QUADRATIC EQUATIONS

EXAM QUESTIONS

Question 1 (**)

By using the quadratic formula, or otherwise, find the exact solutions of the equation

$$\frac{1}{x} = 2x + 3.$$

$$\boxed{x = \frac{-3 \pm \sqrt{17}}{4}}$$

$\frac{1}{x} = 2x+3$

$\Rightarrow 1 = (2x+3)x$

$\Rightarrow 1 = 2x^2+3x$

$\Rightarrow 0 = 2x^2+3x-1$

BY QUADRATIC FORMULA

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-3 \pm \sqrt{3^2-4 \times 2 \times (-1)}}{2 \times 2}$

$x = \frac{-3 \pm \sqrt{9+8}}{4}$

$x = \frac{-3 \pm \sqrt{17}}{4}$

Question 2 (**)

$$f(x) = x^2 - 4x - 16, \quad x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Hence solve the equation $f(x) = 0$, giving the answers as exact surds.

$\boxed{}$, $\boxed{a = -2}$, $\boxed{b = -20}$, $\boxed{x = 2 \pm 2\sqrt{5}}$

(a) $f(x) = x^2-4x-16$

$f(x) = (x-2)^2-2^2-16$

$f(x) = (x-2)^2-4-16$

$f(x) = (x-2)^2-20$

(b) $f(x) = 0$

$(x-2)^2-20 = 0$

$(x-2)^2 = 20$

$x-2 = \pm\sqrt{20}$

$x-2 = \pm\sqrt{4 \times 5}$

$x = 2 \pm 2\sqrt{5}$

Question 3 **(**)**

Find the solutions of the equation

$$3x-\frac{5}{x}=2.$$

C1 , $\boxed{x=-1,\frac{5}{3}}$

$\bullet\ 3x-\frac{5}{x}=2 \quad (\times x)$
$\Rightarrow 3x^2-5=2x$
$\Rightarrow 3x^2-2x-5=0$
$\Rightarrow (3x-5)(x+1)=0$
$\therefore x = -1,\ \frac{5}{3}$

Question 4 **(**)**

$$f(x)=x^2-14x+50.$$

Show that $f(x)$ is positive for all values of x.

proof

$f(x)=x^2-14x+50$
$f(x)=(x-7)^2-7^2+50$
$f(x)=(x-7)^2-49+50$
$f(x)=(x-7)^2+1 \geqslant 1$
$\therefore f(x)>0$ FOR ALL x

Question 5 **(**)**

Find the coordinates of any points of intersection between the graphs of

$$y = x^2 - 4x + 2 \quad \text{and} \quad y = -x^2 - 8x\,.$$

$\boxed{}$, $\boxed{(-1,7)}$

SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} y = x^2 - 4x + 2 \\ y = -x^2 - 8x \end{array}\right\} \Rightarrow \begin{array}{l} x^2 - 4x + 2 = -x^2 - 8x \\ 2x^2 + 4x + 2 = 0 \\ x^2 + 2x + 1 = 0 \\ (x+1)^2 = 0 \\ x = -1 \end{array}$$

$\therefore\ y = (-1)^2 - 4(-1) + 2$
$y = 1 + 4 + 2$
$y = 7$
$\therefore\ (-1, 7)$

Question 6 **(**)**

$$f(x) = x^2 + 6x + 10,\ x \in \mathbb{R}\,.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Describe geometrically the transformations which map the graph of x^2 onto the graph of $f(x)$.

$\boxed{a=3}$, $\boxed{b=1}$, $\boxed{\text{translation by} \begin{pmatrix} -3 \\ 1 \end{pmatrix}}$

(a) $x^2 + 6x + 10 = (x+3)^2 - 3^2 + 10 = (x+3)^2 - 9 + 10$
$= (x+3)^2 + 1$

(b) $x^2 \xmapsto[\text{"LEFT", BY 3 UNITS}]{\text{TRANSLATION,}} (x+3)^2 \xmapsto[\text{"UPWARDS", BY 1 UNIT}]{\text{TRANSLATION,}} (x+3)^2 + 1$

OR TRANSLATION BY VECTOR $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Question 7 (**)

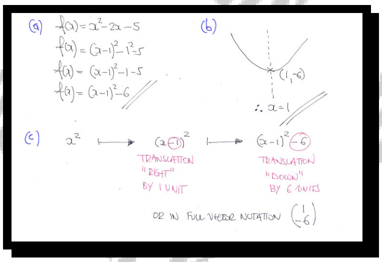
$$f(x) = x^2 - 2x - 5, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) State the equation of the line of symmetry of the graph of $f(x)$.

c) Describe geometrically the transformations which map the graph of x^2 onto the graph of $f(x)$.

$\boxed{a=-1}$, $\boxed{b=-6}$, $\boxed{x=1}$, $\boxed{\text{translation by} \begin{pmatrix} 1 \\ -6 \end{pmatrix}}$



Question 8 (**+)

$$f(x) = x^2 + 4x + 12, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Determine the greatest value of $\dfrac{1}{f(x)}$.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=8}$, $\boxed{\dfrac{1}{8}}$

(a)
$f(x) = x^2+4x+12$
$f(x) = (x+2)^2-2^2+12$
$f(x) = (x+2)^2-4+12$
$f(x) = (x+2)^2+8$

ie $a=2$
$b=8$

(b)
$\dfrac{1}{f(x)} = \dfrac{1}{x^2+4x+12}$
$\dfrac{1}{f(x)} = \dfrac{1}{(x+2)^2+8}$

TO BE AS LARGE AS POSSIBLE WE NEED THE SMALLEST POSSIBLE DENOMINATOR

BUT SMALLEST VALUE OF $(x+2)^2+8$ IS 8

$\therefore \left.\dfrac{1}{f(x)}\right|_{MAX} = \dfrac{1}{8}$

Question 9 (**+)

$$f(x) = x^2 - 4x + 9, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

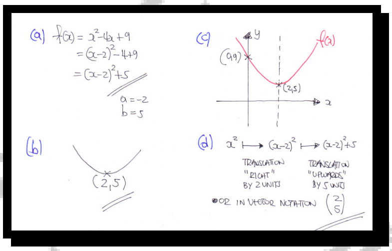
b) State the coordinates of the minimum point of the graph of $f(x)$.

c) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

d) Describe geometrically the transformations which map the graph of x^2 onto the graph of $f(x)$.

$\boxed{a=-2}$, $\boxed{b=5}$, $\boxed{(2,5)}$, $\boxed{\text{translation by } \begin{pmatrix} 2 \\ 5 \end{pmatrix}}$



Question 10 (**+)

The curve C has equation

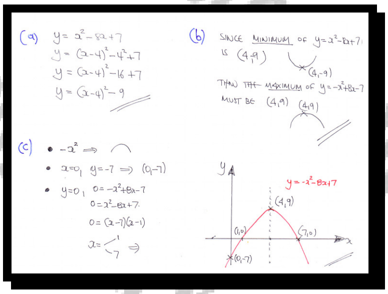
$$y = -x^2 + 8x - 7.$$

a) Express $x^2 - 8x + 7$ in the form $(x+a)^2 + b$, where a and b are constants.

b) Hence write down the coordinates of the **maximum** point of C.

c) Sketch the graph of C, indicating clearly all the points where C meets the coordinate axes.

$\boxed{}$, $\boxed{(x-4)^2 - 9}$, $\boxed{(4,9)}$



Question 11 **(**+)**

The curve C has equation

$$y = (x-a)^2 + b,$$

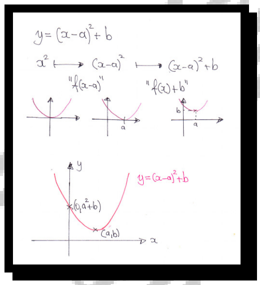
where a, b are positive constants.

By considering the two transformations that map the graph of $y = x^2$ onto the graph of C, or otherwise, sketch the graph of C.

The sketch must include the coordinates, in terms of a, b, of

- … all the points where the curve meets the coordinate axes.
- … the maximum point of the curve.

graph



Question 12 **(**+)**

The quadratic equation

$$x^2 + ax + b = 0,$$

where a and b are constants,

is satisfied by $x = -2$ and $x = 5$.

Determine the values of a and b.

$\boxed{a = -3}$, $\boxed{b = -10}$

BY INSPECTION

$(x+2)(x-5) = 0$

$x^2 - 5x + 2x - 10 = 0$

$x^2 - 3x - 10 = 0$

$\therefore a = -3$

$b = -10$

Question 13 (**+)

$$f(x) = x^2 + 8x + 20, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

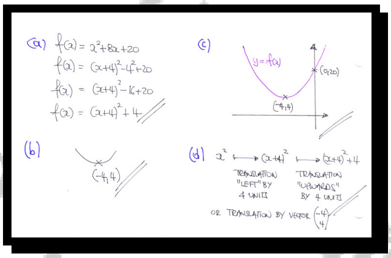
b) State the coordinates of the minimum point of the graph of $f(x)$.

c) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

d) Describe geometrically the transformations which map the graph of x^2 onto the graph of $f(x)$.

$\boxed{a=4}$, $\boxed{b=4}$, $\boxed{(-4,4)}$, $\boxed{\text{translation by } \begin{pmatrix} -4 \\ 4 \end{pmatrix}}$



Question 14 (**+)

$$f(x) = x^2 + 4x - 12, \ x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Hence solve the equation

$$x^4 + 4x^2 - 12 = 0.$$

C1G, $\boxed{x = -6, 2}$, $\boxed{x = \pm\sqrt{2}}$

(a) $f(x) = 0$
$\Rightarrow x^2 + 4x - 12 = 0$
$\Rightarrow (x+6)(x-2) = 0$
$\Rightarrow x = -6, \ 2$

(b) $x^4 + 4x^2 - 12 = 0$
$\Rightarrow (x^2)^2 + 4(x^2) - 12 = 0$
$\Rightarrow (x^2+6)(x^2-2) = 0$
$\Rightarrow x^2 = \cancel{-6}, \ 2$
$x^2 = 2$
$x = \sqrt{2}, \ -\sqrt{2}$

Question 15 (**+)

Find in exact form where appropriate the solutions of the equation

$$2(3x^2 - 5) - (x+2)(x-3) = 0.$$

$\boxed{x = -1, \frac{4}{5}}$

$2(3x^2 - 5) - (x+2)(x-3) = 0$
$\Rightarrow 6x^2 - 10 - (x^2 - 3x + 2x - 6) = 0$
$\Rightarrow 6x^2 - 10 - (x^2 - x - 6) = 0$
$\Rightarrow 6x^2 - 10 - x^2 + x + 6 = 0$
$\Rightarrow 5x^2 + x - 4 = 0$
$\Rightarrow (5x-4)(x+1) = 0$
$\Rightarrow x = -1, \ \frac{4}{5}$

Question 16 (**+)

$$f(x) = x^2 + 6x + 7, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Hence find the exact coordinates of the points where the graph of $f(x)$ meets the x axis.

$\boxed{}$, $\boxed{a=3}$, $\boxed{b=-2}$, $\boxed{\left(-3 \pm \sqrt{2}, 0\right)}$

(a) $f(x) = x^2+6x+7$
$f(x) = (x+3)^2 - 9 + 7$
$f(x) = (x+3)^2 - 2$
∴ $a=3$
$b=-2$

(b) "MEETS THE x AXIS" $\Rightarrow y=0$
$0 = (x+3)^2 - 2$
$2 = (x+3)^2$
$\pm\sqrt{2} = x+3$
$-3 \pm \sqrt{2} = x$
$\therefore \left(-3+\sqrt{2}, 0\right)$ & $\left(-3-\sqrt{2}, 0\right)$

Question 17 (**+)

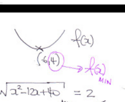
$$f(x) = x^2 - 12x + 40, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Hence state the minimum value of $\sqrt{x^2 - 12x + 40}$.

$\boxed{a=-6}$, $\boxed{b=4}$, $\boxed{2}$

(a) $f(x) = x^2 - 12x + 40$
$f(x) = (x-6)^2 - 36 + 40$
$f(x) = (x-6)^2 + 4$
∴ $a=-6$
$b=40$

(b)


$\therefore \sqrt{x^2-12x+40}_{\text{MIN}} = 2$

Question 18 (***)

$$f(x) = x^2 - 6x + 16, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = (x+a)^2 + b$, where a and b are constants.

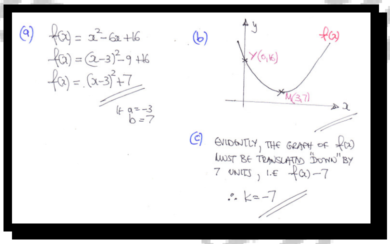
The graph of $f(x)$ has a minimum point at M and meets the y axis at Y.

b) Sketch the graph of $f(x)$, indicating the coordinates of the points M and Y.

The graph of $f(x)+k$, where k is a constant, **touches** the x axis.

c) State the value of k.

$\boxed{f(x) = (x-3)^2 + 7}$, $\boxed{k = -7}$

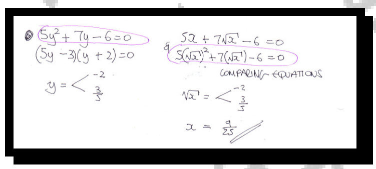


Question 19 (***)

By considering the factorization of the equation $5y^2 + 7y - 6 = 0$, solve the equation

$$5x + 7\sqrt{x} - 6 = 0.$$

$\boxed{x = \frac{9}{25}}$



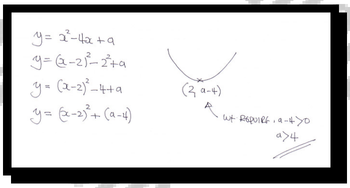
Question 20 (***)

Find the range of values of k for which

$$x^2 - 4x + a$$

is positive for all values of x.

$\boxed{a > 4}$



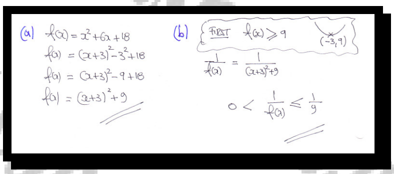
Question 21 (***)

$$f(x) = x^2 + 6x + 18, \ x \in \mathbb{R}.$$

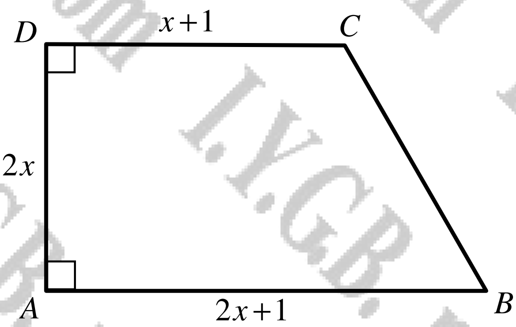
a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are integers.

b) Hence state the minimum and maximum values of $\dfrac{1}{f(x)}$.

$\boxed{a = 3}$, $\boxed{b = 9}$, $\boxed{0 < \dfrac{1}{f(x)} \le \dfrac{1}{9}}$



Question 22 (***)

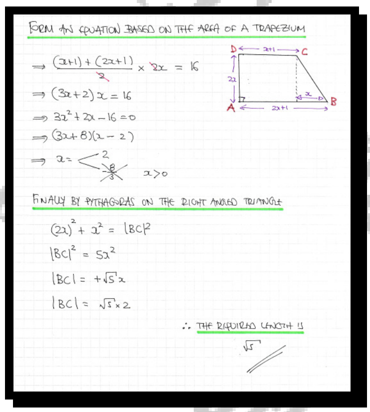


A right angled trapezium $ABCD$ is shown in the figure above.

The trapezium has parallel sides AB and CD of lengths $(2x+1)$ cm and $(x+1)$ cm. The height of the trapezium AD is $2x$ cm.

Given that the area of the trapezium is 16 cm^2, determine the exact length of BC.

$\boxed{}$, $\boxed{|BC| = 2\sqrt{5}}$



FORM AN EQUATION BASED ON THE AREA OF A TRAPEZIUM

$\Rightarrow \frac{(x+1)+(2x+1)}{\cancel{2}} \times \cancel{2}x = 16$

$\Rightarrow (3x+2)x = 16$

$\Rightarrow 3x^2 + 2x - 16 = 0$

$\Rightarrow (3x+8)(x-2)$

$\Rightarrow x = \begin{cases} 2 \\ \cancel{-\frac{8}{3}} \end{cases} \quad x > 0$

FINALLY BY PYTHAGORAS ON THE RIGHT ANGLED TRIANGLE

$(2x)^2 + x^2 = |BC|^2$

$|BC|^2 = 5x^2$

$|BC| = +\sqrt{5}x$

$|BC| = \sqrt{5} \times 2$

$\therefore$ THE REQUIRED LENGTH IS

$\sqrt{5}$

Question 23 (***) (non calculator)

$$x^2 - 1.6x - 3.36 = 0$$

Solve the above equation giving the answers in decimal form.

$\boxed{x = -1.2, 2.8}$

$x^2 - 1.6x - 3.36 = 0$
$\Rightarrow (x-0.8)^2 - 0.8^2 - 3.36 = 0$
$\Rightarrow (x-0.8)^2 - 0.64 - 3.36 = 0$
$\Rightarrow (x-0.8)^2 - 4 = 0$
$\Rightarrow (x-0.8)^2 = 4$
$\Rightarrow x - 0.8 = \begin{cases} -2 \\ 2 \end{cases}$
$\Rightarrow x = \begin{cases} -1.2 \\ 2.8 \end{cases}$

Question 24 (***)

$$f(x) = 2x^2 + 5x + 3, \; x \in \mathbb{R}.$$

a) Express $f(x)$ as a product of two linear factors.

b) Hence, express 253 as a product of two prime factors.

$\boxed{f(x) = (2x+3)(x+1)}$, $\boxed{253 = 23 \times 11}$

(a) $f(x) = 2x^2 + 5x + 3$
$f(x) = (2x+3)(x+1)$

(b) LET $x=10$ & USE THE FACT
$\boxed{2x^2 + 5x + 3 = (2x+3)(x+1)}$
$200 + 50 + 3 = 23 \times 11$
$253 = 23 \times 11$

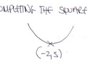
Question 25 (***)

A quadratic curve has equation $y = x^2 + bx + c$, where a and b are constants.

Given that the coordinates of the minimum point of the quadratic is $(-2,5)$ determine the values of a and b.

$\boxed{a=4}$, $\boxed{b=9}$

USING COMPLETING THE SQUARE "BACKWARDS" $\Longrightarrow y = (x+2)^2 + 5$



$y = x^2 + 4x + 4 + 5$

$y = x^2 + 4x + 9$ // $a = 4$, $b = 9$

Question 26 (***)

If $f(x) = x^2 - 7x + 6$, solve the equation $f(x) = f(x+2)$.

$\boxed{}$, $\boxed{x = \frac{5}{2}}$

$f(x) = f(x+2)$

$\Rightarrow x^2 - 7x + \cancel{6} = (x+2)^2 - 7(x+2) + \cancel{6}$

$\Rightarrow \cancel{x^2} - \cancel{7x} = \cancel{x^2} + 4x + 4 - \cancel{7x} - 14$

$\Rightarrow 10 = 4x$

$\Rightarrow x = \frac{5}{2}$ //

Question 27 (***+)

It is given that for all values of x

$$5x^2 + Ax - 7 \equiv B(x+2)^2 + C,$$

where A, B and C are constants.

Determine the values of A, B and C.

$\boxed{A = 20}$, $\boxed{B = 5}$, $\boxed{C = -27}$

$5x^2 + Ax - 7 \equiv B(x+2)^2 + C$
$5x^2 + Ax - 7 \equiv B(x^2 + 4x + 4) + C$
$5x^2 + Ax - 7 \equiv Bx^2 + 4Bx + 4B + C$

- $[x^2]$: $5 = B$
- $[x]$: $A = 4B$
  $A = 20$
- $[x^0]$: $-7 = 4B + C$
  $-7 = 20 + C$
  $C = -27$

$\therefore$ $A = 20$, $B = 5$, $C = -27$

Question 28 (***+)

$$f(x) = 4x^2 + 12kx, \quad x \in \mathbb{R},$$

where k is a constant.

a) Show clearly that the equation $f(x) = 9$ has two distinct real roots for all values of k.

b) Hence find the solutions of the equation $f(x) = 9$, giving the answers in the form $pk \pm p\sqrt{k^2 + 1}$, where p is a constant to be found.

$\boxed{x = \frac{3}{2}k \pm \frac{3}{2}\sqrt{k^2 + 1}}$

(a) $4x^2 + 12kx = 9$
$\Rightarrow 4x^2 + 12kx - 9 = 0$
$b^2 - 4ac = (4k)^2 - 4 \times 4 \times (-9)$
$= 144k^2 + 144$
$\geqslant 144$
> 0
$\therefore$ ALWAYS TWO REAL ROOTS

(b) SOLUTIONS BY QUADRATIC FORMULA
$x = \frac{-12k \pm \sqrt{144k^2 + 144}}{2 \times 4}$
$x = \frac{-12k \pm 12\sqrt{k^2 + 1}}{8}$
$x = -\frac{3}{2}k \pm \frac{3}{2}\sqrt{k^2 + 1}$

Question 29 (***+)

$$f(x) = 11 + 8x - x^2, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = A - (x+B)^2$, where A and B are constants.

b) State the maximum value of $f(x)$.

c) Solve the equation $f(x) = 0$, giving the answers in the form $p \pm q\sqrt{3}$, where p and q are constants

$\boxed{A = 27}$, $\boxed{B = -4}$, $\boxed{f(x)_{\text{max}} = 27}$, $\boxed{x = -4 \pm 3\sqrt{3}}$

(a) $f(x) = 11 + 8x - x^2$

$-f(x) = x^2 - 8x - 11$

$-f(x) = (x-4)^2 - 16 - 11$

$-f(x) = (x-4)^2 - 27$

$f(x) = 27 - (x-4)^2$

$\therefore A = 27$

$B = -4$

ALTERNATIVE

$f(x) = -x^2 + 8x + 11$

$f(x) = -[x^2 - 8x - 11]$

$f(x) = -[(x+4)^2 - 16 - 11]$

$f(x) = -[(x+4)^2 - 27]$

$f(x) = -(x+4)^2 + 27$

$f(x) = 27 - (x+4)^2$

(b) MAXIMUM VALUE IS 27

(c) $f(x) = 0$

$\Rightarrow 27 - (x+4)^2 = 0$

$\Rightarrow 27 = (x+4)^2$

$\Rightarrow x+4 = \pm\sqrt{27}$

$\Rightarrow x+4 = \pm 3\sqrt{3}$

$\Rightarrow x = -4 \pm 3\sqrt{3}$

$p = -4$

$q = 3$

Question 30 (***+)

$$x - \frac{14}{x} = 6\sqrt{2}, \ x \neq 0.$$

Solve the above equation giving the answers in the form $p\sqrt{2}$, where p is a constant.

$\boxed{x = -\sqrt{2} \quad \text{or} \quad x = 7\sqrt{2}}$

START BY MULTIPLYING BY x

$$\Rightarrow x - \frac{14}{x} = 6\sqrt{2}$$
$$\Rightarrow x^2 - 14 = 6\sqrt{2}x$$
$$\Rightarrow x^2 - 6\sqrt{2}x - 14 = 0$$

BY THE QUADRATIC FORMULA OR BY COMPLETING THE SQUARE

$$\Rightarrow x = \frac{6\sqrt{2} \pm \sqrt{(6\sqrt{2})^2 - 4 \times 1 \times (-14)}}{2 \times 1}$$
$$\Rightarrow x = \frac{6\sqrt{2} \pm \sqrt{72 + 56}}{2}$$
$$\Rightarrow x = \frac{6\sqrt{2} \pm \sqrt{128}}{2}$$
$$\Rightarrow x = \frac{6\sqrt{2} \pm 8\sqrt{2}}{2} = \begin{cases} \frac{6\sqrt{2} + 8\sqrt{2}}{2} = 7\sqrt{2} \\ \frac{6\sqrt{2} - 8\sqrt{2}}{2} = -\sqrt{2} \end{cases}$$

OR BY COMPLETING THE SQUARE

$$\Rightarrow x^2 - 6\sqrt{2}x - 14 = 0$$
$$\Rightarrow (x - 3\sqrt{2})^2 - (3\sqrt{2})^2 - 14 = 0$$
$$\Rightarrow (x - 3\sqrt{2})^2 - 18 - 14 = 0$$
$$\Rightarrow (x - 3\sqrt{2})^2 = 32$$
$$\Rightarrow x - 3\sqrt{2} = \begin{cases} \sqrt{32} = 4\sqrt{2} \\ -\sqrt{32} = -4\sqrt{2} \end{cases}$$
$$\Rightarrow x = \begin{cases} 7\sqrt{2} \\ -\sqrt{2} \end{cases}$$

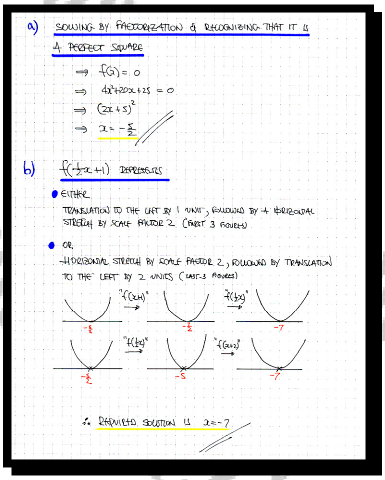
Question 31 **(***+)**

$$f(x) = 4x^2 + 20x + 25, \quad x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Hence, or otherwise, solve the equation $f\left(\frac{1}{2}x + 1\right) = 0$.

$\boxed{}$, $\boxed{x = -\frac{5}{2}}$, $\boxed{x = -7}$



a) SOLVING BY FACTORIZATION & RECOGNIZING THAT IT IS A PERFECT SQUARE

$\Rightarrow f(x) = 0$

$\Rightarrow 4x^2 + 20x + 25 = 0$

$\Rightarrow (2x+5)^2$

$\Rightarrow x = -\frac{5}{2}$

b) $f\left(\frac{1}{2}x+1\right)$ REPRESENTS

- EITHER

TRANSLATION TO THE LEFT BY 1 UNIT, FOLLOWED BY A HORIZONTAL STRETCH BY SCALE FACTOR 2 (FIRST 3 FIGURES)

- OR

HORIZONTAL STRETCH BY SCALE FACTOR 2, FOLLOWED BY TRANSLATION TO THE LEFT BY 2 UNITS (LAST 3 FIGURES)

$\therefore$ REQUIRED SOLUTION IS $x = -7$

Question 32 (***+)

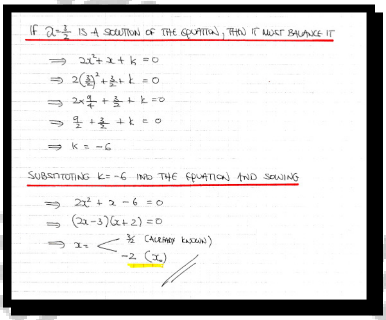
The quadratic equation

$$2x^2 + x + k = 0,$$

where k is a constant, has solutions $x = \frac{3}{2}$ and $x = x_0$.

Find the value of $x = x_0$.

$\boxed{}$, $\boxed{x_0 = -2}$



IF $x=\frac{3}{2}$ IS A SOLUTION OF THE EQUATION, THEN IT MUST BALANCE IT

$\Rightarrow 2x^2+x+k=0$

$\Rightarrow 2\left(\frac{3}{2}\right)^2+\frac{3}{2}+k=0$

$\Rightarrow 2\times\frac{9}{4}+\frac{3}{2}+k=0$

$\Rightarrow \frac{9}{2}+\frac{3}{2}+k=0$

$\Rightarrow k=-6$

SUBSTITUTING $k=-6$ INTO THE EQUATION AND SOLVING

$\Rightarrow 2x^2+x-6=0$

$\Rightarrow (2x-3)(x+2)=0$

$\Rightarrow x = \begin{cases} \frac{3}{2} & \text{(ALREADY KNOWN)} \\ -2 & (x_0) \end{cases}$

Question 33 (***+)

$$f(x) = 9x^2 + 18x - 7, \; x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Express $f(x)$ in the form

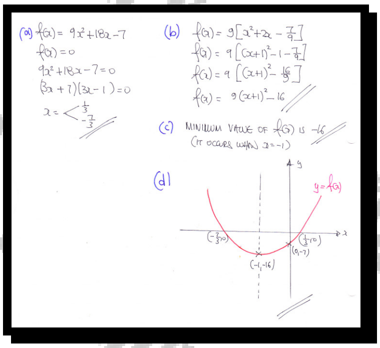
$$f(x) = 9(x + A)^2 + B,$$

where A and B are integer constants.

c) State the minimum value of $f(x)$.

d) Sketch the graph of $f(x)$, indicating clearly the coordinates of the points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{}$, $\boxed{x = -\frac{7}{3}, \frac{1}{3}}$, $\boxed{A = 1}$, $\boxed{B = -16}$, $\boxed{f(x)_{\min} = -16}$



Question 34 (***+)

$$f(x) = \left(x - 4 - \sqrt{3}\right)\left(x - 4 + \sqrt{3}\right), \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form …

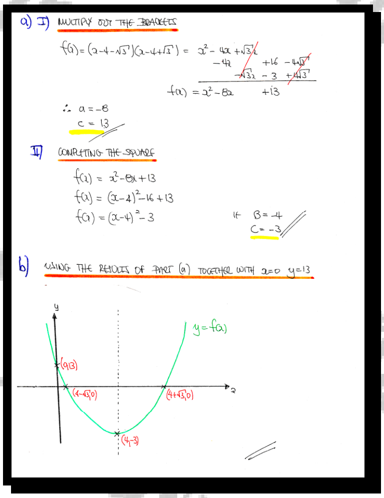
i. … $f(x) = x^2 + bx + c$, where b and c are constants.

ii. … $f(x) = (x + B)^2 + C$, where B and C are constants.

b) Sketch the graph of the curve C with equation $y = f(x)$.

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes, and the coordinates of the minimum point of C.

$\boxed{}$, $\boxed{f(x) = x^2 - 8x + 13}$, $\boxed{f(x) = (x-4)^2 - 3}$



Question 35 **(***+)**

A curve C and a straight line L have respective equations

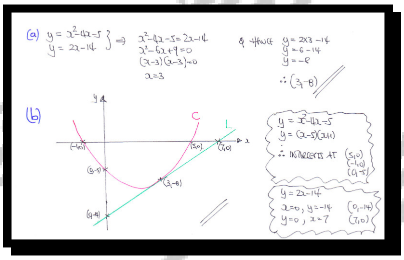
$$y = x^2 - 4x - 5 \quad \text{and} \quad y = 2x - 14 .$$

a) Find the coordinates of any points of intersection between C and L.

b) Sketch in the same diagram the graph of C and the graph of L.

The sketch must include of any points of intersection between the graph of C and the coordinate axes, and any points of intersection between the graph of L and the coordinate axes.

$\boxed{}$, $\boxed{(3,-8)}$



Question 36 (***+)

$$f(x) = x^2 + 2kx + c,$$

where k and c are constants.

a) Express $f(x)$ in "completed the square" form.

b) Hence, or otherwise, solve the equation $f(x) = 0$, giving the answer in terms of k and c.

The equation $f(x) = 0$ has repeated roots.

c) Express c in terms of k.

$\boxed{(x+k)^2 - k^2 + c}$, $\boxed{x = -k \pm \sqrt{c - k^2}}$, $\boxed{c = k^2}$

(a) $f(x) = x^2 + 2kx + c$
$f(x) = (x+k)^2 - k^2 + c$

(b) $f(x) = 0$
$x^2 + 2kx + c = 0$
$(x+k)^2 - k^2 + c = 0$
$(x+k)^2 = k^2 - c$
$x + k = \pm\sqrt{k^2 - c}$
$x = -k \pm \sqrt{k^2 - c}$

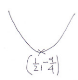
(c) FOR REPEATED ROOTS
$\sqrt{k^2 - c} = 0$
$k^2 - c = 0$
$c = k^2$

Question 37 (***+)

$$f(x) = x^2 + Ax + B, \quad x \in \mathbb{R}.$$

Given that the graph of $f(x)$ has a minimum at the point $\left(\frac{1}{2}, -\frac{9}{4}\right)$, determine the values of the constants A and B.

$\boxed{A = -1}$, $\boxed{B = -2}$



$\therefore f(x) = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$
$f(x) = x^2 - \cancel{2} \times \frac{1}{\cancel{2}} \times x + \frac{1}{4} - \frac{9}{4}$
$f(x) = x^2 - x - \frac{8}{4}$
$f(x) = x^2 - x - 2$

$\therefore A = -1$
$B = -2$

Question 38 (***+)

$$f(x) = x^2 - 2x - 47, \ x \in \mathbb{R}.$$

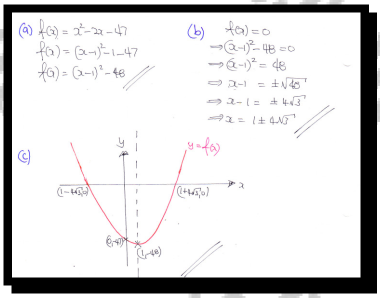
a) Express $f(x)$ in the form $f(x) = (x+a)^2 + b$, where a and b are constants.

b) Solve the equation $f(x) = 0$, giving the answers in exact form in terms of $\sqrt{3}$.

c) Sketch the graph of $f(x)$.

The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes, and the coordinates of the minimum point of $f(x)$.

$\boxed{f(x) = (x-1)^2 - 48}$, $\boxed{x = 1 \pm 4\sqrt{3}}$



Question 39 **(***+) non calculator**

$$f(x) = 5 + 9x - 2x^2,\ x \in \mathbb{R}.$$

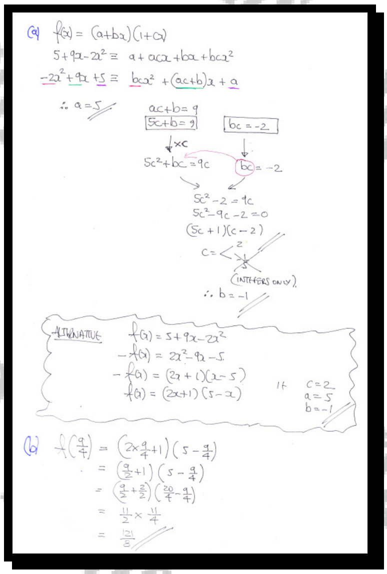
a) Given that

$$f(x) \equiv (a + bx)(1 + cx),$$

determine the values of the **integer** constants a, b and c.

b) Evaluate $f\left(\frac{9}{4}\right)$.

$\boxed{a = 5}$, $\boxed{b = -1}$, $\boxed{c = 2}$, $\boxed{f\left(\frac{9}{4}\right) = \frac{121}{8}}$



(a) $f(x) = (a+bx)(1+cx)$

$5 + 9x - 2x^2 \equiv a + acx + bx + bcx^2$

$-2x^2 + 9x + 5 \equiv bcx^2 + (ac+b)x + a$

$\therefore a = 5$

$ac + b = 9$

$\boxed{5c + b = 9}$ $\boxed{bc = -2}$

$\times c$

$5c^2 + bc = 9c$ $bc = -2$

$5c^2 - 2 = 9c$

$5c^2 - 9c - 2 = 0$

$(5c+1)(c-2)$

$c = 2$, $\cancel{-\frac{1}{5}}$ (INTEGERS ONLY)

$\therefore b = -1$

ALTERNATIVE

$f(x) = 5 + 9x - 2x^2$

$-f(x) = 2x^2 - 9x - 5$

$-f(x) = (2x+1)(x-5)$

$f(x) = (2x+1)(5-x)$ if $c = 2$, $a = 5$, $b = -1$

(b) $f\left(\frac{9}{4}\right) = \left(2 \times \frac{9}{4} + 1\right)\left(5 - \frac{9}{4}\right)$

$= \left(\frac{9}{2} + 1\right)\left(5 - \frac{9}{4}\right)$

$= \left(\frac{9}{2} + \frac{2}{2}\right)\left(\frac{20}{4} - \frac{9}{4}\right)$

$= \frac{11}{2} \times \frac{11}{4}$

$= \frac{121}{8}$

Question 40 **(***+)**

$$f(x) = 3x^2 + 12x + 8, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are integers.

b) State the minimum value of $f(x)$.

c) Solve the equation $f(x) = 0$, giving the answers as exact simplified surds.

$\boxed{}$, $\boxed{a=3}$, $\boxed{b=2}$, $\boxed{c=-4}$, $\boxed{-4}$, $\boxed{x=-2\pm\frac{2}{3}\sqrt{3}}$

(a) $f(x) = 3x^2 + 12x + 8$
$\Rightarrow f(x) = 3\left[x^2 + 4x + \frac{8}{3}\right]$
$\Rightarrow f(x) = 3\left[(x+2)^2 - 4 + \frac{8}{3}\right]$
$\Rightarrow f(x) = 3(x+2)^2 - 12 + 8$
$\Rightarrow f(x) = 3(x+2)^2 - 4$

(b) $f(x)_{MIN} = -4$
(occurs when $x=-2$)

(c) • $f(x) = 0$
$\Rightarrow 3x^2 + 12x + 8 = 0$
$\Rightarrow 3(x+2)^2 - 4 = 0$
$\Rightarrow 3(x+2)^2 = 4$
$\Rightarrow (x+2)^2 = \frac{4}{3}$
$\Rightarrow x+2 = \pm\sqrt{\frac{4}{3}}$
$\Rightarrow x+2 = \pm\frac{2}{\sqrt{3}}$
$\Rightarrow x+2 = \pm\frac{2}{3}\sqrt{3}$
$\Rightarrow x = -2 \pm \frac{2}{3}\sqrt{3}$

Question 41 (***+)

A curve C and a line L have respective equation

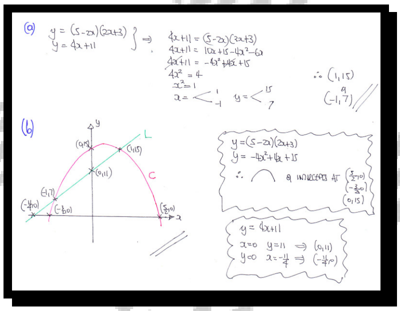
$$y = (5-2x)(2x+3) \quad \text{and} \quad y = 4x+11.$$

a) Find the coordinates of any points of intersection between C and L.

b) Sketch in the same diagram the graphs of C and L.

The sketch must include of any points of intersection between the graph of C and the coordinate axes, and any points of intersection between the graph of L and the coordinate axes.

$$\boxed{(-1,7),\ (1,15)}$$



Question 42 (***+)

$$f(x) \equiv 8 + 2x - x^2, \; x \in \mathbb{R}.$$

a) Find the values of the constants A and B so that $f(x) \equiv A - (x + B)^2$.

b) Sketch the graph of $f(x)$.

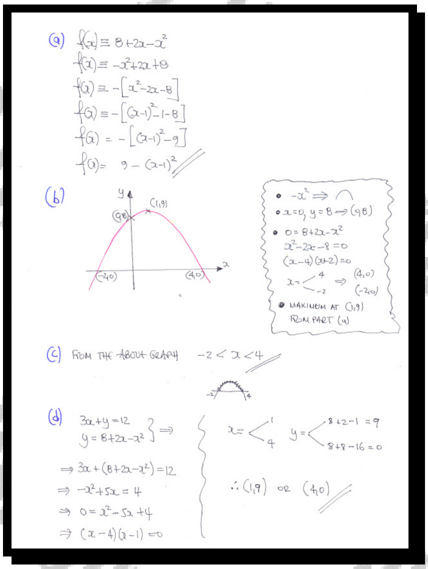
The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes, and the coordinates of the maximum point of $f(x)$.

c) Hence, solve the inequality

$$8 + 2x - x^2 > 0.$$

d) Find the coordinates of the points of intersection between the graph of $f(x)$ and the line with equation $3x + y = 12$.

$\boxed{A = 9}$, $\boxed{B = -1}$, $\boxed{-2 < x < 4}$, $\boxed{(1,9) \text{ or } (4,0)}$



Question 43 **(***+)**

It is given that for all values of x

$$5x^2 + Ax + 7 = B(x-2)^2 + C\,,\ x \in \mathbb{R}\,.$$

Determine the values of each of the constants A, B and C.

$\boxed{\text{SYN-C}}$, $\boxed{A=-20}$, $\boxed{B=5}$, $\boxed{C=-13}$

EXPAND AND COMPARE COEFFICIENTS ON BOTH SIDES

$$\Rightarrow 5x^2 + Ax + 7 \equiv B(x-2)^2 + C$$
$$\Rightarrow 5x^2 + Ax + 7 \equiv B(x^2-4x+4) + C$$
$$\Rightarrow 5x^2 + Ax + 7 \equiv Bx^2 - 4Bx + 4B + C$$
$$\Rightarrow 5x^2 + Ax + 7 \equiv Bx^2 - 4Bx + (4B + C)$$

HENCE WE HAVE

$[x^2]$: $B = 5$ //

$[x]$: $A = -4B$

$A = -4 \times 5$

$A = -20$ //

$[x^0]$ $4B + C = 7$

$4 \times 5 + C = 7$

$C = -13$ //

Question 44 (***+)

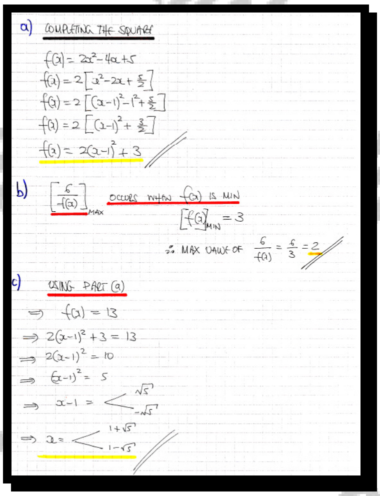
$$f(x) \equiv 2x^2 - 4x + 5\,,\ x \in \mathbb{R}\,.$$

a) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are integers.

b) State the maximum value of $\dfrac{6}{f(x)}$.

c) Solve the equation $f(x) = 13$, giving the answers as exact simplified surds.

$\boxed{\text{SYN-I}}$, $\boxed{a=2}$, $\boxed{b=-1}$, $\boxed{c=3}$, $\boxed{2}$, $\boxed{x = 1 \pm \sqrt{5}}$



a) COMPLETING THE SQUARE

$f(x) = 2x^2 - 4x + 5$

$f(x) = 2\left[x^2 - 2x + \frac{5}{2}\right]$

$f(x) = 2\left[(x-1)^2 - 1^2 + \frac{5}{2}\right]$

$f(x) = 2\left[(x-1)^2 + \frac{3}{2}\right]$

$f(x) = 2(x-1)^2 + 3$

b) $\left[\frac{6}{f(x)}\right]_{MAX}$ OCCURS WHEN $f(x)$ IS MIN

$\left[f(x)\right]_{MIN} = 3$

$\therefore$ MAX VALUE OF $\frac{6}{f(x)} = \frac{6}{3} = 2$

c) USING PART (a)

$\Rightarrow f(x) = 13$

$\Rightarrow 2(x-1)^2 + 3 = 13$

$\Rightarrow 2(x-1)^2 = 10$

$\Rightarrow (x-1)^2 = 5$

$\Rightarrow x - 1 = \begin{cases} \sqrt{5} \\ -\sqrt{5} \end{cases}$

$\Rightarrow x = \begin{cases} 1+\sqrt{5} \\ 1-\sqrt{5} \end{cases}$

Question 45 (***+)

The line straight L and the curve C have respective equations

$$L: \ 2y = 7x + 10 .$$

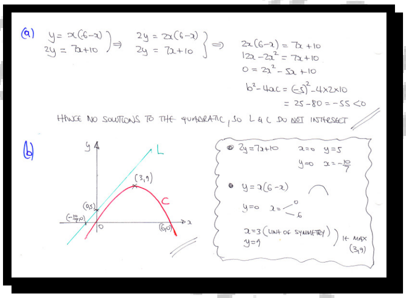
$$C: \ y = x(6 - x) .$$

a) Show that L and C do not intersect.

b) Find the coordinates of the maximum point of C

c) Sketch on the same diagram the graphs of L and C, showing clearly the coordinates of any points where the graphs meet the coordinate axes.

$\boxed{\max(3,9)}$



Question 46 **(***+)**

$$f(x) = 7 + 6x - x^2, \; x \in \mathbb{R}.$$

a) Factorize $f(x)$.

b) Express $f(x)$ in the form $A - (x+B)^2$, where A and B are constants.

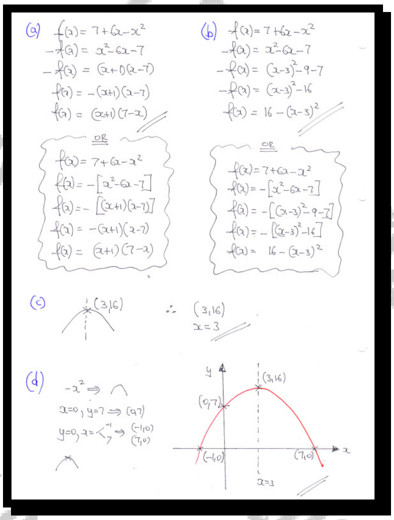
c) State …

i. … the coordinates of the vertex of the curve.

ii. … the equation of the line of symmetry of the curve.

d) Sketch the graph of $f(x)$, indicating clearly the coordinates of the points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{f(x) = (7-x)(x+1)}$, $\boxed{f(x) = 16 - (x-3)^2}$



Question 47 (***+)

$$f(x) = x^2 + 10x + 27, \quad x \in \mathbb{R}.$$

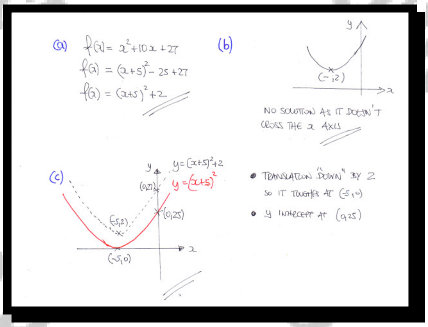
a) Express $f(x)$ in the form $(x+b)^2 + c$, where b and c are constants.

b) Show that the equation $f(x) = 0$ has no real solutions.

The graph of $f(x) - k$, where k is a positive constant, touches the x axis.

c) Sketch the graph of $f(x) - k$, indicating clearly the coordinates of the points where the graph of $f(x) - k$ meets the coordinate axes.

$$\boxed{f(x) = (x+5)^2 + 2}$$



Question 48 (***+)

$$f(x) = \frac{169}{8} - 2\left(x + \frac{7}{4}\right)^2, \ x \in \mathbb{R}.$$

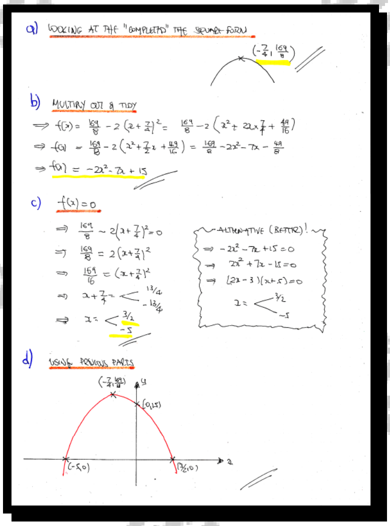
a) State the coordinates of the maximum point of $f(x)$.

b) Express $f(x)$ in the form $ax^2 + bx + c$, where a, b and c are constants.

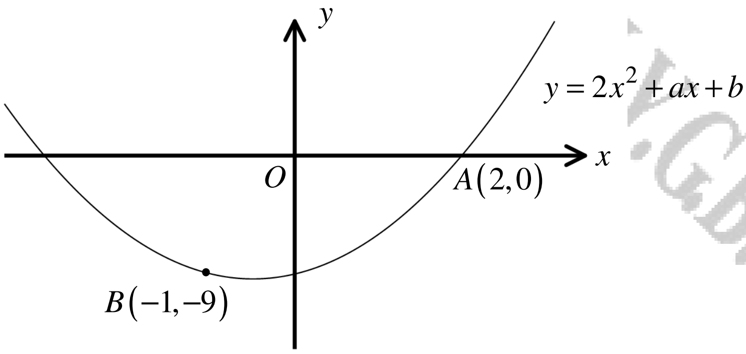
c) Solve the equation $f(x) = 0$.

d) Sketch the graph of $f(x)$, indicating clearly the coordinates of the points where the graph of $f(x)$ meets the coordinate axes.

SYN-W, $\boxed{\left(-\frac{7}{4}, \frac{169}{8}\right)}$, $\boxed{f(x) = -2x^2 - 7x + 15}$, $\boxed{x = -5 \cup x = \frac{3}{2}}$



Question 49 (***+)



The figure above shows the graph of the curve with equation

$$y = 2x^2 + ax + b,$$

where a and b are constants.

The curve crosses the x axis at the point $A(2,0)$ and the point $B(-1,-9)$ also lies on the curve.

Determine the values of a and b.

$\boxed{a=1}$, $\boxed{b=-10}$

$y = 2x^2 + ax + b$

• $A(2,0) \Rightarrow 0 = 2 \times 2^2 + a \times 2 + b$
$\Rightarrow 0 = 8 + 2a + b$
$\boxed{-8 = 2a + b}$

• $B(-1,-9) \Rightarrow -9 = 2(-1)^2 + a(-1) + b$
$\Rightarrow -9 = 2 - a + b$
$\Rightarrow \boxed{-11 = -a + b}$

$(\times 2)$

$2a + b = -8$
$-2a + 2b = -22$
ADD $3b = -30$
$b = -10$

$2a + b = -8$
$2a - 10 = -8$
$2a = 2$
$a = 1$

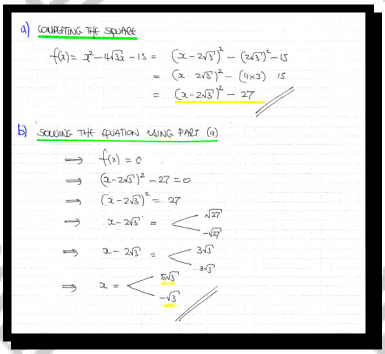
Question 50 (***+)

$$f(x) \equiv x^2 - 4\sqrt{3}x - 15, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = (x+a)^2 + b$, where a and b are constants.

b) Hence find the exact solutions of the equation $f(x) = 0$.

$\boxed{}$, $\boxed{a = -2\sqrt{3}}$, $\boxed{b = -27}$, $\boxed{x = -\sqrt{3}, \ 5\sqrt{3}}$



Question 51 (***+)

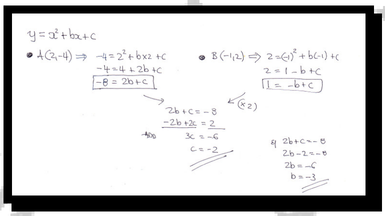
The quadratic curve C has equation

$$f(x) = x^2 + bx + c,$$

where b and c are constants.

Given that the graph of C passes through the points $A(2,-4)$ and $B(-1,2)$ determine the values of b and c.

$\boxed{b = -3, c = -2}$



Question 52 (***+)

$$2x^2 - xy - y^2.$$

Factorize the above quadratic expression.

You may factorize by inspection, or by using the quadratic formula or by completing the square.

$\boxed{}$, $\boxed{(2x+y)(x-y)}$

BY INSPECTION

$$2x^2 - xy - y^2 = (2x+y)(x-y)$$

BY THE QUADRATIC FORMULA – TREAT x AS "THE VARIABLE"

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad a = 2,\ b = -y,\ c = -y^2$$

$$x = \frac{y \pm \sqrt{y^2 - 4 \times 2 \times (-y^2)}}{4}$$

$$x = \frac{y \pm \sqrt{9y^2}}{4} = \begin{cases} \frac{y+3y}{4} = y \\ \frac{y-3y}{4} = -\frac{1}{2}y \end{cases}$$

EITHER $x = y \Rightarrow x - y = 0$

OR $x = -\frac{1}{2}y$

$2x = -y \Rightarrow 2x + y = 0$

$\therefore (x-y)(2x+y)$ AS BEFORE

BY COMPLETING THE SQUARE TREATING y AS A VARIABLE

$$2x^2 - xy - y^2 = -\left[y^2 + xy - 2x^2\right]$$

$$= -\left[\left(y+\tfrac{1}{2}x\right)^2 - \tfrac{1}{4}x^2 - 2x^2\right]$$

$$= -\left[\left(y+\tfrac{1}{2}x\right)^2 - \tfrac{9}{4}x^2\right]$$

$$= \tfrac{9}{4}x^2 - \left(y+\tfrac{1}{2}x\right)^2$$

$$= \left(\tfrac{3}{2}x\right)^2 - \left(y+\tfrac{1}{2}x\right)^2 \quad \leftarrow A^2 - B^2 \equiv (A+B)(A-B)$$

$$= \left[\tfrac{3}{2}x + \left(y+\tfrac{1}{2}x\right)\right]\left[\tfrac{3}{2}x - \left(y+\tfrac{1}{2}x\right)\right]$$

$$= (2x+y)(x-y)$$

AS BEFORE

Question 53 (***+)

Find the solutions of the equation

$$\left(x^2-x-3\right)^2-12\left(x^2-x-3\right)+27=0.$$

$\boxed{}$, $\boxed{x=-3,-2,3,4}$

LET $y=x^2-x-3$

$\Rightarrow (x^2-x-3)^2-12(x^2-x-3)+27=0$

$\Rightarrow y^2-12y+27=0$

$\Rightarrow (y-3)(y-9)=0$

$\Rightarrow y=\begin{cases}3\\9\end{cases}$

$\Rightarrow x^2-x-3=\begin{cases}3\\9\end{cases}$

$\Rightarrow \begin{cases}x^2-x-6=0\\x^2-x-12=0\end{cases}$

$\Rightarrow \begin{cases}(x+2)(x-3)=0\\(x-4)(x+3)=0\end{cases}$

$\therefore x=\begin{cases}-3\\-2\\3\\4\end{cases}$ //

Question 54 (***+)

Solve the following quadratic equation.

$$(2x+3)^2-(4-x)^2=45.$$

$\boxed{}$, $\boxed{x=2,\quad x=-\frac{26}{3}}$

EXPAND AND SIMPLIFY

$\Rightarrow (2x+3)^2-(4-x)^2=45$

$\Rightarrow 4x^2+12x+9-(16-8x+x^2)=45$

$\Rightarrow 4x^2+12x+9-16+8x-x^2=45$

$\Rightarrow 3x^2+20x-7=45$

$\Rightarrow 3x^2+20x-52=0$

FACTORIZING NOTING THAT 1×52, 2×26, 4×13

$\Rightarrow (3x+26)(x-2)=0$

$\Rightarrow x=\begin{cases}2\\-\frac{26}{3}\end{cases}$ //

ALTERNATIVE

$\Rightarrow (2x+3)^2-(4-x)^2=45$

$\Rightarrow [(2x+3)+(4-x)][(2x+3)-(4-x)]=45$

$\Rightarrow (x+7)(3x-1)=45$

BY INSPECTION $x=2$ IS A SOLUTION

$\Rightarrow 3x^2+20x-7=45$

$\Rightarrow 3x^2+20x-52=0$

$\Rightarrow (x-2)(3x+26)=0$

(FROM ABOVE)

$\therefore x=\begin{cases}2\\-\frac{26}{3}\end{cases}$ //

Question 55 (***+)

The curve C has equation

$$y = 9 - (x-2)^2 .$$

a) Describe geometrically the three transformations that map the graph of $y = x^2$ onto the graph of C.

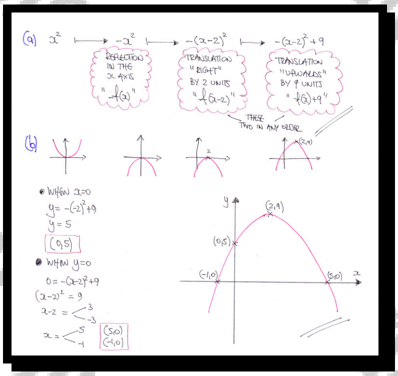
b) Hence, sketch the graph of C.

The sketch must include the coordinates of

- … all the points where the curve meets the coordinate axes.
- … the coordinates of the maximum point of the curve.

$\boxed{\text{reflection in the } x \text{ axis}}$, $\boxed{\text{translation "right" by 2 units}}$,

$\boxed{\text{translation "upwards" by 9 units}}$



Question 56 **(***+)**

$$f(x) \equiv 5x^2 - 30x + 50, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are constants.

b) Hence write down the minimum value of $f(x)$.

The point A has coordinates $(5,6)$.

The variable point B has coordinates $(x, 2x+1)$.

c) Show clearly that

$$|AB|^2 = 5x^2 - 30x + 50.$$

d) Use part (b) to determine the shortest distance between A and B.

e) Hence write down the coordinates of B when the distance between A and B is shortest.

$\boxed{}$, $\boxed{f(x) \equiv 5(x-3)^2 + 5}$, $\boxed{f(x)_{\text{min}} = 5}$, $\boxed{|AB|_{\text{min}} = \sqrt{5}}$, $\boxed{B(3,7)}$

a) $f(x) = 5x^2 - 30x + 50 = 5\left[x^2 - 6x + 10\right] = 5\left[(x-3)^2 - 9 + 10\right]$
$= 5\left[(x-3)^2 + 1\right] = 5(x-3)^2 + 5$

b) $f(x)_{\text{MIN}}$ IS 5

c) $A(5,6)$ $B(x, 2x+1)$

$\Rightarrow |AB| = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

$\Rightarrow |AB| = \sqrt{(2x+1-6)^2 + (x-5)^2}$

$\Rightarrow |AB| = \sqrt{(2x-5)^2 + (x-5)^2}$

$\Rightarrow |AB|^2 = (2x-5)^2 + (x-5)^2$

$\Rightarrow |AB|^2 = 4x^2 - 20x + 25 + x^2 - 10x + 25$

$\Rightarrow |AB|^2 = 5x^2 - 30x + 50$

d) $|AB|^2_{\text{MIN}} = 5$ from part (b)

$\therefore |AB|_{\text{MIN}} = \sqrt{5}$

e) MIN VALUE OF $f(x)$ OCCURS WHEN $x = 3$

MIN VALUE OF $\sqrt{f(x)}$ ALSO OCCURS WHEN $x = 3$

$\therefore B(3,7)$

Question 57 (***+)

A quadratic curve meets the coordinate axes at $(-2,0)$, $(4,0)$ and $(0,-20)$.

Determine the equation of the curve in the form $y = ax^2 + bx + c$, where a, b and c are constants.

$$\boxed{y = \frac{5}{2}x^2 - 5x - 20}$$

"CROSSES THE x AXIS AT $(-2,0)$ & $(4,0)$" $\Rightarrow y = k(x+2)(x-4)$

$\therefore y = k(x^2 - 2x - 8)$

THEN $x=0$ $y=-20$ $\Longrightarrow$ $-20 = k(-8)$

$k = \frac{5}{2}$

$\therefore y = \frac{5}{2}(x^2 - 2x - 8)$

$y = \frac{5}{2}x^2 - 5x - 20$ //

Question 58 (***+)

$$f(x) = 4x^2 + 4x - 1, \ x \in \mathbb{R}.$$

a) Express $f(x)$ in completed the square form.

b) Hence find, as exact surds, the roots of the equation $f(x) = 0$.

$\boxed{}$, $\boxed{f(x) = 4\left(x - \frac{1}{2}\right)^2 - 2}$, $\boxed{x = \frac{-1 \pm \sqrt{2}}{2}}$

a) COMPLETING THE SQUARE

$f(x) = 4x^2 + 4x - 1$

$f(x) = 4\left[x^2 + x - \frac{1}{4}\right]$

$f(x) = 4\left[\left(x+\frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{4}\right]$

$f(x) = 4\left(x+\frac{1}{2}\right)^2 - 1 - 1$

$f(x) = 4\left(x+\frac{1}{2}\right)^2 - 2$ //

ALTERNATIVE

$f(x) = 4x^2 + 4x - 1$

$f(x) = (4x^2 + 4x + 1) - 2$

$f(x) = (2x+1)^2 - 2$ //

b) SOLVING THE EQUATION USING PART (a)

$f(x) = 0$

$4x^2 + 4x - 1 = 0$

$4\left(x+\frac{1}{2}\right)^2 - 2 = 0$

$4\left(x+\frac{1}{2}\right)^2 = 2$

$\left(x+\frac{1}{2}\right)^2 = \frac{1}{2}$

$x + \frac{1}{2} = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

$x + \frac{1}{2} = \pm\frac{\sqrt{2}}{2}$

$x = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$ //

ALTERNATIVE

$f(x) = 0$

$(2x+1)^2 - 2 = 0$

$(2x+1)^2 = 2$

$2x + 1 = \pm\sqrt{2}$

$2x = -1 \pm \sqrt{2}$

$x = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$ //

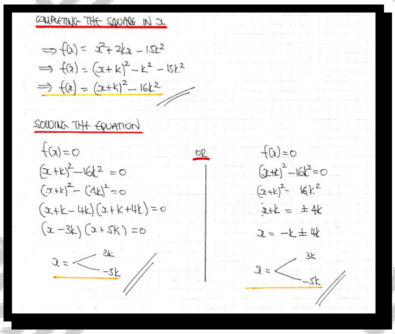
Question 59 **(***+)**

$$f(x) = x^2 + 2kx - 15k^2, \text{ where } k \text{ is a constant.}$$

a) Express $f(x)$ in completed the square form.

b) Hence solve the equation $f(x) = 0$.

$\boxed{}$, $\boxed{f(x) = (x-k)^2 - 16k^2}$, $\boxed{x = -5k,\ 3k}$


COMPLETING THE SQUARE IN x

$\Rightarrow f(x) = x^2 + 2kx - 15k^2$

$\Rightarrow f(x) = (x+k)^2 - k^2 - 15k^2$

$\Rightarrow f(x) = (x+k)^2 - 16k^2$

SOLVING THE EQUATION

$f(x) = 0$

$(x+k)^2 - 16k^2 = 0$

$(x+k)^2 - (4k)^2 = 0$

$(x+k-4k)(x+k+4k) = 0$

$(x-3k)(x+5k) = 0$

$x = 3k, -5k$

OR

$f(x) = 0$

$(x+k)^2 - 16k^2 = 0$

$(x+k)^2 = 16k^2$

$x+k = \pm 4k$

$x = -k \pm 4k$

$x = 3k, -5k$

Question 60 (***+)

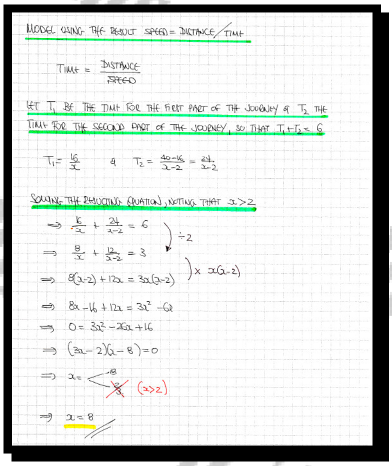
A runner took part in a 40 km walk.

He walked the first 16 km at an average speed x km h^{-1}.

He walked the rest of the race at an average speed of 2 km h^{-1} less than the average speed of his the first 16 km.

Given that the **total** time for the walk was 6 hours, determine the value of x.

$\boxed{}$, $\boxed{x=8}$



MODEL USING THE RESULT SPEED = DISTANCE/TIME

$$\text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}}$$

LET T_1 BE THE TIME FOR THE FIRST PART OF THE JOURNEY & T_2 THE TIME FOR THE SECOND PART OF THE JOURNEY, SO THAT $T_1 + T_2 = 6$

$$T_1 = \frac{16}{x} \quad \& \quad T_2 = \frac{40-16}{x-2} = \frac{24}{x-2}$$

SOLVING THE RESULTING EQUATION, NOTING THAT $x > 2$

$$\Rightarrow \frac{16}{x} + \frac{24}{x-2} = 6 \quad \div 2$$

$$\Rightarrow \frac{8}{x} + \frac{12}{x-2} = 3 \quad \times\, x(x-2)$$

$$\Rightarrow 8(x-2) + 12x = 3x(x-2)$$

$$\Rightarrow 8x - 16 + 12x = 3x^2 - 6x$$

$$\Rightarrow 0 = 3x^2 - 26x + 16$$

$$\Rightarrow (3x-2)(x-8) = 0$$

$$\Rightarrow x = 8 \text{ or } \cancel{\tfrac{2}{3}} \quad (x>2)$$

$$\Rightarrow x = 8$$

Question 61 **(***+)**

Find, in exact simplified surd form, the roots of the following equation.

$$\sqrt{3}\left(x+\frac{6}{x}\right)=9\,,\ x\neq 0\,.$$

Detailed workings must be shown in this question.

$\boxed{}$, $\boxed{x=\sqrt{3},\ x=2\sqrt{3}}$

TIDY THE EQUATION

$$\Rightarrow \sqrt{3}\left(x+\frac{6}{x}\right)=9$$

$$\Rightarrow x+\frac{6}{x}=\frac{9}{\sqrt{3}}$$

$$\frac{9}{\sqrt{3}}=\frac{9\sqrt{3}}{\sqrt{3}\sqrt{3}}=\frac{9\sqrt{3}}{3}=3\sqrt{3}$$

$$\Rightarrow x+\frac{6}{x}=3\sqrt{3}$$

$$\Rightarrow x^2+6=3\sqrt{3}x$$

$$\Rightarrow x^2-3\sqrt{3}x+6=0$$

BY THE QUADRATIC FORMULA

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{3\sqrt{3}\pm\sqrt{(-3\sqrt{3})^2-4\times1\times6}}{2\times1}$$

$$\Rightarrow x=\frac{3\sqrt{3}\pm\sqrt{9\times3-24}}{2}$$

$$\Rightarrow x=\frac{3\sqrt{3}\pm\sqrt{3}}{2}$$

$$\Rightarrow x=\begin{cases}\frac{4\sqrt{3}}{2}\\ \frac{2\sqrt{3}}{2}\end{cases}$$

$$\Rightarrow x=\begin{cases}2\sqrt{3}\\ \sqrt{3}\end{cases}$$

OR BY COMPLETING THE SQUARE

$$\Rightarrow \left(x-\frac{3\sqrt{3}}{2}\right)^2-\left(\frac{3\sqrt{3}}{2}\right)^2+6=0$$

$$\Rightarrow \left(x-\frac{3\sqrt{3}}{2}\right)^2-\frac{9\times3}{4}+6=0$$

$$\Rightarrow \left(x-\frac{3\sqrt{3}}{2}\right)^2-\frac{27}{4}+6=0$$

$$\Rightarrow \left(x-\frac{3\sqrt{3}}{2}\right)^2=\frac{27}{4}-\frac{24}{4}$$

$$\Rightarrow \left(x-\frac{3\sqrt{3}}{2}\right)^2=\frac{3}{4}$$

$$\Rightarrow x-\frac{3}{2}\sqrt{3}=\begin{cases}\sqrt{\frac{3}{4}}\\ -\sqrt{\frac{3}{4}}\end{cases}$$

$$\Rightarrow x-\frac{3}{2}\sqrt{3}=\begin{cases}\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2}\end{cases}$$

$$\Rightarrow x=\begin{cases}2\sqrt{3}\\ \sqrt{3}\end{cases}$$

Question 62 (****)

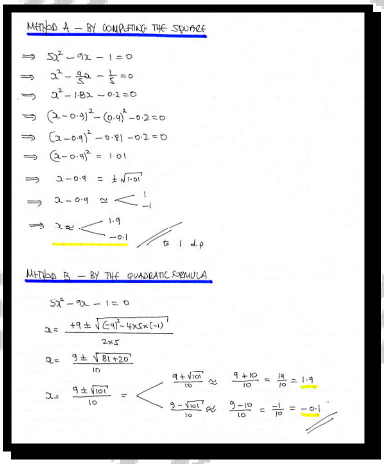
Solve, without the use of any calculating aid, the quadratic equation

$$5x^2 - 9x - 1 = 0,$$

giving the answers correct to one decimal place.

Detailed workings must be shown in this question.

$\boxed{x \approx -0.1 \;\cup\; x \approx 1.9}$



METHOD A – BY COMPLETING THE SQUARE

$$\Rightarrow 5x^2 - 9x - 1 = 0$$
$$\Rightarrow x^2 - \frac{9}{5}x - \frac{1}{5} = 0$$
$$\Rightarrow x^2 - 1.8x - 0.2 = 0$$
$$\Rightarrow (x-0.9)^2 - (0.9)^2 - 0.2 = 0$$
$$\Rightarrow (x-0.9)^2 - 0.81 - 0.2 = 0$$
$$\Rightarrow (x-0.9)^2 = 1.01$$
$$\Rightarrow x - 0.9 = \pm\sqrt{1.01}$$
$$\Rightarrow x - 0.9 \simeq \begin{cases} 1 \\ -1 \end{cases}$$
$$\Rightarrow x \approx \begin{cases} 1.9 \\ -0.1 \end{cases} \quad \text{to 1 d.p}$$

METHOD B – BY THE QUADRATIC FORMULA

$$5x^2 - 9x - 1 = 0$$
$$x = \frac{+9 \pm \sqrt{(-9)^2 - 4\times5\times(-1)}}{2\times5}$$
$$x = \frac{9 \pm \sqrt{81+20}}{10}$$
$$x = \frac{9 \pm \sqrt{101}}{10} = \begin{cases} \dfrac{9+\sqrt{101}}{10} \approx \dfrac{9+10}{10} = \dfrac{19}{10} = 1.9 \\[2mm] \dfrac{9-\sqrt{101}}{10} \approx \dfrac{9-10}{10} = \dfrac{-1}{10} = -0.1 \end{cases}$$

Question 63 (***+)

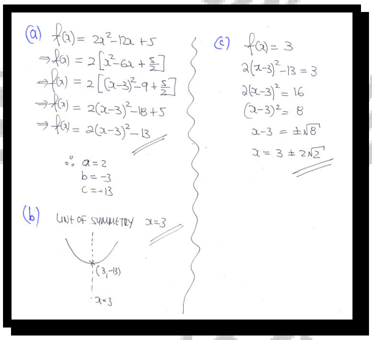
$$f(x) = 2x^2 - 12x + 5, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = A(x+B)^2 + C$, where A, B and C are integer constants.

b) State the line of symmetry of $f(x)$.

c) Solve the equation $f(x) = 3$, giving the answers in the form $p \pm q\sqrt{2}$, where p and q are constants.

$\boxed{f(x) = 2(x-3)^2 - 13}$, $\boxed{x = 3}$, $\boxed{x = 3 \pm 2\sqrt{2}}$


(a) $f(x) = 2x^2 - 12x + 5$
$\Rightarrow f(x) = 2\left[x^2 - 6x + \frac{5}{2}\right]$
$\Rightarrow f(x) = 2\left[(x-3)^2 - 9 + \frac{5}{2}\right]$
$\Rightarrow f(x) = 2(x-3)^2 - 18 + 5$
$\Rightarrow f(x) = 2(x-3)^2 - 13$
$\therefore a = 2$
$b = -3$
$c = -13$

(b) LINE OF SYMMETRY $x = 3$

(c) $f(x) = 3$
$2(x-3)^2 - 13 = 3$
$2(x-3)^2 = 16$
$(x-3)^2 = 8$
$x - 3 = \pm\sqrt{8}$
$x = 3 \pm 2\sqrt{2}$

Question 64 (***+)

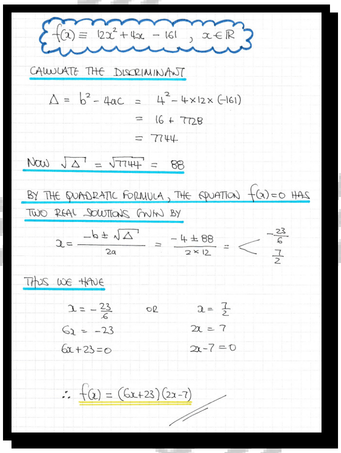
A quadratic curve has equation

$$f(x) \equiv 12x^2 + 4x - 161, \ x \in \mathbb{R}.$$

Express the above equation as the product of two linear factors.

A detailed method must be shown in this question.

$\boxed{}$, $\boxed{f(x) \equiv (6x+23)(2x-7)}$



Question 65 (****)

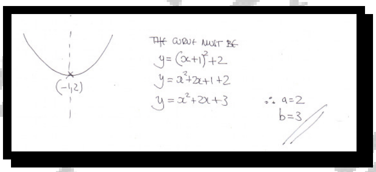
The curve C has equation

$$y = x^2 + ax + b,$$

where a and b are non zero constants.

Given that C has a minimum at $(-1, 2)$, determine the value of a and the value of b.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=3}$



Question 66 (****)

$$f(x) = 3x^2 + 5x - 2, \ x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Sketch the graph of $f(x)$.

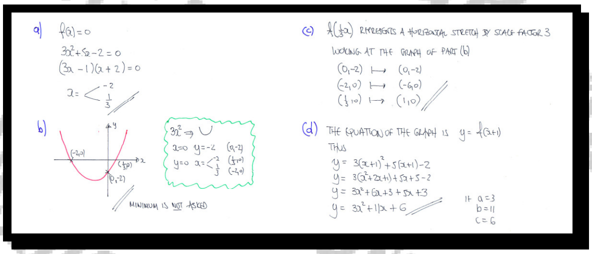
The sketch must include the coordinates of any points where the graph of $f(x)$ meets the coordinate axes.

c) Find the coordinates of any points where the graph of the curve with equation $y = f\left(\frac{1}{3}x\right)$ meets the coordinate axes.

The graph of $y = f(x)$ is translated by 1 unit in the negative x direction onto the graph of the curve with equation $y = ax^2 + bx + c$, where a, b and c are constants.

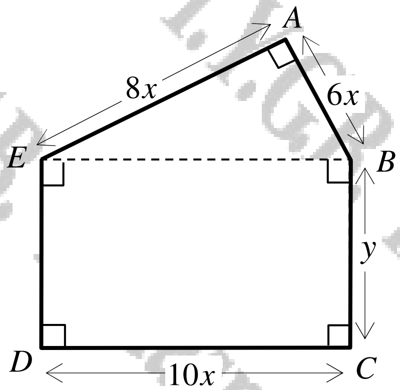
d) Determine the value of a, b and c.

$\boxed{\quad}$, $\boxed{x = -2, \ x = \frac{1}{3}}$, $\boxed{(-2,0), \left(\frac{1}{3},0\right), (0,-2)}$, $\boxed{(-6,0), (1,0), (0,-2)}$, $\boxed{a = 3, \ b = 11, \ c = 6}$



Question 67 (****)

The figure below shows a pentagon $ABCDE$ whose measurements, in cm, are given in terms of x and y.



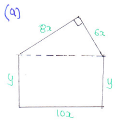
a) If the perimeter of the pentagon is 120 cm, show its area A cm^2 is given by

$$A = 600x - 96x^2.$$

b) Determine, without the use of calculus, the maximum value for the area of the pentagon and the corresponding value of x which produces this maximum area.

$\boxed{x = 3.125}$, $\boxed{A_{max} = 937.5}$

(a)



$P = 120$

$\Rightarrow 2y + 24x = 120$

$\Rightarrow \boxed{y + 12x = 60}$ ⊛

$A = (10xy) + \frac{1}{2}(8x)(6x)$

$A = 10xy + 24x^2$

$A = 10x(60 - 12x) + 24x^2$

$A = 600x - 120x^2 + 24x^2$

$A = 600x - 96x^2$

AS REQUIRED

(b) $A = -96x^2 + 600x$

$\Rightarrow A = -96\left[x^2 - \frac{600}{96}x\right]$

$\Rightarrow A = -96\left[x^2 - \frac{25}{4}x\right]$

$\Rightarrow A = -96\left[\left(x - \frac{25}{8}\right)^2 - \frac{625}{64}\right]$

$\Rightarrow A = -96\left(x - \frac{25}{8}\right)^2 + \frac{1875}{2}$

$\Rightarrow A = \frac{1875}{2} - 96\left(x - \frac{25}{8}\right)^2$

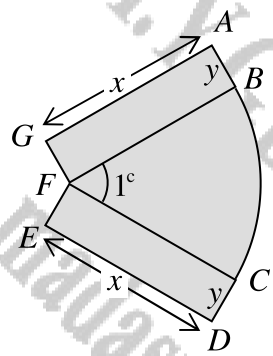
$\Rightarrow A = 937.5 - 96(x - 3.125)^2$

$\therefore A_{MAX} = 937.5$

$x_{MAX} = 3.125$

Question 68 (****)

The figure below shows a clothes design consisting of two identical rectangles attached to either straight side of a circular sector of radius x cm.



The rectangles measure x cm by y cm and the circular sector subtends an angle of one radian at the centre.

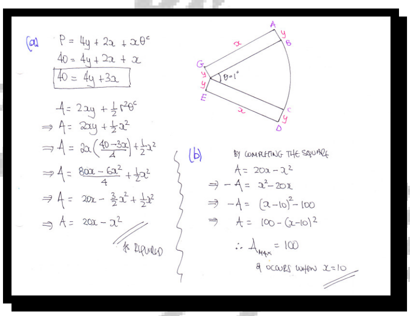
The perimeter of the design is 40 cm.

a) Show that the area, A cm^2, of the design is given by

$$A = 20x - x^2.$$

b) Determine, without the use of calculus, the maximum value for the area of the design and the corresponding value of x which produces this maximum area.

$\boxed{x_{\text{max}} = 10}$, $\boxed{A_{\text{max}} = 100}$



Question 69 (****)

$$f(x) = x^2 - 2x - 8, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $f(x) = (x+a)^2 + b$, where a and b are integers.

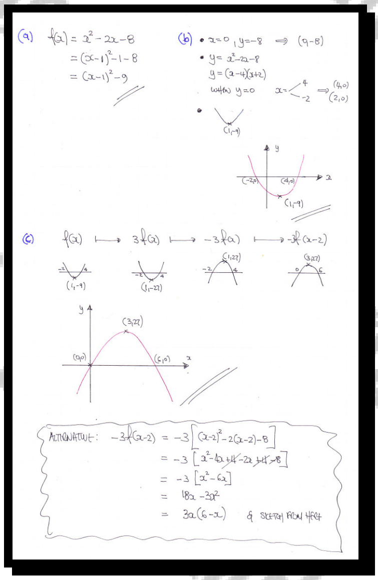
b) Sketch the graph of $f(x)$.

a) By considering a series of three geometrical transformations, sketch the graph of $y = -3f(x-2)$.

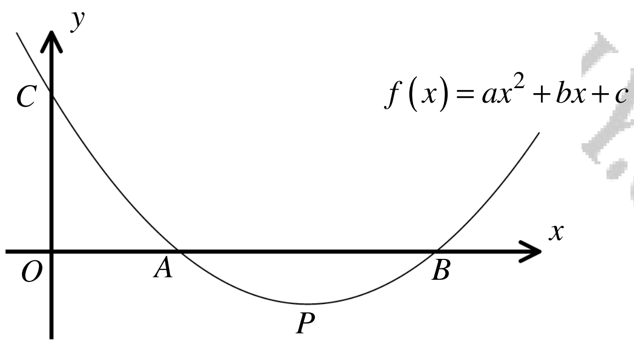
Both sketches must include the coordinates of …

- … all the points where the curves meets the coordinate axes.
- … the minimum or maximum points of the curves.

$\boxed{a=-1}$, $\boxed{b=-9}$



Question 70 (****)



The figure above shows the graph of the curve with equation

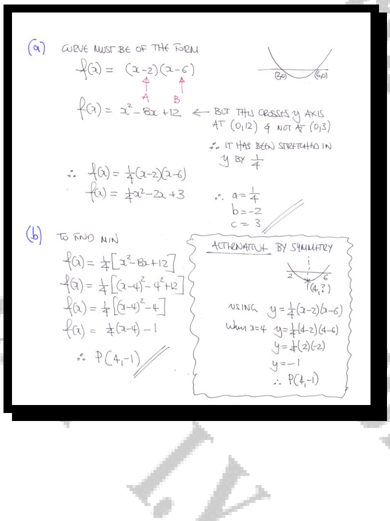
$$f(x) = ax^2 + bx + c, \; x \in \mathbb{R}.$$

The graph meets the axes at $A(2,0)$, $B(6,0)$ and $C(0,3)$, and has a minimum at P.

a) Determine the value of a, b and c.

b) Find the coordinates of P.

$\boxed{a = \frac{1}{4}}$, $\boxed{b = -2}$, $\boxed{c = 3}$, $\boxed{P(4,-1)}$



Question 71 (****)

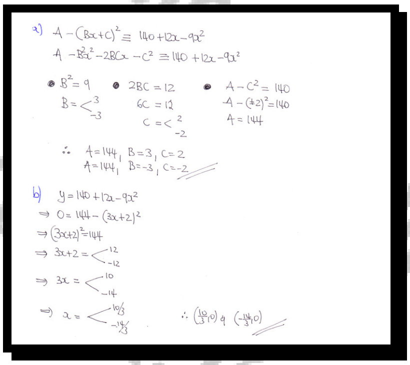
$$A-(Bx+C)^2 \equiv 140+12x-9x^2, \quad x \in \mathbb{R}.$$

a) Find the value of each of the constants A, B and C in the above identity.

b) Hence or otherwise determine the x intercepts of the curve with equation

$$y=140+12x-9x^2, \quad x \in \mathbb{R}.$$

$\boxed{A=144,\ B=\pm 3,\ C=\pm 2}$, $\boxed{\left(\frac{10}{3},0\right),\ \left(-\frac{14}{3},0\right)}$



a) $A-(Bx+C)^2 \equiv 140+12x-9x^2$

$A-B^2x^2-2BCx-C^2 \equiv 140+12x-9x^2$

- $B^2=9$, $B=\pm 3$
- $2BC=12$, $6C=12$, $C=\pm 2$
- $A-C^2=140$, $A-(\pm 2)^2=140$, $A=144$

$\therefore$ $A=144$, $B=3$, $C=2$; $A=144$, $B=-3$, $C=-2$

b) $y=140+12x-9x^2$

$\Rightarrow 0=144-(3x+2)^2$

$\Rightarrow (3x+2)^2=144$

$\Rightarrow 3x+2=12$ or -12

$\Rightarrow 3x=10$ or -14

$\Rightarrow x=10/3$ or $-14/3$ $\quad \therefore \left(\frac{10}{3},0\right)$ & $\left(-\frac{14}{3},0\right)$

Question 72 **(****)**

The curve C has equation

$$f(x) = (x-a)(x+b), \; x \in \mathbb{R},$$

where a and b are constants such that $a > b > 0$.

Sketch, in separate sets of axes, the graph of ...

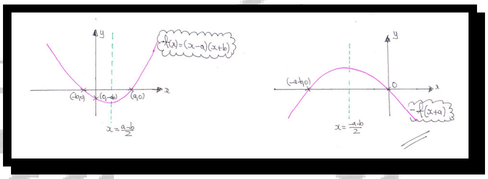
a) … $y = f(x)$.

b) … $y = -f(x+a)$.

Each of the graphs must show clearly …

- … the coordinates of any points where the curve meets the coordinates axes.
- … the equation of the line of symmetry of the curve.

graph



Question 73 (****)

In case of an emergency, the typical stopping distance of a car, y metres, when travelling at a speed x miles per hour is given by

$$y = ax^2 + bx + c,$$

where a, b and c are constants.

A typical car takes

- 12 metres to stop if travelling at 20 miles per hour.
- 23 metres to stop if travelling at 30 miles per hour.
- 36 metres to stop if travelling at 40 miles per hour.

a) Determine the value of a, b and c.

b) Find the speed of car that has a total stopping distance of 183 metres.

(you may find the fact $11 \times 17 = 187$ useful in this part.)

$\boxed{a = \frac{1}{100}}$, $\boxed{b = \frac{3}{5}}$, $\boxed{c = -4}$, $\boxed{x = 110}$

(a) $y = ax^2 + bx + c$

$$\left.\begin{array}{l}(20,12) \Rightarrow 12 = 400a + 20b + c \\ (30,23) \Rightarrow 23 = 900a + 30b + c \\ (40,36) \Rightarrow 36 = 1600a + 40b + c\end{array}\right\} \Rightarrow \boxed{c = 12 - 400a - 20b}$$

Thus:

$$\left.\begin{array}{l}23 = 900a + 30b + 12 - 400a - 20b \\ 36 = 1600a + 40b + 12 - 400a - 20b\end{array}\right\} \Rightarrow \begin{array}{l}11 = 500a + 10b \\ 24 = 1200a + 20b\end{array}$$

Hence:

$\boxed{10b = 11 - 500a}$

So
$$24 = 1200a + 2(11 - 500a)$$
$$24 = 1200a + 22 - 1000a$$
$$2 = 200a$$
$$a = \frac{1}{100}$$

- $10b = 11 - 500a$
  $10b = 11 - 5$
  $10b = 6$
  $b = \frac{3}{5}$
- $c = 12 - 400a - 20b$
  $c = 12 - 4 - 12$
  $c = -4$

(b) Now $y = \frac{1}{100}x^2 + \frac{3}{5}x - 4$

$$183 = \frac{1}{100}x^2 + \frac{3}{5}x - 4$$
$$18300 = x^2 + 60x - 400$$
$$0 = x^2 + 60x - 18700$$
$$0 = (x - 110)(x + 170)$$
$$x = \begin{cases} 110 \\ \cancel{-170} \end{cases}$$

$\therefore 110 \text{ km h}^{-1}$

Question 74 (****+)

The curve C has equation

$$y = 4x^2 + 24x + A,$$

where A is a non zero constant.

a) Express y in the form $p(x+q)^2 + r$, where p, q and r are constants.

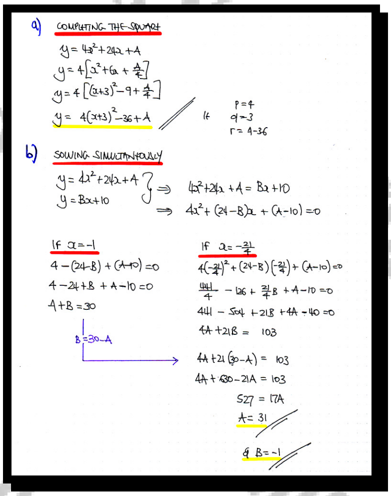
The straight line L has equation

$$y = Bx + 10,$$

where B is a non zero constant.

b) Given that C and L meet at the points with $x = -1$ and $x = -\frac{21}{4}$, determine the value of A and the value of B.

$\boxed{\quad}$, $\boxed{y = 4(x+3)^2 - 36 + A}$, $\boxed{A = 31, B = -1}$



a) COMPLETING THE SQUARE

$y = 4x^2 + 24x + A$

$y = 4\left[x^2 + 6x + \frac{A}{4}\right]$

$y = 4\left[(x+3)^2 - 9 + \frac{A}{4}\right]$

$y = 4(x+3)^2 - 36 + A$

if $p = 4$, $q = 3$, $r = A - 36$

b) SOLVING SIMULTANEOUSLY

$\left.\begin{array}{l} y = 4x^2 + 24x + A \\ y = Bx + 10 \end{array}\right\} \Rightarrow 4x^2 + 24x + A = Bx + 10$

$\Rightarrow 4x^2 + (24 - B)x + (A - 10) = 0$

IF $x = -1$

$4 - (24 - B) + (A - 10) = 0$

$4 - 24 + B + A - 10 = 0$

$A + B = 30$

$B = 30 - A$

IF $x = -\frac{21}{4}$

$4\left(-\frac{21}{4}\right)^2 + (24 - B)\left(-\frac{21}{4}\right) + (A - 10) = 0$

$\frac{441}{4} - 126 + \frac{21}{4}B + A - 10 = 0$

$441 - 504 + 21B + 4A - 40 = 0$

$4A + 21B = 103$

$4A + 21(30 - A) = 103$

$4A + 630 - 21A = 103$

$527 = 17A$

$A = 31$

& $B = -1$

Question 75 (****+)

The sum of £840 is to be shared equally amongst n qualifying individuals.

It was later found that 6 of those n individuals did not actually qualify so the share of the rest increased by £45.

Find the value of n.

$\boxed{}$, $\boxed{n = 14}$

SETTING UP AN EQUATION AS FOLLOWS

"SHARE BEFORE" $= \frac{840}{n}$

"SHARE AFTER" $= \frac{840}{n-6}$

HENCE WE NOW HAVE

DIFFERENCE IN THE SHARES $= 45$

$$\frac{840}{n-6} - \frac{840}{n} = 45$$

$\div 15$

$$\frac{56}{n-6} - \frac{56}{n} = 3$$

$\times n(n-6)$

$$56n - 56(n-6) = 3n(n-6)$$

$$\cancel{56n} - \cancel{56n} + 336 = 3n^2 - 18n$$

$$0 = 3n^2 - 18n - 336$$

$\div 3$

$$0 = n^2 - 6n - 112$$

BY INSPECTION, QUADRATIC FORMULA OR COMPLETING THE SQUARE

$$(n+8)(n-14) = 0$$

$$n = \begin{cases} 14 \\ \cancel{-8} \end{cases}$$

$$\therefore n = 14$$

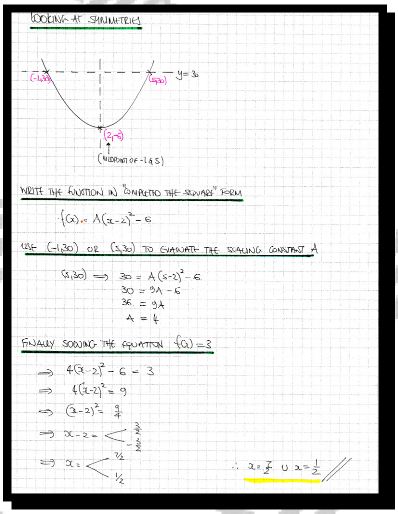
Question 76 (****+)

$$f(x) = ax^2 + bx + c,$$

where a, b and c are non zero constants.

Given that $f(-1) = f(5) = 30$ and that the minimum value of $f(x)$ is -6, solve the equation $f(x) = 3$.

$\boxed{}$, $\boxed{x = \frac{1}{2} \cup x = \frac{7}{2}}$



LOOKING AT SYMMETRIES

$y=30$, $(-1,30)$, $(5,30)$, $(2,-6)$

(MIDPOINT OF -1 & 5)

WRITE THE FUNCTION IN "COMPLETED THE SQUARE" FORM

$f(x) = A(x-2)^2 - 6$

USE $(-1,30)$ OR $(5,30)$ TO EVALUATE THE SCALING CONSTANT A

$(5,30) \Rightarrow 30 = A(5-2)^2 - 6$

$30 = 9A - 6$

$36 = 9A$

$A = 4$

FINALLY SOLVING THE EQUATION $f(x) = 3$

$\Rightarrow 4(x-2)^2 - 6 = 3$

$\Rightarrow 4(x-2)^2 = 9$

$\Rightarrow (x-2)^2 = \frac{9}{4}$

$\Rightarrow x - 2 = \frac{3}{2}, -\frac{3}{2}$

$\Rightarrow x = \frac{7}{2}, \frac{1}{2}$

$\therefore x = \frac{7}{2} \cup x = \frac{1}{2}$

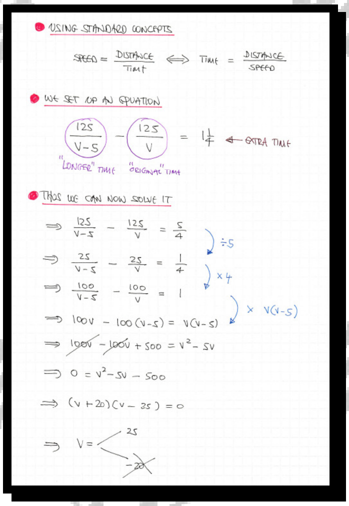
Question 77 (****+)

A cyclist travelling at **constant** speed V km/h covers a distance of 125 km.

If he was to decrease his speed by 5 km/h it would have taken him an extra $1\frac{1}{4}$ hours to cover the same distance.

Find the value of V.

$\boxed{}$, $\boxed{V = 25}$



USING STANDARD CONCEPTS

$$\text{SPEED} = \frac{\text{DISTANCE}}{\text{TIME}} \Longleftrightarrow \text{TIME} = \frac{\text{DISTANCE}}{\text{SPEED}}$$

WE SET UP AN EQUATION

$$\underbrace{\frac{125}{V-5}}_{\text{"LONGER" TIME}} - \underbrace{\frac{125}{V}}_{\text{"ORIGINAL" TIME}} = 1\tfrac{1}{4} \leftarrow \text{EXTRA TIME}$$

THUS WE CAN NOW SOLVE IT

$$\Rightarrow \frac{125}{V-5} - \frac{125}{V} = \frac{5}{4}$$

÷5

$$\Rightarrow \frac{25}{V-5} - \frac{25}{V} = \frac{1}{4}$$

×4

$$\Rightarrow \frac{100}{V-5} - \frac{100}{V} = 1$$

× V(V−5)

$$\Rightarrow 100V - 100(V-5) = V(V-5)$$

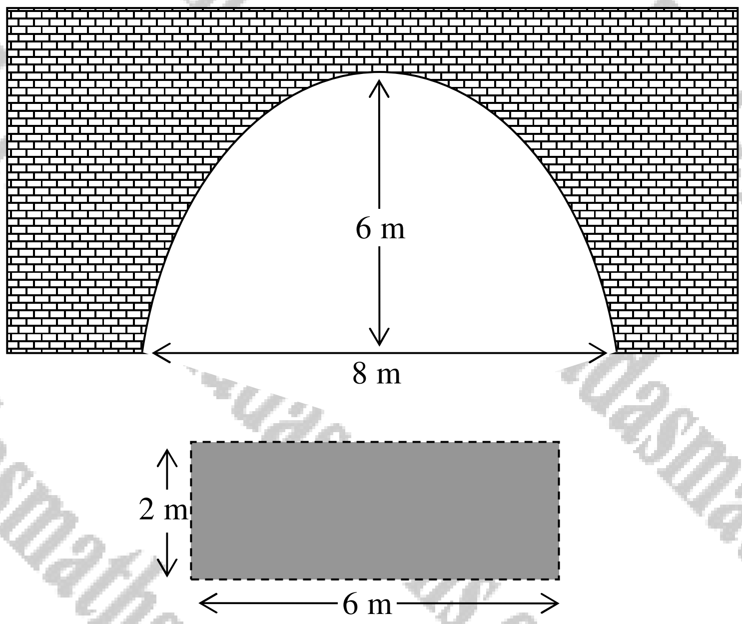
$$\Rightarrow \cancel{100V} - \cancel{100V} + 500 = V^2 - 5V$$

$$\Rightarrow 0 = V^2 - 5V - 500$$

$$\Rightarrow (V+20)(V-25) = 0$$

$$\Rightarrow V = 25 \text{ or } \cancel{-20}$$

Question 78 (****+)

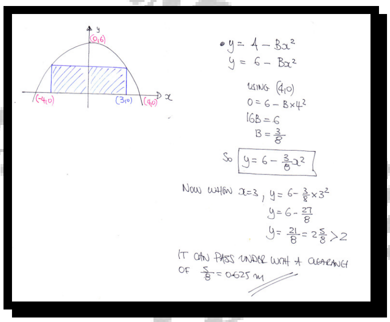


The figure above shows the parabolic arch under a railway bridge.

The width of the arch at its lowest level is 8 metres and the highest point of the arch is 6 metres from the ground.

Determine, showing a clear algebraic method whether a lorry with a wide load of width 6 metres and height 2 metres can pass through this parabolic arch.

[ ] , it does with a clearance of height of 0.625 metres



$y = A - Bx^2$

$y = 6 - Bx^2$

USING $(4,0)$

$0 = 6 - B \times 4^2$

$16B = 6$

$B = \frac{3}{8}$

So $\boxed{y = 6 - \frac{3}{8}x^2}$

NOW WHEN $x = 3$, $y = 6 - \frac{3}{8} \times 3^2$

$y = 6 - \frac{27}{8}$

$y = \frac{21}{8} = 2\frac{5}{8} > 2$

IT CAN PASS UNDER WITH A CLEARANCE OF $\frac{5}{8} = 0.625$ m

Question 79 (****+)

$$f(x) \equiv x^2 - 10x + 50, \; x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2 + b$, where a and b are constants.

b) Hence write down the minimum value of $f(x)$.

The point A has coordinates $(20,-3)$.

The variable point B lies on the straight line with equation

$$y = 3x - 13.$$

c) Show clearly that

$$|AB|^2 = 10x^2 - 100x + 500.$$

d) Use parts (a) and (b) to determine the shortest distance between A and B.

e) Hence write down the coordinates of B when the distance between A and B is shortest.

$\boxed{}$, $\boxed{f(x) \equiv (x-5)^2 + 25}$, $\boxed{f(x)_{\text{min}} = 25}$, $\boxed{|AB|_{\text{min}} = 5\sqrt{10}}$, $\boxed{B(5,2)}$

a) $f(x) = x^2 - 10x + 50 = (x-5)^2 - 25 + 50 = (x-5)^2 + 25$

b) $f(x)_{\text{MIN}}$ IS 25

c) $A(20,-3)$ $\quad B(x, 3x-13)$

$\Rightarrow |AB| = \sqrt{(3x-13+3)^2 + (x-20)^2}$

$\Rightarrow |AB| = \sqrt{(3x-10)^2 + (x-20)^2}$

$\Rightarrow |AB|^2 = (3x-10)^2 + (x-20)^2$

$\Rightarrow |AB|^2 = 9x^2 - 0x + 100 + x^2 - 40x + 400$

$\Rightarrow |AB|^2 = 10x^2 - 100x + 500$

d) $|AB|^2 = 10\left[x^2 - 10x + 50\right]$

$\therefore |AB|^2_{\text{MIN}} = 10 \times 25$

$|AB|_{\text{MIN}} = \sqrt{10 \times 25} = 5\sqrt{10}$

e) MIN OCCURS WHEN $x = 5$ (FROM COMPLETING THE SQUARE)

$\therefore B(5,2)$

Question 80 (****+)

A quadratic equation has two real roots differing by k, where k is a positive constant.

Determine, in terms of k, an exact simplified expression for the discriminant of this quadratic.

You may assume that the coefficient of the quadratic term of the equation is one.

$\boxed{}$, $\boxed{b^2 - 4ac = k^2}$

LET THE TWO ROOTS BE α & $\alpha + k$

THEN
$$(x-\alpha)[x-(\alpha+k)] = 0$$
$$x^2 - (\alpha+k)x - \alpha x + \alpha(\alpha+k) = 0$$
$$x^2 - (2\alpha+k)x + (\alpha^2 + k\alpha) = 0$$

THUS $b = -(2\alpha+k)$ & $c = \alpha^2 + \alpha k$

$$\Rightarrow b^2 - 4ac = [-(2\alpha+k)]^2 - 4 \times 1 \times (\alpha^2 + \alpha k)$$
$$= (2\alpha+k)^2 - 4(\alpha^2 + \alpha k)$$
$$= \cancel{4\alpha^2} + \cancel{4\alpha k} + k^2 - \cancel{4\alpha^2} - \cancel{4\alpha k}$$
$$= k^2$$

Question 81 (****+)

A quadratic curve has equation

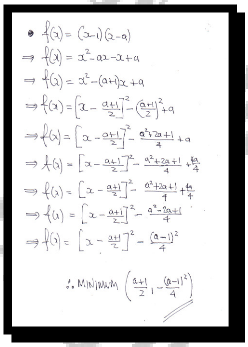
$$f(x)=(x-1)(x-a),$$

where a is a constant.

Show, **without** a calculus method, that the coordinates of the minimum point of the curve are

$$\left(\frac{a+1}{2},-\frac{(a-1)^2}{4}\right).$$

C1, proof



- $f(x)=(x-1)(x-a)$

$\Rightarrow f(x)=x^2-ax-x+a$

$\Rightarrow f(x)=x^2-(a+1)x+a$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\left(\frac{a+1}{2}\right)^2+a$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\frac{a^2+2a+1}{4}+a$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\frac{a^2+2a+1}{4}+\frac{4a}{4}$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\frac{a^2+2a+1}{4}+\frac{4a}{4}$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\frac{a^2-2a+1}{4}$

$\Rightarrow f(x)=\left[x-\frac{a+1}{2}\right]^2-\frac{(a-1)^2}{4}$

$\therefore$ MINIMUM $\left(\frac{a+1}{2},-\frac{(a-1)^2}{4}\right)$

Question 82 (****+)

The point P has coordinates $(0,2)$.

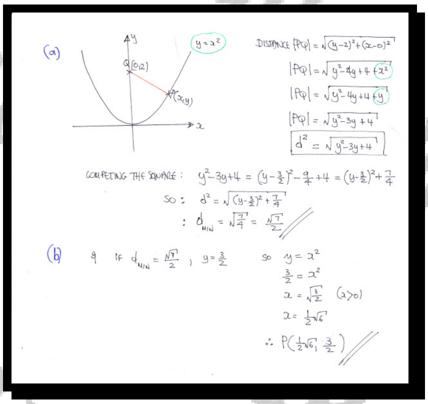
The point Q, with $x>0$, lies on the curve with equation $y=x^2$.

Use a **non calculus** algebraic method to find …

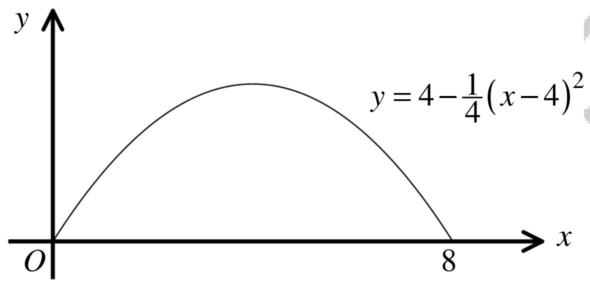
a) … the shortest distance between P and Q

b) … the coordinates of Q.

$$\boxed{d_{\min}=\frac{1}{2}\sqrt{7}}, \quad \boxed{Q\left(\frac{\sqrt{6}}{2},\frac{3}{2}\right)}$$



Question 83 (****+)



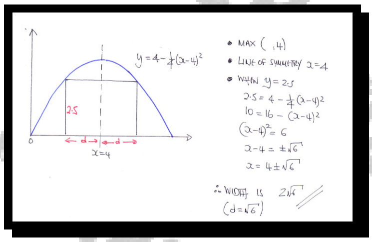
The figure above shows the cross section of a tunnel modelled by the parabolic arc with equation

$$y = 4 - \frac{1}{4}(x-4)^2, \; 0 \le x \le 8 .$$

A wide lorry load whose cross section is modelled as a rectangle of height 2.5 metres can just pass through this tunnel.

Given that 1 unit on the graph represents 1 metre, determine the width of the lorry load, giving the answer in exact surd form.

$\boxed{}$, $\boxed{\text{width} = 2\sqrt{6}}$



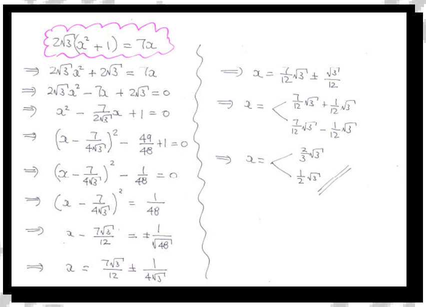
Question 84 (****+)

Find the solutions of the quadratic equation

$$2\sqrt{3}\left(x^2+1\right)=7x.$$

Give the answers in the form $k\sqrt{3}$, where k is a constant.

$\boxed{}$, $\boxed{x=\frac{2}{3}\sqrt{3},\ x=\frac{1}{2}\sqrt{3}}$



$2\sqrt{3}(x^2+1)=7x$

$\Rightarrow 2\sqrt{3}x^2+2\sqrt{3}=7x$

$\Rightarrow 2\sqrt{3}x^2-7x+2\sqrt{3}=0$

$\Rightarrow x^2-\frac{7}{2\sqrt{3}}x+1=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2-\frac{49}{48}+1=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2-\frac{1}{48}=0$

$\Rightarrow \left(x-\frac{7}{4\sqrt{3}}\right)^2=\frac{1}{48}$

$\Rightarrow x-\frac{7\sqrt{3}}{12}=\pm\frac{1}{\sqrt{48}}$

$\Rightarrow x=\frac{7\sqrt{3}}{12}\pm\frac{1}{4\sqrt{3}}$

$\Rightarrow x=\frac{7}{12}\sqrt{3}\pm\frac{\sqrt{3}}{12}$

$\Rightarrow x=\frac{7}{12}\sqrt{3}+\frac{1}{12}\sqrt{3}$ or $\frac{7}{12}\sqrt{3}-\frac{1}{12}\sqrt{3}$

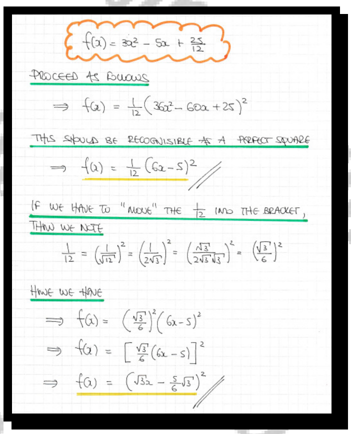
$\Rightarrow x=\frac{2}{3}\sqrt{3}$ or $\frac{1}{2}\sqrt{3}$

Question 85 (****+)

$$f(x)\equiv 3x^2-5x+\frac{25}{12},\quad x\in\mathbb{R}.$$

Factorize fully $f(x)$.

$\boxed{}$, $\boxed{f(x)=\frac{1}{12}(6x-5)^2 \quad \text{or} \quad f(x)=\left(\sqrt{3}\,x-\frac{5}{6}\sqrt{3}\right)^2}$



$f(x)=3x^2-5x+\frac{25}{12}$

PROCEED AS FOLLOWS

$\Rightarrow f(x)=\frac{1}{12}\left(36x^2-60x+25\right)^2$

THIS SHOULD BE RECOGNISIBLE AS A PERFECT SQUARE

$\Rightarrow f(x)=\frac{1}{12}(6x-5)^2$

IF WE HAVE TO "MOVE" THE $\frac{1}{12}$ INTO THE BRACKET, THEN WE NOTE

$\frac{1}{12}=\left(\frac{1}{\sqrt{12}}\right)^2=\left(\frac{1}{2\sqrt{3}}\right)^2=\left(\frac{\sqrt{3}}{2\sqrt{3}\sqrt{3}}\right)^2=\left(\frac{\sqrt{3}}{6}\right)^2$

HENCE WE HAVE

$\Rightarrow f(x)=\left(\frac{\sqrt{3}}{6}\right)^2(6x-5)^2$

$\Rightarrow f(x)=\left[\frac{\sqrt{3}}{6}(6x-5)\right]^2$

$\Rightarrow f(x)=\left(\sqrt{3}x-\frac{5}{6}\sqrt{3}\right)^2$

Question 86 **(****+)**

A quadratic curve has equation

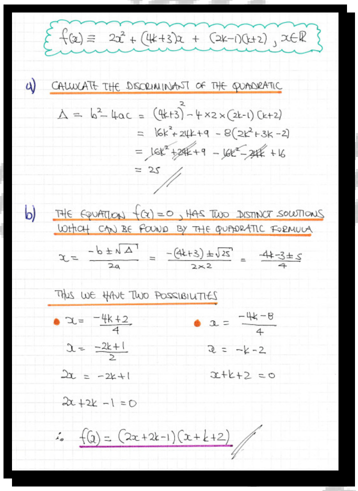
$$f(x) \equiv 2x^2 + (4k+3)x + (2k-1)(k+2), \ x \in \mathbb{R},$$

where k is a constant.

a) Evaluate the discriminant of $f(x)$.

b) Express $f(x)$ as the product of two linear factors.

$\boxed{}$, $\boxed{b^2 - 4ac = 25}$, $\boxed{f(x) \equiv (2x+2k-1)(x+k+2)}$



$f(x) \equiv 2x^2 + (4k+3)x + (2k-1)(k+2), \ x \in \mathbb{R}$

a) CALCULATE THE DISCRIMINANT OF THE QUADRATIC

$$\Delta = b^2 - 4ac = (4k+3)^2 - 4 \times 2 \times (2k-1)(k+2)$$
$$= 16k^2 + 24k + 9 - 8(2k^2 + 3k - 2)$$
$$= \cancel{16k^2} + \cancel{24k} + 9 - \cancel{16k^2} - \cancel{24k} + 16$$
$$= 25$$

b) THE EQUATION $f(x) = 0$, HAS TWO DISTINCT SOLUTIONS WHICH CAN BE FOUND BY THE QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(4k+3) \pm \sqrt{25}}{2 \times 2} = \frac{-4k - 3 \pm 5}{4}$$

THUS WE HAVE TWO POSSIBILITIES

- $x = \frac{-4k+2}{4}$

  $x = \frac{-2k+1}{2}$

  $2x = -2k+1$

  $2x + 2k - 1 = 0$

- $x = \frac{-4k-8}{4}$

  $x = -k-2$

  $x + k + 2 = 0$

$\therefore \ f(x) = (2x+2k-1)(x+k+2)$

Question 87 (*****)

A quadratic curve has equation

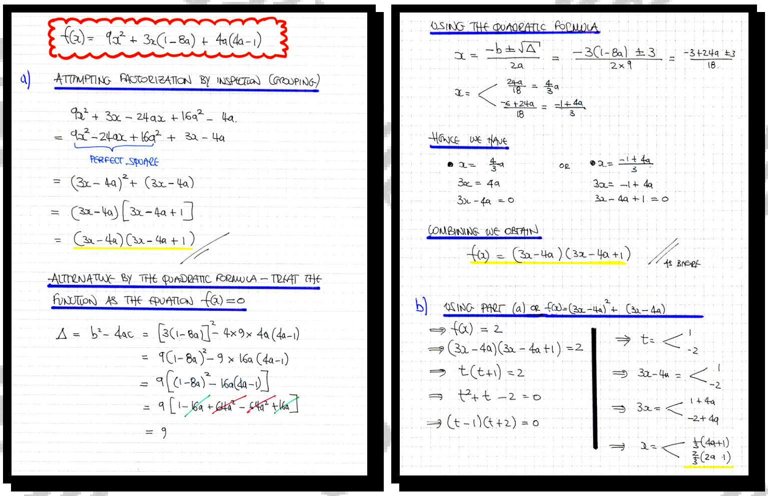
$$f(x) \equiv 9x^2 + 3(1-8a)x + 4a(4a-1), \ x \in \mathbb{R},$$

where a is a constant.

a) Express $f(x)$ as the product of two linear factors.

b) Solve the equation $f(x) = 2$, giving the answers in terms of a.

$\boxed{}$, $\boxed{f(x) \equiv (3x-4a)(3x-4a+1)}$, $\boxed{x = \frac{1}{3}(4a+1) \ \cup \ x = \frac{2}{3}(2a-1)}$



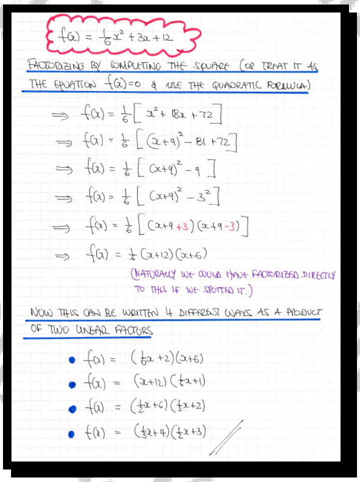
Question 88 (*****)

$$f(x) \equiv \frac{1}{6}x^2 + 3x + 12, \quad x \in \mathbb{R}.$$

Determine the four possible ways of expressing $f(x)$ as product of two linear factors.

$$\boxed{},$$

$$\boxed{f(x) = \left(\tfrac{1}{6}x+2\right)(x+6) = \left(\tfrac{1}{6}x+1\right)(x+12) = \left(\tfrac{1}{2}x+6\right)\left(\tfrac{1}{3}x+2\right) = \left(\tfrac{1}{2}x+3\right)\left(\tfrac{1}{3}x+4\right)}$$



$f(x) = \frac{1}{6}x^2 + 3x + 12$

FACTORIZING BY COMPLETING THE SQUARE (OR TREAT IT AS THE EQUATION $f(x)=0$ & USE THE QUADRATIC FORMULA)

$\Rightarrow f(x) = \frac{1}{6}\left[x^2 + 18x + 72\right]$

$\Rightarrow f(x) = \frac{1}{6}\left[(x+9)^2 - 81 + 72\right]$

$\Rightarrow f(x) = \frac{1}{6}\left[(x+9)^2 - 9\right]$

$\Rightarrow f(x) = \frac{1}{6}\left[(x+9)^2 - 3^2\right]$

$\Rightarrow f(x) = \frac{1}{6}\left[(x+9+3)(x+9-3)\right]$

$\Rightarrow f(x) = \frac{1}{6}(x+12)(x+6)$

(NATURALLY WE COULD HAVE FACTORIZED DIRECTLY TO THIS IF WE SPOTTED IT.)

NOW THIS CAN BE WRITTEN 4 DIFFERENT WAYS AS A PRODUCT OF TWO LINEAR FACTORS

- $f(x) = \left(\frac{1}{6}x+2\right)(x+6)$
- $f(x) = (x+12)\left(\frac{1}{6}x+1\right)$
- $f(x) = \left(\frac{1}{2}x+6\right)\left(\frac{1}{3}x+2\right)$
- $f(x) = \left(\frac{1}{3}x+4\right)\left(\frac{1}{2}x+3\right)$

Question 89 (*****)

A curve has equation

$$y = 2x^2 + 5x + c,$$

where c is a non zero constant.

Given that the roots of the equation differ by 3, determine the value of c.

$$\boxed{c = -\frac{11}{8}}$$

LET THE SMALLER ROOT OF THE QUADRATIC BE α

- THE SUM OF THE ROOTS: $\alpha + (\alpha+3) = -\frac{b}{a} = -\frac{5}{2}$

I.E $2\alpha + 3 = -\frac{5}{2}$

$2\alpha = -\frac{11}{2}$

$\alpha = -\frac{11}{4}$

- THE PRODUCT OF THE ROOTS: $\alpha(\alpha+3) = \frac{c}{a} = \frac{c}{2}$

I.E. $c = 2\alpha(\alpha+3)$

$c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4}+3\right)$

$c = -\frac{11}{2} \times \frac{1}{4}$

$c = -\frac{11}{8}$

ALTERNATIVE – WITHOUT USING DIRECTLY RESULTS ON THE SUM AND PRODUCT OF ROOTS OF A QUADRATIC

- LET THE SMALLER OF THE TWO ROOTS BE α

THEN $2x^2 + 5x + c = 0$

$\Rightarrow x^2 + \frac{5x}{2} + \frac{c}{2} = 0$

$\Rightarrow (x-\alpha)(x-(\alpha+3)) = 0$

$\Rightarrow x^2 - (\alpha+3)x - \alpha x + \alpha(\alpha+3) = 0$

$\Rightarrow x^2 - (2\alpha+3)x + \alpha(\alpha+3) = 0$

- BY COMPARISON WE HAVE

$\frac{5}{2} = -(2\alpha+3)$

$\Rightarrow 2\alpha + 3 = -\frac{5}{2}$

$\Rightarrow 4\alpha + 6 = -5$

$\Rightarrow 4\alpha = -11$

$\Rightarrow \alpha = -\frac{11}{4}$

&

$\frac{c}{2} = \alpha(\alpha+3)$

$\Rightarrow c = 2\alpha(\alpha+3)$

$\Rightarrow c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4}+3\right)$

$\Rightarrow c = -\frac{11}{2} \times \frac{1}{4}$

$\Rightarrow c = -\frac{11}{8}$

Question 90 (*****)

The function f is defined as

$$f(A,B) \equiv A^4 + 4B^4, \quad A \in \mathbb{R}, \quad B \in \mathbb{R}.$$

a) By completing the square, or otherwise, factorize f into 2 quadratic factors.

b) Hence factorize $x^4 + 64$.

$\boxed{\text{SYN-S}}$, $\boxed{A^4 + 4B^4 \equiv \left(A^2 - 2AB + 2B^2\right)\left(A^2 + 2AB + 2B^2\right)}$,

a) COMPLETING THE SQUARE AS FOLLOWS

BRACKET IS $x^2+2xy+y^2$

$$\begin{aligned} A^4 + 4B^4 &= (A^2)^2 + (2B^2)^2 \\ &= \left[(A^2)^2 + 2(A^2)(2B^2) + (2B^2)^2\right] - 2(A^2)(2B^2) \\ &= (A^2+2B^2)^2 - 4A^2B^2 \\ &= (A^2+2B^2)^2 - (2AB)^2 \\ &= (A^2+2B^2-2AB)(A^2+2B^2+2AB) \\ &= (A^2-2AB+2B^2)(A^2+2AB+2B^2) \end{aligned}$$

DIFFERENCE OF SQUARES

b) USING PART (a)

$$\begin{aligned} x^4 + 64 = x^4 + 4\times 16 &= x^4 + 4\times 2^4 \\ &= (x^2 - 2\times 2x + 2\times 2^2)(x^2 + 2\times 2x + 2\times 2^2) \\ &= (x^2-4x+8)(x^2+4x+8) \end{aligned}$$

Question 91 (*****)

A function has equation

$$f(x) = x^2 + 6x + 20 + k\left(x^2 - 3x - 12\right), \ x \in \mathbb{R},$$

where k is a non zero constant.

a) State the value of k if $f(x)$ represents a straight line.

b) Find the value of k if the equation $f(x) = 0$ two equal in magnitude roots, but of opposite signs.

c) Determine the value of k and the value of p, given that $f(x)$ has a maximum at $(2, p)$.

$\boxed{}$, $\boxed{k = -1}$, $\boxed{k = 2}$, $\boxed{k = 2 \quad \text{and} \quad p = 176}$

$\boxed{f(x) = x^2+6x+20+k(x^2-3x-12), \ x \in \mathbb{R}}$

a) BY INSPECTION $k = -1$ (SO THAT THE TERMS IN x^2 CANCEL)

b) $f(x) = (k+1)x^2 + (6-3k)x + (20-12k)$

FOR TWO REAL ROOTS EQUAL IN MAGNITUDE BUT OPPOSITE SIGN, THE COEFFICIENT OF x MUST BE ZERO.

$6 - 3k = 0$
$6 = 3k$
$k = 2$

c) IF $f(x)$ HAS A MAXIMUM AT $(2, p)$

$\Rightarrow p - A(x-2)^2 \equiv (k+1)x^2 + (6-3k)x + (20-12k)$

WHERE $A > 0$

$\Rightarrow p - Ax^2 + 4Ax - 4A \equiv (k+1)x^2 + (6-3k) + (20-12k)$
$\Rightarrow -Ax^2 + 4Ax + (p-4A) \equiv (k+1)x^2 + (6-3k) + (20-12k)$

- $-A = k+1$
- $4A = 6-3k$
- $p - 4A = 20 - 12k$

$\Rightarrow \boxed{A = -k-1} \Rightarrow 4(-k-1) = 6-3k$
$\Rightarrow -4k - 4 = 6 - 3k$
$\Rightarrow -10 = k$
$\Rightarrow k = -10$

HENCE $A = -k-1$
$A = -(-10)-1$
$A = 9$

AND $p - 4A = 20 - 12k$
$p - 4 \times 9 = 20 - 12(-10)$
$p - 36 = 20 + 120$
$p = 176$

$\therefore k = -10$ & $p = 176$

Question 92 (*****)

Solve the following quadratic equation

$$\left(\sqrt{3}-1\right)x^2 - 2\sqrt{3}x = 3 + 3\sqrt{3}.$$

Give one of the roots in the form $p+q\sqrt{3}$ and the other root in the form $r\sqrt{3}$, where p, q and r are integers.

$\boxed{x=-\sqrt{3}, \quad x = 3+2\sqrt{3}}$

$$(\sqrt{3}-1)x^2 - 2\sqrt{3}x = 3+3\sqrt{3}$$

$$\Rightarrow x^2 - \frac{2\sqrt{3}}{\sqrt{3}-1}x = \frac{3+3\sqrt{3}}{\sqrt{3}-1}$$

$$\Rightarrow x^2 - \frac{2\sqrt{3}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}x = \frac{(3+3\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$\Rightarrow x^2 - \frac{6+2\sqrt{3}}{2}x = \frac{3\sqrt{3}+3+9+3\sqrt{3}}{2}$$

$$\Rightarrow x^2 - (3+\sqrt{3})x = \frac{12+6\sqrt{3}}{2}$$

$$\Rightarrow x^2 - (3+\sqrt{3})x - (6+3\sqrt{3}) = 0$$

BY THE QUADRATIC FORMULA

$$x = \frac{3+\sqrt{3} \pm \sqrt{(3+\sqrt{3})^2 + 4\times 1\times(6+3\sqrt{3})}}{2\times 1}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm \sqrt{12+6\sqrt{3}+24+12\sqrt{3}}}{2}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm \sqrt{36+18\sqrt{3}}}{2}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm 3\sqrt{4+2\sqrt{3}}}{2}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm 3\sqrt{\sqrt{3}^2+1+2\times1\times\sqrt{3}}}{2}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm 3\sqrt{(1+\sqrt{3})^2}}{2}$$

$$\Rightarrow x = \frac{3+\sqrt{3} \pm 3(1+\sqrt{3})}{2}$$

$$\Rightarrow x = \begin{cases} \dfrac{3+\sqrt{3}+3+3\sqrt{3}}{2} = \dfrac{6+4\sqrt{3}}{2} \\ \dfrac{\cancel{3}+\sqrt{3}-\cancel{3}-3\sqrt{3}}{2} = \dfrac{-2\sqrt{3}}{2} \end{cases}$$

$$\Rightarrow x = \begin{cases} 3+2\sqrt{3} \\ -\sqrt{3} \end{cases}$$

Question 93 (*****)

The quadratic curve C, has equation

$$y = 4x - 2x^2 - \tfrac{1}{2}kx^2,$$

where k is a non zero constant.

Express y in the form

$$\frac{8}{f(k)} - \frac{1}{2}f(k)\left[x - \frac{4}{f(k)}\right]^2,$$

where $f(k)$ is a function to be found.

$\boxed{}$, $\boxed{f(k) = k + 4}$

$\boxed{y = 4x - 2x^2 - \tfrac{1}{2}kx^2}$

$\Rightarrow 2y = 8x - 4x^2 - kx^2$

$\Rightarrow -2y = kx^2 + 4x^2 - 8x$

$\Rightarrow -2y = (k+4)x^2 - 8x$

$\Rightarrow -\frac{2y}{k+4} = x^2 - \frac{8}{k+4}x$

$\Rightarrow -\frac{2y}{k+4} = \left(x - \frac{4}{k+4}\right)^2 - \frac{16}{(k+4)^2}$

$\Rightarrow -2y = (k+4)\left(x - \frac{4}{k+4}\right)^2 - \frac{16}{k+4}$

$\Rightarrow 2y = \frac{16}{k+4} - (k+4)\left[x - \frac{4}{k+4}\right]^2$

$\Rightarrow y = \frac{8}{k+4} - \frac{1}{2}(k+4)\left(x - \frac{4}{k+4}\right)^2$ //

i.e. $f(k) = k+4$

Question 94 (*****)

$$f(x) = b - (x-a)^2, \ x \in \mathbb{R}$$

$$g(x) = a + (x-b)^2, \ x \in \mathbb{R}.$$

The graph of $f(x)$ has a maximum at P and the graph of $g(x)$ has a minimum at Q, where P and Q are distinct points.

a) Given that $f(x)$ passes through Q, show that $g(x)$ passes through P.

b) Given further that $f(x)$ touches the x axis sketch both graphs in the same set of axes.

$\boxed{}$, $\boxed{\text{proof/graph}}$

a) $\boxed{\begin{array}{ll} f(x) = b-(x-a)^2 & \text{HAS MAX AT } P(a,b) \\ g(x) = a+(x-b)^2 & \text{HAS MAX AT } Q(b,a) \end{array}}$

- $f(x)$ PASSES THROUGH $Q(b,a)$ [GIVEN]

$\Rightarrow f(b) = a$

$\Rightarrow b - (b-a)^2 = a$

$\Rightarrow (b-a) = (b-a)^2$

- AS $b \neq a$, SINCE THIS WOULD HAVE IMPLIED THAT $P(a,a)$ AND $Q(a,a)$, WHICH WE ARE TOLD THEY ARE UNIQUE

$\Rightarrow 1 = b-a$

$\Rightarrow \boxed{b = a+1}$

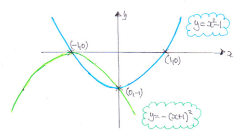
- NOW WE OBTAIN THROUGH $g(x)$

$g(a) = a + (a-b)^2 = a + (b-a)^2 = a+1 = b$

$\therefore g(a) = b$, SO $g(x)$ PASSES THROUGH $P(a,b)$

b) - IF $f(x)$ TOUCHES THE x AXIS $b=0$ & $a=-1$. $\Rightarrow \boxed{\begin{array}{l} f(x) = -(x+1)^2 \\ g(x) = x^2-1 \end{array}}$

- HENCE WE OBTAIN THE FOLLOWING GRAPH



Question 95 (*****)

Heron's formula for the area of a triangle asserts that

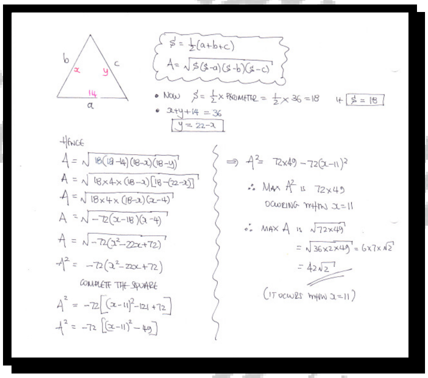
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}\,,$$

where a, b and c are the lengths of the 3 sides of the triangle and $s = \frac{1}{2}(a+b+c)$.

A given triangle has a perimeter of 36 cm and one of its sides is 14 cm.

Show with full justification that the largest area of this triangle is $42\sqrt{2}$ cm^2.

$\boxed{}$, $\boxed{\text{proof}}$



$s = \frac{1}{2}(a+b+c)$

$A = \sqrt{s(s-a)(s-b)(s-c)}$

- NOW $s = \frac{1}{2} \times$ PERIMETER $= \frac{1}{2} \times 36 = 18$ $\boxed{s = 18}$
- $x + y + 14 = 36$

$\boxed{y = 22 - x}$

HENCE

$A = \sqrt{18(18-14)(18-x)(18-y)}$

$A = \sqrt{18 \times 4 \times (18-x)[18-(22-x)]}$

$A = \sqrt{18 \times 4 \times (18-x)(x-4)}$

$A = \sqrt{-72(x-18)(x-4)}$

$A = \sqrt{-72(x^2-22x+72)}$

$A^2 = -72(x^2-22x+72)$

COMPLETE THE SQUARE

$A^2 = -72\left[(x-11)^2 - 121 + 72\right]$

$A^2 = -72\left[(x-11)^2 - 49\right]$

$\Rightarrow A^2 = 72 \times 49 - 72(x-11)^2$

$\therefore$ MAX A^2 IS 72×49

OCCURING WHEN $x = 11$

$\therefore$ MAX A IS $\sqrt{72 \times 49}$

$= \sqrt{36 \times 2 \times 49} = 6 \times 7 \times \sqrt{2}$

$= 42\sqrt{2}$

(IT OCCURS WHEN $x = 11$)

Question 96 (*****)

A mobile phone wholesaler buys a certain brand of phone for £35 a unit and sells it to shops for £100 a unit. In a typical week the wholesaler expects to sell 500 of these phones.

However research showed that on a typical week for every £1 reduced of the selling price of this phone an extra 20 sales can be achieved.

Let £P be the **weekly** profit of this brand phones and £x the **reduction** in the selling price from £100 .

a) Show clearly that

$$P = -20\left(x^2 - 40x - 1625\right).$$

b) Hence, or otherwise, determine the **selling** price for this phone if the weekly profit is to be maximized, and find this maximum weekly profit.

$\boxed{\text{£80, maximum profit £40500}}$

(a) LET THE PROFIT BE P

$P = 500 \times (100-35)$

$P = 520 \times (99-35)$

$P = 540 \times (98-35)$

SO IN GENERAL...

$P = (500+20x)[(100-x)-35]$

$P = (500+20x)(65-x)$

$P = 20(x+25)(65-x)$

$P = -20(x+25)(x-65)$

$P = -20(x^2-40x-1625)$

AS REQUIRED

(b) BY COMPLETING THE SQUARE

$P = -20\left[(x-20)^2 - 400 - 1625\right]$

$P = -20\left[(x-20)^2 - 2025\right]$

$P = 40500 - 20(x-20)^2$

∴ MAX P IS 40500 AND OCCURS WHEN $x=20$

∴ MAX PROFIT OF £40500 WHEN SELLING AT £80 EACH

Question 97 (*****)

The quadratic curve C with equation

$$y = x^2 - 6x + c,$$

passes through the points with coordinates (a,b), (b,a) and $(-a,27)$, where a, b and c are constants.

Find an equation for C, given that …

i. … $a = b$.

ii. … $a \neq b$.

$\boxed{\text{P-Q}}$, $\boxed{y = x^2 - 6x + \frac{1728}{169}}$, $\boxed{y = x^2 - 6x + 11}$

- START BY USING EACH OF THE THREE POINTS GIVEN TO FORM EQUATIONS
  - $(a,b) \Rightarrow b = a^2 - 6a + c$ — (I)
  - $(b,a) \Rightarrow a = b^2 - 6b + c$ — (II)
  - $(-a,27) \Rightarrow 27 = a^2 + 6a + c$ — (III)
- SUBTRACTING THE FIRST TWO EQUATIONS YIELDS

$\Rightarrow b - a = a^2 - b^2 - 6a + 6b$

$\Rightarrow b - a = (a-b)(a+b) - 6(a-b)$

$\Rightarrow (a-b)(a+b) - 6(a-b) + (a-b) = 0$

$\Rightarrow \boxed{(a-b)(a+b) - 5(a-b) = 0}$ — (IV)

- NOW WE NEED TO CONSIDER THE TWO CASES SEPARATELY

$\boxed{\text{IF } a = b}$

- THE FIRST TWO EQUATIONS BECOME

(I): $a = a^2 - 6a + c$

(III): $27 = a^2 + 6a + c$

- SUBTRACTING "UPWARDS"

$\Rightarrow 27 - a = 12a$

$\Rightarrow 13a = 27$

$\Rightarrow a = \frac{27}{13}$

- AND $c = 7a - a^2$

$\Rightarrow c = 7\left(\frac{27}{13}\right) - \left(\frac{27}{13}\right)^2$

$\Rightarrow c = \frac{27}{13}\left[7 - \frac{27}{13}\right]$

$\Rightarrow c = \frac{27}{13}\left[\frac{7 \times 13 - 27}{13}\right]$

$\Rightarrow c = \frac{27}{169}\left[91 - 27\right]$

$\Rightarrow c = \frac{27}{169} \times 64$

$$\begin{array}{r} 27 \\ \times\ 64 \\ \hline 108 \\ 162 \\ \hline 1728 \end{array}$$

$\Rightarrow c = \frac{1728}{169}$

$\therefore y = x^2 - 6x + \frac{1728}{169}$

$\boxed{\text{IF } a \neq b}$

- EQUATION (IV) CAN NOW BE DIVIDED

$\Rightarrow (a-b)(a+b) = 5(a-b)$

$\Rightarrow a + b = 5$

- THE FIRST TWO EQUATIONS NOW GIVE

(I) $b = a^2 - 6a + c$

$5 - a = a^2 - 6a + c$

$\left.\begin{array}{l} a^2 - 5a - 5 + c = 0 \\ \text{(III)}\ a^2 + 6a - 27 + c = 0 \end{array}\right\}$ SUBTRACT

$\therefore 11a - 22 = 0$

$11a = 22$

$a = 2$

(& SINCE $a + b = 5$

$b = 3$)

AND FINALLY

$a^2 + 6a - 27 + c = 0$

$4 + 12 - 27 + c = 0$

$c = 11$

$\therefore y = x^2 - 6x + 11$

Question 98 (*****)

A curve C, has equation

$$(x-1)y^2 - 2xy + x = 0, \; x \geq 0.$$

By completing the square in the above equation, express y in terms of x.

$\boxed{}$, $\boxed{y = \dfrac{\sqrt{x}}{1+\sqrt{x}}}$

$(x-1)y^2 - 2xy + x = 0$

$\Rightarrow y^2 - \frac{2x}{x-1}y + \frac{x}{x-1} = 0$

$\Rightarrow \left[y - \frac{x}{x-1}\right]^2 - \frac{x^2}{(x-1)^2} + \frac{x}{x-1} = 0$

$\Rightarrow \left[y - \frac{x}{x-1}\right]^2 = \frac{x^2}{(x-1)^2} - \frac{x}{x-1}$

$\Rightarrow \left[y - \frac{x}{x-1}\right]^2 = \frac{x^2 - x(x-1)}{(x-1)^2}$

$\Rightarrow \left[y - \frac{x}{x-1}\right]^2 = \frac{x}{(x-1)^2}$

$\Rightarrow y - \frac{x}{x-1} = \frac{\pm\sqrt{x}}{x-1}$

$\Rightarrow y = \frac{x \pm \sqrt{x}}{x-1}$

$\Rightarrow y = \frac{(\sqrt{x})^2 \pm \sqrt{x}}{(\sqrt{x})^2 - 1}$

$\Rightarrow y = \frac{\sqrt{x}(\sqrt{x} \pm 1)}{(\sqrt{x}+1)(\sqrt{x}-1)}$

$\Rightarrow y = \left\langle \begin{array}{l} \frac{\sqrt{x}}{\sqrt{x}+1} \\ \frac{\sqrt{x}}{\sqrt{x}-1} \end{array} \right.$

PICK A SENSIBLE VALUE TO CHECK AGAINST THE ORIGINAL

E.g $x=1 \Rightarrow -2y+1=0$

$y = \frac{1}{2}$

SO THE REARRANGEMENT MUST SATISFY $(x,y) = (1, \frac{1}{2})$

$\therefore y = \frac{\sqrt{x}}{\sqrt{x}+1}$

THE OTHER REARRANGEMENT DOES NOT SATISFY $(1, \frac{1}{2})$

Question 99 (*****)

Solve the following quadratic in x, giving the answers in terms of k.

$$(k+1)x^2-\left(k^2+k+1\right)x+k=0,\ k\neq 1.$$

$\boxed{}$, $\boxed{x=-k}$, $\boxed{x=-\dfrac{1}{k+1}}$

$$\boxed{(k+1)x^2+(k^2+k+1)x+k=0}$$

- BY THE QUADRATIC FORMULA — FORM THE DISCRIMINANT FIRST

$$\Rightarrow b^2-4ac=(k^2+k+1)^2-4(k+1)k$$
$$\Rightarrow b^2-4ac=(k^2+k+1)^2-4k(k+1)$$

$$\boxed{(A+B+C)^2\equiv A^2+B^2+C^2+2AB+2BC+2CA}$$

$$\Rightarrow b^2-4ac=k^4+k^2+1+2k^3+2k+2k^2-4k^2-4k$$
$$\Rightarrow b^2-4ac=k^2+2k^3-k^2-2k+1$$

- LOOK FOR FACTORIZATIONS AS A PERFECT SQUARE

E.g. $(k^2-k+1)^2$, $(k^2+k-1)^2$, $(k^2-k-1)^2$ ETC

$$(k^2-k+1)^2=k^4+k^2+1-2k^3-2k+2k^2$$
$$(k^2+k-1)^2=k^4+k^2+1+2k^3-2k-2k^2=k^4+2k^3-k^2-2k+1$$

$$\Rightarrow b^2-4ac=k^2+2k^3-k^2-2k+1=(k^2+k-1)^2$$

- THUS THE QUADRATIC FORMULA NOW PRODUCES

$$\Rightarrow x=\frac{-(k^2+k+1)\pm\sqrt{(k^2+k-1)^2}}{2(k+1)}$$

- BECAUSE OF THE $\pm$ AT THE FRONT THERE IS NO NEED FOR $\sqrt{a^2}=|a|$

$$\Rightarrow x=\begin{cases}\dfrac{-k^2-k-1+k^2+k-1}{2(k+1)}=\dfrac{-2}{2(k+1)}=-\dfrac{1}{k+1}\\[2ex]\dfrac{-k^2-k-1-k^2-k+1}{2(k+1)}=\dfrac{-2k^2-2k}{2(k+1)}=\dfrac{-2k(k+1)}{2(k+1)}=-k\end{cases}$$

$$\Rightarrow x=\begin{cases}-\dfrac{1}{k+1}\\-k\end{cases}$$

ALTERNATIVE BY INSPECTION

$$(k+1)x^2+(k^2+k+1)x+k=0$$
$$\Rightarrow x^2+\frac{k^2+k+1}{k+1}x+\frac{k}{k+1}=0$$
$$\Rightarrow x^2+\frac{k(k+1)+1}{k+1}x+\frac{k}{k+1}=0$$
$$\Rightarrow x^2+\left(k+\frac{1}{k+1}\right)x+k\times\frac{1}{k+1}=0$$
$$\Rightarrow (x+k)\left(x+\frac{1}{k+1}\right)=0$$
$$\Rightarrow x=\begin{cases}-k\\-\dfrac{1}{k+1}\end{cases}$$

Question 100 (*****)

The quadratic equation

$$ax^2 + bx + c = 0, \ x \in \mathbb{R},$$

where a, b and c are constants, $a \neq 0$, has real roots which differ by 1.

Determine a simplified relationship between a, b and c.

$\boxed{}$, $\boxed{b^2 - 4ac = a^2}$

$\boxed{ax^2 + bx + c = 0 \ \text{, SOLUTIONS DIFFER BY 1}}$

- LET THE TWO SOLUTIONS BE x_2 & x_1, $x_2 > x_1$

$\Rightarrow x_2 - x_1 = 1$

$\Rightarrow \frac{-b + \sqrt{b^2-4ac}}{2a} - \frac{-b - \sqrt{b^2-4ac}}{2a} = 1$

$\Rightarrow \frac{\cancel{2}\sqrt{b^2-4ac}}{\cancel{2}a} = 1$

$\Rightarrow \sqrt{b^2-4ac} = a$

$\Rightarrow b^2 - 4ac = a^2$

ALTERNATIVE APPROACH

- LET THE TWO ROOTS BE α & β, WHERE $\beta = \alpha + 1$

$\left.\begin{array}{l} \alpha + \beta = -\frac{b}{a} \\ \alpha\beta = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \alpha + (\alpha+1) = -\frac{b}{a} \\ \alpha(\alpha+1) = \frac{c}{a} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 2\alpha + 1 = -\frac{b}{a} \\ \alpha^2 + \alpha = \frac{c}{a} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\alpha = -\frac{b}{a} - 1 \\ 4\alpha^2 + 4\alpha = \frac{4c}{a} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} 4\alpha^2 = \left(-\frac{b}{a} - 1\right)^2 = \left(\frac{b}{a}+1\right)^2 \\ 4\alpha^2 + 4\alpha = \frac{4c}{a} \end{array}\right\} \Rightarrow \left(\frac{b}{a}+1\right)^2 + 2\left(-\frac{b}{a} - 1\right) = \frac{4c}{a}$

$\Rightarrow \left(\frac{b+a}{a}\right)^2 - \frac{2(b+a)}{a} = \frac{4c}{a}$

$\Rightarrow \frac{(b+a)^2}{a^2} - \frac{2(b+a)}{a} = \frac{4c}{a}$

$\Rightarrow (b+a)^2 - 2a(b+a) = 4ac$

$\Rightarrow b^2 + \cancel{2ab} + a^2 - \cancel{2ab} - 2a^2 = 4ac$

$\Rightarrow b^2 - a^2 = 4ac$

$\Rightarrow b^2 - 4ac = a^2$

AS BEFORE

Question 101 (*****)

Solve the following quadratic in x, giving the answers in terms of k.

$$k^2x^2 - \left(k^3 + k + 1\right)x + k^2 + k = 0, \; k \neq 0.$$

$\boxed{}$, $\boxed{x = k}$, $\boxed{x = \frac{k+1}{k^2}}$

$\boxed{k^2x^2 - (k^3+k+1)x + k^2 + k = 0}$

- START BY THE QUADRATIC FORMULA – FORM THE DISCRIMINANT

$\Rightarrow b^2 - 4ac = \left[-(k^3+k+1)\right]^2 - 4k^2 \times (k^2+k)]$

$\Rightarrow b^2 - 4ac = (k^3+k+1)^2 - 4k(k^2+k)$

$\boxed{(A+B+C)^2 \equiv A^2 + B^2 + C^2 + 2AB + 2BC + 2CA}$

$\Rightarrow b^2 - 4ac = k^6 + k^2 + 1 + 2k^4 + 2k + 2k^3 - 4k^4 - 4k^3$

$\Rightarrow b^2 - 4ac = k^6 - 2k^4 - 2k^3 + k^2 + 2k + 1$

- AS THE QUADRATIC FACTORIZES INTO RATIONAL FACTORS, THE DISCRIMINANT MUST BE A PERFECT SQUARE

e.g. $(k^3-k+1)^2, (k^3-k-1)^2, (k^3+k-1)^2$ ETC

- BY INSPECTION $(k^3-k-1)^2 = k^6 + k^2 + 1 - 2k^4 + 2k - 2k^3$

- THUS THE QUADRATIC FORMULA YIELDS

$$x = \frac{k^3+k+1 \pm \sqrt{(k^3-k-1)^2}}{2k^2}$$

- BECAUSE OF THE $\pm$ AT THE FRONT OF THE SQUARE ROOT WE DON'T NEED $\sqrt{a^2} = |a|$

$$x = \begin{cases} \dfrac{(k^3+k+1)+(k^3-k-1)}{2k^2} = \dfrac{2k^3}{2k^2} = k \\[2ex] \dfrac{(k^3+k+1)-(k^3-k-1)}{2k^2} = \dfrac{2k+2}{2k^2} = \dfrac{k+1}{k^2} \end{cases}$$

- $\therefore x = \begin{cases} k \\ \dfrac{k+1}{k^2} \end{cases}$

ALTERNATIVE BY INSPECTION

$\boxed{k^2x^2 - (k^3+k+1)x + k^2 + k = 0}$

$\Rightarrow k^2x^2 - (k^3+k+1)x + k(k+1) = 0$

- BY INSPECTION THE "BRACKETS" SHOULD LOOK AS FOLLOWS

$(k^2x \; \ldots)(x \; \ldots)$

OR

$(kx \; \ldots)(kx \; \ldots)$

WITH THE "BLANK" SPACES FILLED BY $\pm k$ $\pm(k+1)$

- BUT SINCE ONE OF THE BRACKETS MUST HAVE "k" AS A CONSTANT THEN $(kx \ldots)(kx \ldots)$ CANNOT BE POSSIBLE AS THE k WOULD BE A COMMON FACTOR

- THUS $(k^2x \; \ldots)(x - k)$

k CANNOT GO HERE (COMMON FACTOR ALSO)

$\left[k^2x - (k+1)\right](x-k) = 0$

$(k^2x - k - 1)(x - k) = 0$

$x = \begin{cases} k \\ \dfrac{k+1}{k^2} \end{cases}$