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VECTOR INTEGRALS

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TYPE $\int_C \varphi \, dr$

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Question 1

$$V(x, y, z) = 60xyz^2.$$

Evaluate the following integral along C , from $(3,1,1)$ to $(4,3,2)$.

$$\int_C V \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t + 2, \quad y = 2t - 1, \quad z = t.$$

$$\boxed{1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}}$$

Handwritten solution for the line integral problem:

$$V(x,y,z) = 60xyz^2 \quad \begin{matrix} x = t+2 & dx = dt \\ y = 2t-1 & dy = 2dt \\ z = t & dz = dt \end{matrix}$$

$$\begin{aligned} \int_C V \, d\mathbf{r} &= \int_{(3,1,1)}^{(4,3,2)} 60xyz^2 (dx, dy, dz) = \int_{t=1}^{t=2} 60(t+2)(2t-1)t^2 (dt, 2dt, dt) \\ &= \int_{t=1}^2 60(t+2)(2t-1)t^2 (1, 2, 1) dt \\ &= 60(1, 2, 1) \int_{t=1}^2 (2t^3 + 3t^2 - 2t^3) dt \\ &= 60(1, 2, 1) \left[\frac{2}{4}t^4 + \frac{3}{3}t^3 - \frac{2}{2}t^3 \right] \\ &= 60(1, 2, 1) \left[\left(\frac{2}{4}(2^4 - 1^4) + \frac{3}{3}(2^3 - 1^3) - \frac{2}{2}(2^3 - 1^3) \right) \right] \\ &= 1139(1, 2, 1)^T \\ &= 1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k} \end{aligned}$$

Question 2

$$\phi(x, y, z) \equiv 3x + 2y + z.$$

Evaluate the following integral along C , from $(1, 0, 0)$ to $(2, 2, 1)$,

$$\int_C \phi \, d\mathbf{r}, \quad d\mathbf{r} = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = t + 1, \quad y = 2t, \quad z = t^2.$$

$$\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}$$

Handwritten solution for the line integral problem:

$$\begin{aligned} \phi(x, y, z) &= 3x + 2y + z & x = t + 1 &\Rightarrow dx = dt \\ & & y = 2t &\Rightarrow dy = 2dt \\ & & z = t^2 &\Rightarrow dz = 2t dt \end{aligned}$$

$$\begin{aligned} \text{Then } \int_C \phi \, d\mathbf{r} &= \int_{(1,0,0)}^{(2,2,1)} (3x + 2y + z) (dx, dy, dz) = \int_{t=0}^{t=1} (3(t+1) + 2(2t) + t^2) (dt, 2dt, 2t dt) \\ &= \int_{t=0}^{t=1} (3t + 3 + 4t + t^2) (1, 2, 2t) dt \\ &= \int_{t=0}^{t=1} (7t + 3) (1, 2, 2t) dt \\ &= \int_0^1 [7t + 3, 14t + 6, 14t^2 + 6t] dt \\ &= \left[\frac{7}{2}t^2 + 3t, 7t^2 + 6t, \frac{14}{3}t^3 + 3t^2 \right]_0^1 \\ &= \left(\frac{7}{2} + 3, 7 + 6, \frac{14}{3} + 3 \right) = \left(\frac{13}{2}, 13, \frac{25}{3} \right) \\ &= \frac{13}{2}\mathbf{i} + 13\mathbf{j} + \frac{25}{3}\mathbf{k} \end{aligned}$$

Question 3

$$F(x, y, z) = xyz.$$

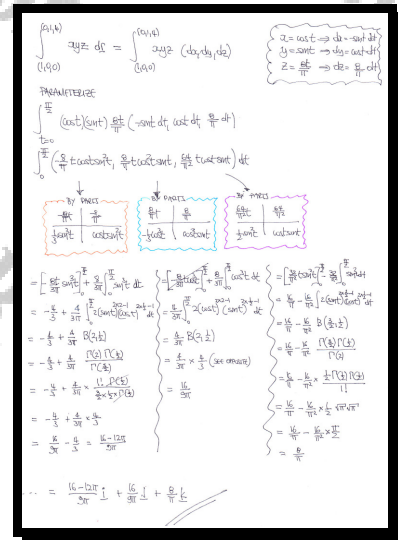
Evaluate the following integral along C , from $(1,0,0)$ to $(0,1,4)$,

$$\int_C F \, dr, \quad dr = (dx, dy, dz)^T,$$

where C is the curve with parametric equations

$$x = \cos t, \quad y = \sin t, \quad z = \frac{8t}{\pi}.$$

$$\frac{16-12\pi}{9\pi} \mathbf{i} + \frac{16}{9\pi} \mathbf{j} + \frac{8}{\pi} \mathbf{k}$$



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TYPE $\int_V F dV$

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Question 1

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} \, dV,$$

where V is the finite region in the first octant bounded by the planes with equations

$$x = 2, \quad y = 3 \quad \text{and} \quad z = 4.$$

$$\boxed{36\mathbf{i} + 48\mathbf{j} - 32\mathbf{k}}$$

Handwritten solution for the vector integral problem:

$$\begin{aligned} \int_V (xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}) \, dV &= \int_{z=0}^4 \int_{y=0}^3 \int_{x=0}^2 (xy\mathbf{i} + z\mathbf{j} - x^2\mathbf{k}) \, dx \, dy \, dz \\ &= \int_{z=0}^4 \int_{y=0}^3 [xy^2\mathbf{i} + zxy\mathbf{j} - \frac{1}{3}x^3\mathbf{k}]_{x=0}^2 \, dy \, dz \\ &= \int_{z=0}^4 \int_{y=0}^3 (2y^2\mathbf{i} + 2zy\mathbf{j} - \frac{8}{3}\mathbf{k}) \, dy \, dz = \int_{z=0}^4 [2y^3\mathbf{i} + zy^2\mathbf{j} - \frac{8y}{3}\mathbf{k}]_{y=0}^3 \, dz \\ &= \int_{z=0}^4 (9z\mathbf{i} + 9z\mathbf{j} - 8\mathbf{k}) \, dz = [\frac{9}{2}z^2\mathbf{i} + \frac{9}{2}z^2\mathbf{j} - 8z\mathbf{k}]_{z=0}^4 \\ &= (36\mathbf{i} + 36\mathbf{j} - 32\mathbf{k}) \end{aligned}$$

Question 2

$$\mathbf{F}(x, y, z) \equiv z\mathbf{i} + \mathbf{j} + y\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} dV,$$

where V is the finite region in the first octant bounded by the plane with equation

$$2x + y + z = 6.$$

$$\boxed{27\mathbf{i} + 18\mathbf{j} + 27\mathbf{k}}$$

Handwritten solution for the vector integral problem. The solution shows the triple integral of the vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + \mathbf{j} + y\mathbf{k}$ over the volume V bounded by the plane $2x + y + z = 6$ in the first octant. The solution includes a diagram of the tetrahedron and the final result $27\mathbf{i} + 18\mathbf{j} + 27\mathbf{k}$.

$$\int_V \mathbf{F} dV = \int_V (z\mathbf{i} + \mathbf{j} + y\mathbf{k}) dV = \int_0^3 \int_0^{6-2x} \int_0^{6-2x-y} (z\mathbf{i} + \mathbf{j} + y\mathbf{k}) dz dy dx$$

$$= \int_0^3 \int_0^{6-2x} \left[\frac{1}{2}z^2\mathbf{i} + z\mathbf{j} + yz\mathbf{k} \right]_0^{6-2x-y} dy dx$$

$$= \int_0^3 \int_0^{6-2x} \left(\frac{1}{2}(6-2x-y)^2\mathbf{i} + (6-2x-y)\mathbf{j} + y(6-2x-y)\mathbf{k} \right) dy dx$$

$$= \int_0^3 \left[\frac{1}{6}(6-2x-y)^3\mathbf{i} + \frac{1}{2}(6-2x-y)^2\mathbf{j} + \frac{1}{2}y(6-2x-y)^2\mathbf{k} \right]_0^{6-2x} dx$$

$$= \int_0^3 \left[\frac{1}{6}(6-2x)^3\mathbf{i} + \frac{1}{2}(6-2x)^2\mathbf{j} + \frac{1}{2}(6-2x)^2\mathbf{k} \right] dx$$

$$= \left[\frac{1}{24}(6-2x)^4\mathbf{i} + \frac{1}{6}(6-2x)^3\mathbf{j} + \frac{1}{6}(6-2x)^3\mathbf{k} \right]_0^3$$

$$= \left[\frac{1}{24}(2)^4\mathbf{i} + \frac{1}{6}(2)^3\mathbf{j} + \frac{1}{6}(2)^3\mathbf{k} \right] - \left[\frac{1}{24}(6)^4\mathbf{i} + \frac{1}{6}(6)^3\mathbf{j} + \frac{1}{6}(6)^3\mathbf{k} \right]$$

$$= \left[\frac{16}{24}\mathbf{i} + \frac{8}{6}\mathbf{j} + \frac{8}{6}\mathbf{k} \right] - \left[\frac{1296}{24}\mathbf{i} + \frac{216}{6}\mathbf{j} + \frac{216}{6}\mathbf{k} \right]$$

$$= \left[\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{4}{3}\mathbf{k} \right] - \left[54\mathbf{i} + 36\mathbf{j} + 36\mathbf{k} \right]$$

$$= \left[\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + \frac{4}{3}\mathbf{k} - 54\mathbf{i} - 36\mathbf{j} - 36\mathbf{k} \right]$$

$$= \left[-\frac{160}{3}\mathbf{i} - \frac{100}{3}\mathbf{j} - \frac{100}{3}\mathbf{k} \right]$$

Question 3

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + 2z\mathbf{j} + y\mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} \, dV,$$

where V is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 9, \quad 0 \leq z \leq 2.$$

$$18\pi(\mathbf{i} + 2\mathbf{j})$$

$\mathbf{F}(x,y,z) = (1, 2z, y)$
 $\int_V \mathbf{F} \, dV = \int (1, 2z, y) \, dV$
 convert into cylindrical coordinates
 $= \int_0^2 \int_0^{2\pi} \int_0^3 (1, 2z, r \sin \theta) \, r \, dr \, d\theta \, dz$
 $= \int_0^2 \int_0^{2\pi} \int_0^3 (1, 2z, r \sin \theta) \, r \, dr \, d\theta \, dz = \int_0^2 \int_0^{2\pi} [4r^2, 2zr^2, r^2 \sin \theta]_0^3 \, d\theta \, dz$
 $= \int_0^2 \int_0^{2\pi} (12, 6z, r^2 \sin \theta) \, d\theta \, dz = \int_0^2 \int_0^{2\pi} (12, 6z, 9 \sin \theta) \, d\theta \, dz$
 $= \int_0^2 [12\theta, 6z\theta, -9 \cos \theta]_0^{2\pi} \, dz = \int_0^2 (24\pi, 12z \cdot 2\pi, -9 \cos 2\pi + 9 \cos 0) \, dz$
 $= \int_0^2 (24\pi, 24\pi z, 0) \, dz = [24\pi z, 12\pi z^2, 0]_0^2 = (48\pi, 48\pi, 0)$
 $\therefore \int_V \mathbf{F} \, dV = 18\pi(\mathbf{i} + 2\mathbf{j})$

Question 4

$$\mathbf{F}(x, y, z) \equiv \frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k}.$$

Evaluate the vector integral

$$\int_V \mathbf{F} \, dV,$$

where V is the finite region enclosed by the cylinder with equation

$$x^2 + y^2 = 4, \quad 0 \leq z \leq 3.$$

$2\mathbf{i} + \mathbf{j}$

Handwritten solution for the vector integral problem:

$$\int_V \mathbf{F} \, dV = \int \left(\frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right) dV$$

SKOTUH INTO CYLINDRICAL POLAR (r, θ, z)

$$= \int_{z=0}^3 \int_{\theta=0}^{2\pi} \int_{r=0}^2 \left(\frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right) (r \, dr \, d\theta \, dz)$$

$$= \int_{z=0}^3 \int_{\theta=0}^{2\pi} \int_{r=0}^2 \left(\frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{r \sin \theta}{9\pi} \mathbf{k} \right) r \, dr \, d\theta \, dz$$

$$= \int_{z=0}^3 \int_{\theta=0}^{2\pi} \left[\frac{1}{12\pi} r^2 \mathbf{i} + \frac{z}{36\pi} r^2 \mathbf{j} + \frac{r^3 \sin \theta}{27\pi} \mathbf{k} \right]_{r=0}^2 d\theta \, dz$$

$$= \int_{z=0}^3 \int_{\theta=0}^{2\pi} \left(\frac{1}{3} \mathbf{i} + \frac{z}{9\pi} \mathbf{j} + \frac{8 \sin \theta}{27\pi} \mathbf{k} \right) d\theta \, dz$$

$$= \int_{z=0}^3 \left[\frac{1}{3} \theta \mathbf{i} + \frac{z}{9\pi} \theta \mathbf{j} + \frac{8 \cos \theta}{27\pi} \mathbf{k} \right]_{\theta=0}^{2\pi} dz$$

$$= \int_{z=0}^3 \left(\frac{2}{3} \mathbf{i} + \frac{2z}{9\pi} \mathbf{j} + 0 \right) dz$$

$$= \left[\frac{2}{3} z \mathbf{i} + \frac{2z^2}{18\pi} \mathbf{j} \right]_0^3$$

$$= \left(\frac{2}{3} \cdot 3 \mathbf{i} + \frac{2 \cdot 9}{18\pi} \mathbf{j} \right)$$

$$= (2\mathbf{i} + \mathbf{j})$$

Diagram: A cylinder with radius 2 and height 3. The vector field $\mathbf{F} = \left(\frac{1}{6\pi} \mathbf{i} + \frac{z}{18\pi} \mathbf{j} + \frac{y}{9\pi} \mathbf{k} \right)$ is shown. The volume element $dV = r \, dr \, d\theta \, dz$ is indicated.

Question 5

$$\mathbf{F}(x, y, z) \equiv 3i + -yj + 6zk.$$

Evaluate the vector integral

$$\int_V \mathbf{F} \, dV,$$

where V is the finite region enclosed by the hemisphere with equation

$$x^2 + y^2 + z^2 = 4, \quad z \geq 0.$$

16πi

$\mathbf{F} = (3, -y, 6z)$
 $x = 2 \cos \theta \sin \phi$
 $y = 2 \sin \theta \sin \phi$
 $z = 2 \cos \phi$
 $dV = r^2 \sin \phi \, dr \, d\theta \, d\phi$

$\int_V \mathbf{F} \, dV = \int_V (3i - yj + 6zk) \, dV$
 $= \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 (3 - 2 \sin \theta \sin^2 \phi + 12 \cos \phi) r^2 \sin \phi \, dr \, d\theta \, d\phi$
 $= \int_0^{2\pi} \int_0^{\pi/2} (3r^3 - \frac{2}{3} r^3 \sin^2 \phi + 4r^3 \cos \phi) \sin \phi \, d\theta \, d\phi$
 $= \int_0^{2\pi} \int_0^{\pi/2} (24 \cos \phi - \frac{16}{3} \sin^4 \phi + 32 \cos \phi) \, d\theta \, d\phi$
 $= \int_0^{2\pi} (24 \sin \phi - \frac{16}{3} \cos \phi + 32 \sin \phi) \, d\theta$
 $= \int_0^{2\pi} (56 \sin \phi - \frac{16}{3} \cos \phi) \, d\theta$
 $= [-56 \cos \phi - \frac{16}{3} \sin \phi]_0^{\pi/2} \cdot 2\pi$
 $= (0 - \frac{16}{3}) - (-56 - 0) \cdot 2\pi$
 $= (112 - \frac{32}{3}) \pi = 16\pi i$

Question 6

The finite region V in the first octant, is bounded by the surfaces with equations

$$y = 4 - x^2 \quad \text{and} \quad y = 4 - z^2.$$

Given that $\mathbf{F} = \frac{1}{8}\mathbf{i} + 3y^2\mathbf{j} - \frac{1}{4}\mathbf{k}$ determine

$$\int_V \mathbf{F} \, dv.$$

$$\mathbf{i} + 64\mathbf{j} - 2\mathbf{k}$$

The handwritten solution shows the following steps:

- Diagram:** A 3D coordinate system with x, y, and z axes. The region V is bounded by the surfaces $y = 4 - x^2$ and $y = 4 - z^2$ in the first octant.
- Divergence:** $\text{div } \mathbf{F} = \frac{\partial}{\partial x}(\frac{1}{8}) + \frac{\partial}{\partial y}(3y^2) + \frac{\partial}{\partial z}(-\frac{1}{4}) = 0 + 6y + 0 = 6y$.
- Volume Integral:** $\int_V \mathbf{F} \cdot d\mathbf{v} = \int_0^2 \int_0^{4-x^2} \int_0^{4-z^2} 6y \, dz \, dy \, dx$.
- Integration:** The integral is evaluated by first integrating with respect to z , then y , and finally x . A substitution $u = 4 - x^2$ is used to simplify the final integral.
- Final Answer:** $\mathbf{i} + 64\mathbf{j} - 2\mathbf{k}$.

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TYPE $\int_S F \, dS$

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Question 1

$$F(x, y, z) \equiv x + y + z.$$

Evaluate the integral

$$\int_S F \, dS,$$

where S is the plane surface with equation

$$2x + y + 2z = 6, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

$$36\mathbf{i} + 18\mathbf{j} + 36\mathbf{k}$$

Handwritten solution for the surface integral problem. The solution includes a 3D diagram of the triangular surface in the first octant, a normal vector \mathbf{n} , and a detailed step-by-step calculation of the surface integral.

Diagram: A 3D coordinate system showing the plane $2x + y + 2z = 6$ in the first octant. The vertices are at $(3, 0, 0)$, $(0, 6, 0)$, and $(0, 0, 3)$. The normal vector \mathbf{n} is shown pointing upwards. The surface is shaded blue.

Equation: $F(x, y, z) = x + y + z$

Equation: $2x + y + 2z = 6$

Equation: $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Equation: $|\mathbf{n}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$

Equation: $\frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

Equation: $\frac{dS}{dx dy} = \frac{1}{\sqrt{1 + 4 + 4}} = \frac{1}{3}$

Equation: $\int_S F \, dS = \int_R (x + y + z) \frac{1}{3} \, dx \, dy$

Equation: $= \frac{1}{3} \int_0^3 \int_0^{6-2x} (x + y + \frac{6-2x-y}{2}) \, dy \, dx$

Equation: $= \frac{1}{3} \int_0^3 \left[xy + \frac{1}{2}y^2 + \frac{6y}{2} - \frac{2xy}{2} - \frac{y^2}{2} \right]_0^{6-2x} \, dx$

Equation: $= \frac{1}{3} \int_0^3 \left[x(6-2x) + \frac{1}{2}(6-2x)^2 + 3(6-2x) - x(6-2x) - \frac{(6-2x)^2}{2} \right] \, dx$

Equation: $= \frac{1}{3} \int_0^3 \left[6x - 2x^2 + \frac{1}{2}(36 - 24x + 4x^2) + 18 - 6x - x(6-2x) - \frac{1}{2}(36 - 24x + 4x^2) \right] \, dx$

Equation: $= \frac{1}{3} \int_0^3 \left[6x - 2x^2 + 18 - 12x + 2x^2 + 18 - 6x - 6x + 2x^2 - 18 + 12x - 2x^2 \right] \, dx$

Equation: $= \frac{1}{3} \int_0^3 \left[36 - 2x^2 \right] \, dx$

Equation: $= \frac{1}{3} \left[36x - \frac{2}{3}x^3 \right]_0^3$

Equation: $= \frac{1}{3} \left[36 \cdot 3 - \frac{2}{3} \cdot 27 \right]$

Equation: $= \frac{1}{3} [108 - 18]$

Equation: $= \frac{1}{3} [90] = 30$

Equation: $\mathbf{F} = 30\mathbf{i} + 15\mathbf{j} + 30\mathbf{k}$

Question 2

$$\varphi(x, y, z) \equiv \frac{3}{4}xyz.$$

Evaluate the integral

$$\int_S \varphi \, dS,$$

where S is the curved surface of the cylinder with equation

$$x^2 + y^2 = 4, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2.$$

4i + 4j

$\varphi(x, y, z) = \frac{3}{4}xyz$
 $\int_S \varphi \, dS = \int_S \frac{3}{4}xyz \, dS$
 WE NEED THE UNIT NORMAL TO THE CURVED SURFACE
 LET $f(x, y, z) = x^2 + y^2 - 4$ (3-D CURVED SURFACE)
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 0)$
 TAKE $\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y, 0)}{\sqrt{4x^2 + 4y^2}} = \frac{(x, y, 0)}{\sqrt{x^2 + y^2}} = \frac{(x, y, 0)}{2}$
 $\dots = \int_S \frac{3}{4}xyz \cdot \frac{1}{2}(x, y, 0) \, dS = \frac{3}{8} \int_S (x^2y, xy^2, 0) \, dS$
 SWITCH NOW TO POLAR COORDINATES IN CIRCULAR PLANE (WE CAN ALSO PARAMETERISE THE CURVE OR USE POLAR)
 $= \frac{3}{8} \int_0^2 \int_0^{\pi/2} (r^2 \cos^2 \theta \cdot r \sin^2 \theta \cdot r) \cdot 2 \, d\theta \, dz$
 $= \frac{3}{4} \int_0^2 \int_0^{\pi/2} r^4 \cos^2 \theta \sin^2 \theta \, d\theta \, dz$
 $= 6 \int_0^2 \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta \, d\theta \, dz = 6 \int_0^2 \left[\frac{1}{8} \sin 4\theta \right]_0^{\pi/2} dz$
 $= 6 \int_0^2 (0 + \frac{1}{8} \sin 2\theta) \, dz = 2 \int_0^2 \left(\frac{1}{4} \sin 2\theta \right) dz$
 $= 2 \left[\frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = 2(2 - 0) = 4i + 4j$

THE END OF COURSE TO THE SURFACE INTEGRAL IS CALCULATED BY PROJECTING ONTO THE CURVE (OR USE POLAR)
 $\int_S \varphi \, dS = \int_S \frac{3}{4}xyz \, dS$
 THE NORMAL OF THE CURVE POINTS IN THE CURVED SURFACE
 $dS = \frac{dx \, dy}{\sqrt{1 - \frac{\partial z}{\partial x}^2 - \frac{\partial z}{\partial y}^2}}$
 $dS = \frac{dx \, dy}{1}$
 $= \frac{3}{4} \int_0^2 \int_0^{\pi/2} (x^2y, xy^2, 0) \cdot \frac{1}{2}(x, y, 0) \, dx \, dy \, dz$
 $= \frac{3}{8} \int_0^2 \int_0^{\pi/2} (x^3y, x^2y^2, 0) \, dx \, dy \, dz$
 $= \frac{3}{8} \int_0^2 \int_0^{\pi/2} \left[\frac{1}{4}x^4y, \frac{1}{3}x^2y^3, 0 \right]_0^{\pi/2} dy \, dz$
 $= \frac{3}{8} \int_0^2 \left(\frac{1}{4} \cdot \frac{1}{3} \right) dy \, dz$
 $= 2 \int_0^2 \left(\frac{1}{4} \cdot \frac{1}{3} \right) dz$
 $= 2 \left[\frac{1}{12} \cdot \frac{1}{3} \right]_0^2$
 $= 4i + 4j$

Question 3

$$\varphi(x, y, z) \equiv \frac{1}{2}xyz^2.$$

Evaluate the integral

$$\int_S \varphi \, dS,$$

where S is the curved surface of the cylinder with equation

$$x^2 + y^2 = 9, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 2.$$

$$\boxed{12\mathbf{i} + 12\mathbf{j}}$$

Handwritten solution for the surface integral problem:

$\int_S \varphi \, dS = \int_S \varphi \, dS = \int_S \frac{1}{2}xyz^2 \, dS$
 $= \int_S \left(\frac{1}{2}x^2y^2 + \frac{1}{2}xy^2z^2 \right) \, dS = \dots$

SWITCH INTO CYLINDRICAL PLANE $\begin{cases} x = 3\cos\theta \\ y = 3\sin\theta \\ z = z \end{cases}$
 $dS = 9 \, d\theta \, dz$

$= \int_{z=0}^2 \int_{\theta=0}^{\pi/2} \left[\frac{1}{2}(9\cos^2\theta)(9\sin^2\theta) + \frac{1}{2}(9\cos\theta)(9\sin\theta)z^2 \right] (9 \, d\theta \, dz)$
 $= \int_{z=0}^2 \int_{\theta=0}^{\pi/2} \left[\frac{81}{2}\cos^2\theta\sin^2\theta + \frac{81}{2}\cos\theta\sin\theta z^2 \right] \, d\theta \, dz$
 $= \int_{z=0}^2 \left[\frac{81}{2}\cos^2\theta\sin^2\theta + \frac{81}{2}\cos\theta\sin\theta z^2 \right]_{\theta=0}^{\pi/2} \, dz$
 $= \int_{z=0}^2 \left(0 + \frac{81}{2}z^2 - 0 \right) \, dz = \int_{z=0}^2 \left(\frac{81}{2}z^2 \right) \, dz$
 $= \left[\frac{81}{2} \cdot \frac{2}{3}z^3 \right]_0^2 = (12, 12, 0) = 12\mathbf{i} + 12\mathbf{j}$

$\varphi(x, y, z) = \frac{1}{2}xyz^2$
 $\frac{\partial \varphi}{\partial x} = \frac{1}{2}yz^2$
 $\frac{\partial \varphi}{\partial y} = \frac{1}{2}xz^2$
 $\frac{\partial \varphi}{\partial z} = xyz$
 $\mathbf{n} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = \left(\frac{1}{2}yz^2, \frac{1}{2}xz^2, xyz \right)$

Question 4

φ(x, y, z) ≡ 2x + 2y.

Evaluate the integral

∫_S φ ds,

where S is the curved surface of the sphere with equation

x^2 + y^2 + z^2 = 1, x ≥ 0, y ≥ 0, z ≥ 0.

[] , [1/3 [(π+2)i + (π+2)j + 4k]]

SPOT BY TRYING UP THE INTEGRAL - NONUSUAL CURVE A SPHERICAL SECTION

$$\int_S \phi ds = \int_S (2x+2y) ds$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} (2\sin\theta \cos\theta + 2\cos\theta \sin\theta) d\theta d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} 2\sin\theta \cos\theta (1 + \cos\theta) d\theta d\phi$$

NOTE THAT THE 'θ SECTION' CAN BE DONE WITH 2x1/2 cosθ - 1/2 cos³θ

$$\int_0^{\pi/2} 2\sin\theta \cos\theta (1 + \cos\theta) d\theta = \int_0^{\pi/2} (2\sin\theta \cos\theta + 2\sin\theta \cos^2\theta) d\theta$$

$$= \int_0^{\pi/2} 2\sin\theta \cos\theta d\theta + \int_0^{\pi/2} 2\sin\theta \cos^2\theta d\theta$$

$$= \int_0^{\pi/2} -\cos\theta d\theta + \int_0^{\pi/2} -\frac{2}{3}\cos^3\theta d\theta$$

$$= \left[-\cos\theta - \frac{2}{9}\cos^3\theta\right]_0^{\pi/2} = \left[0 - \left(-1 - \frac{2}{9}\right)\right] = 1 + \frac{2}{9}$$

∴ ∫_S φ ds = (1 + 2/9) * 2π = (11/9) * 2π = 22π/9

FOR THE VECTOR PART:

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(2, 2, 0)}{\sqrt{8}} = \frac{1}{\sqrt{2}}(1, 1, 0)$$

∴ ∫_S φ ds = ∫_S (2x+2y) * (1/√2)(1, 1, 0) dA = ∫_S (2x+2y)/√2 dA = √2 ∫_S (x+y) dA

∴ ∫_S φ ds = √2 * [∫_0^{\pi/2} ∫_0^{\pi/2} (\sin\theta \cos\theta + \cos\theta \sin\theta) d\theta d\phi] = √2 * [2π * (1 + 2/9)] = 22√2π/9

NOTE THAT THE 'θ SECTION' CAN BE DONE WITH 2x1/2 cosθ - 1/2 cos³θ

$$\int_0^{\pi/2} 2\sin\theta \cos\theta (1 + \cos\theta) d\theta = \int_0^{\pi/2} (2\sin\theta \cos\theta + 2\sin\theta \cos^2\theta) d\theta$$

$$= \int_0^{\pi/2} -\cos\theta d\theta + \int_0^{\pi/2} -\frac{2}{3}\cos^3\theta d\theta$$

$$= \left[-\cos\theta - \frac{2}{9}\cos^3\theta\right]_0^{\pi/2} = \left[0 - \left(-1 - \frac{2}{9}\right)\right] = 1 + \frac{2}{9}$$

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∴ ∫_S φ ds = ∫_S (2x+2y) * (1/√2)(1, 1, 0) dA = ∫_S (2x+2y)/√2 dA = √2 ∫_S (x+y) dA

∴ ∫_S φ ds = √2 * [∫_0^{\pi/2} ∫_0^{\pi/2} (\sin\theta \cos\theta + \cos\theta \sin\theta) d\theta d\phi] = √2 * [2π * (1 + 2/9)] = 22√2π/9

NOTE THAT THE 'θ SECTION' CAN BE DONE WITH 2x1/2 cosθ - 1/2 cos³θ

$$\int_0^{\pi/2} 2\sin\theta \cos\theta (1 + \cos\theta) d\theta = \int_0^{\pi/2} (2\sin\theta \cos\theta + 2\sin\theta \cos^2\theta) d\theta$$

$$= \int_0^{\pi/2} -\cos\theta d\theta + \int_0^{\pi/2} -\frac{2}{3}\cos^3\theta d\theta$$

$$= \left[-\cos\theta - \frac{2}{9}\cos^3\theta\right]_0^{\pi/2} = \left[0 - \left(-1 - \frac{2}{9}\right)\right] = 1 + \frac{2}{9}$$

∴ ∫_S φ ds = (1 + 2/9) * 2π = (11/9) * 2π = 22π/9

FOR THE VECTOR PART:

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{(2, 2, 0)}{\sqrt{8}} = \frac{1}{\sqrt{2}}(1, 1, 0)$$

∴ ∫_S φ ds = ∫_S (2x+2y) * (1/√2)(1, 1, 0) dA = ∫_S (2x+2y)/√2 dA = √2 ∫_S (x+y) dA

∴ ∫_S φ ds = √2 * [∫_0^{\pi/2} ∫_0^{\pi/2} (\sin\theta \cos\theta + \cos\theta \sin\theta) d\theta d\phi] = √2 * [2π * (1 + 2/9)] = 22√2π/9

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TYPE $\int_S \mathbf{F} \, dS$

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Question 1

The Cartesian equation of a surface S is

$$z = x^2 + y^2, \quad z \leq 1.$$

Evaluate the surface integral

$$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS,$$

where $\hat{\mathbf{n}}$ is an outward normal unit vector field to S , and ϕ is the function with Cartesian equation

$$\phi(x, y, z) = y.$$

$\pi\mathbf{i}$

$\nabla \phi = (0, 1, 0)$
 $\mathbf{n} = (2x, 2y, -1)$
 $|\mathbf{n}| = \sqrt{4x^2 + 4y^2 + 1}$
 $\hat{\mathbf{n}} = \frac{(2x, 2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$
 $\phi(x, y, z) = y$
 $\nabla \phi = (0, 1, 0)$

$\hat{\mathbf{n}} \wedge \nabla \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & 2y & -1 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2x & 2y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$
 $= \begin{bmatrix} 1 & -2x & 2y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS = \dots$ Project onto the $x-y$ plane $dS = \frac{dx \, dy}{\sqrt{1 - 4x^2 - 4y^2}}$
 $= \int_R \begin{bmatrix} 1 & -2x & 2y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{dx \, dy}{\sqrt{1 - 4x^2 - 4y^2}}$
 $= \int_R \begin{bmatrix} 1 & -2x & 2y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{1 - 4x^2 - 4y^2}} \, dx \, dy = \int_R (1, 0, 2x) \frac{dx \, dy}{\sqrt{1 - 4x^2 - 4y^2}}$

$= \int_R (1, 0, 2x) \, dx \, dy$
 Switch the order of axes
 $= \int_0^1 \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} (1, 0, 2x) \, dx \, dy$
 $= \int_0^1 \left[x, 0, x^2 \right]_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \, dy$
 $= \int_0^1 \left[\frac{1}{2} x^2 \right]_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \, dy = \int_0^1 \frac{1}{2} \, dy = \frac{\pi}{2}$

Question 2

The Cartesian equation of a surface S is

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

Evaluate the surface integral

$$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS,$$

where $\hat{\mathbf{n}}$ is an outward unit normal vector field to S , and ϕ is the function with Cartesian equation

$$\phi(x, y, z) = 1 - 2x^2 y.$$

$$\frac{\pi \mathbf{i}}{2}$$

FIND NORMAL TO THE SURFACE
 LET $f(x,y,z) = z - 1 + x^2 + y^2 = 0$
 $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 1)$
 (NORMAL TO THE SURFACE AS WE WANT POSITIVE INTO THE SURFACE)
 $\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$
 $\nabla \phi(x,y,z) = (-4xy, -2x^2, 1)$
 $\hat{\mathbf{n}} \wedge \nabla \phi = \frac{1}{|\nabla f|} \nabla f \wedge \nabla \phi = \frac{1}{|\nabla f|} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & 2y & 1 \\ -4xy & -2x^2 & 1 \end{vmatrix}$
 $= \frac{1}{|\nabla f|} [2x^2 - 4xy, 2xy^2 - 4x^2, 1]$

PROJECT ONTO THE XY PLANE
 INTO THE REGION R SUCH
 THAT $x^2 + y^2 \leq 1$

$\int_S \hat{\mathbf{n}} \wedge \nabla \phi \, dS = \iint_R \left[\frac{2x^2 - 4xy}{\sqrt{4x^2 + 4y^2 + 1}}, \frac{2xy^2 - 4x^2}{\sqrt{4x^2 + 4y^2 + 1}}, \frac{1}{\sqrt{4x^2 + 4y^2 + 1}} \right] \frac{dx \, dy}{\sqrt{4x^2 + 4y^2 + 1}}$
 $= \iint_R \left[\frac{2x^2 - 4xy}{4x^2 + 4y^2 + 1}, \frac{2xy^2 - 4x^2}{4x^2 + 4y^2 + 1}, \frac{1}{4x^2 + 4y^2 + 1} \right] dx \, dy$
 $= \iint_R \left[\frac{2x^2 - 4xy}{4x^2 + 4y^2 + 1}, \frac{2xy^2 - 4x^2}{4x^2 + 4y^2 + 1}, \frac{1}{4x^2 + 4y^2 + 1} \right] dx \, dy$

NOW R IS A CIRCULAR DISK IN THE xy PLANE, SO USE POLAR COORDINATES
 IN THE xy PLANE — SO POLAR COORDINATES ARE SUITABLE
 $= \int_0^{2\pi} \int_0^1 \left[\frac{2(\cos^2 \theta) - 4(\cos \theta \sin \theta)}{4(\cos^2 \theta + \sin^2 \theta) + 1}, \frac{2(\sin^2 \theta) - 4(\sin \theta \cos \theta)}{4(\cos^2 \theta + \sin^2 \theta) + 1}, \frac{1}{4(\cos^2 \theta + \sin^2 \theta) + 1} \right] r \, dr \, d\theta$
 $= \int_0^{2\pi} \int_0^1 \left[\frac{2\cos^2 \theta - 4\cos \theta \sin \theta}{5}, \frac{2\sin^2 \theta - 4\sin \theta \cos \theta}{5}, \frac{1}{5} \right] r \, dr \, d\theta$
 $= \int_0^{2\pi} \left[\frac{1}{5} + \frac{1}{5} \cos 2\theta \right] d\theta = \frac{1}{5} \int_0^{2\pi} (1 + \cos 2\theta) d\theta = \frac{\pi \mathbf{i}}{2}$

Created by T. Madas

TYPE $\int_S \mathbf{F} \cdot d\mathbf{S}$

Created by T. Madas

Question 1

Evaluate the surface integral

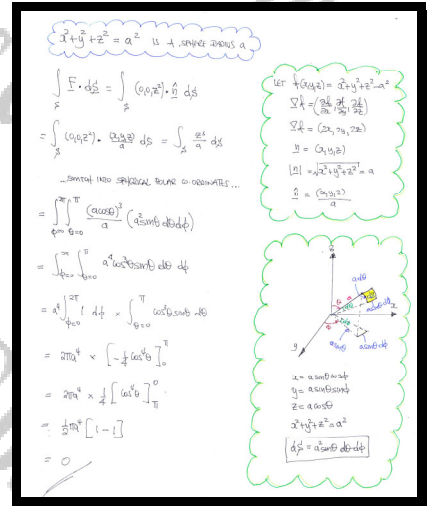
$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with equation

$$x^2 + y^2 + z^2 = a^2, \quad a > 0,$$

and $\mathbf{F} = z^2 \mathbf{k}$.

0



Question 2

$$\mathbf{F}(x, y, z) \equiv x^2\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the plane surface with equation

$$2x + 2y + z = 2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

$\frac{7}{6}$

$\mathbf{F}(x,y,z) = (x^2, -2y, -2z)$
 Plane: $2x + 2y + z = 2$
 Normal: $\mathbf{n} = (2, 2, 1)$
 $\hat{\mathbf{n}} = \frac{1}{3}(2, 2, 1)$
 If $z=0$: $2x + 2y = 2$
 $x + y = 1$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_S (x^2 - 2y - 2z) \cdot \frac{1}{3}(2, 2, 1) \, dS = \frac{1}{3} \int_S (2x^2 - 4y - 2z) \, dS$
 Project surface onto the xy plane
 $\dots = \frac{1}{3} \int_R (2x^2 - 4y - 2(2 - 2x - 2y)) \, dx \, dy$
 Value of z on S : $z = 2 - 2x - 2y$
 $= \frac{1}{3} \int_0^1 \int_0^{1-x} (2x^2 - 4y - 4 + 4x + 4y) \, dy \, dx = \frac{1}{3} \int_0^1 (2x^2 - 4y) \, dy \, dx$
 $= \int_0^1 (2x^2 - 4y) \, dy \, dx = \int_0^1 (2x^2 - 4y) \, dy \, dx$
 $= \int_0^1 (2x^2 - 4(1-x)) \, dx = \int_0^1 (2x^2 - 4 + 4x) \, dx$
 $= \int_0^1 (-2x^2 + 4x - 4) \, dx = \left[-\frac{2}{3}x^3 + 2x^2 - 4x \right]_0^1$
 $= -\frac{2}{3} + 2 - 4 = -\frac{7}{3}$

Question 4

Evaluate the surface integral

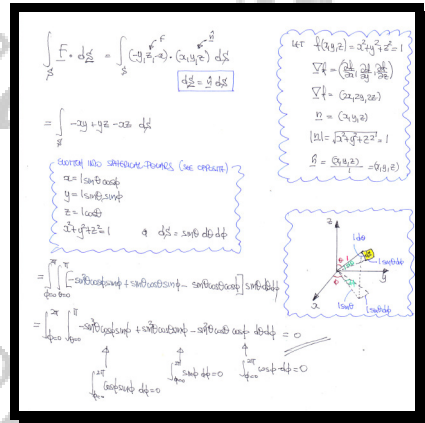
$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with equation

$$x^2 + y^2 + z^2 = 1,$$

and $\mathbf{F} = -y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$.

0



Question 5

$$\mathbf{F}(x, y, z) \equiv \mathbf{i} + \frac{1}{2}y\mathbf{j} + z^2\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the curved cylindrical surface with equation

$$x^2 + y^2 = 4, \quad x \geq 0, \quad 0 \leq z \leq 3.$$

$$3\pi + 12$$

The curved surface has equation $x^2 + y^2 = 4$
 let $(x, y, z) = (2\cos\theta, 2\sin\theta, z)$
 $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j} + z\mathbf{k}$
 Then the normal $\mathbf{n} = (\mathbf{r}_\theta \times \mathbf{r}_z)$
 $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j} + 0\mathbf{k}$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_S (\frac{1}{2}y\mathbf{j} + z^2\mathbf{k}) \cdot (2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}) \, dS$
 $= \int_S (z^2 \sin\theta) \, dS$
 Surface area element $dS = 2 \, d\theta \, dz$
 $= \int_0^3 \int_0^{\frac{\pi}{2}} (z^2 \sin\theta) \cdot 2 \, d\theta \, dz$
 $= 2 \int_0^3 z^2 \left[-\cos\theta \right]_0^{\frac{\pi}{2}} dz$
 $= 2 \int_0^3 z^2 (0 - (-1)) dz$
 $= 2 \int_0^3 z^2 dz$
 $= 2 \left[\frac{z^3}{3} \right]_0^3$
 $= 2 \left(\frac{27}{3} - 0 \right)$
 $= 2(9)$
 $= 18$

Wait, the final answer in the image is $3(\pi + 4)$. Let me re-check the calculation in the image.
 The image calculation shows:
 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S (\frac{1}{2}y\mathbf{j} + z^2\mathbf{k}) \cdot (2\cos\theta\mathbf{i} + 2\sin\theta\mathbf{j}) \, dS$
 $= \int_S (z^2 \sin\theta) \, dS$
 Surface area element $dS = 2 \, d\theta \, dz$
 $= \int_0^3 \int_0^{\frac{\pi}{2}} (z^2 \sin\theta) \cdot 2 \, d\theta \, dz$
 $= 2 \int_0^3 z^2 \left[-\cos\theta \right]_0^{\frac{\pi}{2}} dz$
 $= 2 \int_0^3 z^2 (0 - (-1)) dz$
 $= 2 \int_0^3 z^2 dz$
 $= 2 \left[\frac{z^3}{3} \right]_0^3$
 $= 2 \left(\frac{27}{3} - 0 \right)$
 $= 2(9)$
 $= 18$

There is a discrepancy between the handwritten calculation and the boxed answer. The handwritten calculation results in 18, while the boxed answer is $3(\pi + 4)$. The handwritten solution might be missing a term or has a typo. The correct answer is $3(\pi + 4)$.

Question 6

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z^4\mathbf{k}.$$

Calculate the flux of \mathbf{F} through the open surface with equation

$$z = \sqrt{x^2 + y^2}, \quad z \leq 1,$$

in the direction of z decreasing.

$$\boxed{-\frac{1}{3}\pi}$$

$\int_S (x, y, z^4) \cdot d\mathbf{S}$
 $= \int_S (x, y, z^4) \cdot \hat{n} \, dS$
 PROVE TO THE OX PLANE
 ON THE REGION R, WITH
 EQUATION $x^2 + y^2 = 1$
 $= \int_R (x, y, z^4) \cdot \hat{n} \, \frac{dx \, dy}{\sqrt{1+z^2}}$
 $= \int_R (x, y, z^4) \cdot \frac{-x}{\sqrt{1+z^2}} \mathbf{i} - \frac{y}{\sqrt{1+z^2}} \mathbf{j} + \frac{z^4}{\sqrt{1+z^2}} \mathbf{k} \, dx \, dy$
 $= \int_R (x, y, z^4) \cdot \left[\frac{-x}{\sqrt{1+z^2}} \mathbf{i} - \frac{y}{\sqrt{1+z^2}} \mathbf{j} + \frac{z^4}{\sqrt{1+z^2}} \mathbf{k} \right] \, dx \, dy$
 $= \int_R \left[-\frac{x^2}{\sqrt{1+z^2}} - \frac{y^2}{\sqrt{1+z^2}} + \frac{z^4}{\sqrt{1+z^2}} \right] \, dx \, dy$
 $= \int_R \left[-\frac{x^2 + y^2}{\sqrt{1+z^2}} + \frac{z^4}{\sqrt{1+z^2}} \right] \, dx \, dy$
 SIMPLIFY INTO POLAR FORMS NEXT

$f(x, y, z) = (x^2 + y^2)^2 - z$
 $\nabla f = (2x(x^2 + y^2), 2y(x^2 + y^2), -1)$
 $\hat{n} = \left[\frac{2x}{\sqrt{4(x^2 + y^2)^2 + 1}}, \frac{2y}{\sqrt{4(x^2 + y^2)^2 + 1}}, -1 \right]$

$= \int_0^{2\pi} \int_0^1 (-r + r^4)(r \, dr \, d\theta)$
 $= \int_0^{2\pi} \int_0^1 -r^2 + r^5 \, dr \, d\theta$
 $= 2\pi \left[-\frac{1}{3}r^3 + \frac{1}{6}r^6 \right]_0^1$
 $= 2\pi \left[\left(-\frac{1}{3}\right) - 0 \right]$
 $= -\frac{2\pi}{3}$

Question 7

The surface S has Cartesian equation

$$z = 1 - x^2 - y^2, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0.$$

Evaluate the surface integral

$$\int_S 15z \mathbf{i} \cdot d\mathbf{S}.$$

4

$z = 1 - x^2 - y^2$
 $z + x^2 + y^2 - 1 = 0$
 let $f(x,y,z) = z + x^2 + y^2 - 1$
 $\nabla f = (2x, 2y, 1)$
 $|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$
 $= \sqrt{4(x^2 + y^2) + 1}$
 $= \sqrt{4(1-z) + 1}$
 $= \sqrt{5 - 4z}$
 $\hat{n} = \frac{(2x, 2y, 1)}{\sqrt{5 - 4z}}$

$d\mathbf{S} = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} du dv \mathbf{i} + \dots$
 $d\mathbf{S} = \frac{\partial x}{\partial x} \frac{\partial x}{\partial y} dx dy \mathbf{i} + \dots$
 $d\mathbf{S} = \frac{1}{\sqrt{5 - 4z}} dx dy \mathbf{i} + \dots$

This
 $\int_S 15z \mathbf{i} \cdot d\mathbf{S} = \int_R 15z \mathbf{i} \cdot \hat{n} dx dy$... PROJECT ONTO THE XY PLANE
 i.e. REGION R IN XY PLANE
 $= \int_R 15z \mathbf{i} \cdot \frac{1}{\sqrt{5 - 4z}} dx dy = \int_R \frac{15z \mathbf{i} \cdot \mathbf{i}}{\sqrt{5 - 4z}} dx dy$
 $= \int_R \frac{15(z, 0, 0) \cdot (2x, 2y, 1)}{\sqrt{5 - 4z}} dx dy = \int_R \frac{30xz}{\sqrt{5 - 4z}} dx dy$
 switch into polar coords ... NOTE THAT $z = 1 - x^2 - y^2 = 1 - r^2$
 $= \int_0^{\pi/2} \int_0^1 \frac{30 \cos(\theta) (1 - r^2) (r dr d\theta)}{\sqrt{5 - 4(1 - r^2)}} = \int_0^{\pi/2} \int_0^1 \frac{30r^2 \cos \theta}{\sqrt{1 + 4r^2}} dr d\theta$
 $= \left[\sin \theta \right]_0^{\pi/2} \left[\frac{10}{3} (1 + 4r^2)^{3/2} \right]_0^1 = 1 \times (10 - 4) = 4$

Question 8

$$\mathbf{F}(x, y, z) \equiv -y\mathbf{i} + x\mathbf{j} + 3z\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface of the hemisphere with equation

$$x^2 + y^2 + z^2 = 9, \quad z \geq 0,$$

contained inside the cylinder with equation

$$x^2 + y^2 = 4, \quad z \geq 0,$$

$$2\pi [27 - 5\sqrt{5}]$$

$$\int_S (-y, x, 3z) \cdot d\mathbf{S}$$

$$= \int_R (-y, x, 3z) \cdot \hat{n} \, dS$$
 Project onto the xy plane onto the region R, on the opposite z-plane.

$$= \int_R (-y, x, 3z) \cdot \hat{n} \frac{dx \, dy}{\delta \cdot \hat{F}}$$
 Let $\rho = \sqrt{x^2 + y^2}$
 $\Sigma^2 = (x, y, z)$
 $\Sigma = (2, 0, 2)$

$$= \int_R (-y, x, 3z) \cdot (x, y, z) \frac{dx \, dy}{(x^2 + y^2 + 9)}$$

$$= \int_R \frac{-xy + xy + 3z^2}{x^2 + y^2 + 9} \, dx \, dy$$

$$= \int_R 3z \, dx \, dy$$

$$= \int_0^2 \int_0^{2\pi} 3\sqrt{9 - \rho^2} \, d\rho \, d\theta$$
 Switch into polar form.

$$= \int_0^{2\pi} \int_0^2 3(9 - \rho^2) \, (\rho \, d\rho \, d\theta)$$

$$= 2\pi \int_0^2 3(9 - \rho^2) \, d\rho = 2\pi \left[-\frac{1}{3}\rho^3 \right]_0^2 = 2\pi \left[\frac{1}{3}\rho^3 \right]_2^0$$

$$= 2\pi [27 - 5\sqrt{5}]$$

Question 9

Space is filled uniformly by the constant vector field $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

A square lamina whose vertices are at $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(0,1,0)$ is rotated by $\frac{1}{4}\pi$, anticlockwise, about the y axis.

determine the magnitude of the flux of the field through the rotated lamina.

$4\sqrt{2}$

$$|\text{Flux}| = \left| \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS \right|$$

$$= \left| \iint_R (3, 4, 5) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \, dS \right|$$

PROJECT onto the xy plane

$$= \left| \iint_R (3, 4, 5) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx \, dy}{\sqrt{z^2}} \right|$$

$$= \left| \iint_R (3, 4, 5) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx \, dy}{\sqrt{z^2}} \right|$$

$$= \left| \iint_R (3, 4, 5) \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \frac{dx \, dy}{\sqrt{z^2}} \right|$$

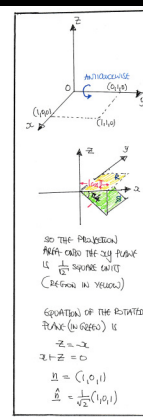
$$= \left| \iint_R (3+0+5) \frac{dx \, dy}{1} \right|$$

$$= 8 \int_0^1 \int_0^1 1 \, dx \, dy$$

$$= 8 \times (\text{Area of } R)$$

$$= 8 \times \frac{1}{2}$$

$$= 8 \times \frac{1}{2}$$

$$= 4\sqrt{2}$$


Question 10

The surface S has Cartesian equation

$$z = 2 - x^2 - y^2, \quad x^2 + y^2 \leq 1.$$

- a) Sketch the graph of S .
- b) Given that $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$, evaluate the integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

$\frac{3\pi}{2}$

1) $z = 2 - x^2 - y^2$ $|x| \leq 1$ $|y| \leq 1$

• If $x=0$, $z = 2 - y^2$ • If $y=0$, $z = 2 - x^2$

Have an "extra zero" PROCEEDED

2) NEW $\mathbf{F} = (y, -x, z)$

• LET SURFACE BE $f(x,y,z) = 2 - x^2 - y^2 - z$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (-2x, -2y, -1)$

$|\nabla f| = \sqrt{4x^2 + 4y^2 + 1}$

$\hat{n} = \frac{(-2x, -2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}}$

• $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dS$

$= \int_S (y, -x, z) \cdot \frac{(-2x, -2y, -1)}{\sqrt{4x^2 + 4y^2 + 1}} \, dS$

$= \int_S \frac{-2xy - 2xy + z}{\sqrt{4x^2 + 4y^2 + 1}} \, dS$

• NEXT PROJECT SURFACE ONTO THE XY PLANE OVER THE CIRCLE $x^2 + y^2 = 1$, $z=0$ (AND IS WINDING) R

$dS = \frac{dx dy}{\hat{n} \cdot \mathbf{k}}$ For PROJECTION ONTO THE XY PLANE

$= \iint_R \frac{z}{\sqrt{4x^2 + 4y^2 + 1}} \, dx dy$ (where R is $x^2 + y^2 \leq 1, z=0$)

$= \iint_R \frac{z}{\sqrt{4x^2 + 4y^2 + 1}} \, dx dy$

$= \int_0^{2\pi} \int_0^1 \frac{2 - r^2}{\sqrt{4r^2 + 1}} \, r \, dr \, d\theta$

$= 2\pi \times \left[r - \frac{1}{4} \right]_0^1 = \frac{3\pi}{2}$

Question 12

$$\mathbf{F}(x, y, z) \equiv 3x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1.$$

4π

$\mathbf{F} = (3x, y^2, z^2)$ over the unit sphere $x^2 + y^2 + z^2 = 1$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, ds$

$= \int_S (3x, y^2, z^2) \cdot (x, y, z) \, ds$

$= \int_S (3x^2 + y^2 + z^2) \, ds$

Use spherical coordinates

$= \int_0^{2\pi} \int_0^\pi (3\cos^2\phi + \sin^2\phi + \cos^2\phi) r^2 \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (3\cos^2\phi + 2\cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (5\cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (5\cos^2\phi - \cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (4\cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (4\cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi (4\cos^2\phi + \sin^2\phi) \sin\phi \, d\phi \, d\theta$

$= 4\pi$

$\mathbf{F} = (3x, y^2, z^2)$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, dV$ (By Gauss's Theorem)

$= \int_V (3 + 2y + 2z) \, dV$

Use spherical coordinates

$= \int_0^{2\pi} \int_0^\pi \int_0^1 (3 + 2r\cos\theta + 2r\cos\phi) r^2 \sin\phi \, dr \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi \int_0^1 (3r^3 + 2r^3\cos\theta + 2r^3\cos\phi) \sin\phi \, dr \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi \left[\frac{3r^4}{4} + \frac{2r^4\cos\theta}{4} + \frac{2r^4\cos\phi}{4} \right]_0^1 \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi \left(\frac{3}{4} + \frac{1}{2}\cos\theta + \frac{1}{2}\cos\phi \right) \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} \int_0^\pi \left(\frac{3}{4}\sin\phi - \frac{1}{2}\cos\theta\cos\phi + \frac{1}{4}\sin^2\phi \right) \, d\phi \, d\theta$

$= \int_0^{2\pi} \left[-\frac{3}{4}\cos\phi - \frac{1}{2}\cos\theta\sin\phi + \frac{1}{4}(-\cos\phi) \right]_0^\pi \, d\theta$

$= \int_0^{2\pi} \left(-\frac{3}{4}(-1) - \frac{1}{2}\cos\theta(0) + \frac{1}{4}(-1) \right) \, d\theta$

$= \int_0^{2\pi} \left(-\frac{3}{4} + \frac{1}{4} \right) \, d\theta$

$= \int_0^{2\pi} -\frac{1}{2} \, d\theta$

$= -\frac{1}{2} \times 2\pi = -\pi$

Account for the inward normal

$= 3 \times \text{Volume of a unit sphere}$

$= 3 \times \frac{4}{3}\pi = 4\pi$

Question 13

$$\mathbf{F}(x, y, z) \equiv (x + y)\mathbf{i} + (x - y)\mathbf{j} + (x + z)\mathbf{k}.$$

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{s},$$

where S is the surface with Cartesian equation

$$z = 1 - x^2 - y^2, \quad z \geq 0.$$

$\frac{\pi}{2}$

Handwritten solution for the surface integral problem:

Left Column:

- Vector field: $\mathbf{F} = (x+y, x-y, x+z)$
- Surface equation: $z = 1 - x^2 - y^2$
- Normal vector calculation: $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (2x, 2y, 1)$
- Unit normal vector: $\hat{n} = \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$
- Differential area element: $ds = \frac{dx dy}{\sqrt{4x^2 + 4y^2 + 1}}$
- Surface integral setup: $\int_S \mathbf{F} \cdot d\mathbf{s} = \int_R (x+y, x-y, x+z) \cdot \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}} dx dy$
- Final integral: $\int_R (2x^2 - 2y^2 + 2x + 1 - 2x^2 - y^2) dx dy$

Right Column:

- Conversion to polar coordinates: $x = r \cos\theta, y = r \sin\theta, z = 1 - r^2$
- Integral in polar coordinates: $\int_0^{2\pi} \int_0^1 (2r^2 \cos^2\theta - 2r^2 \sin^2\theta + 2r \cos\theta + 1 - r^2) r dr d\theta$
- Evaluation: $\int_0^{2\pi} \left[\frac{1}{2} r^3 (2 \cos^2\theta - 2 \sin^2\theta + 2 \cos\theta + 1 - r) \right]_0^1 d\theta$
- Final result: $\frac{1}{2} \times 2\pi = \frac{\pi}{2}$

Question 14

$$\mathbf{F}(x, y, z) \equiv (x + z + xy)\mathbf{i} + (z^2 - 2xz - y)\mathbf{j} + \mathbf{k}.$$

Evaluate the surface integral

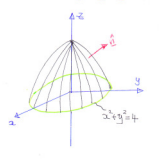
$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 4, \quad z \geq 0.$$

4π

$\mathbf{F} = (x+z+xy)\mathbf{i} + (z^2-2xz-y)\mathbf{j} + \mathbf{k}$



$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$
 $\nabla\phi = (2x, 2y, 2z)$
 $\mathbf{n} = (x, y, z)$
 $|\mathbf{n}| = \sqrt{x^2 + y^2 + z^2} = 2$
 $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{2}(x, y, z)$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S (x+z+xy, z^2-2xz-y, 1) \cdot \frac{1}{2}(x, y, z) \, dS$$

$$= \frac{1}{2} \int_S (x^2 + xz + xy + yz + yz + z^3 - 2xz - yz - yz + z) \, dS$$

Recall (to the 2D plane) $dz = \frac{dx dy}{2}$
 $dS = \frac{dx dy}{\sqrt{4-x^2-y^2}}$
 $dz = \frac{z}{2} dx dy$

$$= \int_R \left(\frac{x^2}{2} + \frac{z^3}{2} + \frac{yz}{2} + \frac{xy}{2} + \frac{z^3}{2} + \frac{z}{2} \right) dx dy$$

where R is $x^2 + y^2 \leq 4$

NOTE: R IS A SEMICIRCLE (SHOWN) IN x AND y (SO ALL COORDINATES OF x AND y HAVE NO RESTRICTIONS)

$$= \int_R \left(\frac{x^2}{2} + \frac{z^3}{2} + \frac{yz}{2} + \frac{xy}{2} + \frac{z^3}{2} + \frac{z}{2} \right) dx dy \quad (z = \sqrt{4-x^2-y^2})$$

$$= \int_R \left(\frac{x^2}{2} + \frac{z^3}{2} + 1 \right) dx dy = \int_R \frac{x^2 + z^3 + 2}{2} dx dy + \int_R 1 dx dy$$

SWITCH INTO POLAR COORDINATES

$$= \int_0^{2\pi} \int_0^2 \left(\frac{r^2 \cos^2 \theta + r^3 \sin^3 \theta}{\sqrt{4-r^2}} + (1 \times r^2) \right) (r dr d\theta) + 2\pi$$

$$= \int_0^{2\pi} \int_0^2 \left(\frac{r^3 \cos^2 \theta - r^3 \sin^3 \theta}{\sqrt{4-r^2}} \right) dr d\theta + 4\pi$$

$$= \int_0^{2\pi} \int_0^2 \frac{r^3 \cos^2 \theta}{\sqrt{4-r^2}} dr d\theta + 4\pi$$

NO INDEPENDENT θ (THE θ INTEGRATION) WITH THESE VALUES

$$= 4\pi$$

Question 15

$$\mathbf{F}(x, y, z) \equiv -xy\mathbf{i} + (yz - xy)\mathbf{k}.$$

Show that there is zero net flux of \mathbf{F} through the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 25, \quad z \geq 3.$$

proof

$\mathbf{F} = (-xy, 0, yz - xy)$
 let $f(x,y,z) = x^2 + y^2 + z^2 = 25$
 $S^1 = (x, y, z)$
 $\mathbf{n} = (x, y, z)$

THIS Flux IN THE DIRECTION OF z INCREASING

$$\int_S \mathbf{F} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} dS = \int_S (-xy, 0, yz - xy) \cdot \frac{(x, y, z)}{5} dS = \dots$$

Project onto the xy plane onto the circle $R: x^2 + y^2 \leq 16$

$$dS = \frac{dx dy}{\sqrt{5}}$$

$$= \int_R (-xy, 0, yz - xy) \cdot \frac{(x, y, z)}{5} \frac{dx dy}{\sqrt{5}}$$

$$= \int_R (-xy^2 + yz^2 - xy^2) \frac{dx dy}{5}$$

$$= \int_R \frac{-2xy^2 + yz^2}{5} dx dy$$

$x^2 + y^2 + z^2 = 25$
 $z^2 = 25 - x^2 - y^2$
 $z = \sqrt{25 - x^2 - y^2}$

$$= \int_R \left[\frac{-2xy^2}{5} + \frac{y\sqrt{25 - x^2 - y^2} \cdot z^2}{5} - \frac{xy^2}{5} \right] dx dy$$

BUT R IS A SYMMETRICAL REGION IN x & y , SO ANY ODD TERMS IN x OR y WILL BE ZERO.

$= 0$

\therefore Zero flux

Question 16

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

- a) Given that S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

show that

$$\int_S \mathbf{F} \cdot d\mathbf{S} = 4 \int_R \left[\frac{x^2}{\sqrt{1-x^2-y^2}} + 1 - x^2 - y^2 \right] dx dy,$$

where R is the region in the first quadrant with Cartesian equation

$$x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0.$$

- b) Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}.$$

$$\boxed{\frac{7}{6}\pi}$$

a) $\mathbf{F} = (x, y^2, z^2)$ over the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$

$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dA = \int_S (x\hat{i} + y^2\hat{j} + z^2\hat{k}) \cdot (\frac{x}{\sqrt{1-x^2-y^2}}\hat{i} + \frac{y}{\sqrt{1-x^2-y^2}}\hat{j} + \frac{z}{\sqrt{1-x^2-y^2}}\hat{k}) \, dA$

$= \int_S (x^2 + y^2 + z^2) \, dA$

PROJECT ONTO THE XY PLANE, OVER THE CIRCLE (AND ITS QUADRANT)

THEY EQUATION $x^2 + y^2 = 1$, QUADRANT R'

$\hat{n} \cdot \hat{k} = \frac{z}{\sqrt{1-x^2-y^2}} = \frac{z}{\sqrt{1-x^2-y^2}}$

$d\mathbf{S} = \frac{z}{\sqrt{1-x^2-y^2}} \, dx dy$

$= \int_R (x^2 + y^2 + z^2) \frac{z}{\sqrt{1-x^2-y^2}} \, dx dy$ where $z = \sqrt{1-x^2-y^2}$

$= \int_R (x^2 + y^2 + z^2) \, dx dy$

$= \int_R \frac{x^2}{\sqrt{1-x^2-y^2}} + \frac{y^2}{\sqrt{1-x^2-y^2}} + (1-x^2-y^2) \, dx dy$

$= 4 \int_R \left[\frac{x^2}{\sqrt{1-x^2-y^2}} + 1 - x^2 - y^2 \right] \, dx dy$

where R is the QUADRANT $x^2 + y^2 \leq 1$ in the FIRST QUADRANT

LET THE SURFACE BE $\mathcal{S}(x,y,z) = x^2 + y^2 + z^2 = 1$
 $\nabla \mathcal{S} = (2x, 2y, 2z)$
 $\hat{n} = \frac{(2x, 2y, 2z)}{\sqrt{4x^2 + 4y^2 + 4z^2}}$
 $\hat{n} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$
 $\hat{n} = (x, y, z)$

b) SWITCH INTO POLAR COORDS

$4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 \left[\frac{r^2 \cos^2 \theta}{\sqrt{1-r^2}} + (1-r^2) \right] r \, dr \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^1 \left[\frac{r^3 \cos^2 \theta}{\sqrt{1-r^2}} + r - r^3 \right] \, dr \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2} \cos^2 \theta \int_{r=0}^1 \frac{r^3}{\sqrt{1-r^2}} \, dr + \frac{1}{2} \int_{r=0}^1 (2r - 2r^3) \, dr \right] \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2} \cos^2 \theta \left(-\frac{1}{2} \sqrt{1-r^2} + \frac{1}{2} \arcsin r \right) + \frac{1}{2} (r^2 - \frac{2}{3} r^3) \right]_{r=0}^1 \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2} \cos^2 \theta \left(-\frac{1}{2} \sqrt{1-1} + \frac{1}{2} \arcsin 1 \right) + \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} \right) \right] \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{2} \cos^2 \theta \left(\frac{1}{2} \arcsin 1 \right) + \frac{1}{2} \left(\frac{1}{6} \right) \right] \, d\theta$

$= 4 \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{1}{4} \cos^2 \theta \arcsin 1 + \frac{1}{12} \cos^2 \theta \right] \, d\theta$

$= 4 \left[\frac{1}{4} \arcsin 1 \int_{\theta=0}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta + \frac{1}{12} \int_{\theta=0}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \right]$

$= 4 \left[\frac{1}{4} \arcsin 1 \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + \frac{1}{12} \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \right]_{\theta=0}^{\frac{\pi}{2}}$

$= 4 \left[\frac{1}{4} \arcsin 1 \left(\frac{\pi}{4} + \frac{\sin \pi}{4} \right) + \frac{1}{12} \left(\frac{\pi}{4} + \frac{\sin \pi}{4} \right) \right]$

$= 4 \left[\frac{1}{4} \arcsin 1 \left(\frac{\pi}{4} \right) + \frac{1}{12} \left(\frac{\pi}{4} \right) \right]$

$= \frac{1}{3} \pi + \frac{1}{3} \pi = \frac{2}{3} \pi$

Question 17

$$\mathbf{F} = x^2y^3\mathbf{i} + z\mathbf{j} + x\mathbf{k}.$$

Show by **direct** evaluation that

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = 0,$$

where S is the sphere with equation

$$x^2 + y^2 + z^2 = 1,$$

and $\hat{\mathbf{n}}$ is an outward unit normal to S .

proof

$\mathbf{F} = (x^2y^3, z, x)$

- $\nabla \wedge \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & z & x \end{vmatrix} = [0-1, 0-1, 0-3xy^2] = [-1, -1, -3xy^2]$
- $\nabla(x^2+y^2+z^2) = (2x, 2y, 2z)$
- $\mathbf{n} = (x, y, z)$
- $\hat{\mathbf{n}} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = (x, y, z)$

THIS INTEGRATE OVER THE SURFACE OF THE SPHERE $x^2+y^2+z^2=1$

$$\int_S \nabla \wedge \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \int_S (-1-1-3xy^2) \cdot (x, y, z) \, dS$$

$$= \int_S (-2 - 3xy^2z) \, dS = - \int_S (2 + 3xy^2z) \, dS$$

SWITCH INTO SPHERICAL COORDINATES

- $x = \sin\theta \cos\phi$
- $y = \sin\theta \sin\phi$
- $z = \cos\theta$
- $x^2+y^2+z^2 = 1$
- $dS = \sin\theta \, d\theta \, d\phi$
- $0 \leq \theta \leq \pi$
- $0 \leq \phi \leq 2\pi$

$$\dots = - \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} [2\cos\theta + 3\sin^3\theta \cos\theta \sin^2\phi \cos\phi] \sin\theta \, d\theta \, d\phi$$

$$= - \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} [2\sin\theta \cos\theta + 3\sin^4\theta \cos\theta \sin^2\phi \cos\phi] \, d\theta \, d\phi$$

$$= - \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} [2\sin\theta \cos\theta + 3\sin^4\theta \cos\theta \sin^2\phi \cos\phi] \, d\theta \, d\phi$$

LOOKING AT THESE INTEGRATIONS, WITHOUT NECESSARILY ORDERING THEM AT

- $\int_{\theta=0}^{\pi} 2\sin\theta \cos\theta \, d\theta = 0$
- $\int_{\theta=0}^{\pi} \sin^4\theta \cos\theta \, d\theta = 0$
- $\int_{\phi=0}^{2\pi} 3\sin^4\theta \cos\theta \sin^2\phi \cos\phi \, d\phi = \left[\frac{1}{2} \sin^3\phi \right]_0^{2\pi} = 0$

$\therefore 0$

As required

Question 18

$$\mathbf{F}(x, y, z) \equiv (x + yz)\mathbf{i} + (y^3z + x)\mathbf{j} + (z + xyz)\mathbf{k}.$$

Calculate the magnitude of the flux of \mathbf{F} through the open cylindrical surface with equation

$$x^2 + y^2 = 1, 0 \leq z \leq 4.$$

10π

$\mathbf{F} = (x + yz)\mathbf{i} + (y^3z + x)\mathbf{j} + (z + xyz)\mathbf{k}$

flux = $\int_S \mathbf{F} \cdot d\mathbf{s} = \int_C \mathbf{F} \cdot \hat{\mathbf{t}} ds$

SWITCH INTO PARAMETRIC

$$\dots = \int_C \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) du dv$$
$$\dots = \int_C \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \right) d\theta dz$$

PARAMETERISE CURVE

$$= \int_{z=0}^4 \int_{\theta=0}^{2\pi} \left((x + yz)\mathbf{i} + (y^3z + x)\mathbf{j} + (z + xyz)\mathbf{k} \right) \cdot (y\mathbf{i} - x\mathbf{j}) d\theta dz$$

NO CONTRIBUTION FROM θ , $0 \leq \theta < 2\pi$

$$= \int_{z=0}^4 \int_{\theta=0}^{2\pi} (y^2z + \frac{1}{2}z^2 \sin^2 \theta) d\theta dz$$
$$= \int_{z=0}^4 (4y^2z + 8z^2) dz$$

PARAMETERISE VIA POLARS

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

LET $\mathbf{r}(u, v, w) = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos v, \sin v, 0)$$
$$\frac{\partial \mathbf{r}}{\partial v} = (-\sin v, \cos v, 0)$$
$$\frac{\partial \mathbf{r}}{\partial w} = (0, 0, 1)$$

$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \mathbf{i} \cos^2 v + \mathbf{j} \sin^2 v + \mathbf{k} \sin v \cos v$

$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w} = \sin v \cos v$

$\text{Flux} = \int_{z=0}^4 \int_{\theta=0}^{2\pi} \mathbf{F} \cdot \mathbf{n} d\theta dz$

$$= \int_{z=0}^4 \int_{\theta=0}^{2\pi} (y^2z + \frac{1}{2}z^2 \sin^2 \theta) d\theta dz$$

SWITCH INTO SIN AND COS FUNCTIONS

$$= \int_{z=0}^4 \int_{\theta=0}^{2\pi} (y^2z + \frac{1}{4}z^2 (1 - \cos 2\theta)) d\theta dz$$
$$= \int_{z=0}^4 \left(2\pi y^2z + \frac{1}{4}z^2 \left(\theta - \frac{\sin 2\theta}{2} \right) \right) dz$$
$$= \int_{z=0}^4 \left(2\pi y^2z + \frac{1}{4}z^2 \left(\frac{1}{2} \theta - \frac{\sin 2\theta}{4} \right) \right) dz$$
$$= \int_{z=0}^4 \left(2\pi y^2z + \frac{1}{4}z^2 \left(\frac{1}{2} \cdot 2\pi - \frac{\sin 2\pi}{4} - \left(\frac{1}{2} \cdot 0 - \frac{\sin 0}{4} \right) \right) \right) dz$$
$$= \int_{z=0}^4 (4\pi y^2z + 2\pi y^2z) dz$$
$$= \int_{z=0}^4 6\pi y^2z dz$$
$$= 6\pi \int_{z=0}^4 z dz = 6\pi \left[\frac{z^2}{2} \right]_0^4 = 6\pi \cdot 8 = 48\pi$$

ALTERNATIVE BY DIVERGENCE THEOREM (DIVERGENCE)

THIS USE THE CLOSED SURFACE BE $\partial V(x, y, z) = x^2 + y^2 = 1$

$$\mathbf{S}_G = (x, y, 0)$$
$$\mathbf{n} = (x, y, 0)$$
$$|\mathbf{n}| = \sqrt{x^2 + y^2} = 1 \Rightarrow \hat{\mathbf{n}} = (x, y, 0)$$

flux = $\int_V \text{div} \mathbf{F} dV = \int_V (x + yz + y^3z + x + z + xyz) dV$

SWITCH INTO CYLINDRICAL POLARS

$$= \int_{z=0}^4 \int_{\theta=0}^{2\pi} (x + yz + y^3z + x + z + xyz) r d\theta dz$$

NO CONTRIBUTION FROM θ , $0 \leq \theta < 2\pi$

$$= \int_{z=0}^4 \int_{r=0}^1 (2x + 2yz + z) r dr dz$$
$$= \int_{z=0}^4 \left(z + \frac{2}{3}y^3z^2 \right) dz$$
$$= \int_{z=0}^4 (z + \frac{2}{3}z^2) dz = \left[\frac{z^2}{2} + \frac{2}{9}z^3 \right]_0^4 = 8 + \frac{128}{9} = \frac{140}{9}$$

OTHER METHODS WITH POLARS METHOD

Question 19

$$\mathbf{F}(x, y, z) \equiv y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}.$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (u+v)\mathbf{k}, \quad 0 \leq u \leq 1, \quad 1 \leq v \leq 4.$$

All integrations must be carried out in parametric.

□, $\frac{1}{2}$

$\mathbf{F}(x,y,z) = \begin{pmatrix} y \\ x^2 \\ z \end{pmatrix}$ $\mathbf{I}(u,v) = \begin{pmatrix} u \\ v \\ u+v \end{pmatrix}$ $0 \leq u \leq 1$
 $1 \leq v \leq 4$

FIND AN EXPRESSION FOR THE "AREA FLUX ELEMENT" ds

- $\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
- $\frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$
- NORMAL = $\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (-1, -1, 1)$
- UNIT NORMAL $\hat{n} = \frac{\mathbf{n}}{|\mathbf{n}|}$
- $\hat{n} = \frac{\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{(-1)^2 + (-1)^2 + 1^2}}$

COLLECTING THESE RESULTS

$ds = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} du dv$

$\hat{n} ds = \hat{n} \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv$

$ds = \left(\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right) du dv$

$ds = (-1, -1, 1) du dv$

FIND THE FLUX CAN BE CALCULATED

$$\begin{aligned} \text{flux} &= \int_S \mathbf{F} \cdot d\mathbf{s} = \int \mathbf{F}(\mathbf{r}) \cdot \hat{n} ds \\ &= \int_1^4 \int_0^1 (y, x^2, z) \cdot (-1, -1, 1) du dv \\ &= \int_1^4 \int_0^1 (-v - u^2 + u + v) du dv \\ &= \int_1^4 \int_0^1 (0 - u^2) du dv \\ &= \int_1^4 \left[-\frac{1}{3}u^3 - \frac{1}{2}u^2 \right]_0^1 dv \\ &= \int_1^4 \left(-\frac{1}{3} - \frac{1}{2} \right) dv \\ &= \int_1^4 -\frac{5}{6} dv \\ &= \left[-\frac{5}{6}v \right]_1^4 \\ &= -\frac{5}{6}(4) - \left(-\frac{5}{6}(1) \right) \\ &= -\frac{20}{6} + \frac{5}{6} \\ &= -\frac{15}{6} \\ &= -\frac{5}{2} \end{aligned}$$

Question 20

Evaluate the surface integral

$$\int_S z \mathbf{k} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad 0 \leq \theta \leq \frac{1}{2}\pi, \quad 0 \leq \varphi \leq \frac{1}{2}\pi.$$

$$\boxed{\frac{1}{6}\pi}$$

$\mathbf{r}(\theta, \varphi) = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq \frac{\pi}{2}$

$\iint_S z \mathbf{k} \cdot d\mathbf{S} = \iint_S z \mathbf{k} \cdot \mathbf{n} \, dS$

FIND THE UNIT NORMAL TO THE PARAMETRIZED SURFACE & SCALAR THE NORMAL INTO PARAMETRIC

$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix}$ $\frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{bmatrix}$

$\therefore \mathbf{n} = \begin{vmatrix} \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \cos \theta \cos \varphi & -\sin \theta \sin \varphi \\ \cos \theta \sin \varphi & \sin \theta \cos \varphi \end{vmatrix}$

$= \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \cos \varphi \cos \varphi + \cos \theta \sin \varphi \sin \varphi \\ \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \sin \varphi \cos \varphi - \cos \theta \cos \varphi \sin \varphi \\ \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \sin \varphi \end{bmatrix}$

$= \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi & \cos \theta \\ \sin \theta \cos \varphi & \sin \theta \sin \varphi & 0 \\ \sin \theta \cos \varphi & \sin \theta \sin \varphi & 0 \end{bmatrix}$

STRONG WITH A DISK

SPHERE: $x^2 + y^2 + z^2 = a^2$
 CYLINDER: $x^2 + y^2 = b^2$
 ($a > b$)

AREA OF THE INNER CYLINDRICAL FACE IS GIVEN BY

" $2\pi r h = 2\pi b(2h)$
 $= 4\pi b h$
 $= 4\pi b(a^2 - b^2)^{\frac{1}{2}}$

NEXT WE FIND THE AREA OF ONE OF THE SPHERICAL CAPS (SHOWN IN YELLOW) - PROJECT THE "TOP" CAP ($z > 0$) ONTO THE XY PLANE

$\Rightarrow z = \sqrt{a^2 - x^2 - y^2}$
 $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2 - x^2 - y^2}}$ $\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2 - x^2 - y^2}}$

$\Rightarrow dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dx \, dy$
 $\Rightarrow dS = \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} \, dx \, dy$

ALTERNATIVELY BY SPHERICAL COORDS & PROJECTION ONTO THE XY PLANE

$\iint_S z \mathbf{k} \cdot d\mathbf{S} = \iint_S (r \cos \theta) \mathbf{k} \cdot \mathbf{n} \, dS$

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (r \cos \theta) \cdot (r \sin \theta) \, d\theta \, d\varphi$
(USE BRACKETS EVERYWHERE)

$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2 \cos \theta \sin \theta \, d\theta \, d\varphi$

NEED SWITCH INTO SPHERICAL COORDS & PROJECTIONS

$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (r \cos \theta) \sin \theta \, d\theta \, d\varphi$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{dz \, dx \, dy}{r \cdot r}$

$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{2}} \, d\varphi$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{z \, dx \, dy}{r^2}$

$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{2}} \, d\varphi$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{z \, dx \, dy}{\sqrt{1-x^2-y^2}}$ (r=1)

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \, d\varphi$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{z \, dx \, dy}{\sqrt{1-x^2-y^2}}$

$= \frac{1}{2} \times \frac{\pi}{2}$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{z \, dx \, dy}{\sqrt{1-x^2-y^2}}$

$= \frac{\pi}{8}$ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{z \, dx \, dy}{\sqrt{1-x^2-y^2}}$

$= \frac{\pi}{8}$

$\frac{\partial \mathbf{r}}{\partial \theta} = \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix}$
 $\frac{\partial \mathbf{r}}{\partial \varphi} = \begin{bmatrix} -\sin \theta \sin \varphi \\ \sin \theta \cos \varphi \\ 0 \end{bmatrix}$

$d\mathbf{S} = \sin \theta \, d\theta \, d\varphi \, \mathbf{n}$

$\mathbf{n} = \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}$

$\mathbf{r} = r \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}$

$\mathbf{r} \cdot \mathbf{n} = r^2 \sin \theta \cos \theta$

$dS = r^2 \sin \theta \, d\theta \, d\varphi$

$\mathbf{k} \cdot \mathbf{n} = \cos \theta$

$d\mathbf{S} \cdot \mathbf{k} = \sin \theta \cos \theta \, d\theta \, d\varphi$

$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta \, d\theta \, d\varphi$

$\int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} \cos 2\theta \right]_0^{\frac{\pi}{2}} \, d\varphi$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} \, d\varphi$

$= \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{\pi}{8}$

Question 21

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface represented parametrically by

$$\mathbf{r}(u, v) = \begin{bmatrix} u + v \\ u - v \\ u \end{bmatrix}, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 3,$$

and \mathbf{F} is the vector field

$$x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

All integrations must be carried out in parametric.

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• PREPARE ALL THE AUXILIARY ITEMS

- $\mathbf{F}(x, y, z) = (x^2, y^2, z^2)$
- $\mathbf{I}(u, v) = (u+v, u-v, u), 0 \leq u \leq 2, 0 \leq v \leq 3$
(THIS IS IN FACT A PLANE THROUGH O)
- $\frac{\partial \mathbf{I}}{\partial u} = (1, 1, 1), \frac{\partial \mathbf{I}}{\partial v} = (1, -1, 0)$
- $\frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1, 1, -2)$
- $\therefore \frac{\mathbf{n}}{dS} = \frac{\frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v}}{\left\| \frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v} \right\|} = \frac{(1, 1, -2)}{dS}$

• HENCE WE KNOW HAVE IN PARAMETRIC

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \frac{\mathbf{n}}{dS} dS$$

$$= \int_S \mathbf{F} \cdot \left(\frac{\frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v}}{\left\| \frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v} \right\|} \right) \left\| \frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v} \right\| du dv$$

$$= \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{I}}{\partial u} \wedge \frac{\partial \mathbf{I}}{\partial v} \right) du dv$$

• SUBSTITUTING EVERY INTO THE REQUIRED SURFACE INTEGRAL

$$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{u=0}^2 \int_{v=0}^3 [(u+v)^2, (u-v)^2, u^2] \cdot (1, 1, -2) du dv$$

$$= \int_{u=0}^2 \int_{v=0}^3 [(u+v)^2 + (u-v)^2 - 2u^2] du dv$$

$$= \int_{u=0}^2 \int_{v=0}^3 [u^2 + 2uv + v^2 + u^2 - 2uv + v^2 - 2u^2] du dv$$

$$= \int_{u=0}^2 \int_{v=0}^3 [2v^2] du dv$$

$$= \int_{v=0}^3 [2v^2]_{u=0}^2 dv$$

$$= \int_{v=0}^3 4v^2 dv$$

$$= \left[\frac{4}{3}v^3 \right]_0^3$$

$$= \frac{4}{3} \times 27$$

$$= 36$$

Question 22

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (u \sin v)\mathbf{j} + u\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\frac{2}{3}\pi$$

Handwritten solution for Question 22:

$\mathbf{r}(u, v) = [u \cos v, u \sin v, u]$ $0 \leq u \leq 1$
 $0 \leq v \leq 2\pi$
 $\mathbf{F}(u, v) = (u, u, 2u)$

FIND THE "JACOBIAN" AND THE NORMAL
 $\frac{\partial \mathbf{r}}{\partial u} = [\cos v, \sin v, 1]$
 $\frac{\partial \mathbf{r}}{\partial v} = [-u \sin v, u \cos v, 0]$
 $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix}$
 $= [-u \cos v, -u \sin v, u]$
 $= [-u \cos v, -u \sin v, u]$ ← NORMAL

$\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| = \sqrt{(-u \cos v)^2 + (-u \sin v)^2 + u^2} = u \sqrt{\cos^2 v + \sin^2 v + 1} = u \sqrt{2}$ ← "JACOBIAN"
 $d\mathbf{S} = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv$

NOW THE FLUX INTEGRAL CAN BE COMPUTED
 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$

$\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 [u \cos v, u \sin v, 2u] \cdot [-u \cos v, -u \sin v, u] du dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 (-u^2 \cos^2 v - u^2 \sin^2 v + 2u^2) du dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 -u^2 (\cos^2 v + \sin^2 v) + 2u^2 du dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \int_{u=0}^1 u^2 du dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \left[\frac{1}{3} u^3 \right]_0^1 dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \int_{v=0}^{2\pi} \frac{1}{3} dv$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{1}{3} \times 2\pi$
 $\Rightarrow \int_S \mathbf{F} \cdot d\mathbf{S} = \frac{2\pi}{3}$

Question 23

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (1 + \sin u \cos v)\mathbf{i} + (\sin u \sin v)\mathbf{j} + (\cos u)\mathbf{k}$$

such that $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

4π

$$\mathbf{r}(u, v) = [1 + \sin u \cos v, \sin u \sin v, \cos u]$$

$$\mathbf{F}(\mathbf{r}(u, v)) = (x, y, z)$$

- $$\frac{\partial \mathbf{r}}{\partial u} = [\cos u \cos v, \cos u \sin v, -\sin u]$$

$$\frac{\partial \mathbf{r}}{\partial v} = [-\sin u \sin v, \sin u \cos v, 0]$$
- $$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} 0 + \sin^2 u \cos^2 v - \sin^2 u \sin^2 v & -\sin u \cos^2 v & \sin u \cos v \sin u \\ \sin u \cos^2 v & \sin^2 u \cos v & 0 \\ \sin u \cos v \sin u & 0 & \sin^2 u \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2 u (\cos^2 v - \sin^2 v) & -\sin u \cos^2 v & \sin^2 u \cos v \\ \sin u \cos^2 v & \sin^2 u \cos v & 0 \\ \sin u \cos v \sin u & 0 & \sin^2 u \end{bmatrix}$$
- $$\hat{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|} = \frac{\mathbf{n}}{|\mathbf{n}|}$$

NO NEED TO NORMALISE $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ HERE AS IT WILL CANCEL

HENCE

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dS = \int_S \mathbf{F}(\mathbf{r}(u, v)) \cdot \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right] \, du \, dv$$

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} [1 + \sin u \cos v, \sin u \sin v, \cos u] \cdot [\sin^2 u (\cos^2 v - \sin^2 v), -\sin u \cos^2 v, \sin^2 u \cos v] \, du \, dv$$

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \sin^2 u (\cos^2 v - \sin^2 v) + \sin^2 u \cos v + \sin u \cos^2 v \, du \, dv$$

NO SINE TERM
2nd & 3rd

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \sin^2 u (\cos^2 v - \sin^2 v + \cos^2 v) + \sin u \cos^2 v \, du \, dv$$

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \sin^2 u (2\cos^2 v) + \sin u \cos^2 v \, du \, dv$$

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} \sin^2 u (2\cos^2 v + \cos^2 v) \, du \, dv$$

$$= \int_{v=0}^{2\pi} \int_{u=0}^{\pi} 3\sin^2 u \cos^2 v \, du \, dv$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \int_{u=0}^{\pi} \sin^2 u \, du$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \int_{u=0}^{\pi} \frac{1 - \cos 2u}{2} \, du$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \left[\frac{u}{2} - \frac{\sin 2u}{4} \right]_{u=0}^{\pi}$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \left[\frac{\pi}{2} - \frac{\sin 2\pi}{4} - \left(\frac{0}{2} - \frac{\sin 0}{4} \right) \right]$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \left[\frac{\pi}{2} - 0 - \left(\frac{0}{2} - 0 \right) \right]$$

$$= 3 \int_{v=0}^{2\pi} \cos^2 v \, dv \left[\frac{\pi}{2} \right]$$

$$= \frac{3\pi}{2} \int_{v=0}^{2\pi} \cos^2 v \, dv$$

$$= \frac{3\pi}{2} \int_{v=0}^{2\pi} \frac{1 + \cos 2v}{2} \, dv$$

$$= \frac{3\pi}{4} \int_{v=0}^{2\pi} (1 + \cos 2v) \, dv$$

$$= \frac{3\pi}{4} \left[v + \frac{\sin 2v}{2} \right]_{v=0}^{2\pi}$$

$$= \frac{3\pi}{4} \left[2\pi + \frac{\sin 4\pi}{2} - \left(0 + \frac{\sin 0}{2} \right) \right]$$

$$= \frac{3\pi}{4} \left[2\pi + 0 - \left(0 + 0 \right) \right]$$

$$= \frac{3\pi}{4} \cdot 2\pi$$

$$= 3\pi^2$$

Question 24

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(u, v) = (u \cos v)\mathbf{i} + (1 + u \sin v)\mathbf{j} + (u - 1)\mathbf{k},$$

such that $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$.

All integrations must be carried out in parametric.

$$\frac{1}{3}\pi$$

Handwritten solution for the surface integral problem:

- Parametric equations: $\mathbf{r}(u, v) = [u \cos v, 1 + u \sin v, u - 1]$ for $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.
- Vector field: $\mathbf{F}(x, y, z) = (x, y, z)$.
- Partial derivatives: $\frac{\partial \mathbf{r}}{\partial u} = [\cos v, \sin v, 1]$ and $\frac{\partial \mathbf{r}}{\partial v} = [-u \sin v, u \cos v, 0]$.
- Cross product: $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = [0 - u \cos v - u \sin v, -u \sin^2 v - u \cos^2 v, u \cos^2 v + u \sin^2 v] = [-u \cos v - u \sin v, -u(\sin^2 v + \cos^2 v), u(\cos^2 v + \sin^2 v)] = [-u \cos v - u \sin v, -u, u]$.
- Normal vector: $d\mathbf{S} = [-u \cos v - u \sin v, -u, u] dv du$.
- Surface integral: $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 (u \cos v + u \sin v - u) [-u \cos v - u \sin v, -u, u] dv du$.
- Evaluation: $\int_0^{2\pi} \int_0^1 (-u^2 \cos^2 v - u^2 \sin^2 v - u^2 + u^2 \cos v + u^2 \sin v) dv du = \int_0^{2\pi} \int_0^1 (-u^2(\cos^2 v + \sin^2 v) + u^2(\cos v + \sin v)) dv du = \int_0^{2\pi} \int_0^1 (-u^2 + u^2(\cos v + \sin v)) dv du = \int_0^{2\pi} [-\frac{1}{3}u^3 + \frac{1}{2}u^2(\sin v - \cos v)]_0^1 dv = \int_0^{2\pi} [-\frac{1}{3} + \frac{1}{2}(\sin v - \cos v)] dv = [-\frac{1}{3}v + \frac{1}{2}(-\cos v - \sin v)]_0^{2\pi} = -\frac{1}{3}(2\pi) + \frac{1}{2}(-\cos 2\pi - \sin 2\pi + \cos 0 + \sin 0) = -\frac{2\pi}{3} + \frac{1}{2}(-1 - 0 + 1 + 0) = -\frac{2\pi}{3} + 0 = -\frac{2\pi}{3}$.
- Magnitude: $|\int_S \mathbf{F} \cdot d\mathbf{S}| = \frac{2\pi}{3}$.

Question 25

$$\mathbf{F}(x, y, z) \equiv x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with parametric equations

$$\mathbf{r}(\theta, \varphi) = [(4 + \cos \theta) \cos \varphi] \mathbf{i} + [(4 + \cos \theta) \sin \varphi] \mathbf{j} + (\sin \theta) \mathbf{k},$$

such that $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$.

All integrations must be carried out in parametric.

$$24\pi^2$$

$\mathbf{F}(\theta, \varphi) = [(4 + \cos \theta) \cos \varphi] \mathbf{i} + [(4 + \cos \theta) \sin \varphi] \mathbf{j} + (\sin \theta) \mathbf{k}$

 $0 \leq \theta \leq 2\pi$

 $0 \leq \varphi \leq 2\pi$

 $\mathbf{F}(x, y, z) = (x, y, z)$

$\frac{\partial \mathbf{F}}{\partial \theta} = [-\sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta]$

 $\frac{\partial \mathbf{F}}{\partial \varphi} = [-\cos \theta \sin \varphi, \cos \theta \cos \varphi, 0]$

$\frac{\partial \mathbf{F}}{\partial \theta} \times \frac{\partial \mathbf{F}}{\partial \varphi} = \begin{vmatrix} -\sin \theta \cos \varphi & -\sin \theta \sin \varphi & \cos \theta \\ -\cos \theta \sin \varphi & \cos \theta \cos \varphi & 0 \end{vmatrix}$

 $= [0 - 4 \sin^2 \theta \cos \varphi \sin \varphi, -4 \sin \theta \cos^2 \varphi - 4 \sin \theta \cos^2 \varphi, -\sin \theta \cos \theta \cos^2 \varphi - \cos \theta \sin \theta \sin^2 \varphi]$

 $= [-4 \sin^2 \theta \cos \varphi \sin \varphi, -4 \sin \theta \cos^2 \varphi, -4 \sin \theta \cos \theta (\cos^2 \varphi + \sin^2 \varphi)]$

 $= [-4 \sin^2 \theta \cos \varphi \sin \varphi, -4 \sin \theta \cos^2 \varphi, -4 \sin \theta \cos \theta]$

 $= (4 + \cos \theta) [-\cos \theta \sin \varphi \cos \varphi, -\cos \theta \sin \varphi, -\sin \theta]$

 \rightarrow normal vector is

$\hat{n} = \frac{1}{|\mathbf{n}|} \left[\frac{\partial \mathbf{F}}{\partial \theta} \times \frac{\partial \mathbf{F}}{\partial \varphi} \right]$

 $\hat{n} = \frac{1}{|\mathbf{n}|} \left[\frac{\partial \mathbf{F}}{\partial \theta} \times \frac{\partial \mathbf{F}}{\partial \varphi} \right]$

NOTE: THE EVALUATION OF $\left| \frac{\partial \mathbf{F}}{\partial \theta} \times \frac{\partial \mathbf{F}}{\partial \varphi} \right|$ IS NOT NEEDED IN THIS TYPE OF QUESTION, AS IT WILL CANCEL.

Now

 $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot \hat{n} \, dS = \int_S \mathbf{F}(\theta, \varphi) \cdot \left[\frac{\partial \mathbf{F}}{\partial \theta} \times \frac{\partial \mathbf{F}}{\partial \varphi} \right] \, d\theta \, d\varphi$

 $= \int_0^{2\pi} \int_0^{2\pi} [(4 + \cos \theta) \cos \varphi] \cos \theta \sin \varphi \, d\theta \, d\varphi + \int_0^{2\pi} \int_0^{2\pi} [(4 + \cos \theta) \sin \varphi] \cos \theta \cos \varphi \, d\theta \, d\varphi + \int_0^{2\pi} \int_0^{2\pi} (\sin \theta) (-4 \sin \theta \cos \theta) \, d\theta \, d\varphi$

 $= \int_0^{2\pi} \int_0^{2\pi} [-4 \cos \theta] [(4 + \cos \theta) \cos \theta \sin \varphi \cos \varphi + (4 + \cos \theta) \sin \varphi \cos \theta \sin \varphi + \sin^2 \theta] \, d\theta \, d\varphi$

 $= \int_0^{2\pi} \int_0^{2\pi} -4 \cos \theta [4 \cos \theta + \cos^2 \theta + \sin^2 \theta] \, d\theta \, d\varphi$

 $= \int_0^{2\pi} \int_0^{2\pi} -4 \cos \theta (4 \cos \theta + 1) \, d\theta \, d\varphi = - \int_0^{2\pi} \int_0^{2\pi} (16 \cos^2 \theta + 4 \cos \theta) \, d\theta \, d\varphi$

 $= - \int_0^{2\pi} \int_0^{2\pi} \left[4 + 4 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) \right] \, d\theta \, d\varphi = - \int_0^{2\pi} \int_0^{2\pi} 6 \, d\theta \, d\varphi$

 $= -6 \times 2\pi \times 2\pi = -24\pi^2$

 $\therefore \left| \int_S \mathbf{F} \cdot d\mathbf{S} \right| = 24\pi^2$

Question 26

It is given that the vector field \mathbf{F} satisfies

$$\mathbf{F} = 2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x.$$

You **must** find a suitable parameterization for S , and carry out the **integration in parametric**, without using any integral theorems.

$\frac{\pi}{4}$

$\mathbf{F} = (2y, -2x, 1)$ $x^2 + y^2 + z^2 = 1$
 $x^2 + y^2 = x$

PARAMETERISE THE REGION R — FIRST LOOK AT THE VERY ROW ABOVE

$\Rightarrow x^2 + y^2 = x$
 $\Rightarrow x^2 - x + y^2 = 0$
 $\Rightarrow (x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$
 $\Rightarrow [(x - \frac{1}{2})^2 + (2y)^2] = 1$
 $\Rightarrow (2x - 1)^2 + (2y)^2 = 1$

HENCE THE GOOD CHOICE PARAMETRIZES AS

$2x - 1 = \cos\theta$?
 $2y = \sin\theta$?
 $z = \sqrt{1 - (x - \frac{1}{2})^2 - y^2}$
 $= \sqrt{1 - (\frac{1 + \cos\theta}{2})^2 - (\frac{\sin\theta}{2})^2}$
 $= \sqrt{1 - \frac{1 + 2\cos\theta + \cos^2\theta}{4} - \frac{\sin^2\theta}{4}}$
 $= \sqrt{1 - \frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{4}}$
 $= \sqrt{1 - \frac{1 + 2\cos\theta + 1}{4}}$
 $= \sqrt{1 - \frac{2 + 2\cos\theta}{4}}$
 $= \sqrt{1 - \frac{1 + \cos\theta}{2}}$
 $= \sqrt{\frac{2 - 1 - \cos\theta}{2}}$
 $= \sqrt{\frac{1 - \cos\theta}{2}}$
 $= \sin\frac{\theta}{2}$

WANTED TO ADD A SUBSTITUTION! PRACTICE SO IT BECOMES THE BEST CHOICE THE QUOTE SAY IT

$z = \frac{1}{2}(1 + \cos\theta)$
 $y = \frac{1}{2}\sin\theta$ $0 \leq \theta \leq \pi$
 $0 \leq r \leq 1$

WILL TRACE THE CURVE ONCE AT (3rd) QUANTAL 2x INCREASE ITS INCREASE

FOR $x^2 + y^2 + z^2 = 1$
 $\frac{\partial z}{\partial x} = \frac{-2x}{2z} = -\frac{x}{z}$
 $\frac{\partial z}{\partial y} = \frac{-2y}{2z} = -\frac{y}{z}$
 $z = \sqrt{1 - x^2 - y^2}$
 $z = (1 - \frac{1}{2}(1 + \cos\theta))^{\frac{1}{2}}$

HENCE

$\mathbf{r} = (\frac{1}{2}(1 + \cos\theta), \frac{1}{2}\sin\theta, (1 - \frac{1}{2}(1 + \cos\theta))^{\frac{1}{2}})$
 $0 \leq r \leq 1$
 $0 \leq \theta \leq \pi$

$\frac{\partial \mathbf{r}}{\partial \theta} = [\frac{1}{2}(-\sin\theta), \frac{1}{2}\cos\theta, \frac{1}{2}(-\frac{1}{2}(1 + \cos\theta))^{-\frac{1}{2}}(-\sin\theta)]$
 $\frac{\partial \mathbf{r}}{\partial \theta} = [-\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta, \frac{1}{4}\sin\theta(1 - \frac{1}{2}(1 + \cos\theta))^{-\frac{3}{2}}]$

$\frac{\partial \mathbf{r}}{\partial \theta} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = [\frac{1}{4}\sin^2\theta + \frac{1}{4}\cos^2\theta + \frac{1}{16}\sin^2\theta(1 - \frac{1}{2}(1 + \cos\theta))^{-3}]$
 $= \frac{1}{4} + \frac{1}{16}\sin^2\theta(1 - \frac{1}{2}(1 + \cos\theta))^{-3}$
 $= \frac{1}{4} + \frac{1}{16}\sin^2\theta(\frac{1 - \cos\theta}{2})^{-3}$
 $= \frac{1}{4} + \frac{1}{16}\sin^2\theta \frac{8}{(1 - \cos\theta)^3}$
 $= \frac{1}{4} + \frac{1}{2}\frac{\sin^2\theta}{(1 - \cos\theta)^3}$
 $= \frac{1}{4} + \frac{1}{2}\frac{1 - \cos^2\theta}{(1 - \cos\theta)^3}$
 $= \frac{1}{4} + \frac{1}{2}\frac{(1 - \cos\theta)(1 + \cos\theta)}{(1 - \cos\theta)^3}$
 $= \frac{1}{4} + \frac{1}{2}\frac{1 + \cos\theta}{(1 - \cos\theta)^2}$
 $= \frac{1}{4} + \frac{1}{2}\frac{1 + \cos\theta}{(1 - \cos\theta)^2}$

Now $\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi \int_0^1 \mathbf{F}(\mathbf{r}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial r}) dr d\theta$
 $= \int_0^\pi \int_0^1 \mathbf{F}(\mathbf{r}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial r}) dr d\theta$

Now $\mathbf{F}(x, y, z) = (2y, -2x, 1)$
 $\mathbf{F}(\mathbf{r}) = (\sin\theta, -1 + \cos\theta, 1)$

So $\mathbf{F}(\mathbf{r}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial r}) = \sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta \sin\theta (1 - \cos\theta) [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$

Simplify

$= \frac{1}{2}\sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$
 $= \frac{1}{2}\sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$
 $= \frac{1}{2}\sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$

Simplify

$= \frac{1}{2}\sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$
 $= \frac{1}{2}\sin\theta \cos\theta [1 - \frac{1}{2}(1 + \cos\theta)]^{\frac{3}{2}} + \frac{1}{2}\sin\theta \cos\theta (1 + \cos\theta) + \frac{1}{2}\sin\theta$

Now looking at the above integrand notice the θ variations, $\int_0^\pi \sin\theta \cos\theta d\theta = 0$
 $\int_0^\pi \sin\theta d\theta = [-\cos\theta]_0^\pi = -(-1) - (-1) = 0$
 $\int_0^\pi \sin\theta \cos\theta (1 + \cos\theta) d\theta = \int_0^\pi \sin\theta d\theta = 0$
 $\therefore \int_0^\pi \int_0^1 \mathbf{F} \cdot d\mathbf{S} = \int_0^\pi [\frac{1}{2}\sin\theta] d\theta = \int_0^\pi \frac{1}{2}\sin\theta d\theta = \frac{1}{2}[-\cos\theta]_0^\pi = \frac{1}{2}[-(-1) - (-1)] = \frac{1}{2}[2] = 1$

Created by T. Madas

TYPE \oiint_S **F · dS**

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Question 1

$$\mathbf{F}(x, y, z) \equiv xy\mathbf{i} + y\mathbf{j} + 4z\mathbf{k}$$

Evaluate the integral

$$\oiint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the closed surface enclosing the finite region V , defined by

$$x^2 + y^2 \leq 9, \quad x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 4.$$

$$9\pi + 36$$

Handwritten solution for the divergence theorem problem:

$\int_V \mathbf{F} \cdot d\mathbf{S} = \dots$ DIVERGENCE THEOREM
 $\dots = \int_V \nabla \cdot \mathbf{F} \, dV = \int_V \left(\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(4z) \right) dV$
 $= \int_V (y + 1 + 4) \, dV = \int_V (y + 5) \, dV$
 SLICE WITH CIRCULAR BASES (r, θ, z)
 $= \int_{z=0}^4 \int_{\theta=0}^{\pi/2} \int_{r=0}^3 (r \sin \theta + 5) (r \, dr \, d\theta \, dz)$
 $= \int_{z=0}^4 \left[\int_{\theta=0}^{\pi/2} \left(\frac{r^2 \sin \theta}{2} + 5r \right) dr \, d\theta \right] dz = \int_{z=0}^4 \left[\frac{r^2}{2} \left(\frac{1}{2} \cos \theta + \frac{1}{2} \right) + 5r \right]_{r=0}^3 dz$
 $= \int_{z=0}^4 \left[\frac{9}{2} \left(\frac{1}{2} \cos \theta + \frac{1}{2} \right) + \frac{15}{2} \right] dz = \int_{z=0}^4 \left[-\frac{9}{4} \sin \theta + \frac{9}{4} + \frac{15}{2} \right] dz$
 $= \int_{z=0}^4 \left(0 + \frac{9\pi}{4} - (-9) + 0 \right) dz = \int_{z=0}^4 \left(\frac{9\pi}{4} + 9 \right) dz$
 $= \left[\frac{9\pi}{4}z + 9z \right]_{z=0}^4 = (9\pi + 36) - 0 = 36 + 9\pi$

Question 2

$$\mathbf{F}(x, y, z) \equiv (x + y^2)\mathbf{i} + (2y + xz)\mathbf{j} + (3z + xyz)\mathbf{k}.$$

Evaluate the integral

$$\oiint_S \mathbf{F} \cdot d\mathbf{S},$$

where S is the surface with Cartesian equation

$$4x^2 + 4y^2 + 4z^2 = 1.$$

You may not use the Divergence Theorem in this question.

π

Handwritten solution for the surface integral problem. The left page shows the vector field $\mathbf{F} = (x+y^2)\mathbf{i} + (2y+xz)\mathbf{j} + (3z+xyz)\mathbf{k}$ and the surface equation $4x^2 + 4y^2 + 4z^2 = 1$. It calculates the normal vector $\mathbf{n} = (2x, 2y, 2z)$ and its magnitude $|\mathbf{n}| = 2\sqrt{x^2 + y^2 + z^2} = 1/2$. The surface integral is then set up as $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \oiint_S (x+y^2)(2x) + (2y+xz)(2y) + (3z+xyz)(2z) \, dS$. The right page shows the parameterization of the sphere using spherical coordinates $x = \frac{1}{2}\sin\theta\cos\phi$, $y = \frac{1}{2}\sin\theta\sin\phi$, $z = \frac{1}{2}\cos\theta$ and the evaluation of the surface integral, resulting in π .

Question 3

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{k} \wedge \mathbf{r}, \quad \text{where } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Show by direct integration that

$$\oiint_S \nabla \wedge \mathbf{F} \cdot d\mathbf{S} = 0,$$

where S is the **closed** surface enclosing the finite region V , defined by

$$x^2 + y^2 + z^2 \leq 1, \quad z \geq 0, \quad \text{and} \quad x^2 + y^2 \leq 1.$$

You may not use any Integral Theorems in this question.

proof

$\text{Let } f(\theta, \phi) = x^2 + y^2 + z^2 = 1$
 $\Sigma_1 = (x, y, z)$
 $|\mathbf{n}| = \sqrt{x^2 + y^2 + z^2} = 1$
 $\hat{\mathbf{n}} = (x, y, z)$

$\bullet \mathbf{F} = \mathbf{k} \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = (-y, x, 0)$
 $\bullet \nabla \cdot \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0, 0, 2)$

$\bullet \oiint_S \nabla \cdot \mathbf{F} \cdot d\mathbf{S} = \int_{\Sigma_1} (0, 0, 2) \cdot d\mathbf{S} + \int_{\Sigma_2} (0, 0, 2) \cdot d\mathbf{S}$
 $= \int_{\Sigma_1} 2z \, dS + \int_{\Sigma_2} -2 \, dS$

SPHERE ON PLANE
 $x = \cos\theta \sin\phi$ For θ on Σ_1
 $y = \sin\theta \sin\phi$ $0 \leq \theta \leq \pi$
 $z = \cos\phi$ $0 \leq \phi \leq 2\pi$
 $dS = r^2 \sin\theta \, d\theta \, d\phi$
 $dS = \sin\theta \, d\theta \, d\phi$

$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} (2\cos\theta) (\sin\theta \, d\theta \, d\phi) - 2 \int_{\Sigma_2} 1 \, dS$
 $= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} 2\sin\theta \cos\theta \, d\theta \, d\phi - 2 \times (\text{Area of } \Sigma_2)$
 $= \int_{\phi=0}^{2\pi} \left[\frac{1}{2} \sin^2\theta \right]_0^{\pi/2} d\phi - 2 \times (\pi \times 1^2)$
 $= \int_{\phi=0}^{2\pi} \frac{1}{2} (-\frac{1}{2}) d\phi = -2\pi$
 $= 1 \times 2\pi - 2\pi$
 $= 0$

Question 4

$$\mathbf{F}(x, y, z) \equiv (4yz)\mathbf{i} + (2y^2)\mathbf{j} + (5xyz + 6z^2 + 3z)\mathbf{k}$$

Evaluate the integral

$$\oiint_S \mathbf{F} \cdot d\mathbf{S}$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + 4z^2 = 1.$$

You may not use the Divergence Theorem in this question.

2π

$\mathbf{F} = 4yz\mathbf{i} + 2y^2\mathbf{j} + (5xyz + 6z^2 + 3z)\mathbf{k}$

• LET THE ELIPSOID HAVE EQUATION
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
 $\frac{x^2}{1} + \frac{y^2}{1} + \frac{z^2}{\frac{1}{4}} = 1$
 $\frac{x^2}{1} + \frac{y^2}{1} + 4z^2 = 1$
 $\mathbf{x} = (x, y, 4z)$
 (CHECK $\frac{\partial \mathbf{x}}{\partial z}$ AS WE WILL NEED THIS FOR THE z Y PLANE)

• SPILT THE ELIPSOID INTO 2 + 2 PLANS

$S_1: z = +\frac{1}{2}\sqrt{1-x^2-y^2}$
 $\frac{d\mathbf{s}}{ds} = \frac{dx dy}{\sqrt{1-x^2-y^2}}$
 $D: x^2 + y^2 \leq 1$

$S_2: z = -\frac{1}{2}\sqrt{1-x^2-y^2}$
 $\frac{d\mathbf{s}}{ds} = \frac{dx dy}{\sqrt{1-x^2-y^2}}$
 $D: x^2 + y^2 \leq 1$

Hence
 $\oiint_S \mathbf{F} \cdot \frac{d\mathbf{s}}{ds} = \iint_D (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{dx dy}{\sqrt{1-x^2-y^2}}$

TOP HALF - PROJECTING ONTO THE xy PLANE

$I_1 = \iint_D (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{dx dy}{\sqrt{1-x^2-y^2}}$

$I_2 = \iint_D (4yz, 2y^2, 5xyz + 6z^2 + 3z) \cdot \frac{dx dy}{\sqrt{1-x^2-y^2}}$

$I_3 = \iint_D \frac{4xyz + 2y^2 + 20xyz + 12z^2 + 12z^2}{4z} dx dy$

$I_1 = \iint_D 2y + \frac{y^2}{\sqrt{1-x^2-y^2}} + 5xyz + 6z^2 + 3z dx dy$

$I_2 = \iint_D 2y + \frac{y^2}{\sqrt{1-x^2-y^2}} + 5xy \times \frac{1}{2}\sqrt{1-x^2-y^2} + 6 \times \frac{1}{4}(1-x^2-y^2) + 3 \times \frac{1}{2}(1-x^2-y^2) dx dy$

$I_3 = \iint_D 2y + \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{5}{2}xy(1-x^2-y^2) + \frac{3}{2}(1-x^2-y^2) + \frac{3}{2}(1-x^2-y^2) dx dy$

WITH THE SURROUNDING FRAME, GET AN EXPRESSION FOR I_2 , IN THE FORM $\frac{dx dy}{\sqrt{1-x^2-y^2}}$ OF THE ELIPSOID AND KEEP THAT

$z = \frac{1}{2}\sqrt{1-x^2-y^2}$
 $\frac{dz}{ds} = \frac{dx dy}{\sqrt{1-x^2-y^2}}$ EACH OF THESE WILL BE NEARLY ANNULLUS, SO SOME TERMS WILL CANCEL

$I_2 = \iint_D \left[-2y - \frac{y^2}{\sqrt{1-x^2-y^2}} - \frac{5}{2}xy(1-x^2-y^2) - \frac{3}{2}(1-x^2-y^2) \right] \frac{dx dy}{\sqrt{1-x^2-y^2}}$

$I_3 = \iint_D \left[-2y - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{5}{2}xy(1-x^2-y^2) - \frac{3}{2}(1-x^2-y^2) + \frac{3}{2}(1-x^2-y^2) \right] dx dy$

ADDING I_2 & I_3

$\oiint_S \mathbf{F} \cdot \frac{d\mathbf{s}}{ds} = \iint_D \left[\frac{2y^2}{\sqrt{1-x^2-y^2}} + 5xy(1-x^2-y^2) + 3(1-x^2-y^2) \right] dx dy$

BEFORE WE GO ON TO THE z Y PLANE THAT $z = \frac{1}{2}\sqrt{1-x^2-y^2}$ IS A SIMILAR CASE, BECAUSE IN x, y, z SO THESE TWO POINTS OF z ARE z THE SAME COORDINATES

THIS
 $\dots = \iint_D \left[\frac{2y^2}{\sqrt{1-x^2-y^2}} + 5xy(1-x^2-y^2) + 3(1-x^2-y^2) \right] dx dy$

$= \iint_D 3(1-x^2-y^2) dx dy$

SWITCH INTO POLAR COORDS

$= \int_0^{2\pi} \int_0^1 3(1-r^2) (r dr d\theta)$

$= \int_0^{2\pi} \left[\frac{3}{2}(1-r^2)^2 \right]_{r=0}^1 d\theta$

$= \int_0^{2\pi} \frac{3}{2} d\theta$

$= \frac{3}{2} \times [\theta]_0^{2\pi}$

$= \frac{3}{2} \times 2\pi$

$= 3\pi$

Question 5

The surface Ω is the sphere with Cartesian equation

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$$

Evaluate the surface integral

$$\oiint_{\Omega} \left[(x+y)\mathbf{i} + (x^2+xy)\mathbf{j} + z^2\mathbf{k} \right] \cdot d\mathbf{S},$$

where $d\mathbf{S}$ is a unit surface element on Ω .

You may not use the Divergence Theorem in this question.

$$\frac{16}{3}\pi$$

$\int_{\Omega} \mathbf{F} \cdot d\mathbf{S} = \int_{\Omega} (x+y, x^2+xy, z^2) \cdot \hat{\mathbf{n}} \, dS = \dots$

- Moving the origin does NOT affect the answer, so translate the centre to $(1,1,1)$
- Thus $(x-1)^2 + (y-1)^2 + (z-1)^2 = 1$ becomes $x^2 + y^2 + z^2 = 1$
 $(x+y, x^2+xy, z^2)$ becomes $[(x+1)(y+1), (x+1)(y+1), (z+1)^2]$

Try it:
 $= [x+y+2, x^2+2x+1+xy+2xy+1, z^2+2z+1]$
 $= [2x+y+2, x^2+2xy+2x+2, z^2+2z+1]$

- Let $f(x,y,z) = x^2+y^2+z^2-1$
 $\nabla f = (2x, 2y, 2z)$
 $|\nabla f| = \sqrt{4x^2+4y^2+4z^2} = 2$
 $\hat{\mathbf{n}} = \frac{\nabla f}{|\nabla f|} = (x, y, z)$

$\dots = \int_{\Omega} (2x+y+2, x^2+2xy+2x+2, z^2+2z+1) \cdot (x, y, z) \, dS$
 $= \int_{\Omega} (2x^2+xy+2x+xy+y^2+2xy+2x^2+2z^2+2z^2+z) \, dS$

Now the domain (sphere) is symmetric in x, y and in z (thinking as circular cross sections) - so all odd powers in any variable will have no contribution.

$= \int_{\Omega} (2x^2 + x^2y + 2z^2 + 2xy + 2xy + 2xy + 2x + 2y + z) \, dS$
 $= \int_{\Omega} (2x^2y + 2z^2) \, dS$
 $= \int_{\Omega} (1 + 2z^2) \, dS$

- Switch into spherical coords
 $x = \sin\theta \cos\phi$
 $y = \sin\theta \sin\phi$
 $z = \cos\theta$
 $dS = \sin\theta \, d\theta \, d\phi$
 $0 \leq \theta \leq \pi$
 $0 \leq \phi \leq 2\pi$

$= \int_0^{2\pi} \int_0^{\pi} (1 + 2\cos^2\theta) \sin\theta \, d\theta \, d\phi$
 $= \left[\int_0^{2\pi} 1 \, d\phi \right] \left[\int_0^{\pi} \sin\theta + 2\cos^2\theta \sin\theta \, d\theta \right]$
 $= 2\pi \times \left[-\cos\theta - \frac{1}{3}\cos^3\theta \right]_0^{\pi}$
 $= 2\pi \times \left[\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{\pi}$
 $= 2\pi \times \left[(1 + \frac{1}{3}) - (-1 - \frac{1}{3}) \right]$
 $= 2\pi \times \frac{8}{3}$
 $= \frac{16}{3}\pi$