $\boxed{1452}$

a) LOOKING AT THE FIRST SUMMATION WE REQUIRE THE FIRST 3 TERMS

$$t_{n+1} = at_n + 3n + 2$$

- $t_1 = -2$
- $t_2 = at_1 + 3\times1 + 2 = a(-2) + 3 + 2 = 5 - 2a$
- $t_3 = at_2 + 3\times2 + 2 = a(5-2a) + 6 + 2 = 8 + 5a - 2a^2$

NOW WE HAVE

$$\begin{aligned}\sum_{r=1}^{3}(t_r + r^3) &= (t_1 + 1^3) + (t_2 + 2^3) + (t_3 + 3^3)\\ &= t_1 + 1 + t_2 + 8 + t_3 + 27\\ &= t_1 + t_2 + t_3 + 36\\ &= -2 + (5-2a) + (8 + 5a - 2a^2) + 36\\ &= -2a^2 + 3a + 47\end{aligned}$$

FINALLY WE HAVE

$$\Rightarrow \sum_{r=1}^{3}(t_r + r^3) = 12$$

$$\Rightarrow -2a^2 + 3a + 47 = 12$$

$$\Rightarrow 0 = 2a^2 - 3a - 35$$

$$\Rightarrow (a-5)(2a+7) = 0$$

$$a = \begin{cases} 5 \\ -\frac{7}{2} \end{cases}$$

b) WE PROCEED AS FOLLOWS (THE VALUE OF k IS IRRELEVANT)

$$\begin{aligned}\sum_{r=8}^{31}\left[t_{r+1} - at_r\right] &= \sum_{r=8}^{31}\left[(\cancel{at_r} + 3r + 2) - \cancel{at_r}\right]\\ &= \sum_{r=8}^{31}(3r+2)\\ &= 26 + 29 + 32 + 35 + \dots + 95\end{aligned}$$

THIS IS AN ARITHMETIC PROGRESSION WITH $(31-7) = 24$ TERMS

$$\Rightarrow S_n = \frac{n}{2}\left[a + L\right]$$

$$\Rightarrow S_{24} = \frac{24}{2}\left[26 + 95\right]$$

$$\Rightarrow S_{24} = 12 \times 121$$

$$\Rightarrow S_{24} = \begin{matrix}1210\\ 242\end{matrix}$$

$$\Rightarrow S_{24} = 1452$$

Question 55 **(****+)**

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_n = 2n^2 - 7n - 5 .$$

Find an expression for u_{n+1} as a recurrence relation of the form

$$u_{n+1} = Au_n + Bn + C , \; u_1 = D ,$$

where A, B, C and D are constants to be found.

$$\boxed{u_{n+1} = u_n + 4n - 5, \quad u_1 = -10}$$

- $u_n = 2n^2 - 7n - 5$
- $u_{n+1} = 2(n+1)^2 - 7(n+1) - 5 = 2(n^2+2n+1) - 7n - 7 - 5$

$$= 2n^2 + 4n + 2 - 7n - 12$$
$$= 2n^2 - 3n - 10$$

$\therefore u_{n+1} - u_n = (2n^2 - 3n - 10) - (2n^2 - 7n - 5)$

$u_{n+1} - u_n = 4n - 5$

$u_{n+1} = u_n + 4n - 5$

& $u_1 = 2 \times 1^2 - 7 \times 1 - 5$

$u_1 = 2 - 7 - 5$

$u_1 = -10$

$\therefore u_{n+1} = u_n + 4n - 5$ with $u_1 = -10$

Question 56 (**+)**

A sequence u_1, u_2, u_3, u_4, … is given by

$$u_n = 2^n + 4n .$$

Find an expression for u_{n+1} as a recurrence relation of the form

$$u_{n+1} = Au_n + Bn + C , \ u_1 = D ,$$

where A, B, C and D are constants to be found.

$\boxed{}$, $\boxed{u_{n+1} = 2u_n - 4n + 4, \quad u_1 = 6}$

$$u_n = 2^n + 4n$$

$$\left.\begin{array}{l} u_n = 2^n + 4n \\ u_{n+1} = 2^{n+1} + 4(n+1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} u_n = 2^n + 4n \\ u_{n+1} = 2^n \times 2^1 + 4n + 4 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} u_n - 4n = 2^n \\ u_{n+1} - 4n - 4 = 2 \times 2^n \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2u_n - 8n = 2 \times 2^n \\ u_{n+1} - 4n - 4 = 2 \times 2^n \end{array}\right\} \Rightarrow$$

$$u_{n+1} - 4n - 4 = 2u_n - 8n$$

$$u_{n+1} = 2u_n - 4n + 4$$

& $u_1 = 2^1 + 4 \times 1$

$u_1 = 6$

Question 57 (****+)

A recurrence relation is defined for $n \geq 1$ by

$$t_{n+1} = at_n + b,$$

where a and b are non zero constants.

It is given that $t_2 = 176$, $t_3 = 248$ and $t_4 = 284$.

a) Find the value of a and the value of b.

b) Determine the value of t_1.

The sequence converges to a limit l.

c) Find the value of l.

The n^{th} term of the sequence is given by

$$t_n = p + q\left(\frac{1}{2}\right)^n, \quad \text{where } p \text{ and } q \text{ are constants.}$$

d) Find the value of p and the value of q.

$$\boxed{a = \frac{1}{2}}, \ \boxed{b = 160}, \ \boxed{t_1 = 32}, \ \boxed{l = 320}, \ \boxed{p = 320}, \ \boxed{q = -576}$$

(a) $t_{n+1} = at_n + b$

- $t_3 = at_2 + b$
- $t_4 = at_3 + b$

$\Rightarrow \left.\begin{matrix} 248 = 176a + b \\ 284 = 248a + b \end{matrix}\right\} \Rightarrow$ SUBTRACT $72a = 36$

$a = \frac{1}{2}$

& $248 = 176a + b$

$248 = 88 + b$

$b = 160$

(b) $t_2 = at_1 + b$

$176 = \frac{1}{2}t_1 + 160$

$16 = \frac{1}{2}t_1$

$t_1 = 32$

(c) $\boxed{t_{n+1} = \frac{1}{2}t_n + 160}$

- As $n \to \infty$ $t_n \to t_{n+1} \to l$

So $l = \frac{1}{2}l + 160$

$2l = l + 320$

$l = 320$

(d) $\boxed{t_n = p + q\left(\frac{1}{2}\right)^n}$ As $n \to \infty$ $\left(\frac{1}{2}\right)^n \to 0$ $t_n \to p$

$\therefore p = l = 320$

$\boxed{t_n = 320 + q\left(\frac{1}{2}\right)^n}$ $t_2 = 176 \Rightarrow 176 = 320 + q \times \left(\frac{1}{2}\right)^2$

$\Rightarrow 176 = 320 + \frac{1}{4}q$

$\Rightarrow -144 = \frac{1}{4}q$

$\Rightarrow q = -576$

Question 58 (****+)

A sequence u_1, u_2, u_3, u_4, … is given by

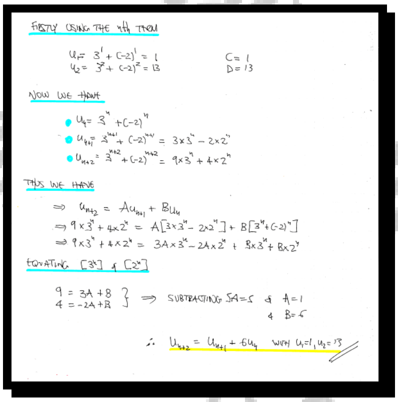
$$u_n = 3^n + (-2)^n .$$

Find an expression for u_{n+2}, as a recurrence relation of the form,

$$u_{n+2} = Au_{n+1} + Bu_n\,,\ u_1 = C\,,\ u_2 = D$$

where A, B, C and D are constants to be found.

$\boxed{}$, $\boxed{u_{n+2} = u_{n+1} + 6u_n, \quad u_1 = 1, \quad u_2 = 13}$



FIRSTLY USING THE nth TERM

$u_1 = 3^1 + (-2)^1 = 1$ $\quad C = 1$
$u_2 = 3^2 + (-2)^2 = 13$ $\quad D = 13$

NOW WE HAVE

- $u_n = 3^n + (-2)^n$
- $u_{n+1} = 3^{n+1} + (-2)^{n+1} = 3\times 3^n - 2\times 2^n$
- $u_{n+2} = 3^{n+2} + (-2)^{n+2} = 9\times 3^n + 4\times 2^n$

THUS WE HAVE

$\Rightarrow u_{n+2} = Au_{n+1} + Bu_n$
$\Rightarrow 9\times 3^n + 4\times 2^n = A[3\times 3^n - 2\times 2^n] + B[3^n + (-2)^n]$
$\Rightarrow 9\times 3^n + 4\times 2^n = 3A\times 3^n - 2A\times 2^n + B\times 3^n + B\times 2^n$

EQUATING $[3^n]$ & $[2^n]$

$\left.\begin{array}{l} 9 = 3A + B \\ 4 = -2A + B \end{array}\right\} \Rightarrow$ SUBTRACTING $5A = 5$ & $A = 1$
& $B = 6$

$\therefore\ u_{n+2} = u_{n+1} + 6u_n$ WITH $u_1 = 1,\ u_2 = 13$

Question 59 (****+)

A sequence u_1, u_2, u_3, u_4, … is given by

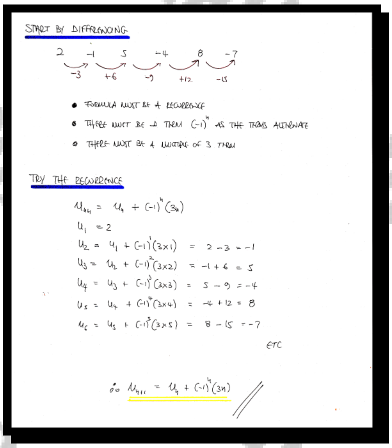
$$u_{n+1} = f\left(n, u_n\right).$$

The first few terms of the sequence are

$$2,\ -1,\ 5,\ -4,\ 8,\ -7, \ldots$$

Find an expression for u_{n+1}, in the form $u_{n+1} = f\left(n, u_n\right)$.

$\boxed{}$, $\boxed{u_{n+1} = u_n + (-1)^n (3n)}$



START BY DIFFERENCING

$$2 \xrightarrow{-3} -1 \xrightarrow{+6} 5 \xrightarrow{-9} -4 \xrightarrow{+12} 8 \xrightarrow{-15} -7$$

- FORMULA MUST BE A RECURRENCE
- THERE MUST BE A TERM $(-1)^n$ AS THE TERMS ALTERNATE
- THERE MUST BE A MULTIPLE OF 3 TERM

TRY THE RECURRENCE

$u_{n+1} = u_n + (-1)^n(3n)$

$u_1 = 2$

$u_2 = u_1 + (-1)^1(3\times1) = 2 - 3 = -1$

$u_3 = u_2 + (-1)^2(3\times2) = -1 + 6 = 5$

$u_4 = u_3 + (-1)^3(3\times3) = 5 - 9 = -4$

$u_5 = u_4 + (-1)^4(3\times4) = -4 + 12 = 8$

$u_6 = u_5 + (-1)^5(3\times5) = 8 - 15 = -7$

ETC

$\therefore\ u_{n+1} = u_n + (-1)^n(3n)$

Question 60 (**+)**

The n^{th} term of the sequence is given by

$$u_n = \frac{n+2}{2n+1}, \ n \in \mathbb{N}, \ n \geq 1.$$

Show that the same sequence can be generated by the recurrence relation

$$u_{n+1} = \frac{Au_n - 1}{Bu_n + 1}, \ u_1 = 1, \ n \in \mathbb{N}, \ n \geq 1,$$

where A and B are integers to be found.

$\boxed{}$, $\boxed{A=5}$, $\boxed{B=4}$

$u_n = \frac{n+2}{2n+1} \Rightarrow u_{n+1} = \frac{(n+1)+2}{2(n+1)+1} = \frac{n+3}{2n+3}$

NOW

REARRANGE FOR n

$2nu_n + u_n = n+2$

$2nu_n - n = 2 - u_n$

$n(2u_n - 1) = 2 - u_n$

$n = \frac{2-u_n}{2u_n - 1}$

$\Longrightarrow n+1 = \frac{2-u_{n+1}}{2u_{n+1}-1}$

$n = \frac{2-u_{n+1}}{2u_{n+1}-1} - 1$

$n = \frac{2-u_{n+1}-2u_{n+1}+1}{2u_{n+1}-1}$

$n = \frac{3-3u_{n+1}}{2u_{n+1}-1}$

$\frac{2-u_n}{2u_n-1} = \frac{3-3u_{n+1}}{2u_{n+1}-1}$

$\Rightarrow 4u_{n+1} - 2 - 2u_nu_{n+1} + u_n = 6u_n - 6u_nu_{n+1} - 3 + 3u_{n+1}$

$\Rightarrow u_{n+1} + 4u_{n+1}u_n = 5u_n - 1$

$\Rightarrow u_{n+1}[1 + 4u_n] = 5u_n - 1$

$\Rightarrow u_{n+1} = \frac{5u_n - 1}{4u_n + 1}$

Question 61 **(****+)**

A sequence is defined for $n \geq 1$ by the recurrence relation

$$u_{n+1} = \frac{5u_n}{1+8u_n}, \quad u_1 = \frac{1}{5}.$$

Determine an expression for u_n, given that it is of the form

$$u_n = \frac{a^{n-1}}{c+ka^{n-1}},$$

where a, c and k are constants to be found.

$\boxed{}$, $\boxed{u_n = \dfrac{5^{n-1}}{3+2\left(5^{n-1}\right)}}$

- START BY GENERATING TERMS FROM THE RECURRENCE RELATION

$$u_{n+1} = \frac{5u_n}{8u_n+1} \qquad u_1 = \frac{1}{5}$$

$$u_2 = \frac{5\times\frac{1}{5}}{8\times\frac{1}{5}+1} = \frac{1}{\frac{8}{5}+1} = \frac{5}{8+5} = \frac{5}{13}$$

$$u_3 = \frac{5\times\frac{5}{13}}{8\times\frac{5}{13}+1} = \frac{25}{40+13} = \frac{25}{53}$$

- NOW FORM SOME EQUATIONS USING THE FIRST 3 TERMS

$$\boxed{u_n = \frac{a^{n-1}}{ka^{n-1}+c}}$$

- $u_1 = \frac{1}{5}$
  $\frac{1}{k+c} = \frac{1}{5}$
  $k+c=5$
  $c = 5-k$

- $u_2 = \frac{5}{13}$
  $\frac{a}{ka+c} = \frac{5}{13}$
  $ka+c = \frac{13a}{5}$
  $ka + 5 - k = \frac{13}{5}a$
  $\boxed{k(a-1) = \frac{13}{5}a - 5}$

- $u_3 = \frac{25}{53}$
  $\frac{a^2}{ka^2+c} = \frac{25}{53}$
  $ka^2+c = \frac{53}{25}a^2$
  $ka^2+5-k = \frac{53}{25}a^2$
  $\boxed{k(a^2-1) = \frac{53}{25}a^2-5}$

- DIVIDING THE TWO EQUATIONS, NOTING $k \neq 0$, $a \neq 1$

$$\Rightarrow \frac{k(a^2-1)}{k(a-1)} = \frac{\frac{53}{25}a^2-5}{\frac{13}{5}a-5}$$

$$\Rightarrow \frac{\cancel{k}\cancel{(a-1)}(a+1)}{\cancel{k}\cancel{(a-1)}} = \frac{53a^2-125}{65a-125}$$

$$\Rightarrow (a+1)(65a-125) = 53a^2-125$$

$$\Rightarrow 65a^2-125a+65a-\cancel{125} = 53a^2-\cancel{125}$$

$$\Rightarrow 12a^2-60a=0$$

$$\Rightarrow 12a(a-5)=0$$

$\therefore a=5 \qquad a \neq 0$

$$\Rightarrow k(a-1) = \frac{13}{5}a-5$$

$$\Rightarrow 4k = 13-5$$

$$\Rightarrow k=2 \quad \& \quad c=3$$

$$\therefore u_n = \frac{5^{n-1}}{2(5^{n-1})+3}$$

Question 62 (*****)

A sequence u_1, u_2, u_3, u_4, u_5 … is given by the recurrence formula

$$u_{n+2} = \frac{3u_n + u_{n+1}}{2}, \quad u_1 = 1, \ u_2 = 1.$$

It is further given that in this sequence **the ratio of consecutive terms** converges to a limit L.

Determine the value of L.

$$\boxed{\lim_{n \to \infty}\left(\frac{u_{n+1}}{u_n}\right) = L = \frac{3}{2}}$$

$$\boxed{u_{n+2} = \frac{3u_n + u_{n+1}}{2}}$$

$\Rightarrow 2u_{n+2} = 3u_n + u_{n+1}$

$\Rightarrow \frac{2u_{n+2}}{u_{n+1}} = \frac{3u_n}{u_{n+1}} + \frac{u_{n+1}}{u_{n+1}}$

$\Rightarrow 2\left(\frac{u_{n+2}}{u_{n+1}}\right) = 3\left(\frac{u_n}{u_{n+1}}\right) + 1$

$\Rightarrow 2\left(\frac{u_{n+2}}{u_{n+1}}\right) = \frac{3}{\left(\frac{u_{n+1}}{u_n}\right)} + 1$

- AS $n \to \infty$
THE RATIO OF SUCCESSIVE TERMS CONVERGES TO A LIMIT L
- THUS $\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = \ldots = L$, AS $n \to \infty$

$\Rightarrow 2L = \frac{3}{L} + 1$

$\Rightarrow 2L^2 = 3 + L$

$\Rightarrow 2L^2 - L - 3 = 0$

$\Rightarrow (2L - 3)(L + 1)$

$\Rightarrow L = \cancel{-1}$ (SEQUENCE HAS POSITIVE TERMS), $\frac{3}{2}$

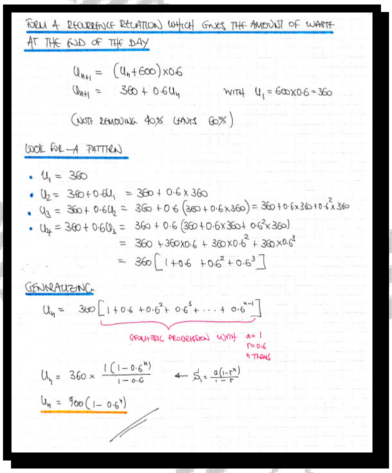
Question 63 (*****)

A factory gets permission to dispose, at the start of every day, 600 kg of waste into a stream of water.

The running stream removes 40% of the any waste present, by the end of the day.

Determine a simplified expression for the amount of waste present in the stream at the end of the n^{th} day.

MP, $\boxed{u_n = 900\left[1-\left(\frac{3}{5}\right)^n\right]}$



FORM A RECURRENCE RELATION WHICH GIVES THE AMOUNT OF WASTE AT THE END OF THE DAY

$$u_{n+1} = (u_n + 600) \times 0.6$$
$$u_{n+1} = 360 + 0.6u_n \qquad \text{WITH } u_1 = 600 \times 0.6 = 360$$

(NOTE REMOVING 40% LEAVES 60%)

LOOK FOR A PATTERN

- $u_1 = 360$
- $u_2 = 360 + 0.6u_1 = 360 + 0.6 \times 360$
- $u_3 = 360 + 0.6u_2 = 360 + 0.6(360 + 0.6 \times 360) = 360 + 0.6 \times 360 + 0.6^2 \times 360$
- $u_4 = 360 + 0.6u_3 = 360 + 0.6(360 + 0.6 \times 360 + 0.6^2 \times 360)$
  $= 360 + 360 \times 0.6 + 360 \times 0.6^2 + 360 \times 0.6^3$
  $= 360\left[1 + 0.6 + 0.6^2 + 0.6^3\right]$

GENERALIZING

$$u_n = 360\Big[\underbrace{1 + 0.6 + 0.6^2 + 0.6^3 + \dots + 0.6^{n-1}}_{\text{GEOMETRIC PROGRESSION WITH } a=1,\ r=0.6,\ n \text{ TERMS}}\Big]$$

$$u_n = 360 \times \frac{1(1-0.6^n)}{1-0.6} \quad \leftarrow S_n = \frac{a(1-r^n)}{1-r}$$

$$u_n = 900(1 - 0.6^n)$$

Question 64 (*****)

The Fibonacci sequence is given by the recurrence formula

$$u_{n+2} = u_{n+1} + u_n, \quad u_1 = 1,\ u_2 = 1.$$

It is further given that in this sequence **the ratio of consecutive terms** converges to a limit ϕ, known as the *Golden Ratio*.

Show, by using the above recurrence formula, that $\phi = \frac{1}{2}\left(1+\sqrt{5}\right)$.

$\boxed{}$, $\boxed{\text{proof}}$

PROCEED AS FOLLOWS

$$u_{n+2} = u_{n+1} + u_n$$

$$\frac{u_{n+2}}{u_{n+1}} = \frac{u_{n+1}}{u_{n+1}} + \frac{u_n}{u_{n+1}}$$

AS ALL THE TERMS ARE POSITIVE WE MAY DIVIDE THROUGH BY u_{n+1}

NOW WE ARE GIVEN THAT AS $r \to \infty$

$$\frac{u_{r+1}}{u_r} = \phi \implies \frac{u_{n+2}}{u_{n+1}} \to \phi \text{ AS } n \to \infty$$

$$\frac{u_n}{u_{n+1}} \to \frac{1}{\phi} \text{ AS } n \to \infty$$

HENCE WE OBTAIN

$$\implies \phi = 1 + \frac{1}{\phi}$$

$$\implies \phi^2 = \phi + 1$$

$$\implies \phi^2 - \phi - 1 = 0$$

$$\implies \left(\phi - \tfrac{1}{2}\right)^2 - \tfrac{5}{4} = 0$$

$$\implies \left(\phi - \tfrac{1}{2}\right)^2 = \tfrac{5}{4}$$

$$\implies \phi - \tfrac{1}{2} = \frac{\pm\sqrt{5}}{2}$$

$$\implies \phi = \frac{1 \pm \sqrt{5}}{2}$$

$$\implies \phi = \begin{cases} \tfrac{1}{2}(1+\sqrt{5}) \\ \cancel{\tfrac{1}{2}(1-\sqrt{5})} < 0 \end{cases}$$

AND RATIO OF POSITIVE TERMS CANNOT BE NEGATIVE

Question 65 (*****)

The n^{th} term of a sequence is given by

$$u_n = 1 + \left(\tfrac{1}{3}\right)^n, \quad \text{where } n \geq 1.$$

a) By expressing u_{n+1} in terms of u_n, or otherwise, define the terms of the sequence as a recurrence relation.

A recurrence relation is defined for $n \geq 1$ by

$$U_{n+1} = 2U_n - 5, \; U_1 = 6.$$

b) By finding the n^{th} term of the sequence, or otherwise, show that

$$u_{31} = 1{,}073{,}741{,}829.$$

$\boxed{}$, $\boxed{u_{n+1} = \dfrac{2+u_n}{3}, \; u_1 = \dfrac{4}{3}}$, $\boxed{U_n = 2^{n-1} + 5}$

(a) $u_n = 1 + \left(\frac{1}{3}\right)^n \Rightarrow \boxed{\left(\frac{1}{3}\right)^n = u_n - 1}$

$u_{n+1} = 1 + \left(\frac{1}{3}\right)^{n+1} = 1 + \left(\frac{1}{3}\right)^n \times \left(\frac{1}{3}\right) = 1 + \frac{1}{3}(u_n - 1) = 1 + \frac{1}{3}u_n - \frac{1}{3} = \frac{2}{3} + \frac{1}{3}u_n$

$\therefore u_{n+1} = \frac{1}{3}(2 + u_n)$ & $u_1 = \frac{4}{3}$

(b) $U_{n+1} = 2U_n - 5$ HAS A "DOUBLING" FEATURE

TRY $\boxed{U_n = A \times 2^n + B}$, A, B CONSTANTS

• NOW $U_1 = 6$ & $U_2 = 7$ FROM RECURRENCE

$\left.\begin{array}{l} 6 = A \times 2^1 + B \\ 7 = A \times 2^2 + B \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2A + B = 6 \\ 4A + B = 7 \end{array}\right\} \Rightarrow$

$2A = 1$

$A = \frac{1}{2}$ & $B = 5$

$\therefore U_n = \frac{1}{2} \times 2^n + 5$

$\boxed{U_n = 2^{n-1} + 5}$

$\therefore U_{31} = 2^{30} + 5 = 1{,}073{,}741{,}829$

Question 66 (*****)

A sequence is defined by the recurrence relation

$$u_n = \frac{2n}{2n+1} u_{n-1}, \; n \in \mathbb{N} \quad u_0 = 1 .$$

Show, by direct manipulation, that

$$u_n = \frac{4^n \times (n!)^2}{(2n+1)!} .$$

[*you may not use proof by induction*]

$\boxed{\text{proof}}$

$$u_n = \frac{2n}{2n+1} u_{n-1}, \quad u_0 = 1$$

$$\Rightarrow u_n = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} u_{n-2}$$

$$\Rightarrow u_n = \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-3} u_{n-3}$$

$$\Rightarrow u_n = \frac{2n(2n-2)(2n-4)\ldots \times 6 \times 4 \times 2}{(2n+1)(2n-1)(2n-3)\ldots \times 7 \times 5 \times 3} u_0$$

$$\Rightarrow u_n = \frac{2^n n(n-1)(n-2)\ldots \times 3 \times 2 \times 1}{(2n+1)(2n-1)(2n-3)\ldots \times 7 \times 5 \times 3} \times 1$$

$$\Rightarrow u_n = \frac{2^n n! \times (2n+2)(2n)(2n-2)\ldots \times 6 \times 4 \times 2}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\ldots \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}$$

$$\Rightarrow u_n = \frac{2^n \times n! \times 2^{n+1}(n+1)(n)(n-1)\ldots \times 3 \times 2 \times 1}{(2n+2)!}$$

$$\Rightarrow u_n = \frac{2^{2n+1} \times n! \times (n+1)!}{(2n+2)!} = \frac{2^{2n} \times n! \times n! \times \cancel{2(n+1)}}{\cancel{(2n+2)}(2n+1)!}$$

$$\Rightarrow u_n = \frac{2^{2n} \times n! \times n!}{(2n+1)!}$$

$$\Rightarrow u_n = \frac{4^n \times (n!)^2}{(2n+1)!}$$

AS REQUIRED

Question 67 (*****)

A sequence of numbers, u_1, u_2, u_3, u_4, ... , is defined by

$$u_{n+1} = 3u_n - 1, \quad u_1 = 2.$$

Determine, in terms of n, a simplified expression for

$$\sum_{r=1}^{n} u_r .$$

$$\boxed{S_n = \frac{1}{4}\left[3^{n+1} + 2n - 3\right]}$$

AS THERE IS NO OBVIOUS PATTERN PROCEED BY GENERATING SOME TERMS

- $u_1 = 2$
- $u_2 = 3u_1 - 1 = 3\times 2 - 1 = 5$
- $u_3 = 3u_2 - 1 = 3\times 5 - 1 = 14$
- $u_4 = 3u_3 - 1 = 3\times 14 - 1 = 41$
- $u_5 = 3u_4 - 1 = 3\times 41 - 1 = 122$

PROCEED BY DIFFERENCING

$$\begin{array}{ccccccccc} 2 & & 5 & & 14 & & 41 & & 122 \\ & 3 & & 9 & & 27 & & 81 & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & 3^1 & & 3^2 & & 3^3 & & 3^4 & \end{array}$$

HENCE WE HAVE

$u_1 = 2$

$u_2 = 2 + 3^1$

$u_3 = 2 + 3^1 + 3^2$

$u_4 = 2 + 3^1 + 3^2 + 3^3$

$\vdots$

$u_n = 2 + \sum_{r=1}^{n-1} 3^r$ OR $u_{n+1} = 2 + \sum_{r=1}^{n} 3^r$

THUS THE nth TERM CAN BE FOUND

$$u_{n+1} = 2 + \sum_{r=1}^{n} 3^r = 2 + \underbrace{3 + 3^2 + 3^3 + \dots + 3^n}_{\text{G.P.}}$$

$$u_{n+1} = 2 + \frac{3(3^n - 1)}{3 - 1} \quad \leftarrow S_n = \frac{a(r^n - 1)}{r - 1}$$

$$u_{n+1} = 2 + \frac{3^{n+1} - 3}{2}$$

$$u_{n+1} = \frac{3^{n+1} + 1}{2} \quad \text{OR} \quad u_n = \frac{3^n + 1}{2}$$

SUMMING UP

$$S_n = \sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{3^r + 1}{2} = \sum_{r=1}^{n} \left[\frac{1}{2} + \frac{1}{2}\times 3^r\right]$$

$$S_n = \frac{1}{2}\sum_{r=1}^{n} 1 + \frac{1}{2}\sum_{r=1}^{n} 3^r$$

$$S_n = \frac{1}{2}n + \frac{1}{2}\left[3 + 3^2 + 3^3 + 3^4 + \dots\right] \leftarrow \text{G.P.}$$

$$S_n = \frac{1}{2}n + \frac{1}{2}\left[\frac{3(3^n - 1)}{3 - 1}\right] \leftarrow S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{n}{2} + \frac{3^{n+1} - 3}{4}$$

$$S_n = \frac{3^{n+1} + 2n - 3}{4}$$

Question 68 (*****)

It is given that

$$\sum_{r=1}^{n} u_r = 6^{n+1} - 10 \times 2^n + 4,$$

where u_n is the n^{th} term of a sequence.

Show clearly that

$$u_{n+2} = Au_{n+1} + Bu_n,$$

where A and B are integers to be found.

$\boxed{}$, $\boxed{u_{n+2} = 8u_{n+1} - 12u_n}$

$$\boxed{\sum_{r=1}^{n} u_r = 6^{n+1} - 10 \times 2^n + 4}$$

- TRYING TO FIND AN EXPRESSION FOR THE nth TERM FIRST

$$\left\{\begin{array}{l} S_n = 6^{n+1} - 10 \times 2^n + 4 \\ S_{n-1} = 6^n - 10 \times 2^{n-1} + 4 \end{array}\right\}$$

$$\Rightarrow u_n = S_n - S_{n-1} = \left[6^{n+1} - 10 \times 2^n + 4\right] - \left[6^n - 10 \times 2^{n-1} + 4\right]$$
$$= 6^{n+1} - 6^n - 10 \times 2^n + 10 \times 2^{n-1}$$
$$= (6 \times 6^n - 6^n) + 10 \times 2^{n-1} - 10 \times 2^n$$
$$= 5 \times 6^n + (5 \times 2 \times 2^{n-1} - 10 \times 2^n)$$
$$= 5 \times 6^n + (5 \times 2^n - 10 \times 2^n)$$
$$= 5 \times 6^n - 5 \times 2^n$$
$$\therefore \boxed{u_n = 5\left[6^n - 2^n\right]}$$

- NOW WE MAY ELIMINATE THE POWERS OF 6 & 2, AS FOLLOWS

$$u_n = 5\left[6^n - 2^n\right]$$
$$u_{n+1} = 5\left[6^{n+1} - 2^{n+1}\right] = 5\left[6 \times 6^n - 2 \times 2^n\right] = 30 \times 6^n - 10 \times 2^n$$
$$u_{n+2} = 30 \times 6^{n+1} - 10 \times 2^{n+1} = 30 \times 6 \times 6^n - 10 \times 2 \times 2^n = 180 \times 6^n - 20 \times 2^n$$

- FOR SIMPLICITY LET $P = 6^n$ & $Q = 2^n$

$$u_n = 5P - 5Q$$
$$\left.\begin{array}{l} u_{n+1} = 30P - 10Q \\ u_{n+2} = 180P - 20Q \end{array}\right\} \Rightarrow \begin{array}{r} -2u_{n+1} = -60P + 20Q \\ u_{n+2} = 180P - 20Q \\ \hline u_{n+2} - 2u_{n+1} = 120P \end{array}$$

ALSO ELIMINATING P FROM THE LAST 2 EQUATIONS

$$\begin{array}{r} -6u_{n+1} = -180P + 60Q \\ u_{n+2} = 180P - 20Q \\ \hline u_{n+2} - 6u_{n+1} = 40Q \end{array}$$

FINALLY TAKE THE FIRST EQUATION & MULTIPLY IT BY 24

$$\Rightarrow 24u_n = 120P - 120Q$$
$$\Rightarrow 24u_n = 120P - 3(40Q)$$
$$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3(u_{n+2} - 6u_{n+1})$$
$$\Rightarrow 24u_n = u_{n+2} - 2u_{n+1} - 3u_{n+2} + 18u_{n+1}$$
$$\Rightarrow 2u_{n+2} = 16u_{n+1} - 24u_n$$
$$\Rightarrow u_{n+2} = 8u_{n+1} - 12u_n$$

& $A = 8$ & $B = -12$

Question 69 (*****)

The function f satisfies the following three relationships

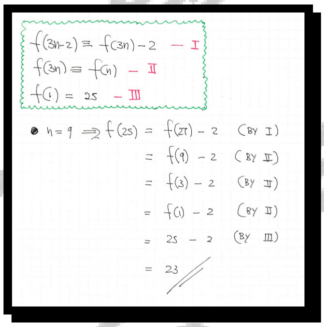
i. $f(3n-2) \equiv f(3n)-2, \ n \in \mathbb{N}$.

ii. $f(3n) \equiv f(n), \ n \in \mathbb{N}$.

iii. $f(1)=25$.

Determine the value of $f(25)$.

$\boxed{\text{SWH}}$, $\boxed{f(25)=23}$



Question 70 (*****)

A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{n}{2n+1} u_n , \quad n \in \mathbb{N} \quad u_1 = 2 .$$

Show, by direct manipulation, that

$$u_n = \frac{2^n \times \left[(n-1)! \right]^2}{(2n-1)!} .$$

[*You may not use proof by induction in this question*]

$\boxed{\text{proof}}$

$$u_{n+1} = \frac{n}{2n+1} u_n \qquad u_1 = 2$$

$$u_{n+1} = \frac{n}{2n+1} \times \frac{n-1}{2n-1} u_{n-1}$$

$$u_{n+1} = \frac{n}{2n+1} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-3} u_{n-2}$$

$$u_{n+1} = \frac{n(n-1)(n-2) \dots \times 3 \times 2 \times 1}{(2n+1)(2n-1)(2n-3) \dots \times 7 \times 5 \times 3} u_1$$

$$u_{n+1} = \frac{n!}{(2n+1)(2n-1)(2n-3) \dots \times 7 \times 5 \times 3} \times 2$$

$$u_{n+1} = \frac{n!\,(2n+2)(2n)(2n-2) \dots \times 6 \times 4 \times 2}{(2n+2)(2n+1)(2n)(2n-1)(2n-2) \dots \times 6 \times 5 \times 4 \times 3 \times 2} \times 2$$

$$u_{n+1} = \frac{n! \times 2^{n+1}(n+1)n(n-1) \dots \times 3 \times 2 \times 1}{(2n+2)!} \times 2$$

$$u_{n+1} = \frac{n! \times 2^{n+2} \times (n+1)!}{(2n+2)(2n+1)!} = \frac{n! \times 2^{n+2} \times (n+1) \times n!}{2(n+1)(2n+1)!}$$

$$u_{n+1} = \frac{n!\, n! \times 2^{n+1} \times \cancel{2(n+1)}}{(2n+1)! \times \cancel{2(n+1)}}$$

$$u_{n+1} = \frac{2^{n+1}(n!)^2}{(2n+1)!}$$

$$\therefore u_n = \frac{2^n [(n-1)!]^2}{(2n-1)!}$$

Question 71 (*****)

Consider the following sequence.

$$\frac{1}{7},\ \frac{1}{2},\ \frac{7}{9},\ 1,\ \frac{13}{11},\ \frac{4}{3},\ \frac{19}{13},\ \frac{11}{7},\ \ldots,\quad x \in \mathbb{R},\quad x < 2.$$

a) Determine the n^{th} of this sequence and hence find a recurrence relation formula for this sequence.

b) Find a **different**, to that given in part **(a)**, recurrence relation formula for the same sequence.

c) Determine a **third** recurrence relation formula for this sequence.

The recurrence relations in this question must be in the form $F\left(u_{n+1},\ u_n,\ n\right)$

$\boxed{}$, $\boxed{u_{n+1} = u_n + \dfrac{20}{(n+6)(n+7)}}$, $\boxed{u_{n+1} = \dfrac{3n^2+19n+6}{3n^2+19n-14}u_n}$, $\boxed{u_{n+1} = \left(\dfrac{1}{u_n}\right)\dfrac{3n^2-3n-2}{n^2+13n+42}}$

a) THE HARDEST THING IS TO SEE THE PATTERN – EASY ONCE YOU SPOT

$$\frac{1}{7},\ \frac{1}{2},\ \frac{7}{9},\ 1,\ \frac{13}{11},\ \frac{4}{3},\ \frac{19}{13},\ \frac{11}{7},\ \ldots$$
$$\frac{1}{7},\ \frac{4}{8},\ \frac{7}{9},\ \frac{10}{10},\ \frac{13}{11},\ \frac{16}{12},\ \frac{19}{13},\ \frac{22}{14},\ \ldots$$

IE $u_n = \dfrac{3n-2}{n+6}$

$u_{n+1} = \dfrac{3n+1}{n+7}$

THUS ONE POSSIBLE EXPRESSION IS

$$u_{n+1} - u_n = \frac{3n+1}{n+7} - \frac{3n-2}{n+6} = \frac{(3n+1)(n+6)-(3n-2)(n+7)}{(n+7)(n+6)}$$
$$= \frac{\cancel{3n^2}+\cancel{18n}+\cancel{n}+6-\cancel{3n^2}-\cancel{21n}+\cancel{2n}+14}{(n+6)(n+7)} = \frac{20}{(n+6)(n+7)}$$

$$\therefore\ u_{n+1} = u_n + \frac{20}{(n+6)(n+7)},\ u_1 = \frac{1}{7}$$

b) ANOTHER POSSIBILITY COULD BE BY DIVISION

$$\frac{u_{n+1}}{u_n} = u_{n+1} \times \frac{1}{u_n} = \frac{3n+1}{n+7} \times \frac{n+6}{3n-2} = \frac{3n^2+19n+6}{3n^2+19n-14}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n^2+19n+6}{3n^2+19n-14}$$

$$\therefore\ u_{n+1} = \frac{3n^2+19n+6}{3n^2+19n-14}u_n,\ u_1 = \frac{1}{7}$$

c) ANOTHER CHOICE COULD BE BY DIRECT MULTIPLICATION

$$u_{n+1}u_n = \frac{3n+1}{n+7} \times \frac{3n-2}{n+6} = \frac{3n^2-3n-2}{n^2+13n+42}$$

$$\therefore\ u_{n+1} = \frac{1}{u_n}\left(\frac{3n^2-3n-2}{n^2+13n+42}\right),\ u_1 = \frac{1}{7}$$

Question 72 (*****)

The n^{th} term of a series is given recursively by

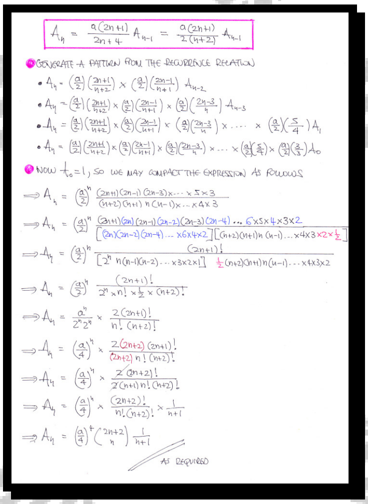
$$A_n = \frac{a(2n+1)}{2n+4} A_{n-1}, \quad n \in \mathbb{N}, \; n \geq 1,$$

where a is a positive constant.

Given further that $A_0 = 1$, show that

$$A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}.$$

proof



$$\boxed{A_n = \frac{a(2n+1)}{2n+4} A_{n-1} = \frac{a(2n+1)}{2(n+2)} A_{n-1}}$$

GENERATE A PATTERN FROM THE RECURRENCE RELATION

- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) A_{n-2}$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) A_{n-3}$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) \times \dots \times \left(\frac{a}{2}\right)\left(\frac{5}{4}\right) A_1$
- $A_n = \left(\frac{a}{2}\right)\left(\frac{2n+1}{n+2}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-1}{n+1}\right) \times \left(\frac{a}{2}\right)\left(\frac{2n-3}{n}\right) \times \dots \times \left(\frac{a}{2}\right)\left(\frac{5}{4}\right) \times \left(\frac{a}{2}\right)\left(\frac{3}{3}\right) A_0$

NOW $A_0 = 1$, SO WE MAY COMPACT THE EXPRESSION AS FOLLOWS

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)(2n-1)(2n-3)\times \dots \times 5 \times 3}{(n+2)(n+1)n(n-1)\times \dots \times 4 \times 3}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4)\dots 6\times 5\times 4\times 3\times 2}{\left[(2n)(2n-2)(2n-4)\dots \times 6\times 4\times 2\right]\left[(n+2)(n+1)n(n-1)\dots \times 4\times 3\times 2\times \frac{1}{2}\right]}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{\left[2^n n(n-1)(n-2)\dots \times 3\times 2\times 1\right] \frac{1}{2}(n+2)(n+1)n(n-1)\dots \times 4\times 3\times 2}$$

$$\Rightarrow A_n = \left(\frac{a}{2}\right)^n \frac{(2n+1)!}{2^n \times n! \times \frac{1}{2} \times (n+2)!}$$

$$\Rightarrow A_n = \frac{a^n}{2^n 2^n} \times \frac{2(2n+1)!}{n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{2(2n+2)(2n+1)!}{(2n+2)n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{\cancel{2}(2n+2)!}{\cancel{2}(n+1)n!(n+2)!}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \times \frac{(2n+2)!}{n!(n+2)!} \times \frac{1}{n+1}$$

$$\Rightarrow A_n = \left(\frac{a}{4}\right)^n \binom{2n+2}{n} \frac{1}{n+1}$$

AS REQUIRED

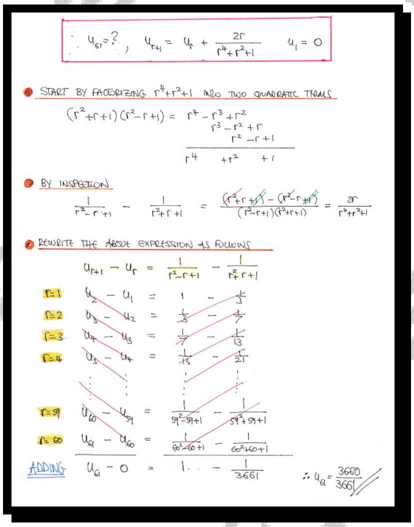
Question 73 (*****)

A sequence is defined as

$$u_{r+1} = u_r + \frac{2r}{r^4 + r^2 + 1}, \quad u_1 = 0, \quad r \in \mathbb{N}.$$

Determine the exact value of u_{61}.

$$\boxed{u_{61} = \frac{3660}{3661}}$$



$$\boxed{u_{61} = ?, \quad u_{r+1} = u_r + \frac{2r}{r^4+r^2+1} \quad u_1 = 0}$$

- START BY FACTORIZING r^4+r^2+1 INTO TWO QUADRATIC TERMS

$$(r^2+r+1)(r^2-r+1) = \begin{array}{l} r^4 - r^3 + r^2 \\ \quad\; r^3 - r^2 + r \\ \qquad\;\; r^2 - r + 1 \\ \hline r^4 \quad + r^2 \quad + 1 \end{array}$$

- BY INSPECTION

$$\frac{1}{r^2-r+1} - \frac{1}{r^2+r+1} = \frac{(\cancel{r^2}+r+\cancel{1}) - (\cancel{r^2}-r+\cancel{1})}{(r^2-r+1)(r^2+r+1)} = \frac{2r}{r^4+r^2+1}$$

- REWRITE THE ABOVE EXPRESSION AS FOLLOWS

$$u_{r+1} - u_r = \frac{1}{r^2-r+1} - \frac{1}{r^2+r+1}$$

$$\begin{array}{lllll}
r=1 & u_2 - u_1 & = & 1 & - \frac{1}{3} \\
r=2 & u_3 - u_2 & = & \frac{1}{3} & - \frac{1}{7} \\
r=3 & u_4 - u_3 & = & \frac{1}{7} & - \frac{1}{13} \\
r=4 & u_5 - u_4 & = & \frac{1}{13} & - \frac{1}{21} \\
 & \vdots & & \vdots & \vdots \\
r=59 & u_{60} - u_{59} & = & \frac{1}{59^2-59+1} & - \frac{1}{59^2+59+1} \\
r=60 & u_{61} - u_{60} & = & \frac{1}{60^2-60+1} & - \frac{1}{60^2+60+1} \\
\hline
\text{ADDING} & u_{61} - 0 & = & 1 & - \frac{1}{3661}
\end{array}$$

$$\therefore u_{61} = \frac{3660}{3661}$$