VECTOR OPERATORS

GRADIENT

$$\text{grad}\,\varphi \equiv \nabla\varphi$$

Question 1

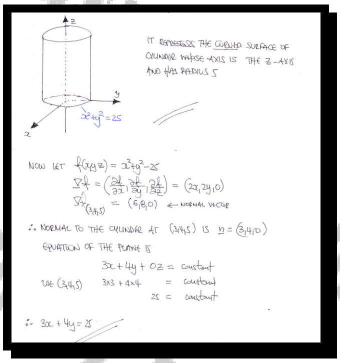
A surface S is given by the Cartesian equation

$$x^2 + y^2 = 25 .$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(3,4,5)$.

$\boxed{3x + 4y = 25}$



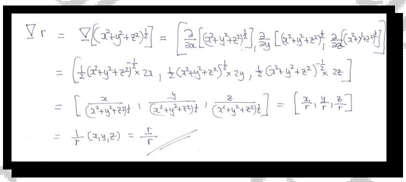
Question 2

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r = \frac{\mathbf{r}}{r} .$$

$\boxed{\text{proof}}$



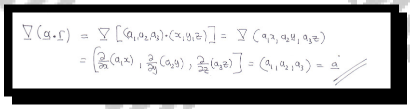
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector find

$$\nabla(\mathbf{a}\cdot\mathbf{r}).$$

$$\boxed{\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}}$$



Question 4

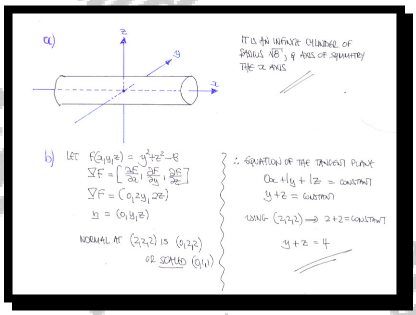
A surface S is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

a) Draw a sketch of S, and describe it geometrically.

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates $(2,2,2)$.

$$\boxed{y + z = 4}$$



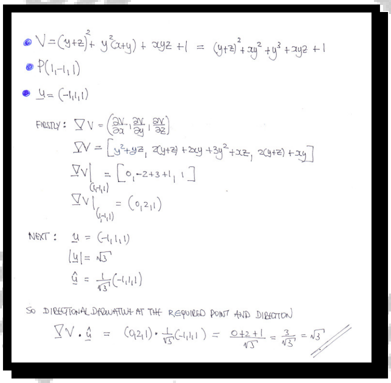
Question 5

The scalar function V is defined as

$$V(x,y,z) = (y+z)^2 + y^2(x+y) + xyz + 1.$$

Determine the value of the directional derivative of V at the point $P(1,-1,1)$, in the direction $-\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$\boxed{\sqrt{3}}$


• $V = (y+z)^2 + y^2(x+y) + xyz + 1 = (y+z)^2 + xy^2 + y^3 + xyz + 1$
• $P(1,-1,1)$
• $\underline{u} = (-1,1,1)$

FIRSTLY: $\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$
$\nabla V = \left[y^2+yz,\ 2(y+z)+2xy+3y^2+xz,\ 2(y+z)+xy\right]$
$\nabla V\big|_{(1,-1,1)} = [0, -2+3+1, 1]$
$\nabla V\big|_{(1,-1,1)} = (0,2,1)$

NEXT: $\underline{u} = (-1,1,1)$
$|\underline{u}| = \sqrt{3}$
$\hat{\underline{u}} = \frac{1}{\sqrt{3}}(-1,1,1)$

SO DIRECTIONAL DERIVATIVE AT THE REQUIRED POINT AND DIRECTION
$\nabla V \cdot \hat{\underline{u}} = (0,2,1)\cdot\frac{1}{\sqrt{3}}(-1,1,1) = \frac{0+2+1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

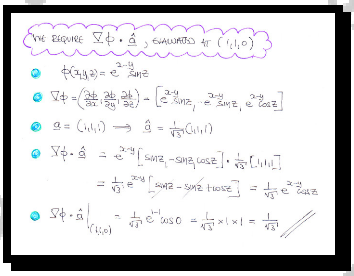
Question 6

The scalar function φ is defined as

$$\varphi(x,y,z) = \mathrm{e}^{x-y}\sin z.$$

Determine the value of the directional derivative of φ at the point $P(1,1,0)$, in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

$\boxed{\frac{1}{\sqrt{3}}}$


WE REQUIRE $\nabla\phi\cdot\hat{\underline{a}}$, EVALUATED AT $(1,1,0)$
• $\phi(x,y,z) = e^{x-y}\sin z$
• $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \left[e^{x-y}\sin z, -e^{x-y}\sin z, e^{x-y}\cos z\right]$
• $\underline{a} = (1,1,1) \Longrightarrow \hat{\underline{a}} = \frac{1}{\sqrt{3}}(1,1,1)$
• $\nabla\phi\cdot\hat{\underline{a}} = e^{x-y}[\sin z, -\sin z, \cos z]\cdot\frac{1}{\sqrt{3}}[1,1,1]$
$= \frac{1}{\sqrt{3}}e^{x-y}[\sin z - \sin z + \cos z] = \frac{1}{\sqrt{3}}e^{x-y}\cos z$
• $\nabla\phi\cdot\hat{\underline{a}}\big|_{(1,1,0)} = \frac{1}{\sqrt{3}}e^{1-1}\cos 0 = \frac{1}{\sqrt{3}}\times1\times1 = \frac{1}{\sqrt{3}}$

Question 7

The point $P(1,2,3)$ lies on the surface with Cartesian equation

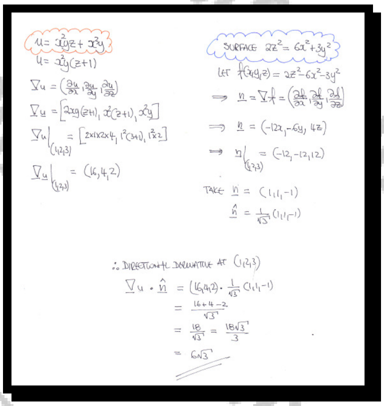
$$2z^2 = 6x^2 + 3y^2.$$

The scalar function u is defined as

$$u(x,y,z) = x^2yz + x^2y.$$

Determine the value of the directional derivative of u at the point P in the direction to the normal at P.

$\boxed{6\sqrt{3}}$



Question 8

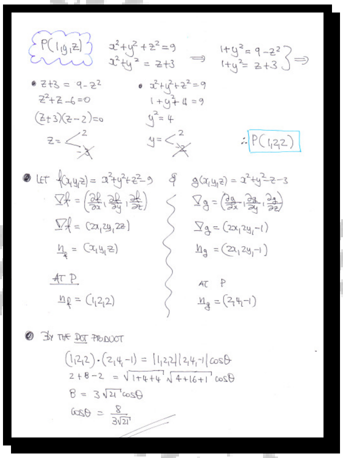
The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3.$$

The two surfaces intersect each other at an angle θ, at the point P.

Given further that P lies in the first octant, determine the exact value of $\cos\theta$.

$$\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$



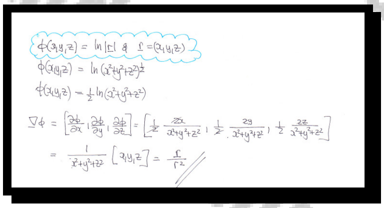
Question 9

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\varphi(r) = \ln r$, show that

$$\nabla \varphi(r) = \frac{\mathbf{r}}{r^2}.$$

proof



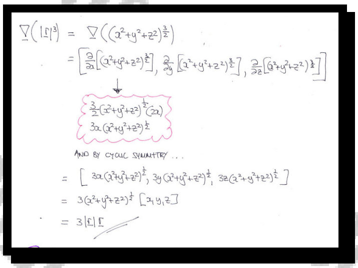
Question 10

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show by detailed workings that

$$\nabla r^3 \equiv 3r\mathbf{r}.$$

proof



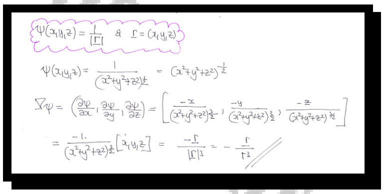
Question 11

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\psi(r) = \dfrac{1}{r}$, show that

$$\nabla \psi(r) = -\frac{\mathbf{r}}{r^3}.$$

proof



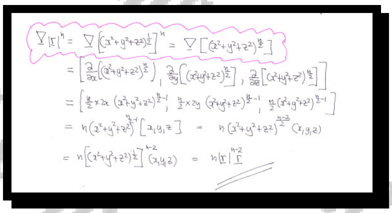
Question 12

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show clearly that

$$\nabla\left(r^n\right) = n r^{n-2}\mathbf{r}.$$

proof



Question 13

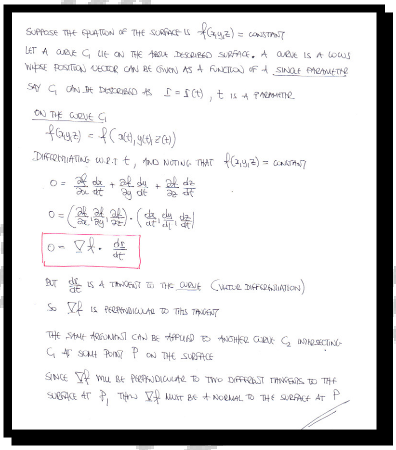
The surface S has Cartesian equation

$$f(x, y, z) = \text{constant},$$

where f is a smooth function.

Given that $\nabla f \neq \mathbf{0}$, show that ∇f is a normal to S.

proof


SUPPOSE THE EQUATION OF THE SURFACE IS $f(x,y,z) = \text{CONSTANT}$

LET A CURVE C_1 LIE ON THE ABOVE DESCRIBED SURFACE. A CURVE IS A LOCUS WHOSE POSITION VECTOR CAN BE GIVEN AS A FUNCTION OF A SINGLE PARAMETER

SAY C_1 CAN BE DESCRIBED AS $\underline{r} = \underline{r}(t)$, t IS A PARAMETER

ON THE CURVE C_1

$$f(x,y,z) = f\left(x(t), y(t), z(t)\right)$$

DIFFERENTIATING W.R.T t, AND NOTING THAT $f(x,y,z) = \text{CONSTANT}$

$$0 = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$0 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$0 = \nabla f \cdot \frac{d\underline{r}}{dt}$$

BUT $\frac{d\underline{r}}{dt}$ IS A TANGENT TO THE CURVE (VECTOR DIFFERENTIATION)

SO ∇f IS PERPENDICULAR TO THIS TANGENT

THE SAME ARGUMENT CAN BE APPLIED TO ANOTHER CURVE C_2 INTERSECTING C_1 AT SOME POINT P ON THE SURFACE

SINCE ∇f WILL BE PERPENDICULAR TO TWO DIFFERENT TANGENTS TO THE SURFACE AT P, THEN ∇f MUST BE A NORMAL TO THE SURFACE AT P

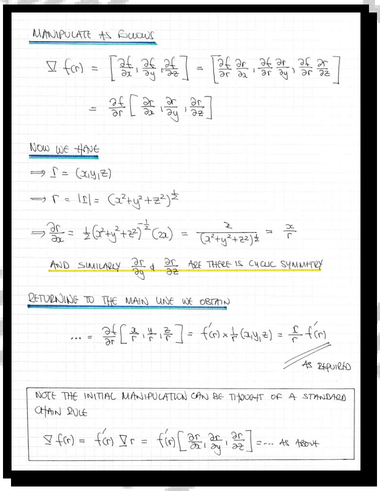
Question 14

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla f(r) = \frac{\mathbf{r}}{r} f'(r).$$

[ ], [proof]



MANIPULATE AS FOLLOWS

$$\nabla f(r) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \left[\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial f}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\right]$$

$$= \frac{\partial f}{\partial r}\left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right]$$

NOW WE HAVE

$$\Rightarrow \underline{r} = (x, y, z)$$

$$\Rightarrow r = |\underline{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}(x^2+y^2+z^2)^{-\frac{1}{2}}(2x) = \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{x}{r}$$

AND SIMILARLY $\frac{\partial r}{\partial y}$ & $\frac{\partial r}{\partial z}$ ARE THERE IS CYCLIC SYMMETRY

RETURNING TO THE MAIN LINE WE OBTAIN

$$\dots = \frac{\partial f}{\partial r}\left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right] = f'(r) \times \frac{1}{r}(x, y, z) = \frac{\underline{r}}{r} f'(r)$$

AS REQUIRED

NOTE THE INITIAL MANIPULATION CAN BE THOUGHT OF A STANDARD CHAIN RULE

$$\nabla f(r) = f'(r)\,\nabla r = f'(r)\left[\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right] = \dots \text{ AS ABOVE}$$

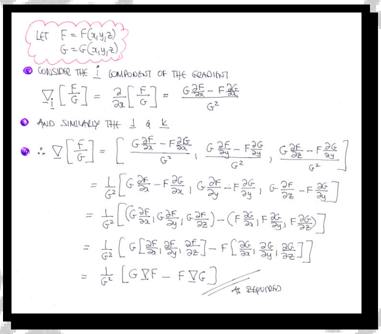
Question 15

The smooth functions $F(x,y,z)$ and $G(x,y,z)$ are given.

Show that

$$\nabla\left[\frac{F}{G}\right]=\frac{G(\nabla F)-F(\nabla G)}{G^2}.$$

proof



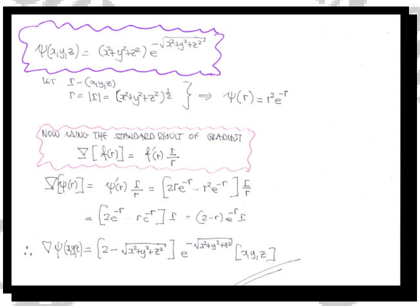
Question 16

$$\Psi(x,y,z)=\left(x^2+y^2+z^2\right)e^{-\sqrt{x^2+y^2+z^2}}$$

Show that

$$\nabla\left[\Psi(x,y,z)\right]=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\left(2-\sqrt{x^2+y^2+z^2}\right)e^{-\sqrt{x^2+y^2+z^2}}.$$

proof



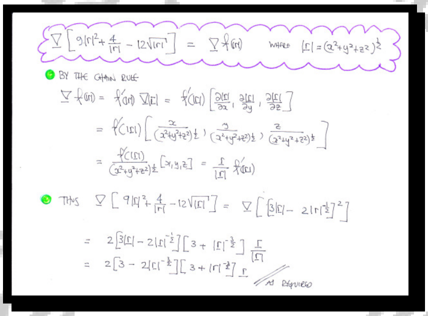
Question 17

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show that

$$\nabla\left(9r^2 + \frac{4}{r} - 12\sqrt{r}\right) = 2\left(3 - 2r^{-\frac{3}{2}}\right)\left(3 + r^{-\frac{3}{2}}\right)\mathbf{r}.$$

proof



Question 18

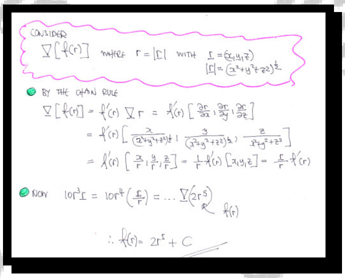
A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

The smooth function $f(r)$ satisfies

$$\nabla\left[f(r)\right] = 10r^3\mathbf{r}.$$

Determine a simplified expression for $f(r)$.

$$f(r) = 2r^5 + C$$



DIVERGENCE

$$\operatorname{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F}$$

Question 1

A Cartesian position vector is denoted by $\mathbf{r}$.

Determine the value of

$$\nabla \cdot \mathbf{r}.$$

$$\boxed{\nabla \cdot \mathbf{r} = 3}$$

$$\nabla \cdot \mathbf{r} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, y, z) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Question 2

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, find

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}).$$

$$\boxed{\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = 0}$$

$$\nabla \cdot (\mathbf{a} \wedge \mathbf{r}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}(a_2 z - a_3 y) + \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_1 y - a_2 x)$$

$$= 0 + 0 + 0$$

$$= 0$$

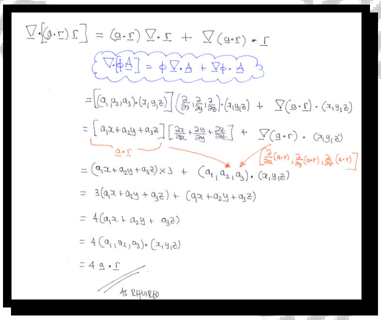
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla \cdot \left[(\mathbf{a} \cdot \mathbf{r}) \mathbf{r} \right] = 4\mathbf{a} \cdot \mathbf{r} .$$

proof



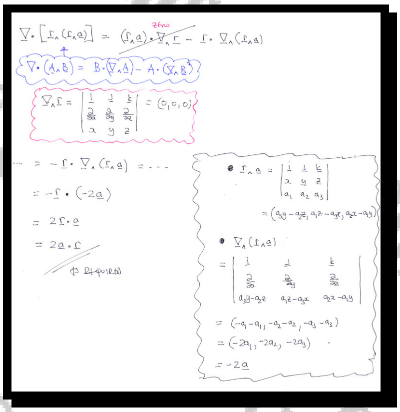
Question 4

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla \cdot \left[\mathbf{r} \wedge (\mathbf{r} \wedge \mathbf{a}) \right] = 2\mathbf{a} \cdot \mathbf{r} .$$

proof



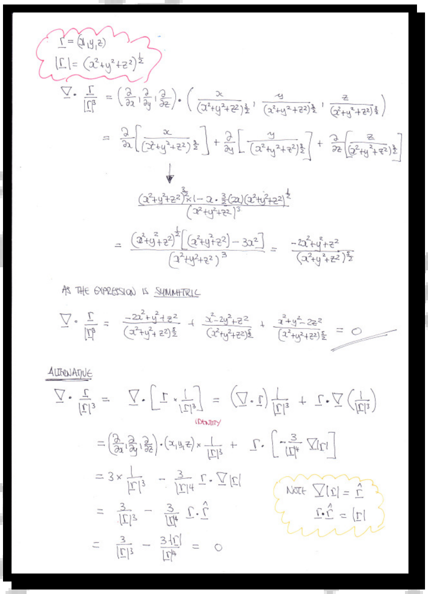
Question 5

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3} \right) = 0\,.$$

proof



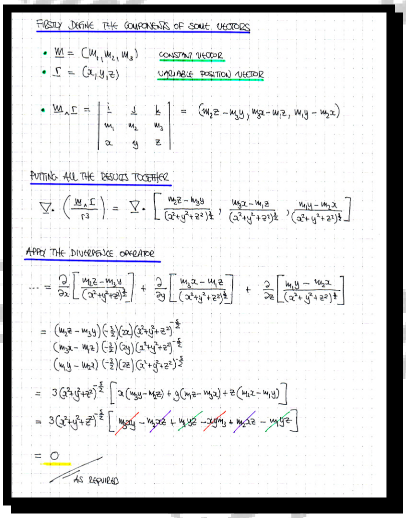
Question 6

A Cartesian position vector is denoted by $\mathbf{r}$.

Given that $\mathbf{m}$ is a constant vector, show that

$$\nabla \cdot \left(\frac{\mathbf{m} \wedge \mathbf{r}}{|\mathbf{r}|^3} \right) = 0 .$$

[ ], [proof]



FIRSTLY DEFINE THE COMPONENTS OF SOME VECTORS

- $\underline{m} = (m_1, m_2, m_3)$ CONSTANT VECTOR
- $\underline{r} = (x, y, z)$ VARIABLE POSITION VECTOR
- $\underline{m} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ m_1 & m_2 & m_3 \\ x & y & z \end{vmatrix} = (m_2 z - m_3 y,\ m_3 x - m_1 z,\ m_1 y - m_2 x)$

PUTTING ALL THE RESULTS TOGETHER

$$\underline{\nabla} \cdot \left(\frac{\underline{m} \wedge \underline{r}}{r^3} \right) = \underline{\nabla} \cdot \left[\frac{m_2 z - m_3 y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{m_3 x - m_1 z}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{m_1 y - m_2 x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right]$$

APPLY THE DIVERGENCE OPERATOR

$$\dots = \frac{\partial}{\partial x}\left[\frac{m_2 z - m_3 y}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial y}\left[\frac{m_3 x - m_1 z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right] + \frac{\partial}{\partial z}\left[\frac{m_1 y - m_2 x}{(x^2+y^2+z^2)^{\frac{3}{2}}} \right]$$

$$= (m_2 z - m_3 y)\left(-\tfrac{3}{2}\right)(2x)(x^2+y^2+z^2)^{-\frac{5}{2}}$$
$$(m_3 x - m_1 z)\left(-\tfrac{3}{2}\right)(2y)(x^2+y^2+z^2)^{-\frac{5}{2}}$$
$$(m_1 y - m_2 x)\left(-\tfrac{3}{2}\right)(2z)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}}\left[x(m_3 y - m_2 z) + y(m_1 z - m_3 x) + z(m_2 x - m_1 y) \right]$$

$$= 3(x^2+y^2+z^2)^{-\frac{5}{2}}\left[m_3 xy - m_2 xz + m_1 yz - xym_3 + m_2 xz - m_1 yz \right]$$

$$= 0$$

AS REQUIRED

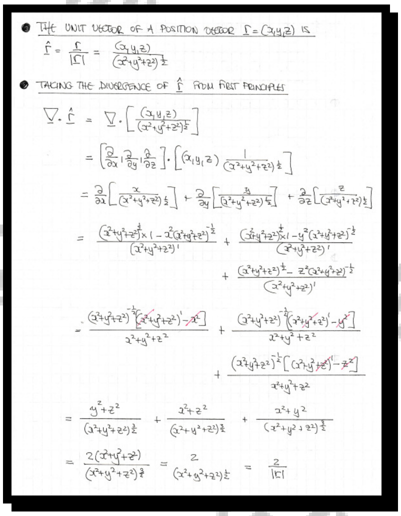
Question 7

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

Show clearly that

$$\nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|} \right) = \frac{2}{|\mathbf{r}|}\,.$$

proof



- THE UNIT VECTOR OF A POSITION VECTOR $\underline{r} = (x,y,z)$ IS

$$\hat{\underline{r}} = \frac{\underline{r}}{|\underline{r}|} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

- TAKING THE DIVERGENCE OF $\hat{\underline{r}}$ FROM FIRST PRINCIPLES

$$\underline{\nabla} \cdot \hat{\underline{r}} = \underline{\nabla} \cdot \left[\frac{(x,y,z)}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right]$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot \left[(x,y,z) \frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} \right]$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}} \times 1 - x^2(x^2+y^2+z^2)^{-\frac{1}{2}}}{(x^2+y^2+z^2)^1} + \frac{(x^2+y^2+z^2)^{\frac{1}{2}} \times 1 - y^2(x^2+y^2+z^2)^{-\frac{1}{2}}}{(x^2+y^2+z^2)^1} + \frac{(x^2+y^2+z^2)^{\frac{1}{2}} - z^2(x^2+y^2+z^2)^{-\frac{1}{2}}}{(x^2+y^2+z^2)^1}$$

$$= \frac{(x^2+y^2+z^2)^{-\frac{1}{2}}\left[(x^2+y^2+z^2)^1 - x^2\right]}{x^2+y^2+z^2} + \frac{(x^2+y^2+z^2)^{-\frac{1}{2}}\left[(x^2+y^2+z^2)^1 - y^2\right]}{x^2+y^2+z^2} + \frac{(x^2+y^2+z^2)^{-\frac{1}{2}}\left[(x^2+y^2+z^2)^1 - z^2\right]}{x^2+y^2+z^2}$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{2}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{2}{|\underline{r}|}$$

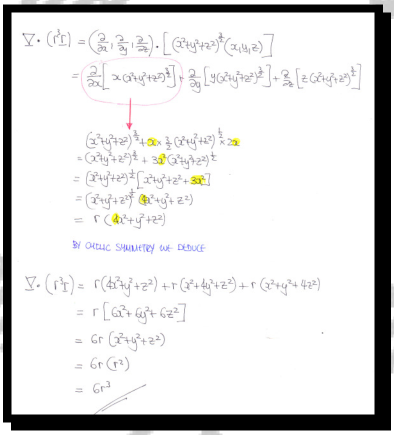
Question 8

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot \left(\mathbf{r} |\mathbf{r}|^3 \right) = 6 |\mathbf{r}|^3.$$

proof



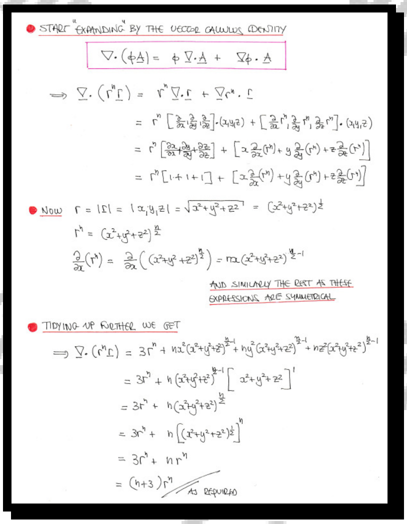
Question 9

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Show that

$$\nabla \cdot \left(r^n \mathbf{r} \right) = (n+3) r^n .$$

[ ], proof



- START "EXPANDING" BY THE VECTOR CALCULUS IDENTITY

$$\nabla \cdot (\phi \underline{A}) = \phi \underline{\nabla} \cdot \underline{A} + \nabla \phi \cdot \underline{A}$$

$$\Rightarrow \underline{\nabla} \cdot (r^n \underline{r}) = r^n \underline{\nabla} \cdot \underline{r} + \underline{\nabla} r^n \cdot \underline{r}$$

$$= r^n \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right] \cdot (x,y,z) + \left[\frac{\partial}{\partial x} r^n, \frac{\partial}{\partial y} r^n, \frac{\partial}{\partial z} r^n \right] \cdot (x,y,z)$$

$$= r^n \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right] + \left[x \frac{\partial}{\partial x}(r^n) + y \frac{\partial}{\partial y}(r^n) + z \frac{\partial}{\partial z}(r^n) \right]$$

$$= r^n [1+1+1] + \left[x \frac{\partial}{\partial x}(r^n) + y \frac{\partial}{\partial y}(r^n) + z \frac{\partial}{\partial z}(r^n) \right]$$

- NOW $r = |\underline{r}| = |x,y,z| = \sqrt{x^2+y^2+z^2} = (x^2+y^2+z^2)^{\frac{1}{2}}$

$$r^n = (x^2+y^2+z^2)^{\frac{n}{2}}$$

$$\frac{\partial}{\partial x}(r^n) = \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{\frac{n}{2}} \right) = nx(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

AND SIMILARLY THE REST AS THESE EXPRESSIONS ARE SYMMETRICAL

- TIDYING UP FURTHER WE GET

$$\Rightarrow \underline{\nabla} \cdot (r^n \underline{r}) = 3r^n + nx^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + ny^2(x^2+y^2+z^2)^{\frac{n}{2}-1} + nz^2(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$= 3r^n + n(x^2+y^2+z^2)^{\frac{n}{2}-1}\left[x^2+y^2+z^2 \right]^1$$

$$= 3r^n + n(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= 3r^n + n\left[(x^2+y^2+z^2)^{\frac{1}{2}} \right]^n$$

$$= 3r^n + nr^n$$

$$= (n+3)r^n \quad \text{AS REQUIRED}$$

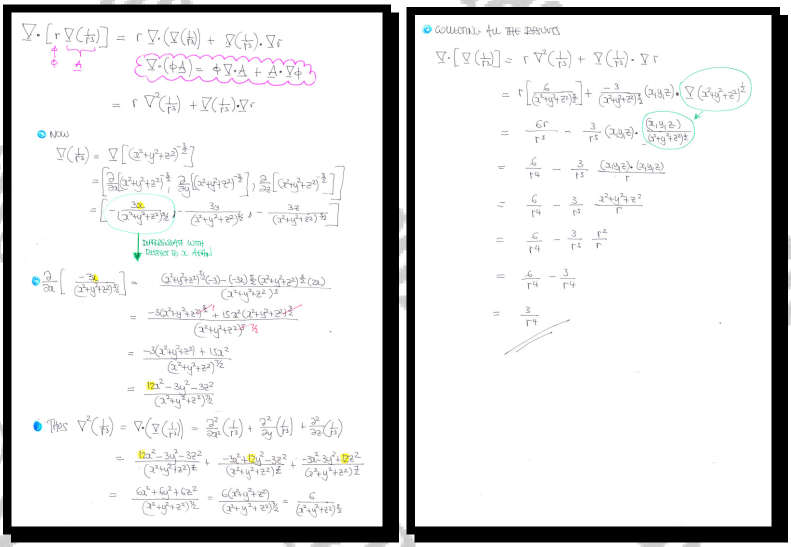
Question 10

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show, with a detailed method, that

$$\nabla \cdot \left[|\mathbf{r}| \nabla \left(\frac{1}{|\mathbf{r}|^3} \right) \right] = \frac{3}{|\mathbf{r}|^4}.$$

proof



$$\nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right] = r \nabla \cdot \left(\nabla \left(\frac{1}{r^3} \right) \right) + \nabla \left(\frac{1}{r^3} \right) \cdot \nabla r$$

$$\nabla \cdot (\phi \mathbf{A}) = \phi \nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla \phi$$

$$= r \nabla^2 \left(\frac{1}{r^3} \right) + \nabla \left(\frac{1}{r^3} \right) \cdot \nabla r$$

NOW

$$\nabla \left(\frac{1}{r^3} \right) = \nabla \left[(x^2+y^2+z^2)^{-\frac{3}{2}} \right]$$

$$= \left[\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-\frac{3}{2}}\right], \frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{-\frac{3}{2}}\right], \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{-\frac{3}{2}}\right] \right]$$

$$= \left[-\frac{3x}{(x^2+y^2+z^2)^{5/2}}, -\frac{3y}{(x^2+y^2+z^2)^{5/2}}, -\frac{3z}{(x^2+y^2+z^2)^{5/2}} \right]$$

DIFFERENTIATE WITH RESPECT TO x AGAIN

$$\frac{\partial}{\partial x}\left[\frac{-3x}{(x^2+y^2+z^2)^{5/2}} \right] = \frac{(x^2+y^2+z^2)^{5/2}(-3) - (-3x)\frac{5}{2}(x^2+y^2+z^2)^{\frac{3}{2}}(2x)}{(x^2+y^2+z^2)^5}$$

$$= \frac{-3(x^2+y^2+z^2)^{\frac{5}{2}} + 15x^2(x^2+y^2+z^2)^{\frac{3}{2}}}{(x^2+y^2+z^2)^{5}}$$

$$= \frac{-3(x^2+y^2+z^2) + 15x^2}{(x^2+y^2+z^2)^{7/2}}$$

$$= \frac{12x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{7/2}}$$

THUS

$$\nabla^2 \left(\frac{1}{r^3} \right) = \nabla \cdot \left(\nabla \left(\frac{1}{r^3} \right) \right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r^3} \right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r^3} \right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r^3} \right)$$

$$= \frac{12x^2 - 3y^2 - 3z^2}{(x^2+y^2+z^2)^{\frac{7}{2}}} + \frac{-3x^2 + 12y^2 - 3z^2}{(x^2+y^2+z^2)^{\frac{7}{2}}} + \frac{-3x^2 - 3y^2 + 12z^2}{(x^2+y^2+z^2)^{\frac{7}{2}}}$$

$$= \frac{6x^2 + 6y^2 + 6z^2}{(x^2+y^2+z^2)^{7/2}} = \frac{6(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}} = \frac{6}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

COLLECTING ALL THE RESULTS

$$\nabla \cdot \left[\nabla \left(\frac{1}{r^3} \right) \right] = r \nabla^2 \left(\frac{1}{r^3} \right) + \nabla \left(\frac{1}{r^3} \right) \cdot \nabla r$$

$$= r \left[\frac{6}{(x^2+y^2+z^2)^{\frac{5}{2}}} \right] + \frac{-3}{(x^2+y^2+z^2)^{\frac{5}{2}}}(x, y, z) \cdot \nabla (x^2+y^2+z^2)^{\frac{1}{2}}$$

$$= \frac{6r}{r^5} - \frac{3}{r^5}(x, y, z) \cdot \frac{(x, y, z)}{(x^2+y^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{6}{r^4} - \frac{3}{r^5} \frac{(x, y, z) \cdot (x, y, z)}{r}$$

$$= \frac{6}{r^4} - \frac{3}{r^5} \frac{x^2+y^2+z^2}{r}$$

$$= \frac{6}{r^4} - \frac{3}{r^5} \frac{r^2}{r}$$

$$= \frac{6}{r^4} - \frac{3}{r^4}$$

$$= \frac{3}{r^4}$$

CURL

$$\operatorname{curl}\mathbf{F} \equiv \nabla \wedge \mathbf{F}$$

Question 1

A Cartesian vector is denoted by

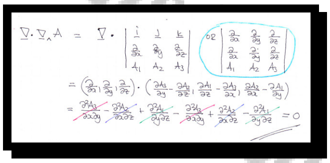
$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \,,$$

where $A_i = f(x, y, z)$, $i = 1, 2, 3$

Given that A_i are differentiable functions, show that

$$\nabla \cdot \nabla \wedge \mathbf{A} = 0 \,.$$

proof



Question 2

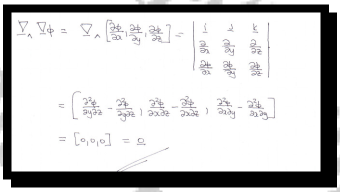
A function φ is denoted by

$$\varphi = \varphi(x, y, z) \,.$$

Given that φ is differentiable show that

$$\nabla \wedge \nabla \varphi = \mathbf{0} \,.$$

proof



Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Determine the value of

$$\nabla \wedge \mathbf{r}.$$

$$\boxed{\nabla \wedge \mathbf{r} = \mathbf{0}}$$

$$\nabla \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}, \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}, \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = (0,0,0)$$

Question 4

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, find

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{r}).$$

$$\boxed{\nabla \wedge (\mathbf{a} \wedge \mathbf{r}) = 2\mathbf{a}}$$

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{r}) = \nabla \wedge \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \nabla \wedge \left[a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x \right]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left[a_1 - (-a_1), a_2 - (-a_2), a_3 - (-a_3) \right] = (2a_1, 2a_2, 2a_3)$$

$$= 2(a_1, a_2, a_3) = 2\mathbf{a}$$

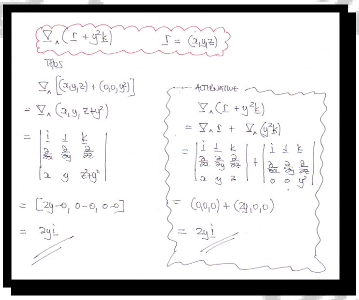
Question 5

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Determine

$$\nabla \wedge \left(\mathbf{r} + y^2 \mathbf{k} \right).$$

$$\boxed{\nabla \wedge \left(\mathbf{r} + y^2 \mathbf{k} \right) = 2y\mathbf{i}}$$



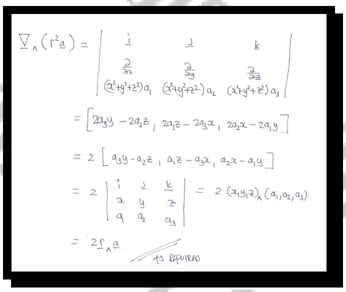
Question 6

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla \wedge \left(r^2 \mathbf{a} \right) \equiv 2\mathbf{r} \wedge \mathbf{a} .$$

$\boxed{\text{proof}}$



Question 7

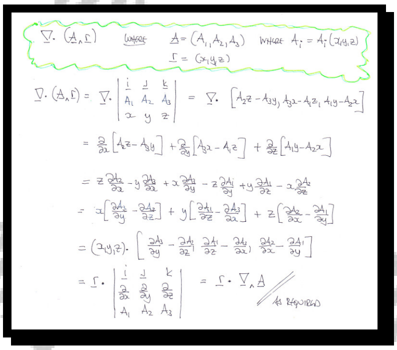
A vector function $\mathbf{A}$ is defined as

$$\mathbf{A} = A_1(x,y,z)\mathbf{i} + A_2(x,y,z)\mathbf{j} + A_3(x,y,z)\mathbf{k}.$$

Given that the standard Cartesian position vector is denoted by $\mathbf{r}$, show that

$$\nabla \cdot (\mathbf{A} \wedge \mathbf{r}) \equiv \mathbf{r} \cdot \nabla \wedge \mathbf{A}.$$

proof

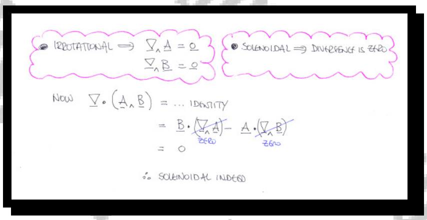


Question 8

The smooth vector functions, $\mathbf{A}$ and $\mathbf{B}$, are both irrotational.

Show that $\mathbf{A} \wedge \mathbf{B}$ is solenoidal.

proof



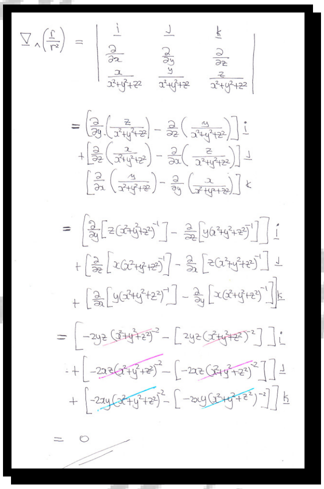
Question 9

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Find the value of

$$\nabla \wedge \left(\frac{\mathbf{r}}{|\mathbf{r}|^2} \right).$$

$\boxed{0}$



$$\nabla \wedge \left(\frac{\mathbf{r}}{r^2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2+z^2} & \frac{y}{x^2+y^2+z^2} & \frac{z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2+z^2}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial x}\left(\frac{z}{x^2+y^2+z^2}\right)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right)\right]\mathbf{k}$$

$$= \left[\frac{\partial}{\partial y}\left[z(x^2+y^2+z^2)^{-1}\right] - \frac{\partial}{\partial z}\left[y(x^2+y^2+z^2)^{-1}\right]\right]\mathbf{i} + \left[\frac{\partial}{\partial z}\left[x(x^2+y^2+z^2)^{-1}\right] - \frac{\partial}{\partial x}\left[z(x^2+y^2+z^2)^{-1}\right]\right]\mathbf{j} + \left[\frac{\partial}{\partial x}\left[y(x^2+y^2+z^2)^{-1}\right] - \frac{\partial}{\partial y}\left[x(x^2+y^2+z^2)^{-1}\right]\right]\mathbf{k}$$

$$= \left[-2yz(x^2+y^2+z^2)^{-2} - \left[2yz(x^2+y^2+z^2)^{-2}\right]\right]\mathbf{i} + \left[-2xz(x^2+y^2+z^2)^{-2} - \left[-2xz(x^2+y^2+z^2)^{-2}\right]\right]\mathbf{j} + \left[-2xy(x^2+y^2+z^2)^{-2} - \left[-2xy(x^2+y^2+z^2)^{-2}\right]\right]\mathbf{k}$$

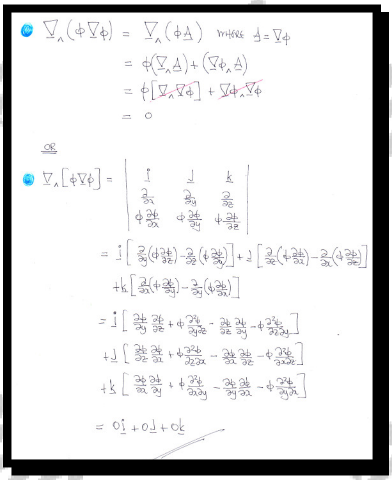
$$= 0$$

Question 10

If $\varphi = \varphi(x, y, z)$ is a smooth function, prove that.

$$\nabla \wedge (\varphi \nabla \varphi) = \mathbf{0}.$$

proof



$\nabla \wedge (\phi \nabla \phi) = \nabla \wedge (\phi \mathbf{A})$ WHERE $\mathbf{A} = \nabla \phi$

$= \phi(\nabla \wedge \mathbf{A}) + (\nabla \phi \wedge \mathbf{A})$

$= \phi[\nabla \wedge \nabla \phi] + \nabla \phi \wedge \nabla \phi$

$= 0$

OR

$\nabla \wedge [\phi \nabla \phi] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi \frac{\partial \phi}{\partial x} & \phi \frac{\partial \phi}{\partial y} & \phi \frac{\partial \phi}{\partial z} \end{vmatrix}$

$= \mathbf{i}\left[\frac{\partial}{\partial y}\left(\phi \frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\phi \frac{\partial \phi}{\partial y}\right)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}\left(\phi \frac{\partial \phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\phi \frac{\partial \phi}{\partial z}\right)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}\left(\phi \frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\phi \frac{\partial \phi}{\partial x}\right)\right]$

$= \mathbf{i}\left[\frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial z} + \phi \frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial y} - \phi \frac{\partial^2 \phi}{\partial z \partial y}\right] + \mathbf{j}\left[\frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial x} + \phi \frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial z} - \phi \frac{\partial^2 \phi}{\partial x \partial z}\right] + \mathbf{k}\left[\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y} + \phi \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial \phi}{\partial y}\frac{\partial \phi}{\partial x} - \phi \frac{\partial^2 \phi}{\partial y \partial x}\right]$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$

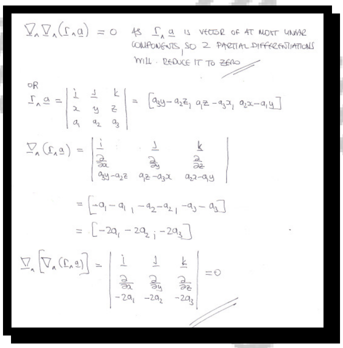
Question 11

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Given that $\mathbf{a}$ is a constant vector, find the value of

$$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}).$$

$\mathbf{0}$



$\nabla \wedge \nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = 0$ AS $\mathbf{r} \wedge \mathbf{a}$ IS VECTOR OF AT MOST LINEAR COMPONENTS, SO 2 PARTIAL DIFFERENTIATIONS WILL REDUCE IT TO ZERO

OR

$\mathbf{r} \wedge \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = [a_3 y - a_2 z, a_1 z - a_3 x, a_2 x - a_1 y]$

$\nabla \wedge (\mathbf{r} \wedge \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$

$= [-a_1 - a_1, -a_2 - a_2, -a_3 - a_3]$

$= [-2a_1, -2a_2, -2a_3]$

$\nabla \wedge [\nabla \wedge (\mathbf{r} \wedge \mathbf{a})] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2a_1 & -2a_2 & -2a_3 \end{vmatrix} = 0$

Question 12

The irrotational vector field $\mathbf{F}$ is given by

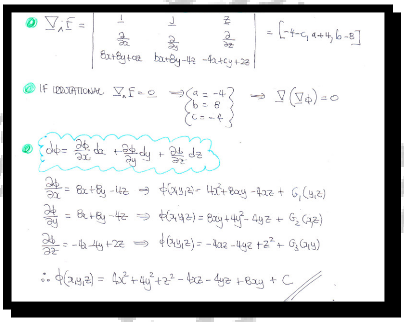
$$\mathbf{F} = (8x+8y+az)\mathbf{i} + (bx+8y-4z)\mathbf{j} + (-4x+cy+2z)\mathbf{k} \, ,$$

where a, b and c are scalar constants.

Determine a smooth scalar function $\varphi(x,y,z)$ such that

$$\nabla\varphi = \mathbf{F} \, .$$

$$\boxed{\varphi(x,y,z) = (2x+2y-x)^2 + \text{constant} = 4x^2+4y^2+z^2-4xz-4yz+8xy+\text{constant}}$$



Question 13

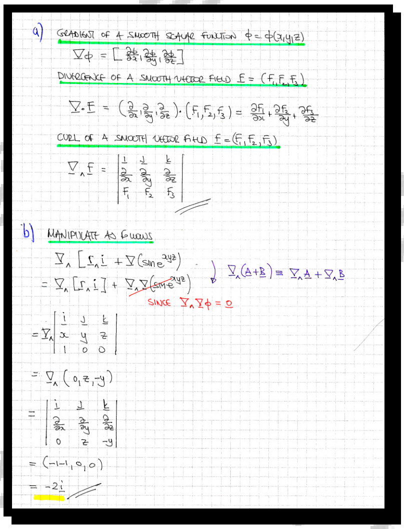
a) Define the vector calculus operators grad, div and curl.

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

b) Determine the vector

$$\nabla \wedge \left[\mathbf{r} \wedge \mathbf{i} + \nabla\left(\sin e^{xyz}\right) \right].$$

$$\boxed{}\,, \quad \boxed{\nabla \wedge \left[\mathbf{r} \wedge \mathbf{i} + \nabla\left(\sin e^{xyz}\right) \right] = -2\mathbf{i}}$$



a) GRADIENT OF A SMOOTH SCALAR FUNCTION $\phi = \phi(x,y,z)$

$\nabla\phi = \left[\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right]$

DIVERGENCE OF A SMOOTH VECTOR FIELD $\underline{F} = (F_1, F_2, F_3)$

$\nabla \cdot \underline{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

CURL OF A SMOOTH VECTOR FIELD $\underline{F} = (F_1, F_2, F_3)$

$\nabla \wedge \underline{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

b) MANIPULATE AS FOLLOWS

$\nabla \wedge \left[\underline{r} \wedge \underline{i} + \nabla(\sin e^{xyz})\right]$ $\quad \nabla \wedge (\underline{A} + \underline{B}) = \nabla \wedge \underline{A} + \nabla \wedge \underline{B}$

$= \nabla \wedge [\underline{r} \wedge \underline{i}] + \nabla \wedge \nabla (\sin e^{xyz})$ $\quad$ SINCE $\nabla \wedge \nabla\phi = 0$

$= \nabla \wedge \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$

$= \nabla \wedge (0, z, -y)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & -y \end{vmatrix}$

$= (-1-1, 0, 0)$

$= -2\mathbf{i}$

Question 14

The smooth functions f and $\mathbf{A}$ are defined as

$$f = f(x,y,z) \quad \text{and} \quad \mathbf{A} = A_1(x,y,z)\,\mathbf{i} + A_2(x,y,z)\,\mathbf{j} + A_3(x,y,z)\,\mathbf{k}.$$

a) Define the vector calculus operators grad, div and curl with reference to f and $\mathbf{A}$.

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

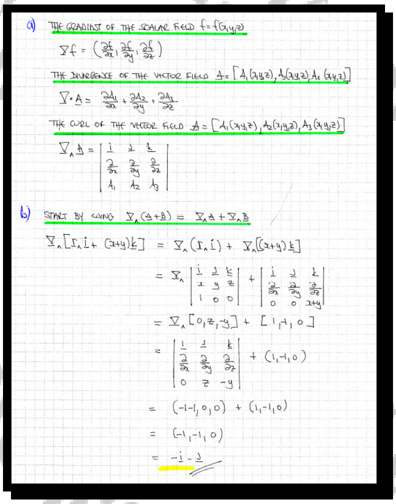
b) Determine the vector

$$\nabla \wedge \left[\mathbf{r} \wedge \mathbf{i} + (x+y)\mathbf{k} \right].$$

c) Show that

$$\nabla \cdot \nabla \left(\frac{1}{r^2} \right) = \frac{2}{r^4}.$$

[ ] , $\boxed{-\mathbf{i} - \mathbf{j}}$



a) THE GRADIENT OF THE SCALAR FIELD $f = f(x,y,z)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

THE DIVERGENCE OF THE VECTOR FIELD $\underline{A} = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \cdot \underline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

THE CURL OF THE VECTOR FIELD $\underline{A} = [A_1(x,y,z), A_2(x,y,z), A_3(x,y,z)]$

$$\nabla \wedge \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

b) START BY USING $\nabla \wedge (\underline{A} + \underline{B}) = \nabla \wedge \underline{A} + \nabla \wedge \underline{B}$

$$\nabla \wedge [\underline{r} \wedge \underline{i} + (x+y)\underline{k}] = \nabla \wedge (\underline{r} \wedge \underline{i}) + \nabla \wedge [(x+y)\underline{k}]$$

$$= \nabla \wedge \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x+y \end{vmatrix}$$

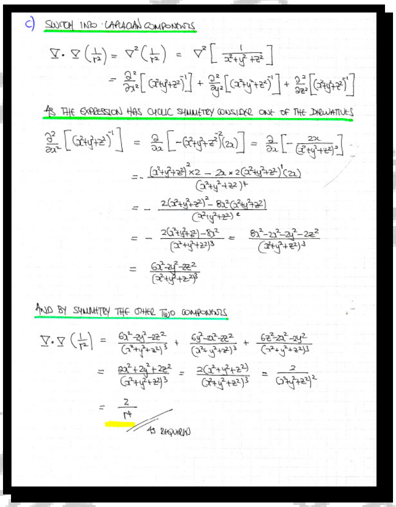
$$= \nabla \wedge [0, z, -y] + [1, -1, 0]$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & z & -y \end{vmatrix} + (1, -1, 0)$$

$$= (-1-1, 0, 0) + (1, -1, 0)$$

$$= (-1, -1, 0)$$

$$= -\underline{i} - \underline{j}$$



c) SWITCH INTO CARTESIAN COMPONENTS

$$\nabla \cdot \nabla\left(\frac{1}{r^2}\right) = \nabla^2\left(\frac{1}{r^2}\right) = \nabla^2\left[\frac{1}{x^2+y^2+z^2}\right]$$

$$= \frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{-1}\right] + \frac{\partial^2}{\partial y^2}\left[(x^2+y^2+z^2)^{-1}\right] + \frac{\partial^2}{\partial z^2}\left[(x^2+y^2+z^2)^{-1}\right]$$

AS THE EXPRESSION HAS CYCLIC SYMMETRY CONSIDER ONE OF THE DERIVATIVES

$$\frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{-1}\right] = \frac{\partial}{\partial x}\left[-(x^2+y^2+z^2)^{-2}(2x)\right] = \frac{\partial}{\partial x}\left[-\frac{2x}{(x^2+y^2+z^2)^2}\right]$$

$$= -\frac{(x^2+y^2+z^2)^2 \times 2 - 2x \times 2(x^2+y^2+z^2)^1(2x)}{(x^2+y^2+z^2)^4}$$

$$= -\frac{2(x^2+y^2+z^2)^2 - 8x^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^4}$$

$$= -\frac{2(x^2+y^2+z^2) - 8x^2}{(x^2+y^2+z^2)^3} = \frac{8x^2-2x^2-2y^2-2z^2}{(x^2+y^2+z^2)^3}$$

$$= \frac{6x^2-2y^2-2z^2}{(x^2+y^2+z^2)^3}$$

AND BY SYMMETRY THE OTHER TWO COMPONENTS

$$\nabla \cdot \nabla\left(\frac{1}{r^2}\right) = \frac{6x^2-2y^2-2z^2}{(x^2+y^2+z^2)^3} + \frac{6y^2-2x^2-2z^2}{(x^2+y^2+z^2)^3} + \frac{6z^2-2x^2-2y^2}{(x^2+y^2+z^2)^3}$$

$$= \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^3} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3} = \frac{2}{(x^2+y^2+z^2)^2}$$

$$= \frac{2}{r^4}$$

AS REQUIRED

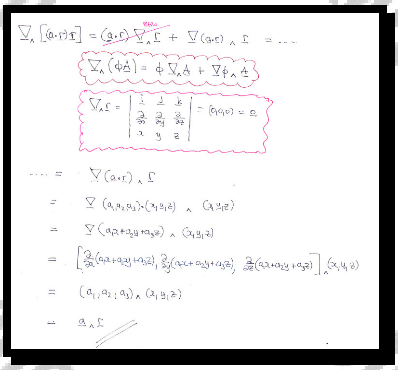
Question 15

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla \wedge \left[(\mathbf{a} \cdot \mathbf{r}) \mathbf{r} \right] = \mathbf{a} \wedge \mathbf{r} .$$

proof



$\underline{\nabla} \wedge [(\underline{a} \cdot \underline{r})\underline{r}] = (\underline{a} \cdot \underline{r}) \underline{\nabla} \wedge \underline{r} + \underline{\nabla}(\underline{a} \cdot \underline{r}) \wedge \underline{r} = \ldots$ (zero)

$\underline{\nabla} \wedge (\phi \underline{A}) = \phi \underline{\nabla} \wedge \underline{A} + \underline{\nabla}\phi \wedge \underline{A}$

$\underline{\nabla} \wedge \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0,0,0) = \underline{0}$

$\ldots = \underline{\nabla}(\underline{a} \cdot \underline{r}) \wedge \underline{r}$

$= \underline{\nabla}\, (a_1,a_2,a_3)\cdot(x,y,z) \wedge (x,y,z)$

$= \underline{\nabla}(a_1x + a_2y + a_3z) \wedge (x,y,z)$

$= \left[\frac{\partial}{\partial x}(a_1x+a_2y+a_3z), \frac{\partial}{\partial y}(a_1x+a_2y+a_3z), \frac{\partial}{\partial z}(a_1x+a_2y+a_3z)\right] \wedge (x,y,z)$

$= (a_1, a_2, a_3) \wedge (x,y,z)$

$= \underline{a} \wedge \underline{r}$

Question 16

The smooth functions $\mathbf{A} = \mathbf{A}(x, y, z)$, $\mathbf{B} = \mathbf{B}(x, y, z)$ and $\varphi = \varphi(x, y, z)$ are defined as

$$\mathbf{A} = yz\,\mathbf{i} - x^2 y\,\mathbf{j} + xz^2\,\mathbf{k}\,, \quad \mathbf{B} = x^2\,\mathbf{i} + yz\,\mathbf{j} - xy\,\mathbf{k} \quad \text{and} \quad \varphi = x\,y\,z\,.$$

Find, in simplified form, expressions for

a) $(\mathbf{A}\cdot\nabla)\varphi$.

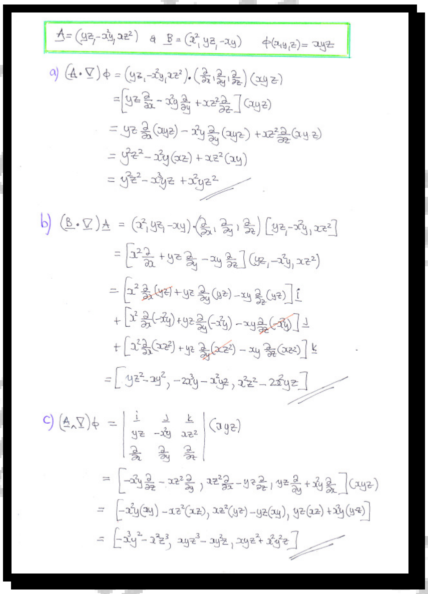
b) $(\mathbf{B}\cdot\nabla)\mathbf{A}$.

c) $(\mathbf{A}\wedge\nabla)\varphi$

$$\boxed{(\mathbf{A}\cdot\nabla)\varphi = y^2z^2 - x^3yz + x^2yz^2}\,,$$

$$\boxed{(\mathbf{B}\cdot\nabla)\mathbf{A} = \left(yz^2 - xy^2\right)\mathbf{i} + \left(-2x^3y - x^2yz\right)\mathbf{j} + \left(x^2z^2 - 2x^2yz\right)\mathbf{k}}\,,$$

$$\boxed{(\mathbf{A}\wedge\nabla)\varphi = \left(-x^3y^2 - x^2z^3\right)\mathbf{i} + \left(xyz^3 - xy^2z\right)\mathbf{j} + \left(xyz^2 + x^2y^2z\right)\mathbf{k}}$$



Question 17

a) Define the vector calculus operators grad, div and curl.

b) Given that $\varphi(x,y,z)$ and $\psi(x,y,z)$ are smooth functions, show that

$$\nabla \wedge [\varphi \nabla \psi] \equiv \nabla \varphi \wedge \nabla \psi .$$

c) Evaluate

$$\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + \left(4x^3 - yz\right)\mathbf{j} + (xyz)\mathbf{k} \right) \right] \right] \right].$$

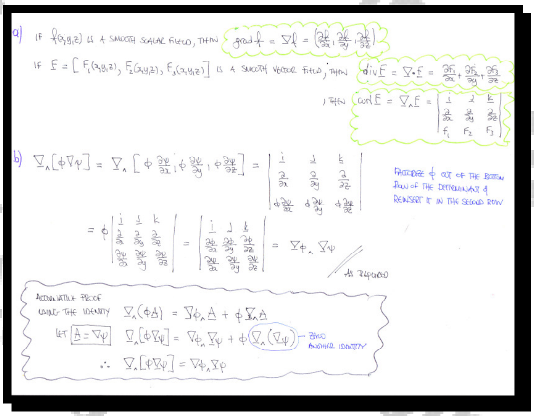
d) Use the vector function

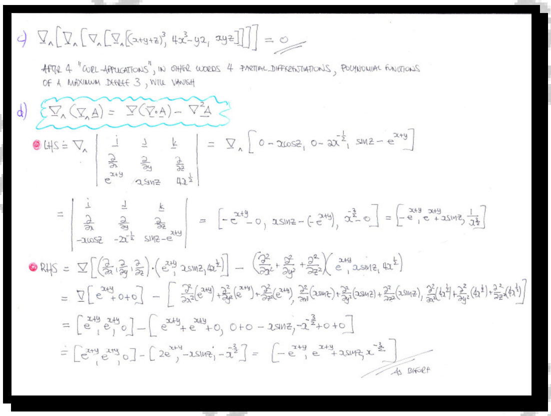
$$\mathbf{A}(x,y,z) = \left(\mathrm{e}^{x+y}\right)\mathbf{i} + (x \sin z)\mathbf{j} + \left(4\sqrt{x}\right)\mathbf{k}$$

to verify the validity of the identity

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} .$$

$$\boxed{\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left[\nabla \wedge \left((x+y+z)^3 \mathbf{i} + \left(4x^3 - yz\right)\mathbf{j} + (xyz)\mathbf{k} \right) \right] \right] \right] = \mathbf{0}}$$





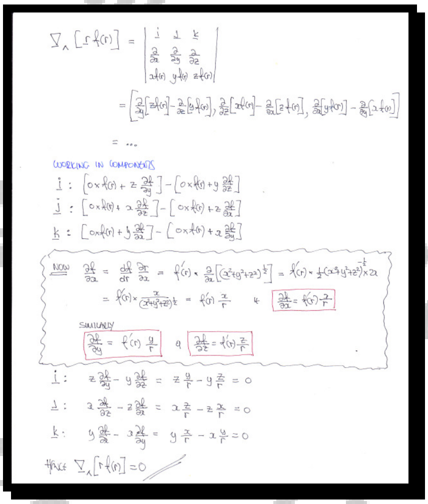
Question 18

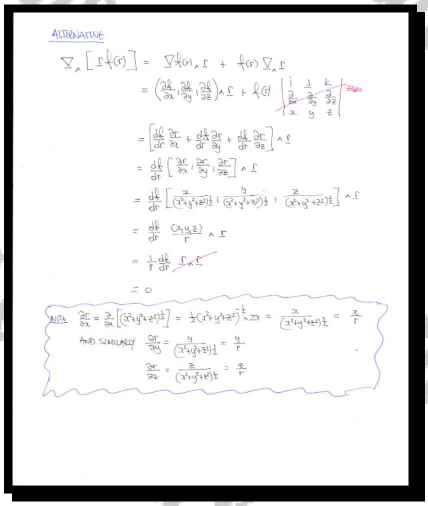
A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $f(r)$ is a differentiable function, show that

$$\nabla \wedge \left[\mathbf{r} \, f(r) \right] = \mathbf{0} .$$

proof



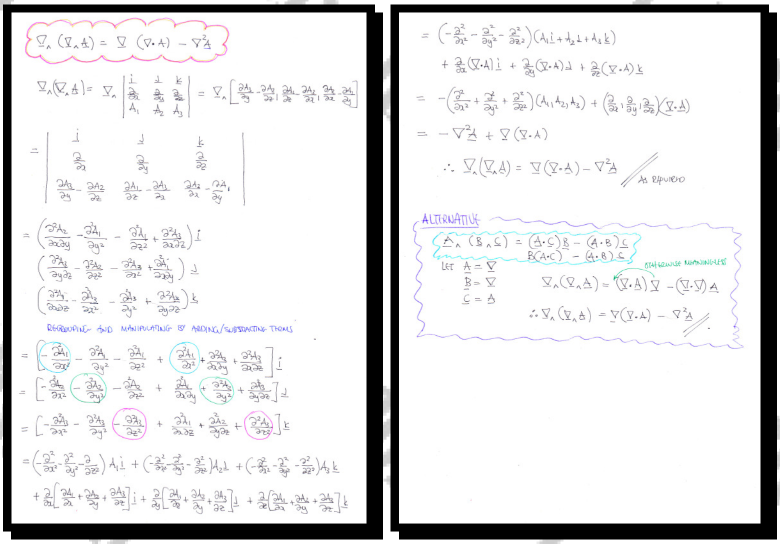


Question 19

Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ is a twice differentiable vector function, show that

$$\nabla \wedge (\nabla \wedge \mathbf{A}) \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} .$$

proof

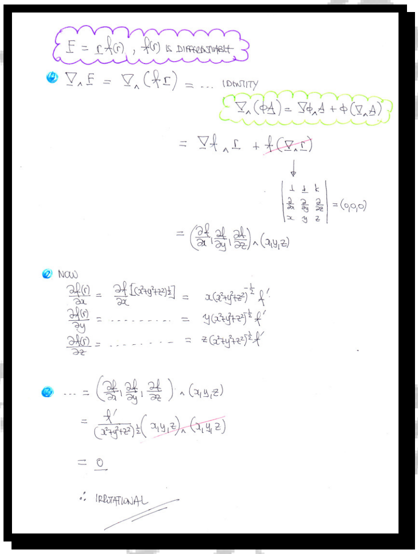


Question 20

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show that $\mathbf{r}\, f(|\mathbf{r}|)$ is irrotational, where $f(|\mathbf{r}|)$ is differentiable function of $|\mathbf{r}|$.

proof



Question 21

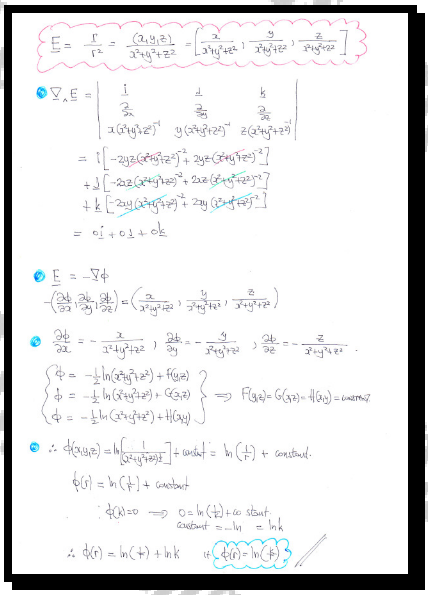
The vector function $\mathbf{E}$ satisfies

$$\mathbf{E} = \frac{\mathbf{r}}{|\mathbf{r}|^2}, \quad |\mathbf{r}| \neq 0,$$

where $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Show that $\mathbf{E}$ is irrotational, and find a smooth scalar function $\varphi(|\mathbf{r}|)$, with $\varphi(k) = 0$, so that $\mathbf{E} = -\nabla\varphi(|\mathbf{r}|)$.

$$\boxed{\varphi(|\mathbf{r}|) = \ln\left(\frac{k}{|\mathbf{r}|}\right)}$$



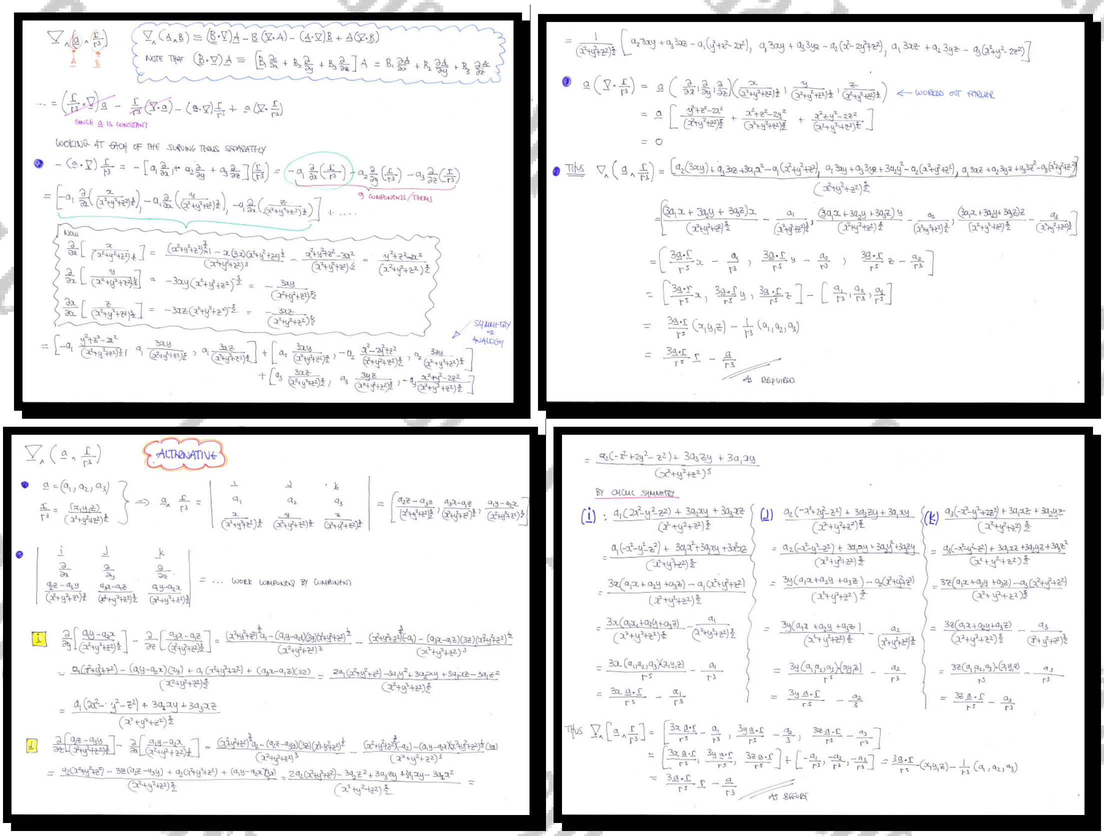
Question 22

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $\mathbf{a}$ is a constant vector, show that

$$\nabla \wedge \left(\mathbf{a} \wedge \frac{\mathbf{r}}{r^3} \right) = \frac{3\mathbf{a}\cdot\mathbf{r}}{r^5}\mathbf{r} - \frac{\mathbf{a}}{r^3} .$$

proof



Question 23

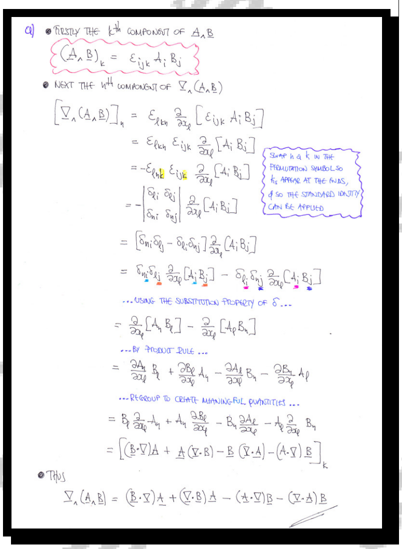
$$\nabla \wedge (\mathbf{A} \wedge \mathbf{B}) \equiv (\mathbf{B} \cdot \nabla)\mathbf{A} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\nabla \cdot \mathbf{A})\mathbf{B} .$$

a) Given that $\mathbf{A} = \mathbf{A}(x, y, z)$ and $\mathbf{B} = \mathbf{B}(x, y, z)$ are smooth vector functions, use index summation notation to prove the validity of the above vector identity.

b) Verify the validity of the vector identity if

$$\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \quad \text{and} \quad \mathbf{B} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

both sides yield $2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$



a) • FIRSTLY THE k^{th} COMPONENT OF $\underline{A} \wedge \underline{B}$

$$(\underline{A} \wedge \underline{B})_k = \varepsilon_{ijk} A_i B_j$$

• NEXT THE n^{th} COMPONENT OF $\underline{\nabla} \wedge (\underline{A} \wedge \underline{B})$

$$[\underline{\nabla} \wedge (\underline{A} \wedge \underline{B})]_n = \varepsilon_{lkn} \frac{\partial}{\partial x_l}[\varepsilon_{ijk} A_i B_j]$$

$$= \varepsilon_{lkn} \varepsilon_{ijk} \frac{\partial}{\partial x_l}[A_i B_j]$$

$$= -\varepsilon_{lnk} \varepsilon_{ijk} \frac{\partial}{\partial x_l}[A_i B_j]$$

SWAP n & k IN THE PERMUTATION SYMBOL SO ks APPEAR AT THE ENDS, & SO THE STANDARD IDENTITY CAN BE APPLIED

$$= -\begin{vmatrix} \delta_{li} & \delta_{lj} \\ \delta_{ni} & \delta_{nj} \end{vmatrix} \frac{\partial}{\partial x_l}[A_i B_j]$$

$$= [\delta_{ni}\delta_{lj} - \delta_{li}\delta_{nj}] \frac{\partial}{\partial x_l}(A_i B_j)$$

$$= \delta_{ni}\delta_{lj} \frac{\partial}{\partial x_l}[A_i B_j] - \delta_{li}\delta_{nj} \frac{\partial}{\partial x_l}[A_i B_j]$$

...USING THE SUBSTITUTION PROPERTY OF δ...

$$= \frac{\partial}{\partial x_l}[A_n B_l] - \frac{\partial}{\partial x_l}[A_l B_n]$$

...BY PRODUCT RULE...

$$= \frac{\partial A_n}{\partial x_l} B_l + \frac{\partial B_l}{\partial x_l} A_n - \frac{\partial A_l}{\partial x_l} B_n - \frac{\partial B_n}{\partial x_l} A_l$$

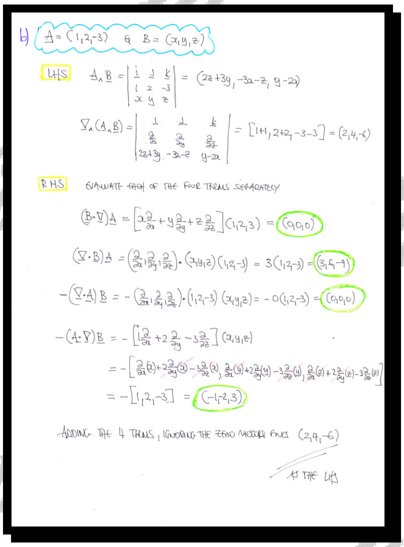
...REGROUP TO CREATE MEANINGFUL QUANTITIES...

$$= B_l \frac{\partial}{\partial x_l} A_n + A_n \frac{\partial B_l}{\partial x_l} - B_n \frac{\partial A_l}{\partial x_l} - A_l \frac{\partial}{\partial x_l} B_n$$

$$= [(\underline{B} \cdot \underline{\nabla})\underline{A} + \underline{A}(\underline{\nabla} \cdot \underline{B}) - \underline{B}(\underline{\nabla} \cdot \underline{A}) - (\underline{A} \cdot \underline{\nabla})\underline{B}]_k$$

• THUS

$$\underline{\nabla} \wedge (\underline{A} \wedge \underline{B}) = (\underline{B} \cdot \underline{\nabla})\underline{A} + (\underline{\nabla} \cdot \underline{B})\underline{A} - (\underline{A} \cdot \underline{\nabla})\underline{B} - (\underline{\nabla} \cdot \underline{A})\underline{B}$$



b) $\underline{A} = (1, 2, -3)$ & $\underline{B} = (x, y, z)$

LHS $\quad \underline{A} \wedge \underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -3 \\ x & y & z \end{vmatrix} = (2z+3y, -3x-z, y-2x)$

$$\underline{\nabla} \wedge (\underline{A} \wedge \underline{B}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+3y & -3x-z & y-2x \end{vmatrix} = [1+1, 2+2, -3-3] = (2, 4, -6)$$

RHS $\quad$ EVALUATE EACH OF THE FOUR TERMS SEPARATELY

$$(\underline{B} \cdot \underline{\nabla})\underline{A} = \left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right](1, 2, 3) = (0, 0, 0)$$

$$(\underline{\nabla} \cdot \underline{B})\underline{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x, y, z)(1, 2, -3) = 3(1, 2, -3) = (3, 6, -9)$$

$$-(\underline{\nabla} \cdot \underline{A})\underline{B} = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (1, 2, -3)(x, y, z) = -0(1, 2, -3) = (0, 0, 0)$$

$$-(\underline{A} \cdot \underline{\nabla})\underline{B} = -\left[1\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} - 3\frac{\partial}{\partial z}\right](x, y, z)$$

$$= -\left[\frac{\partial}{\partial x}(x) + 2\frac{\partial}{\partial y}(x) - 3\frac{\partial}{\partial z}(x), \frac{\partial}{\partial x}(y) + 2\frac{\partial}{\partial y}(y) - 3\frac{\partial}{\partial z}(y), \frac{\partial}{\partial x}(z) + 2\frac{\partial}{\partial y}(z) - 3\frac{\partial}{\partial z}(z)\right]$$

$$= -[1, 2, -3] = (-1, -2, 3)$$

ADDING THE 4 TERMS, IGNORING THE ZERO VECTORS FINDS $(2, 4, -6)$

AS THE LHS

LAPLACIAN

$$\nabla \cdot \nabla \varphi \equiv \nabla^2 \varphi$$

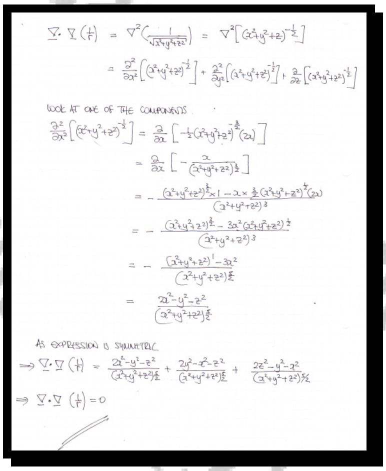
Question 1

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Determine the value of

$$\text{div}\left[\text{grad}\left(\frac{1}{r}\right)\right], \ r \neq 0 .$$

$$\boxed{\text{div}\left[\text{grad}\left(\frac{1}{r}\right)\right] = 0}$$



$$\underline{\nabla}\cdot\underline{\nabla}\left(\tfrac{1}{r}\right) = \nabla^2\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \nabla^2\left[(x^2+y^2+z)^{-\frac{1}{2}}\right]$$

$$= \frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right] + \frac{\partial^2}{\partial y^2}\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right] + \frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right]$$

LOOK AT ONE OF THE COMPONENTS

$$\frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{-\frac{1}{2}}\right] = \frac{\partial}{\partial x}\left[-\tfrac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)\right]$$

$$= \frac{\partial}{\partial x}\left[-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right]$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{3}{2}}\times 1 - x\times\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)}{(x^2+y^2+z^2)^3}$$

$$= -\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}$$

$$= -\frac{(x^2+y^2+z^2)^1 - 3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

AS EXPRESSION IS SYMMETRIC

$$\Rightarrow \underline{\nabla}\cdot\underline{\nabla}\left(\tfrac{1}{r}\right) = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} + \frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow \underline{\nabla}\cdot\underline{\nabla}\left(\tfrac{1}{r}\right) = 0$$

Question 2

A smooth scalar field is denoted by $\varphi = \varphi(x, y, z)$ and a smooth vector field is denoted by $\mathbf{A} = \mathbf{A}(x, y, z)$.

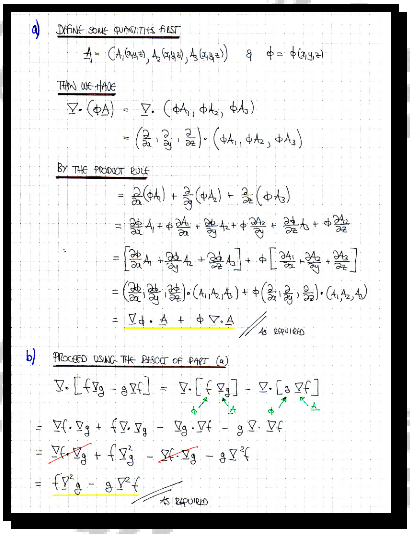
a) Use the standard definitions of vector operators to show that

$$\nabla \cdot (\varphi \mathbf{A}) = \nabla \varphi \cdot \mathbf{A} + \varphi \nabla \cdot \mathbf{A} .$$

b) Given further that f and g are functions of x, y and z, whose second partial derivatives exist, deduce that

$$\nabla \cdot (f \nabla g - g \nabla f) = f \nabla^2 g - g \nabla^2 f .$$

proof



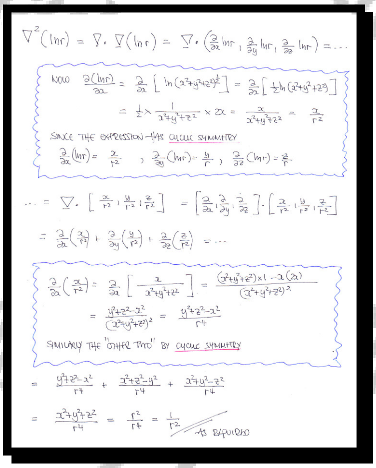
Question 3

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$, $r \neq 0$.

Show that

$$\nabla^2 (\ln r) = \frac{1}{r^2}.$$

proof



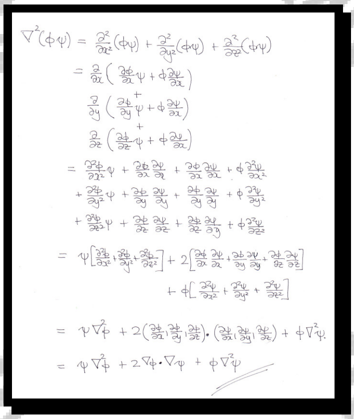
Question 4

It is given that $\varphi = \varphi(x, y, z)$ and $\psi = \psi(x, y, z)$ are twice differentiable functions.

Show, with a detailed method, that

$$\nabla^2(\varphi\psi) = \psi\nabla^2\varphi + 2\nabla\varphi \cdot \nabla\psi + \varphi\nabla^2\psi \; .$$

proof



$$\nabla^2(\phi\psi) = \frac{\partial^2}{\partial x^2}(\phi\psi) + \frac{\partial^2}{\partial y^2}(\phi\psi) + \frac{\partial^2}{\partial z^2}(\phi\psi)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\psi + \phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\psi + \phi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\psi + \phi\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}\psi + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \phi\frac{\partial^2\psi}{\partial x^2}$$
$$+ \frac{\partial^2\phi}{\partial y^2}\psi + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \phi\frac{\partial^2\psi}{\partial y^2}$$
$$+ \frac{\partial^2\phi}{\partial z^2}\psi + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial y} + \phi\frac{\partial^2\psi}{\partial z^2}$$

$$= \psi\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] + 2\left[\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z}\right] + \phi\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right]$$

$$= \psi\nabla^2\phi + 2\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)\cdot\left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right) + \phi\nabla^2\psi$$

$$= \psi\nabla^2\phi + 2\nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi$$

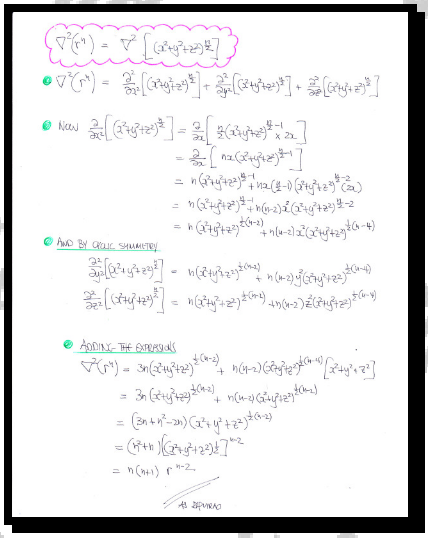
Question 5

$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show clearly that

$$\nabla^2\left(|\mathbf{r}|^n\right) = n(n+1)|\mathbf{r}|^{n-2}.$$

proof



$$\nabla^2(r^n) = \nabla^2\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right]$$

- $\nabla^2(r^n) = \frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right] + \frac{\partial^2}{\partial y^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right] + \frac{\partial^2}{\partial z^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right]$

- Now $\frac{\partial^2}{\partial x^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right] = \frac{\partial}{\partial x}\left[\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1} \times 2x\right]$

$= \frac{\partial}{\partial x}\left[nx(x^2+y^2+z^2)^{\frac{n}{2}-1}\right]$

$= n(x^2+y^2+z^2)^{\frac{n}{2}-1} + nx\left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{\frac{n}{2}-2}(2x)$

$= n(x^2+y^2+z^2)^{\frac{n}{2}-1} + n(n-2)x^2(x^2+y^2+z^2)^{\frac{n}{2}-2}$

$= n(x^2+y^2+z^2)^{\frac{1}{2}(n-2)} + n(n-2)x^2(x^2+y^2+z^2)^{\frac{1}{2}(n-4)}$

- AND BY CYCLIC SYMMETRY

$\frac{\partial^2}{\partial y^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right] = n(x^2+y^2+z^2)^{\frac{1}{2}(n-2)} + n(n-2)y^2(x^2+y^2+z^2)^{\frac{1}{2}(n-4)}$

$\frac{\partial^2}{\partial z^2}\left[(x^2+y^2+z^2)^{\frac{n}{2}}\right] = n(x^2+y^2+z^2)^{\frac{1}{2}(n-2)} + n(n-2)z^2(x^2+y^2+z^2)^{\frac{1}{2}(n-4)}$

- ADDING THE EXPRESSIONS

$\nabla^2(r^n) = 3n(x^2+y^2+z^2)^{\frac{1}{2}(n-2)} + n(n-2)(x^2+y^2+z^2)^{\frac{1}{2}(n-4)}\left[x^2+y^2+z^2\right]$

$= 3n(x^2+y^2+z^2)^{\frac{1}{2}(n-2)} + n(n-2)(x^2+y^2+z^2)^{\frac{1}{2}(n-2)}$

$= (3n+n^2-2n)(x^2+y^2+z^2)^{\frac{1}{2}(n-2)}$

$= (n^2+n)\left[(x^2+y^2+z^2)^{\frac{1}{2}}\right]^{n-2}$

$= n(n+1)\, r^{n-2}$

AS REQUIRED

Question 6

It is given that

$$\varphi(x,y,z) = z + \sinh x \sin y \, .$$

a) Verify that φ is a solution of Laplace's equation

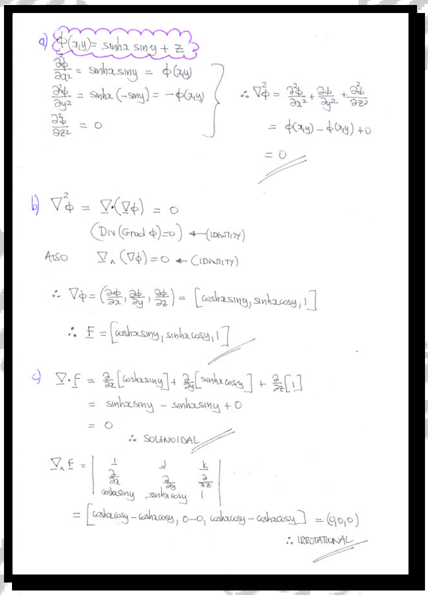
$$\nabla^2 \varphi = 0 \, .$$

b) Hence find a vector field $\mathbf{F}$, so that

$$\nabla \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{F} = \mathbf{0} \, .$$

c) Verify that $\mathbf{F}$ found in part **(b)** is solenoidal and irrotational.

$$\boxed{\mathbf{F} = (\cosh x \sin y)\mathbf{i} + (\sinh x \cos y)\mathbf{j} + \mathbf{k}}$$



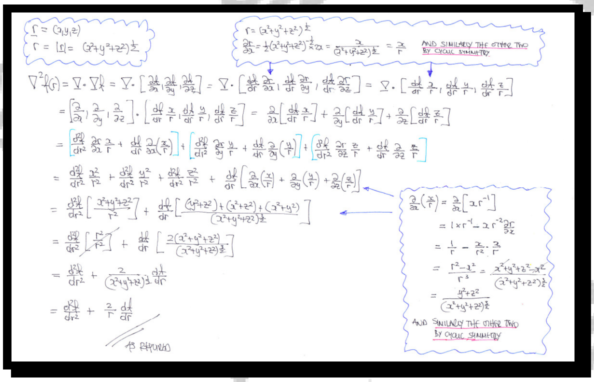
Question 7

A Cartesian position vector is denoted by $\mathbf{r}$ and $r = |\mathbf{r}|$.

Given that $f(r)$ is a twice differentiable function, show that

$$\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}, \quad r \neq 0 .$$

proof



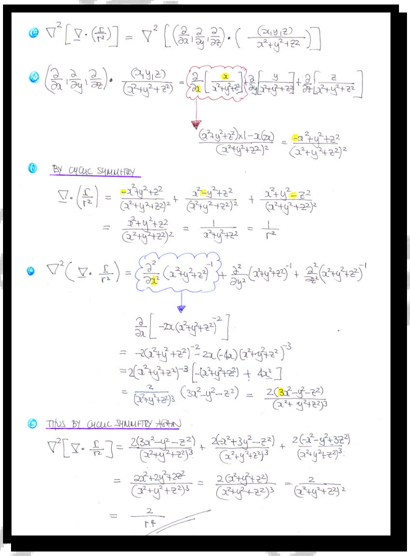
Question 8

$$\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

Show, with a detailed method, that

$$\nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{|\mathbf{r}|^2}\right)\right] = \frac{2}{|\mathbf{r}|^4}\,.$$

proof



Question 9

The scalar function φ is given below in terms of the non zero constants λ and μ.

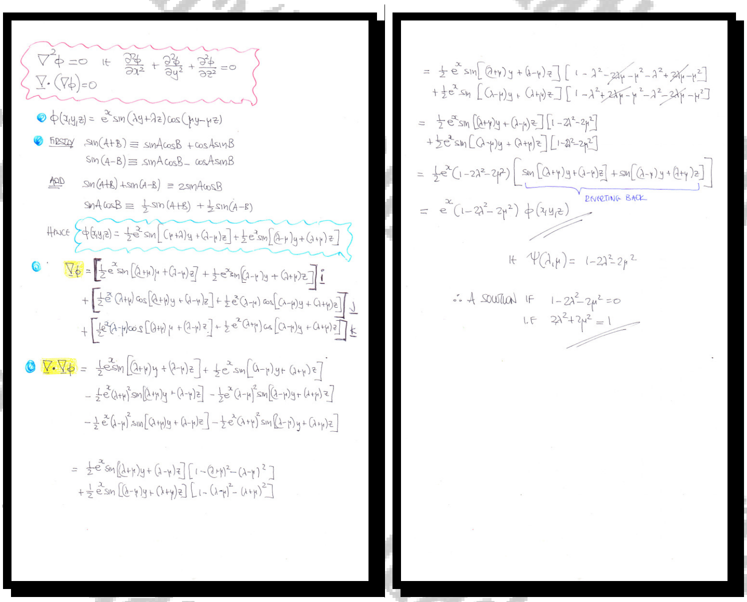
$$\varphi(x,y,z) = \mathrm{e}^{x}\sin(\lambda y + \lambda z)\cos(\mu y - \mu z).$$

Given that

$$\nabla\cdot(\nabla\varphi) = 0,$$

show, with a detailed method, that $\lambda^2 + \mu^2 = \frac{1}{2}$.

proof



Question 10

It is given that

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \equiv \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} .$$

a) Use the above identity to find a simplified expression for $\nabla \wedge (\nabla \wedge \mathbf{F})$, where $\mathbf{F}$ is a smooth vector field.

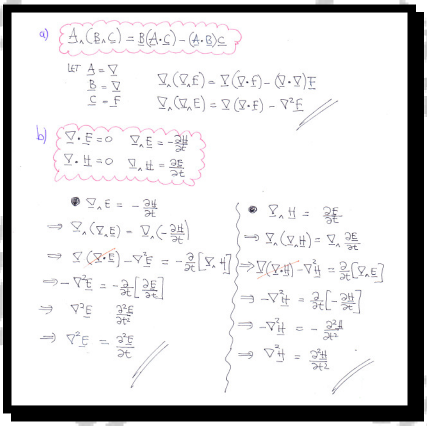
The smooth vector fields $\mathbf{E}$ and $\mathbf{H}$, satisfy the following relationships.

- $\nabla \cdot \mathbf{E} = 0$
- $\nabla \wedge \mathbf{E} = -\dfrac{\partial \mathbf{H}}{\partial t}$
- $\nabla \cdot \mathbf{H} = 0$
- $\nabla \wedge \mathbf{H} = \dfrac{\partial \mathbf{E}}{\partial t}$

b) Show that $\mathbf{E}$ and $\mathbf{H}$, satisfy the wave equation .

$$\nabla^2 \mathbf{U} = \frac{\partial^2 \mathbf{U}}{\partial t^2} .$$

$$\boxed{\nabla \wedge (\nabla \wedge \mathbf{F}) \equiv \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}}$$



a) $\underline{A} \wedge (\underline{B} \wedge \underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - (\underline{A}\cdot\underline{B})\underline{C}$

LET $\underline{A} = \nabla$, $\underline{B} = \nabla$, $\underline{C} = \underline{F}$

$\nabla \wedge (\nabla \wedge \underline{F}) = \nabla(\nabla\cdot\underline{F}) - (\nabla\cdot\nabla)\underline{F}$

$\nabla \wedge (\nabla \wedge \underline{F}) = \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F}$

b) $\nabla\cdot\underline{E} = 0$, $\nabla\wedge\underline{E} = -\frac{\partial \underline{H}}{\partial t}$, $\nabla\cdot\underline{H} = 0$, $\nabla\wedge\underline{H} = \frac{\partial \underline{E}}{\partial t}$

- $\nabla\wedge\underline{E} = -\frac{\partial \underline{H}}{\partial t}$

$\Rightarrow \nabla\wedge(\nabla\wedge\underline{E}) = \nabla\wedge\left(-\frac{\partial \underline{H}}{\partial t}\right)$

$\Rightarrow \nabla(\nabla\cdot\underline{E}) - \nabla^2\underline{E} = -\frac{\partial}{\partial t}[\nabla\wedge\underline{H}]$

$\Rightarrow -\nabla^2\underline{E} = -\frac{\partial}{\partial t}\left[\frac{\partial \underline{E}}{\partial t}\right]$

$\Rightarrow \nabla^2\underline{E} = \frac{\partial^2 \underline{E}}{\partial t^2}$

- $\nabla\wedge\underline{H} = \frac{\partial \underline{E}}{\partial t}$

$\Rightarrow \nabla\wedge(\nabla\wedge\underline{H}) = \nabla\wedge\frac{\partial \underline{E}}{\partial t}$

$\Rightarrow \nabla(\nabla\cdot\underline{H}) - \nabla^2\underline{H} = \frac{\partial}{\partial t}[\nabla\wedge\underline{E}]$

$\Rightarrow -\nabla^2\underline{H} = \frac{\partial}{\partial t}\left[-\frac{\partial \underline{H}}{\partial t}\right]$

$\Rightarrow -\nabla^2\underline{H} = -\frac{\partial^2 \underline{H}}{\partial t^2}$

$\Rightarrow \nabla^2\underline{H} = \frac{\partial^2 \underline{H}}{\partial t^2}$

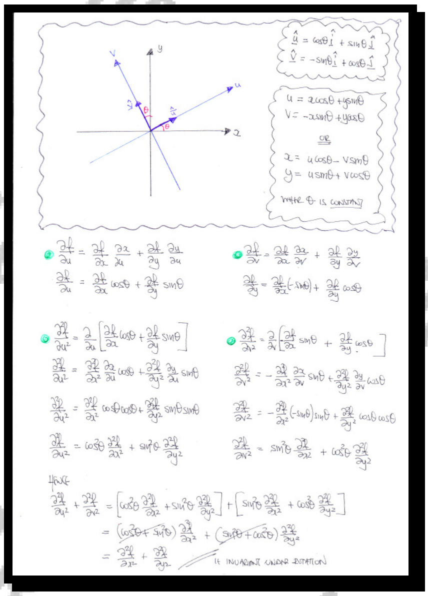
Question 11

The Laplacian operator ∇^2 in the standard two dimensional Cartesian system of coordinates is defined as

$$\nabla^2(\ ) \equiv \frac{\partial^2}{\partial x^2}(\ ) + \frac{\partial^2}{\partial y^2}(\ ).$$

Show that the two dimensional Laplacian operator is invariant under rotation.

proof



Question 12

It is given that if $\mathbf{F}$ is a smooth vector field, then

$$\nabla \wedge (\nabla \wedge \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}.$$

a) By using index summation notation, or otherwise, prove the validity of the above vector identity.

Maxwell's equations for the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, satisfy

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

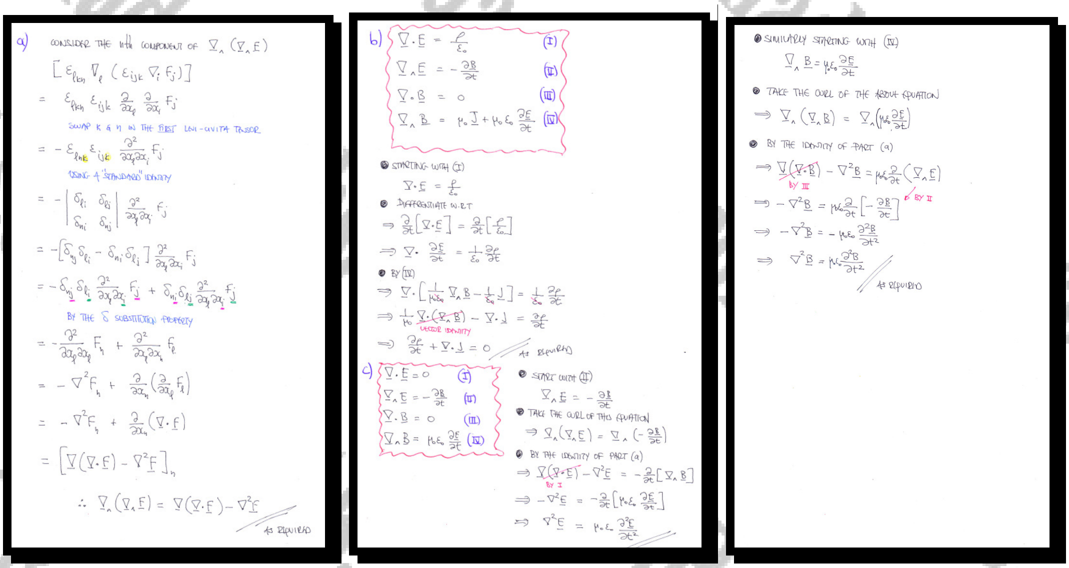
where ρ is the electric charge density, $\mathbf{J}$ is the current density, and μ_0 and ε_0 are positive constants.

b) Show that $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$.

c) Given further that $\rho = 0$ and $\mathbf{J} = \mathbf{0}$, show also that

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

proof

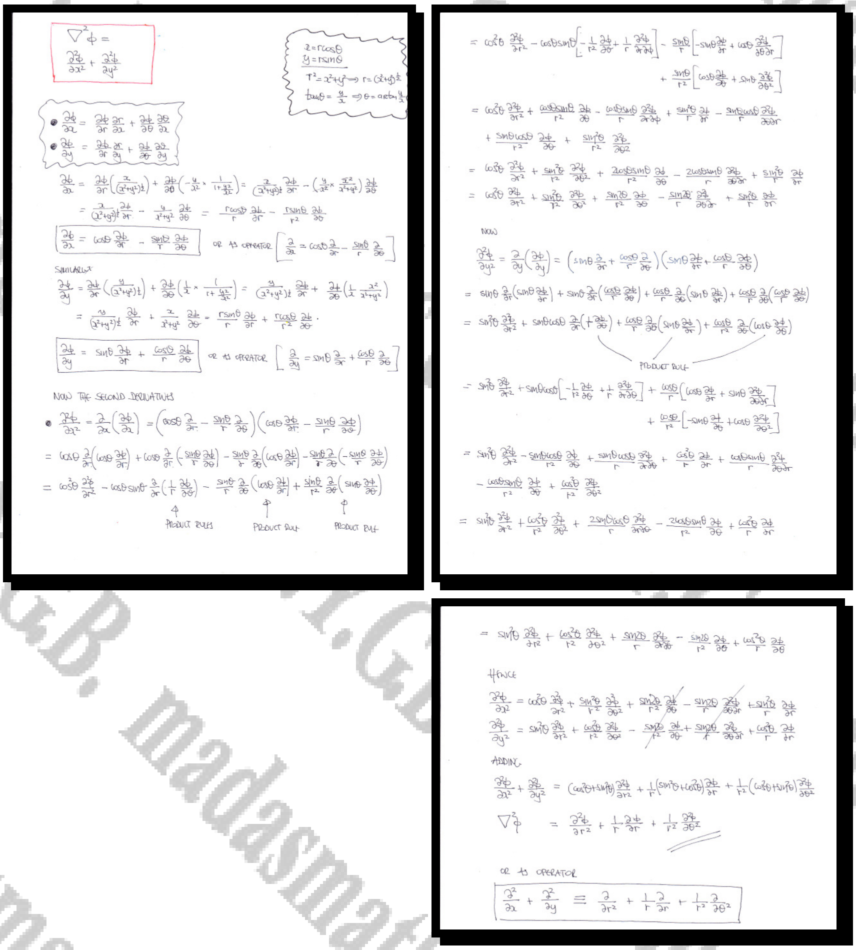


Question 13

Show that the Laplacian operator in the standard two dimensional Polar system of coordinates is given by

$$\nabla^2(\ ) \equiv \frac{\partial^2}{\partial r^2}(\ ) + \frac{1}{r}\frac{\partial}{\partial r}(\ ) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\ ).$$

proof



$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$x = r\cos\theta$, $y = r\sin\theta$

$r^2 = x^2 + y^2 \Rightarrow r = (x^2+y^2)^{\frac{1}{2}}$

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\frac{y}{x}$

- $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial x}$
- $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial y}$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\left(\frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right) + \frac{\partial \phi}{\partial \theta}\left(-\frac{y}{x^2} \times \frac{1}{1+\frac{y^2}{x^2}}\right) = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial \phi}{\partial r} - \left(\frac{y}{x^2} \times \frac{x^2}{x^2+y^2}\right)\frac{\partial \phi}{\partial \theta}$$

$$= \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial \phi}{\partial r} - \frac{y}{x^2+y^2}\frac{\partial \phi}{\partial \theta} = \frac{r\cos\theta}{r}\frac{\partial \phi}{\partial r} - \frac{r\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$\boxed{\frac{\partial \phi}{\partial x} = \cos\theta\frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}} \quad \text{OR AS OPERATOR} \quad \left[\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right]$$

SIMILARLY

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\left(\frac{y}{(x^2+y^2)^{\frac{1}{2}}}\right) + \frac{\partial \phi}{\partial \theta}\left(\frac{1}{x} \times \frac{1}{1+\frac{y^2}{x^2}}\right) = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial \phi}{\partial r} + \frac{\partial \phi}{\partial \theta}\left(\frac{1}{x}\frac{x^2}{x^2+y^2}\right)$$

$$= \frac{y}{(x^2+y^2)^{\frac{1}{2}}}\frac{\partial \phi}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial \phi}{\partial \theta} = \frac{r\sin\theta}{r}\frac{\partial \phi}{\partial r} + \frac{r\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta}.$$

$$\boxed{\frac{\partial \phi}{\partial y} = \sin\theta\frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}} \quad \text{OR AS OPERATOR} \quad \left[\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right]$$

NOW THE SECOND DERIVATIVES

- $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial \phi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial \phi}{\partial r}\right) + \cos\theta\frac{\partial}{\partial r}\left(-\frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial \phi}{\partial r}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} - \cos\theta\sin\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial \phi}{\partial r}\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right)$$

PRODUCT RULES PRODUCT RULE PRODUCT RULE

$$= \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} - \cos\theta\sin\theta\left[-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \phi}\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial \phi}{\partial r} + \cos\theta\frac{\partial^2 \phi}{\partial \theta \partial r}\right]$$
$$+ \frac{\sin\theta}{r^2}\left[\cos\theta\frac{\partial \phi}{\partial \theta} + \sin\theta\frac{\partial^2 \phi}{\partial \theta^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \phi} + \frac{\sin^2\theta}{r}\frac{\partial \phi}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}$$
$$+ \frac{\sin\theta\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} + \frac{\sin^2\theta}{r}\frac{\partial \phi}{\partial r}$$

$$= \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{\sin 2\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} + \frac{\sin^2\theta}{r}\frac{\partial \phi}{\partial r}$$

NOW

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta\frac{\partial \phi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial \phi}{\partial r}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial r}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$= \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} + \sin\theta\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial r}\right) + \frac{\cos\theta}{r^2}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial \phi}{\partial \theta}\right)$$

PRODUCT RULE

$$= \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} + \sin\theta\cos\theta\left[-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right] + \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial \phi}{\partial r} + \sin\theta\frac{\partial^2 \phi}{\partial \theta \partial r}\right]$$
$$+ \frac{\cos\theta}{r^2}\left[-\sin\theta\frac{\partial \phi}{\partial \theta} + \cos\theta\frac{\partial^2 \phi}{\partial \theta^2}\right]$$

$$= \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial \phi}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r}$$
$$- \frac{\cos\theta\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial \phi}{\partial r}$$

$$= \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin 2\theta}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{\sin 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial \phi}{\partial \theta}$$

HENCE

$$\frac{\partial^2 \phi}{\partial x^2} = \cos^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\sin 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} - \frac{\sin 2\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} + \frac{\sin^2\theta}{r}\frac{\partial \phi}{\partial r}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \sin^2\theta\frac{\partial^2 \phi}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\sin 2\theta}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{\sin 2\theta}{r}\frac{\partial^2 \phi}{\partial \theta \partial r} + \frac{\cos^2\theta}{r}\frac{\partial \phi}{\partial r}$$

ADDING

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}(\sin^2\theta + \cos^2\theta)\frac{\partial \phi}{\partial r} + \frac{1}{r^2}(\cos^2\theta + \sin^2\theta)\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

OR AS OPERATOR

$$\boxed{\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} \equiv \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}}$$