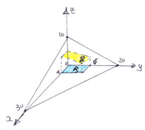
SURFACE INTEGRALS

Question 1

Find the area of the plane with equation

$$2x + 3y + 6z = 60, \; 0 \le x \le 4, \; 0 \le y \le 6.$$

$\boxed{28}$



$f(x,y,z) = 2x+3y+6z-60$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2,3,6)$

$\underline{n} = (2,3,6)$

$|\underline{n}| = \sqrt{4+9+36} = 7$

$\hat{\underline{n}} = \frac{1}{7}(2,3,6)$

$dS = \frac{dx\,dy}{\hat{\underline{n}} \cdot \underline{k}} = \frac{dx\,dy}{\frac{1}{7}(2,3,6)\cdot(0,0,1)}$

$\boxed{dS = \frac{7}{6}\,dx\,dy}$

THUS AREA $= \int_S 1 \, dS = \int_R 1 \times \frac{7}{6}\,dx\,dy = \frac{7}{6}\int 1 \, dx\,dy = \frac{7}{6} \times$ AREA OF R

$= \frac{7}{6} \times (4 \times 6) = 28$

NOTE THAT WE CAN ALSO OBTAIN dS AS FOLLOWS

$z = \frac{1}{6}(60-2x-3y)$

$z = 10 - \frac{1}{3}x - \frac{1}{2}y$

$\frac{\partial z}{\partial x} = -\frac{1}{3}$

$\frac{\partial z}{\partial y} = -\frac{1}{2}$

$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dx\,dy$

$dS = \sqrt{\frac{1}{9} + \frac{1}{4} + 1}\;dx\,dy$

$dS = \sqrt{\frac{49}{36}}\;dx\,dy$

$\boxed{dS = \frac{7}{6}\,dx\,dy}$ AS BEFORE

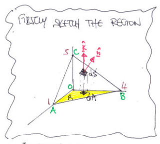
Question 2

A surface has Cartesian equation

$$x+\frac{y}{4}+\frac{z}{5}=1\,.$$

Determine the area of the surface which lies in the first octant.

$$\boxed{\frac{21}{2}}$$



$x+\frac{y}{4}+\frac{z}{5}=1$

$20x+5y+4z=20$

IE A PLANE

$\underline{n}=(20,5,4)$

$|\underline{n}|=\sqrt{200+25+4}$

$\hat{\underline{n}}=\frac{1}{21}(20,5,4)$

PROJECTION ONTO THE xy PLANE (REGION R)

$$dS=\frac{dx\,dy}{\hat{n}\cdot\hat{k}}=\frac{dx\,dy}{\frac{1}{21}(20,5,4)\cdot(0,0,1)}=\frac{21}{4}\,dx\,dy$$

$$\text{THUS AREA}=\iint_{ABC}1\;dS=\iint_{R}1\quad\frac{21}{4}\,dx\,dy$$

$$=\frac{21}{4}\times\text{AREA OF }R$$

$$=\frac{21}{4}\times\underbrace{\left(\tfrac{1}{2}\times1\times4\right)}_{\text{TRIANGLE }\widehat{AOB}}$$

$$=\frac{21}{2}$$

Question 3

The plane with equation

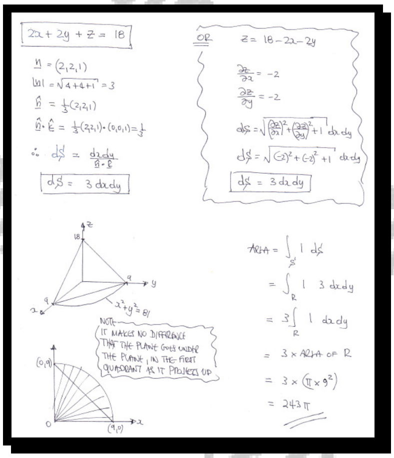
$$2x + 2y + z = 18,$$

intersects the cylinder with equation

$$x^2 + y^2 = 81.$$

Determine the area of the cross-sectional cut.

$\boxed{243\pi}$



Question 4

A tube in the shape of a right circular cylinder of radius 4 m and height 0.5 m, emits heat from its curved surface only.

The heat emission rate, in Wm^{-2}, is given by

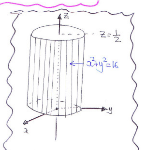
$$\frac{1}{2}\mathrm{e}^{-2z}\sin^2\theta,$$

where θ and z are standard cylindrical polar coordinates, whose origin is at the centre of one of the flat faces of the cylinder.

Given that the cylinder is contained in the part of space for which $z \geq 0$, determine the total heat emission rate from the tube.

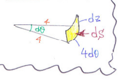
$$\boxed{\pi\left(1-\mathrm{e}^{-1}\right)}$$

HEAT EMISSION RATE $f(r,\theta,z) = \frac{1}{2}e^{-2z}\sin^2\theta$ (Watts/m²)



dS ON THE CURVED SURFACE OF THE CYLINDER IS GIVEN BY

$$\boxed{dS = 4\,d\theta\,dz}$$



TOTAL HEAT EMISSION RATE $= \int_S f(r,\theta,z)\,dS$

$$= \int_{z=0}^{\frac{1}{2}}\int_{\theta=0}^{2\pi}\left(\frac{1}{2}e^{-2z}\sin^2\theta\right)\left(4\,d\theta\,dz\right)$$

$$= \int_{z=0}^{\frac{1}{2}}\int_{\theta=0}^{2\pi} 2e^{-2z}\sin^2\theta \; d\theta\,dz$$

$$= \int_{z=0}^{\frac{1}{2}}\int_{\theta=0}^{2\pi} 2e^{-2z}\left(\frac{1}{2}-\cancel{\frac{1}{2}\cos 2\theta}\right) d\theta\,dz$$

NO CONTRIBUTION FROM THE θ INTEGRATION

$$= \int_{z=0}^{\frac{1}{2}}\int_{\theta=0}^{2\pi} e^{-2z}\,d\theta\,dz$$

$$= 2\pi\int_{z=0}^{\frac{1}{2}} e^{-2z}\,dz$$

$$= 2\pi\left[-\frac{1}{2}e^{-2z}\right]_0^{\frac{1}{2}}$$

$$= \pi\left[e^{-2z}\right]_{\frac{1}{2}}^{0}$$

$$= \pi\left[1-e^{-1}\right]$$

Question 5

A surface has Cartesian equation

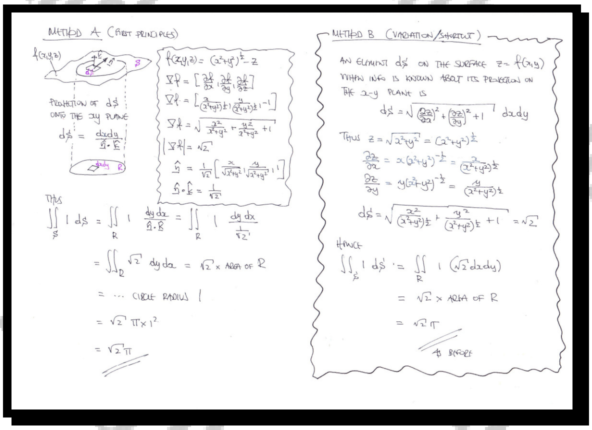
$$z = \sqrt{x^2 + y^2}\ .$$

The projection in the x-y plane of the region S on this surface, is the region R with Cartesian equation

$$x^2 + y^2 = 1\,.$$

Find the area of S.

$$\boxed{\pi\sqrt{2}}$$



Question 6

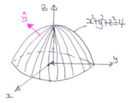
$$I = \int_S z \, dS \, .$$

Find the exact value of I, if S is the surface of the hemisphere with equation

$$x^2 + y^2 + z^2 = 4, \ z \geq 0 \, .$$

You may only use Cartesian coordinates in this question.

$\boxed{8\pi}$



LET $f(x,y,z) = x^2+y^2+z^2-4$

$\underline{n} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$

TAKE (x,y,z) AS THE NORMAL

$\hat{n} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{\sqrt{4}} = \frac{1}{2}(x,y,z)$

THUS

$$\int_S z \, dS = \int_S \sqrt{4-x^2-y^2} \, dS \quad \text{PROJECT ONTO THE } xy \text{ PLANE}$$

$$= \int_R \sqrt{4-x^2-y^2} \, \frac{dx\,dy}{\hat{n} \cdot \underline{k}} = \int_R \sqrt{4-x^2-y^2} \, \frac{dx\,dy}{\frac{1}{2}(x,y,z)\cdot(0,0,1)}$$

$$= \int_R \sqrt{4-x^2-y^2} \, \frac{2}{z} \, dx\,dy = \int_R \cancel{\sqrt{4-x^2-y^2}} \times \frac{2}{\cancel{\sqrt{4-x^2-y^2}}} \, dx\,dy$$

$$= \int_R 2 \, dx\,dy = 2 \times (\text{AREA OF THE CIRCLE } x^2+y^2=4)$$

$$= 2 \times \pi \times 2^2 = 8\pi$$

Question 7

A hemispherical surface, of radius a m, is electrically charged.

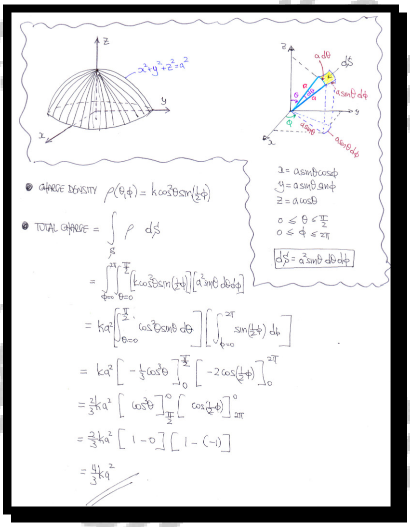
The electric charge density $\rho(\theta,\varphi)$, in C m^{-2}, is given by

$$\rho(\theta,\varphi) = k\cos^2(\theta)\sin\left(\tfrac{1}{2}\varphi\right),$$

where k is a positive constant, and θ and φ are standard spherical polar coordinates, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z \ge 0$, determine the total charge on its surface.

$$\boxed{\frac{4}{3}ka^2}$$



Question 8

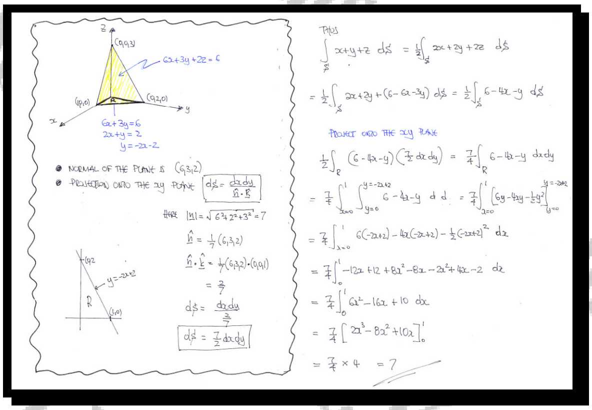
Evaluate the integral

$$\int_S (x+y+z)\,dS\,,$$

where S is the plane with Cartesian equation

$$6x+3y+2z=6,\quad x\geq 0,\quad y\geq 0,\quad z\geq 0.$$

$\boxed{7}$



Question 9

A hemispherical surface, of radius a m, has small indentations due to particle bombardment.

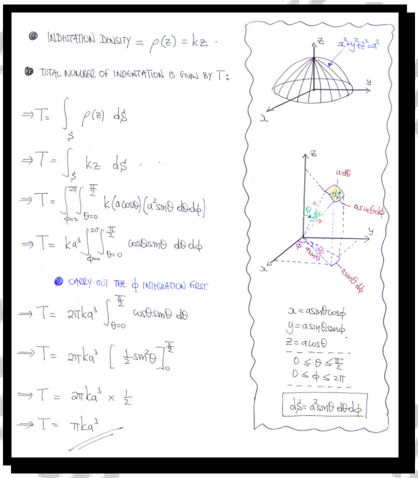
The indentation density $\rho(z)$, in m^{-2}, is given by

$$\rho(z) = kz\,,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat open face of the hemisphere.

Given that the hemisphere is contained in the part of space for which $z \geq 0$, determine the total number of indentations on its surface.

$\boxed{\pi k a^3}$


INDENTATION DENSITY $= \rho(z) = kz$.

TOTAL NUMBER OF INDENTATION IS GIVEN BY T:

$\Rightarrow T = \int_S \rho(z)\, dS$

$\Rightarrow T = \int_S kz\, dS$

$\Rightarrow T = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} k(a\cos\theta)(a^2\sin\theta\, d\theta\, d\phi)$

$\Rightarrow T = ka^3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta\, d\phi$

CARRY OUT THE ϕ INTEGRATION FIRST

$\Rightarrow T = 2\pi k a^3 \int_{\theta=0}^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta$

$\Rightarrow T = 2\pi k a^3 \left[\frac{1}{2}\sin^2\theta\right]_0^{\frac{\pi}{2}}$

$\Rightarrow T = 2\pi k a^3 \times \frac{1}{2}$

$\Rightarrow T = \pi k a^3$

Question 10

Evaluate the integral

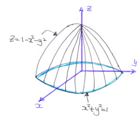
$$\int_S \frac{x^2 - 3y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}} \, dS \,,$$

where S is the surface with Cartesian equation

$$z = 1 - x^2 - y^2 \,, \quad z \geq 0 \,.$$

$$\boxed{\frac{\pi}{2}}$$

$$\int_S \frac{x^2 - 3y^2 + 1}{\sqrt{4x^2 + 4y^2 + 1}} \, dS$$



PROJECT ONTO THE xy REGION R: $x^2 + y^2 \leq 1$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dx\,dy$$

$$\frac{\partial z}{\partial x} = -2x \;;\; \frac{\partial z}{\partial y} = -2y$$

$$dS = \sqrt{(-2x)^2 + (2y)^2 + 1} \; dx\,dy$$

$$dS = \sqrt{4x^2 + 4y^2 + 1} \; dx\,dy$$

$$= \int_R \frac{x^2 - 3y^2 + 1}{\cancel{\sqrt{4x^2 + 4y^2 + 1}}} \cancel{\sqrt{4x^2 + 4y^2 + 1}} \, dx\,dy = \int_R x^2 - 3y^2 + 1 \; dx\,dy$$

SWITCH INTO PLANE POLARS

$$= \int_R \left[r^2\cos^2\theta - 3r^2\sin^2\theta + 1\right]\left[r\,dr\,d\theta\right] = \int_R \left(r^3\cos^2\theta - 3r^3\sin^2\theta + r\right) dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{1} r^3(\cos^2\theta - 3\sin^2\theta) + r \; dr\,d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1}{4}r^4(\cos^2\theta - 3\sin^2\theta) + \frac{1}{2}r^2\right]_{r=0}^{1} d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{4}(\cos^2\theta - 3\sin^2\theta) + \frac{1}{2} \; d\theta = \int_{\theta=0}^{2\pi} \frac{1}{4}\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta - 3\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right] + \frac{1}{2} \; d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{4}\left[-1 + \cancel{\cos 2\theta}\right] + \frac{1}{2} \; d\theta = \int_0^{2\pi} \frac{1}{4} \; d\theta$$

NO CONTRIBUTION FOR $0 \leq \theta \leq 2\pi$

$$= 2\pi \times \frac{1}{4} = \frac{\pi}{2}$$

Question 11

Evaluate the integral

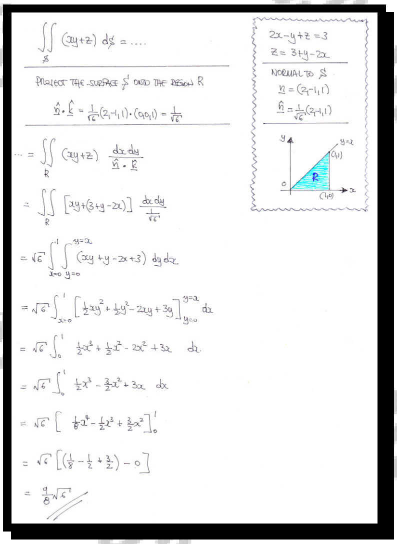
$$\int_S (xy+z)\, dS\,,$$

where S is the plane with Cartesian equation

$$2x - y + z = 3,$$

whose projection onto the plane with equation $z=0$ is the rectilinear triangle with vertices at $(0,0)$, $(1,0)$ and $(1,1)$.

$$\boxed{\frac{9\sqrt{6}}{8}}$$



$$\iint_S (xy+z)\, dS = \dots$$

PROJECT THE SURFACE S ONTO THE REGION R

$$\hat{n}\cdot\hat{k} = \frac{1}{\sqrt{6}}(2,-1,1)\cdot(0,0,1) = \frac{1}{\sqrt{6}}$$

$2x-y+z=3$

$z = 3+y-2x$

NORMAL TO S

$\underline{n} = (2,-1,1)$

$\hat{n} = \frac{1}{\sqrt{6}}(2,-1,1)$

$$\dots = \iint_R (xy+z)\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \iint_R \left[xy+(3+y-2x)\right]\frac{dx\,dy}{\frac{1}{\sqrt{6}}}$$

$$= \sqrt{6}\int_{x=0}^{1}\int_{y=0}^{y=x} (xy+y-2x+3)\,dy\,dx$$

$$= \sqrt{6}\int_{x=0}^{1}\left[\frac{1}{2}xy^2+\frac{1}{2}y^2-2xy+3y\right]_{y=0}^{y=x} dx$$

$$= \sqrt{6}\int_0^1 \frac{1}{2}x^3+\frac{1}{2}x^2-2x^2+3x \;\; dx.$$

$$= \sqrt{6}\int_0^1 \frac{1}{2}x^3-\frac{3}{2}x^2+3x \;\; dx$$

$$= \sqrt{6}\left[\frac{1}{8}x^4-\frac{1}{2}x^3+\frac{3}{2}x^2\right]_0^1$$

$$= \sqrt{6}\left[\left(\frac{1}{8}-\frac{1}{2}+\frac{3}{2}\right)-0\right]$$

$$= \frac{9}{8}\sqrt{6}$$

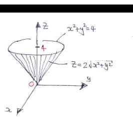
Question 12

$$I = \int_S x^2 + y^2 \; dS \; .$$

Find the exact value of I, if S is the surface of the cone with equation

$$z^2 = 4\left(x^2 + y^2\right), \;\; 0 \le z \le 4 \; .$$

$\boxed{8\pi\sqrt{5}}$



$z=0 \quad x=0, y=0$ (ORIGIN)
$z=4 \quad x^2+y^2=4$ (CIRC

$z = +2(x^2+y^2)^{\frac{1}{2}}$

FOR PROJECTION ON THE xy PLANE $\boxed{dS = \frac{dx\,dy}{\hat{n}\cdot\hat{k}}}$ OR EASIER

$\frac{\partial z}{\partial x} = 2x(x^2+y^2)^{-\frac{1}{2}}$

$\frac{\partial z}{\partial y} = 2y(x^2+y^2)^{-\frac{1}{2}}$

$dS = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dx\,dy$

$dS = \left[1 + \frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2}\right]^{\frac{1}{2}} dx\,dy$

$dS = \left(\frac{x^2+y^2+4x^2+4y^2}{x^2+y^2}\right)^{\frac{1}{2}} dx\,dy$

$dS = \left[\frac{5\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}\right]^{\frac{1}{2}} dx\,dy$

HENCE $dS = \sqrt{5}\,dx\,dy$ WHEN PROJECTED

THUS

$I = \iint_S (x^2+y^2)\,dS = \iint_R x^2+y^2 \left(\sqrt{5}\,dx\,dy\right)$
(CIRCLE $x^2+y^2=4$)

SWITCH INTO PLANE POLARS ONTO $x^2+y^2=4$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{r=2} r^2 \left(\sqrt{5}\,r\,dr\,d\theta\right)$

$= \int_{\theta=0}^{2\pi}\int_{r=0}^{r=2} \sqrt{5}\,r^3\,dr\,d\theta$

$= \int_{\theta=0}^{2\pi}\left[\frac{\sqrt{5}}{4}r^4\right]_0^2 d\theta$

$= \int_{\theta=0}^{2\pi} 4\sqrt{5}\,d\theta$

$= 4\sqrt{5} \times 2\pi$

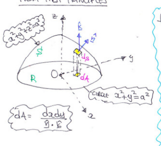
$= 8\pi\sqrt{5}$

Question 13

Show clearly, by a **Cartesian projection** onto the x-y plane, that the surface area of a sphere of radius a, is $4\pi a^2$.

proof

FROM FIRST PRINCIPLES



$f(x,y,z) = x^2+y^2+z^2-a^2$

$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

$\nabla f = (2x, 2y, 2z)$

$\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2} = a$

$\hat{n}\cdot\hat{k} = \frac{1}{a}(x,y,z)\cdot(0,0,1)$

$\hat{n}\cdot\hat{k} = \frac{z}{a}$

THUS THE TOTAL SURFACE AREA

$$2\iint 1\,dS = \iint_R 2\,\frac{dx\,dy}{\frac{z}{a}} = \iint_R \frac{2a}{z}\,dx\,dy = \iint_R \frac{2a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

SINCE R IS A CIRCLE, SWITCH INTO PLANE POLARS

$$\iint_R \frac{2a}{\left[a^2-(x^2+y^2)\right]^{\frac{1}{2}}}\,dx\,dy = \dots \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{2a}{(a^2-r^2)^{\frac{1}{2}}}\,r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} 2ar(a^2-r^2)^{-\frac{1}{2}}\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[-2a(a^2-r^2)^{\frac{1}{2}}\right]_{r=0}^{a} d\theta = \int_{\theta=0}^{2\pi} 2a\left[\sqrt{a^2-r^2}\right]_a^0 d\theta = \int_0^{2\pi} 2a\left[a-0\right] d\theta$$

$$= \int_0^{2\pi} 2a^2\,d\theta = 2a^2\left[\theta\right]_0^{2\pi} = 2a^2\left[2\pi-0\right] = 4\pi a^2$$

VARIATION SHORTCUT

AN ELEMENT dS ON THE SURFACE $z=f(x,y)$ WHEN INFO IS KNOWN ABOUT ITS PROJECTION IN THE xy PLANE IS

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dx\,dy$$

$z = +\sqrt{a^2-x^2-y^2}$ (TOP HALF)

$z = (a^2-x^2-y^2)^{\frac{1}{2}}$

$\frac{\partial z}{\partial x} = -x(a^2-x^2-y^2)^{-\frac{1}{2}}$

$\frac{\partial z}{\partial y} = -y(a^2-x^2-y^2)^{-\frac{1}{2}}$

$dS = \sqrt{\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}+1}\;dx\,dy$

$dS = \sqrt{\frac{x^2}{z^2}+\frac{y^2}{z^2}+1}\;dx\,dy$

$dS = \sqrt{\frac{x^2+y^2+z^2}{z^2}}\;dx\,dy$

$dS = \sqrt{\frac{a^2}{z^2}}\;dx\,dy$

$dS = \frac{a}{z}\,dx\,dy$

& CONTINUE AS OPPOSITE...

Question 14

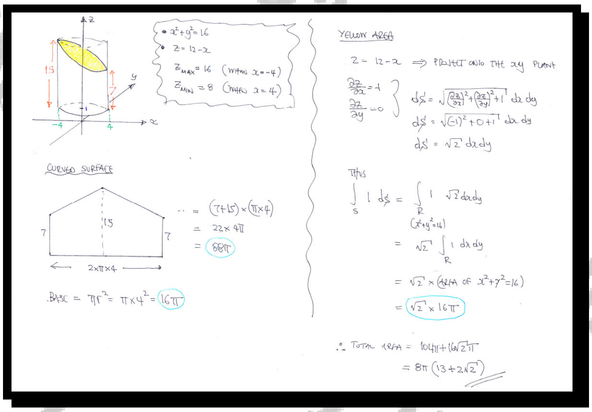
Find in exact form the **total** surface area of the cylinder with equation

$$x^2 + y^2 = 16, \ z \geq 1,$$

cut of by the plane the plane with equation

$$z = 12 - x.$$

$$\boxed{8\pi\left(13 + 2\sqrt{2}\right)}$$



Question 15

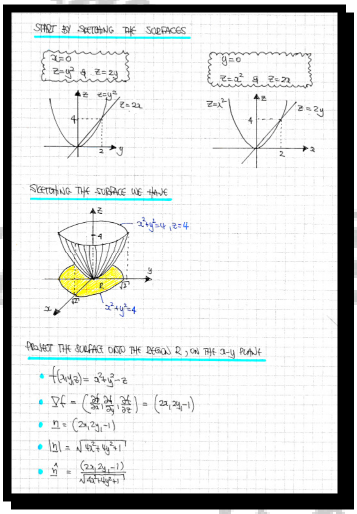
Find the area of finite region on the paraboloid with equation

$$z = x^2 + y^2 ,$$

cut off by the cone with equation

$$\frac{1}{2}z = \sqrt{x^2 + y^2} .$$

$$\boxed{}\,,\ \boxed{\frac{1}{6}\pi\left[17\sqrt{17}-1\right]}$$



START BY SKETCHING THE SURFACES

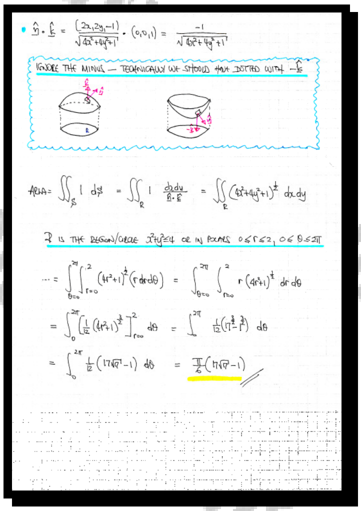
$x=0$
$z=y^2$ & $z=2y$

$y=0$
$z=x^2$ & $z=2x$

SKETCHING THE SURFACE WE HAVE

PROJECT THE SURFACE ONTO THE REGION R, ON THE x-y PLANE

- $f(x,y,z) = x^2+y^2-z$
- $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, -1)$
- $\underline{n} = (2x, 2y, -1)$
- $|\underline{n}| = \sqrt{4x^2+4y^2+1}$
- $\hat{\underline{n}} = \frac{(2x,2y,-1)}{\sqrt{4x^2+4y^2+1}}$



- $\hat{\underline{n}} \cdot \hat{\underline{k}} = \frac{(2x,2y,-1)}{\sqrt{4x^2+4y^2+1}} \cdot (0,0,1) = \frac{-1}{\sqrt{4x^2+4y^2+1}}$

IGNORE THE MINUS — TECHNICALLY WE SHOULD HAVE DOTTED WITH $-\hat{\underline{k}}$

$$\text{AREA} = \iint_S 1\,dS = \iint_R 1\,\frac{dx\,dy}{\hat{\underline{n}}\cdot\hat{\underline{k}}} = \iint_R \left(4x^2+4y^2+1\right)^{\frac{1}{2}}\,dx\,dy$$

R IS THE REGION/CIRCLE $x^2+y^2 \leqslant 4$ OR IN POLARS $0 \leqslant r \leqslant 2,\ 0 \leqslant \theta \leqslant 2\pi$

$$\ldots = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \left(4r^2+1\right)^{\frac{1}{2}}\left(r\,dr\,d\theta\right) = \int_{\theta=0}^{2\pi}\int_{r=0}^{2} r\left(4r^2+1\right)^{\frac{1}{2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{12}\left(4r^2+1\right)^{\frac{3}{2}}\right]_{r=0}^{2} d\theta = \int_0^{2\pi} \frac{1}{12}\left(17^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}\left(17\sqrt{17}-1\right) d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

Question 16

Find a simplified expression for the surface area cut out of the sphere with equation

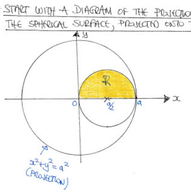
$$x^2 + y^2 + z^2 = a^2 \,,\ a > 0 \,,$$

when it is intersected by the cylinder with equation

$$x^2 + y^2 = ax \,,\ a > 0 \,.$$

$\boxed{}$, $\boxed{A = 2a^2[\pi - 2]}$

START WITH A DIAGRAM OF THE PROJECTION OF THE "TOP HALF" OF THE SPHERICAL SURFACE, PROJECTED ONTO THE xy PLANE



TO "SEE" THE CYLINDER

$x^2+y^2=ax$

$x^2-ax+y^2=0$

$\left(x-\frac{a}{2}\right)^2-\frac{a^2}{4}+y^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$

(MARKED OPPOSITE)

THE SPHERICAL SURFACE FOR WHICH $z \geqslant 0$ (TOP HALF), HAS EQUATION

$z=(a^2-x^2-y^2)^{\frac{1}{2}}$

$\frac{\partial z}{\partial x}=\frac{1}{2}(-2x)(a^2-x^2-y^2)^{-\frac{1}{2}}=-\frac{x}{(a^2-x^2-y^2)^{\frac{1}{2}}}$

$\frac{\partial z}{\partial x}=\frac{1}{2}(-2y)(a^2-x^2-y^2)^{-\frac{1}{2}}=-\frac{y}{(a^2-x^2-y^2)^{\frac{1}{2}}}$

$dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

$dS=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,dx\,dy$

$dS=\sqrt{\frac{a^2-x^2-y^2+x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy$

$\boxed{dS=\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy}$

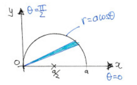
NOTE THAT THE PROJECTION IS ALSO POSSIBLE USING VECTOR METHODS

THE REQUIRED SURFACE IS 4 TIMES THE "PROJECTION" ONTO THE REGION R, SHOWN IN YELLOW, BY SYMMETRY

$\Rightarrow \text{AREA} = 4\int_S 1\,dS = 4\int_R 1\left(\frac{a}{\sqrt{a^2-x^2-y^2}}\right)dx\,dy$

$\Rightarrow \text{AREA} = 4a\int_R \frac{1}{\sqrt{a^2-y^2-x^2}}\,dx\,dy$

SWITCHING THE INTEGRAL INTO PLANE POLARS



$\Rightarrow x^2+y^2=ax$

$\Rightarrow r^2=a(r\cos\theta)$

$\Rightarrow r=a\cos\theta$

WITH

$0 \leqslant r \leqslant a\cos\theta$

$0 \leqslant \theta \leqslant \pi/2$

AND

$dx\,dy = r\,dr\,d\theta$

SO THE INTEGRAL BECOMES

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=a\cos\theta}\frac{1}{\sqrt{a^2-r^2}}(r\,dr\,d\theta)$

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2}\int_{r=0}^{r=a\cos\theta} r(a^2-r^2)^{-\frac{1}{2}}\,dr\,d\theta$

BY INSPECTION (OR SUBSTITUTION $u=a^2-r^2$)

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2}\left[-(a^2-r^2)^{\frac{1}{2}}\right]_{r=0}^{r=a\cos\theta}d\theta$

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2}\left[\sqrt{a^2-r^2}\right]_{r=a\cos\theta}^{r=0}d\theta$

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2} a-\sqrt{a^2-a^2\cos^2\theta}\,d\theta$

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2} a-a\sqrt{1-\cos^2\theta}\,d\theta$

$\Rightarrow \text{AREA} = 4a\int_{\theta=0}^{\pi/2} a-a\sin\theta\,d\theta$

$\Rightarrow \text{AREA} = 4a^2\int_0^{\frac{\pi}{2}} 1-\sin\theta\,d\theta$

$\Rightarrow \text{AREA} = 4a^2\left[\theta+\cos\theta\right]_0^{\pi/2}$

$\Rightarrow \text{AREA} = 4a^2\left[\left(\frac{\pi}{2}+0\right)-(0+1)\right]$

$\Rightarrow \text{AREA} = 4a^2\left(\frac{\pi}{2}-1\right)$

$\Rightarrow \text{AREA} = 2a^2(\pi-2)$

Question 17

Electric charge q is thinly distributed on the surface of a spherical shell with equation

$$x^2 + y^2 + z^2 = a^2, \ a > 0 .$$

Given that $q(x, y) = 2x^2 + y^2$, determine the total charge on the shell.

$\boxed{}$, $\boxed{q = 4\pi a^4}$

THE TOTAL CHARGE ON THE SURFACE IS $\int_S q(x,y)\,dS$

HERE WE HAVE

$$\text{TOTAL CHARGE} = \int_S (2x^2+y^2)\,dS$$

SWITCH INTO SPHERICAL POLARS

TOTAL CHARGE = ...

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[2(a\sin\theta\cos\phi)^2 + (a\sin\theta\sin\phi)^2\right](a^2\sin\theta\,d\theta\,d\phi)$$

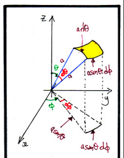
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} a^4\left[2\sin^3\theta\cos^2\phi + \sin^3\theta\sin^2\phi\right] d\theta\,d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta\left[2\cos^2\phi + \sin^2\phi\right] d\theta\,d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^3\theta\,(1+\cos^2\phi)\,d\theta\,d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta(1-\cos^2\theta)\left(1+\tfrac{1}{2}+\tfrac{1}{2}\cos2\phi\right) d\theta\,d\phi$$

$$= a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left(\sin\theta - \sin\theta\cos^2\theta\right)\left(\tfrac{3}{2}+\tfrac{3}{2}\cos2\phi\right) d\theta\,d\phi$$



IN SPHERICAL POLARS

- $x = a\sin\theta\cos\phi$
- $y = a\sin\theta\sin\phi$
- $z = a\cos\theta$

$x^2+y^2+z^2 = a^2$

$\boxed{dS = a^2\sin\theta\,d\theta\,d\phi}$

SPLITTING THE INTEGRAL, AS THERE IS NO DEPENDENCE IN θ & ϕ

$$\text{TOTAL CHARGE} = \left[a^4\int_{\phi=0}^{2\pi} \tfrac{3}{2} + \cancel{\tfrac{1}{2}\cos2\phi}\,d\phi\right]\left[\int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta\,d\theta\right]$$

NO CONTRIBUTION OVER THESE LIMITS

$$= a^4 \times \tfrac{3}{2} \times 2\pi \times \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi}$$

$$= 3\pi a^4\left[\left(1-\tfrac{1}{3}\right) - \left(-1+\tfrac{1}{3}\right)\right]$$

$$= 3\pi a^4 \times \tfrac{4}{3}$$

$$= 4\pi a^4$$

Question 18

A inverted right circular cone, whose vertex is at the origin of a Cartesian axes, lies in the region for which $z \geq 0$. The z axis is the axis of symmetry of the cone. Both the radius and the height of the cone is 6 units.

Electric charge Q is thinly distributed on the **curved** surface of the cone.

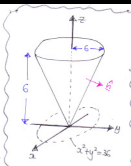
The charge at a given point on the curved surface of the cone satisfies

$$Q(r) = r,$$

where r is the shortest of the point from the z axis.

Determine the total charge on the cone.

$$\boxed{Q = 144\pi\sqrt{2}}$$



THE CURVED SURFACE OF THIS CYLINDER IS
$z^2 = x^2+y^2$, $z \geq 0$ $\quad 0 \leq z \leq 6$

LET THE SURFACE OF THE CONE BE WRITTEN AS

$f(x,y,z) = x^2+y^2-z^2$

$\underline{\nabla} f = (2x, 2y, -2z)$

$\underline{n} = (x,y,z)$

$|\underline{n}| = \sqrt{x^2+y^2+z^2}$

$\hat{\underline{n}} = \dfrac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}}$

- $Q(r) = r$
- HENCE

$$\int_S (x^2+y^2)^{\frac{1}{2}} \, dS$$

PROJECT ONTO THE xy PLANE ONTO THE CIRCLE WITH EQUATION $R: x^2+y^2=36$

$$= \int_R (x^2+y^2)^{\frac{1}{2}} \frac{dx\,dy}{\hat{\underline{n}} \cdot (\underline{k})} \, dS$$

$$= \int_R (x^2+y^2)^{\frac{1}{2}} \left[\frac{dx\,dy}{\frac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}} \cdot (0,0,-1)} \right]$$

$$= \int_R (x^2+y^2)^{\frac{1}{2}} \frac{dx\,dy}{\frac{z}{\sqrt{x^2+y^2+z^2}}}$$

BUT $z^2 = x^2+y^2$

$$= \int_R \cancel{(x^2+y^2)^{\frac{1}{2}}} \frac{dx\,dy}{\frac{\cancel{(x^2+y^2)^{\frac{1}{2}}}}{(2x^2+2y^2)^{\frac{1}{2}}}} \leftarrow z$$

$$= \int_R (2x^2+2y^2)^{\frac{1}{2}} \, dx\,dy$$

$$= \int_R \sqrt{2}\,(x^2+y^2)^{\frac{1}{2}} \, dx\,dy$$

SWITCH INTO PLANE POLARS OVER THE REGION $R: x^2+y^2 \leq 36$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{6} \sqrt{2}\,r \; (r\,dr\,d\theta)$$

$$= \sqrt{2} \int_{\theta=0}^{2\pi} \int_{r=0}^{6} r^2 \, dr\,d\theta$$

$$= \left[\sqrt{2} \int_{\theta=0}^{2\pi} 1 \, d\theta \right] \left[\int_{r=0}^{6} r^2 \, dr \right]$$

$$= \sqrt{2} \times 2\pi \times \left[\tfrac{1}{3} r^3 \right]_0^6$$

$$= \sqrt{2} \times 2\pi \times 72$$

$$= 144\sqrt{2}\,\pi$$

Question 19

A surface S has Cartesian equation

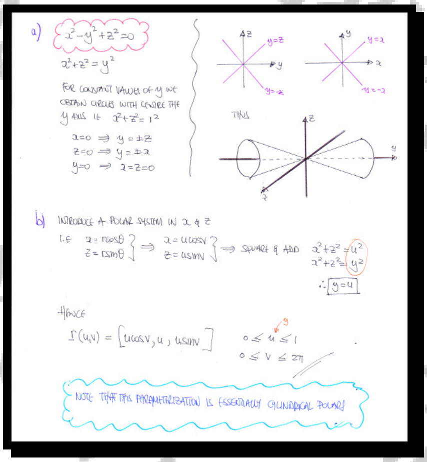
$$x^2 - y^2 + z^2 = 0.$$

a) Sketch the graph of S.

b) Find a parameterization for the equation of S, in terms of the parameters u and v.

c) Use the parameterization of part **(b)** to find the area of S, for $0 \le y \le 1$.

$\boxed{\mathbf{r}(u,v) = \langle u\cos v, u, u\sin v \rangle}$, $\boxed{\text{area} = \pi\sqrt{2}}$



a) $x^2 - y^2 + z^2 = 0$

$x^2 + z^2 = y^2$

FOR CONSTANT VALUES OF y WE OBTAIN CIRCLES WITH CENTRE THE y AXIS IE $x^2+z^2 = 1^2$

$x=0 \Rightarrow y = \pm z$
$z=0 \Rightarrow y = \pm x$
$y=0 \Rightarrow x = z = 0$

b) INTRODUCE A POLAR SYSTEM IN x & z

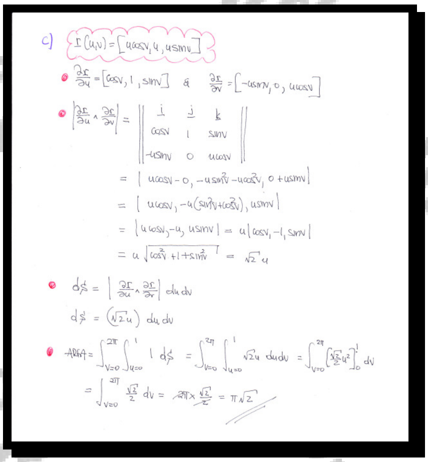
I.E $\left.\begin{array}{l} x = r\cos\theta \\ z = r\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = u\cos v \\ z = u\sin v \end{array}\right\} \Rightarrow$ SQUARE & ADD $\begin{array}{l} x^2+z^2 = u^2 \\ x^2+z^2 = y^2 \end{array}$

$\therefore \boxed{y = u}$

HENCE

$\underline{r}(u,v) = [u\cos v, u, u\sin v]$ $\quad 0 \le u \le 1$, $0 \le v \le 2\pi$

NOTE THAT THIS PARAMETERIZATION IS ESSENTIALLY CYLINDRICAL POLAR!



c) $\underline{r}(u,v) = [u\cos v, u, u\sin v]$

- $\frac{\partial \underline{r}}{\partial u} = [\cos v, 1, \sin v]$ & $\frac{\partial \underline{r}}{\partial v} = [-u\sin v, 0, u\cos v]$

- $\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| = \left\| \begin{array}{ccc} \underline{i} & \underline{j} & \underline{k} \\ \cos v & 1 & \sin v \\ -u\sin v & 0 & u\cos v \end{array} \right\|$

$= |u\cos v - 0, -u\sin^2 v - u\cos^2 v, 0 + u\sin v|$

$= |u\cos v, -u(\sin^2 v + \cos^2 v), u\sin v|$

$= |u\cos v, -u, u\sin v| = u|\cos v, -1, \sin v|$

$= u\sqrt{\cos^2 v + 1 + \sin^2 v} = \sqrt{2}\,u$

- $dS = \left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| du\,dv$

$dS = (\sqrt{2}u)\,du\,dv$

- AREA $= \int_{v=0}^{2\pi}\int_{u=0}^{1} 1\,dS = \int_{v=0}^{2\pi}\int_{u=0}^{1} \sqrt{2}u\,du\,dv = \int_{v=0}^{2\pi}\left[\frac{\sqrt{2}}{2}u^2\right]_0^1 dv$

$= \int_{v=0}^{2\pi} \frac{\sqrt{2}}{2}\,dv = 2\pi \times \frac{\sqrt{2}}{2} = \pi\sqrt{2}$

Question 20

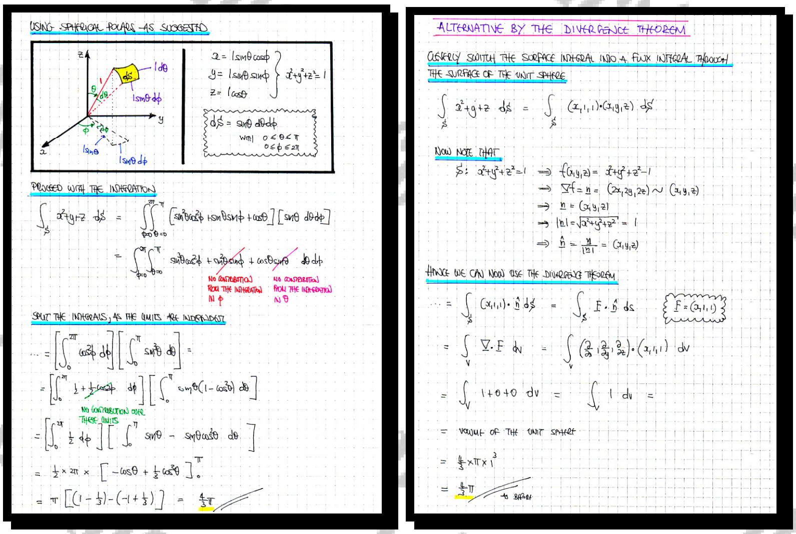
The surface S is the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = 1.$$

By using Spherical Polar coordinates (r, θ, φ), or otherwise, evaluate

$$\oiint_S \left(x^2 + y + z\right) dS.$$

$\boxed{}$, $\boxed{\frac{4}{3}\pi}$



Question 21

A bead is modelled as a sphere with a cylinder, whose axis is a diameter of the sphere, removed from the sphere.

If the respective equations of the sphere and the cylinder are

$$x^2 + y^2 + z^2 = a^2 \quad \text{and} \quad x^2 + y^2 = b^2, \quad 0 < b < a.$$

Show that the total surface area of the bead is

$$4\pi(a+b)\sqrt{a^2 - b^2}.$$

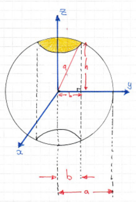
proof

STARTING WITH A DIAGRAM

SPHERE: $x^2+y^2+z^2=a^2$

CYLINDER: $x^2+y^2=b^2$

$(a>b)$



AREA OF THE INNER CYLINDRICAL FACE IS GIVEN BY

$$\text{"}2\pi r H\text{"} = 2\pi b(2h) = 4\pi bh = 4\pi b(a^2-b^2)^{\frac{1}{2}}$$

NEXT WE FIND THE AREA OF ONE OF THE SPHERICAL CAPS, SHOWN IN YELLOW – PROJECT THE "TOP" CAP $(z>0)$ ONTO THE xy PLANE

$$\Rightarrow z = +(a^2-x^2-y^2)^{\frac{1}{2}}$$

$$\frac{\partial z}{\partial x} = -x(a^2-x^2-y^2)^{-\frac{1}{2}} \quad \& \quad \frac{\partial z}{\partial y} = -y(a^2-x^2-y^2)^{-\frac{1}{2}}$$

$$\Rightarrow dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dx\,dy$$

$$\Rightarrow dS = \sqrt{\frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2} + 1}$$

$$\Rightarrow dS = \sqrt{\frac{x^2+y^2+a^2-x^2-y^2}{a^2-x^2-y^2}}$$

$$\Rightarrow \boxed{dS = \frac{a}{\sqrt{a^2-x^2-y^2}}}$$

HENCE THE AREA OF THE TWO CAPS IS GIVEN BY

$$\Rightarrow A = 2\iint_S 1\, dS = 2\iint_R \frac{a}{\sqrt{a^2-(x^2+y^2)}}\, dx\,dy$$

(CIRCLE $x^2+y^2=b^2$)

SWITCH INTO PLANE POLARS

$$= 2\iint_R \frac{a}{\sqrt{a^2-r^2}}\,(r\,dr\,d\theta)$$

$$= 2a\int_{\theta=0}^{2\pi}\int_{r=0}^{b} \frac{r}{(a^2-r^2)^{\frac{1}{2}}}\, dr\,d\theta$$

$$= 2a\left[\int_0^{2\pi} 1\, d\theta\right]\left[\int_{r=0}^{b} r(a^2-r^2)^{-\frac{1}{2}}\, dr\right]$$

$$= 2a \times 2\pi \times \left[-(a^2-r^2)^{\frac{1}{2}}\right]_{r=0}^{b}$$

$$= 4a\pi\left[(a^2-r^2)^{\frac{1}{2}}\right]_b^0$$

$$= 4a\pi\left[a - (a^2-b^2)^{\frac{1}{2}}\right]$$

FINALLY WE HAVE THE AREA OF THE BEAD

$$\underbrace{4\pi a^2}_{\text{(SPHERE)}} - \underbrace{4\pi a\left[a-(a^2-b^2)^{\frac{1}{2}}\right]}_{\text{(TWO CAPS)}} + \underbrace{4\pi b(a^2-b^2)^{\frac{1}{2}}}_{\text{(INNER CYLINDRICAL SURFACE)}}$$

$$= 4\pi\left[a^2 - a\left[a-(a^2-b^2)^{\frac{1}{2}}\right] + b(a^2-b^2)^{\frac{1}{2}}\right]$$

$$= 4\pi\left[a^2 - a^2 + a(a^2-b^2)^{\frac{1}{2}} + b(a^2-b^2)^{\frac{1}{2}}\right]$$

$$= 4\pi\left[a(a^2-b^2)^{\frac{1}{2}} + b(a^2-b^2)^{\frac{1}{2}}\right]$$

$$= 4\pi(a^2-b^2)^{\frac{1}{2}}(a+b)$$

AS REQUIRED

Question 22

A surface S has Cartesian equation

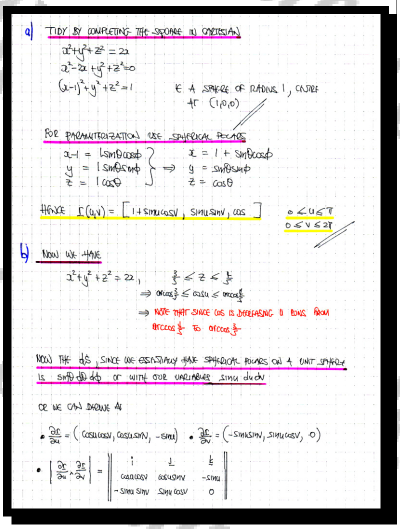
$$x^2 + y^2 + z^2 = 2x .$$

a) Describe fully the graph of S, and hence find a parameterization for its equation in terms of the parameters u and v.

b) Use the parameterization of part **(a)** to find the area for the part of S, for which $\frac{3}{5} \le z \le \frac{4}{5}$.

$\boxed{}$, $\boxed{\mathbf{r}(u,v) = \langle 1+\sin u \cos v, \sin u \sin v, \cos u \rangle, \quad 0 \le u \le \pi, \ 0 \le v \le 2\pi}$,

$\boxed{\text{area} = \frac{2}{5}\pi}$


a) TIDY BY COMPLETING THE SQUARE IN CARTESIAN

$x^2+y^2+z^2=2x$

$x^2-2x+y^2+z^2=0$

$(x-1)^2+y^2+z^2=1$ ← A SPHERE OF RADIUS 1, CENTRE AT $(1,0,0)$

FOR PARAMETRIZATION USE SPHERICAL POLARS

$\left.\begin{array}{l} x-1 = 1\sin\theta\cos\phi \\ y = 1\sin\theta\sin\phi \\ z = 1\cos\theta \end{array}\right\} \Rightarrow \begin{array}{l} x = 1+\sin\theta\cos\phi \\ y = \sin\theta\sin\phi \\ z = \cos\theta \end{array}$

HENCE $\underline{r}(u,v) = [1+\sin u\cos v, \sin u\sin v, \cos]$ $\quad 0 \le u \le \pi$, $0 \le v \le 2\pi$

b) NOW WE HAVE

$x^2+y^2+z^2=2x, \quad \frac{3}{5} \le z \le \frac{4}{5}$

$\Rightarrow \arccos\frac{3}{5} \le \cos u \le \arccos\frac{4}{5}$

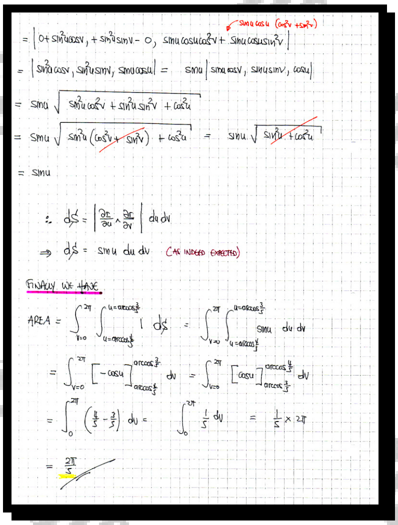
$\Rightarrow$ NOTE THAT SINCE COS IS DECREASING U RUNS FROM $\arccos\frac{4}{5}$ TO $\arccos\frac{3}{5}$

NOW THE dS, SINCE WE ESSENTIALLY HAVE SPHERICAL POLARS ON A UNIT SPHERE IS $\sin\theta\, d\theta\, d\phi$ OR WITH OUR VARIABLES $\sin u\, du\, dv$

OR WE CAN DERIVE AS

• $\frac{\partial r}{\partial u} = (\cos u\cos v, \cos u\sin v, -\sin u)$ • $\frac{\partial r}{\partial v} = (-\sin u\sin v, \sin u\cos v, 0)$

• $\left|\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right| = \left\| \begin{array}{ccc} i & j & k \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin u\sin v & \sin u\cos v & 0 \end{array} \right\|$


$= |0+\sin^2 u\cos v, +\sin^2 u\sin v - 0, \sin u\cos u\cos^2 v + \sin u\cos u\sin^2 v|$ $\quad \sin u\cos u(\cos^2 v+\sin^2 v)$

$= |\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u| = \sin u|\sin u\cos v, \sin u\sin v, \cos u|$

$= \sin u\sqrt{\sin^2 u\cos^2 v + \sin^2 u\sin^2 v + \cos^2 u}$

$= \sin u\sqrt{\sin^2 u(\cancel{\cos^2 v + \sin^2 v}) + \cos^2 u} = \sin u\sqrt{\cancel{\sin^2 u + \cos^2 u}}$

$= \sin u$

$\therefore dS = \left|\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right| du\, dv$

$\Rightarrow dS = \sin u\, du\, dv$ (AS INDEED EXPECTED)

FINALLY WE HAVE

$\text{AREA} = \int_{v=0}^{2\pi}\int_{u=\arccos\frac{4}{5}}^{u=\arccos\frac{3}{5}} 1\, dS = \int_{v=0}^{2\pi}\int_{u=\arccos\frac{4}{5}}^{u=\arccos\frac{3}{5}} \sin u\, du\, dv$

$= \int_{v=0}^{2\pi}\left[-\cos u\right]_{\arccos\frac{4}{5}}^{\arccos\frac{3}{5}} dv = \int_{v=0}^{2\pi}\left[\cos u\right]_{\arccos\frac{3}{5}}^{\arccos\frac{4}{5}} dv$

$= \int_0^{2\pi}\left(\frac{4}{5}-\frac{3}{5}\right) dv = \int_0^{2\pi}\frac{1}{5}\, dv = \frac{1}{5}\times 2\pi$

$= \frac{2\pi}{5}$

Question 23

Evaluate the integral

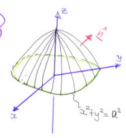
$$\int_S x(x+z+xy)+y\left(z^2-2xz-y\right)+z\;dS\,,$$

where S is the surface with Cartesian equation

$$x^2+y^2+z^2=a^2\,,\quad a>0\,,\quad z\geq 0\,.$$

$$\boxed{\pi a^3}$$

$$\int_S x(x+z+xy)+y(z^2-2xz-y)+z\;dS \qquad (S:\ z^2+x^2+y^2=a^2)$$



- LET $\phi(x,y)=(a^2-x^2-x^2)^{\frac{1}{2}}$

$$\frac{\partial\phi}{\partial x}=-x(a^2-x^2-y^2)^{-\frac{1}{2}}$$

$$\frac{\partial\phi}{\partial y}=-y(a^2-x^2-y^2)^{-\frac{1}{2}}$$

- PROJECTION ONTO THE xy PLANE ONTO THE REGION R, $x^2+y^2\leq a^2$

$$dS=\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+1}\;dx\,dy=\sqrt{\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}+1}\;dx\,dy$$

$$=\sqrt{\frac{\cancel{x^2}+\cancel{y^2}+a^2-\cancel{x^2}-\cancel{y^2}}{a^2-x^2-y^2}}\;dx\,dy=\sqrt{\frac{a^2}{z^2}}\;dx\,dy=\frac{a}{z}\,dx\,dy$$

$$\boxed{dS=\frac{a}{z}\,dx\,dy}$$

- $\displaystyle\int_R\left(x^2+xz+x^2y+yz^2-2xyz-y^2+z\right)\frac{a}{z}\,dx\,dy \qquad \left(\text{WHERE } z^2+y^2+x^2=a^2\right)$

$$=a\int_R\left[\frac{x^2}{z}+\cancel{x}+\cancel{\frac{x^2y}{z}}+\cancel{yz}-\cancel{2xy}-\frac{y^2}{z}+1\right]dx\,dy \qquad \left(\text{WHERE } x^2+y^2+z^2=a^2\right)$$

NOW THE INTEGRATION REGION R, IS A SYMMETRICAL DOMAIN IN x AND IN y, SO ANY ODD POWERS OF x OR y WILL HAVE NO CONTRIBUTION

$$=a\int_R\frac{x^2}{z}-\frac{y^2}{z}+1\;dx\,dy \qquad \left(\text{WHERE } z=\sqrt{a^2-x^2-y^2}\right)$$

$$=a\int_R\frac{x^2-y^2}{z}+1\;dx\,dy \qquad \left(\text{WHERE } z=\sqrt{a^2-(x^2+y^2)}\right)$$

- NEXT SPLIT THE INTEGRAL INTO TWO

$$=a\int_R\frac{x^2-y^2}{z}\,dx\,dy+a\int_R 1\,dx\,dy$$

$$=a\int_R\frac{x^2-y^2}{\sqrt{a^2-(x^2+y^2)}}\,dx\,dy+\left[a\times\text{AREA OF CIRCLE RADIUS } a\right]$$

SWITCH INTO PLANE POLARS

$$=a\int_{\theta=0}^{2\pi}\int_{r=0}^{r=a}\left(\frac{r^2\cos^2\theta-r^2\sin^2\theta}{\sqrt{a^2-r^2}}\right)(r\,dr\,d\theta)+a\times\pi a^2$$

$$=a\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{r^3(\cos^2\theta-\sin^2\theta)}{\sqrt{a^2-r^2}}\,dr\,d\theta+\pi a^3$$

$$=\cancel{a\int_{\theta=0}^{2\pi}\int_{r=0}^{a}\frac{r^3\cos2\theta}{\sqrt{a^2-r^2}}\,dr\,d\theta}+\pi a^3$$

ZERO CONTRIBUTION FROM THE θ INTEGRATION FOR THESE LIMITS

$$=\pi a^3$$

Question 24

A surface S has Cartesian equation

$$x^2 + y^2 - z^2 = 2y + 2z, \quad -1 \le z \le 0 .$$

a) Sketch the graph of S.

b) Find a parameterization for the equation of S, in terms of the parameters u and v.

c) Use the parameterization of part **(b)** to find the area of S.

$\boxed{\mathbf{r}(u,v) = \langle u\cos v, 1 + u\sin v, u-1 \rangle, \ 0 \le u \le 1, \ 0 \le v \le 2\pi}$, $\boxed{\text{area} = \pi\sqrt{2}}$

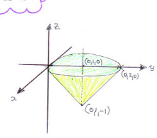
a) $x^2+y^2-z^2 = 2y+2z \ , \ -1 \le z \le 0$

$x^2+y^2-2y = z^2+2z$

$x^2+(y-1)^2-1 = (z+1)^2-1$

$(z+1)^2 = x^2+(y-1)^2$

BY INSPECTION & COMPARING WITH $z^2 = x^2+y^2$, WHICH IS A CONE WITH AXIS THE z AXIS, WE DEDUCE THAT THIS IS ALSO A CONE TRANSLATED BY 1 UNIT "DOWN" THE z AXIS, & 1 UNIT UP THE y AXIS



b) TO PARAMETRIZE WORK ESSENTIALLY IN CYLINDRICAL POLARS

LET $\left.\begin{array}{l} x = r\cos\theta \\ y-1 = r\sin\theta \end{array}\right\} \Rightarrow$ SQUARE & ADD GIVES $x^2+(y-1)^2 = r^2$

SO LET $\boxed{z+1 = r}$

THUS $x = r\cos\theta$
$y = 1 + r\sin\theta$ WITH $0 \le r \le 1$
$z = r-1$ $\quad 0 \le \theta \ 2\pi$

$\therefore$ REWRITING IN u & v

$\mathbf{r}(u,v) = [u\cos v, 1+u\sin v, u-1]$ $\quad 0 \le u \le 1$, $0 \le v \le 2\pi$

c) $\frac{\partial \mathbf{r}}{\partial u} = (\cos v, \sin v, 1)$

$\frac{\partial \mathbf{r}}{\partial v} = (-u\sin v, u\cos v, 0)$

$$\left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{matrix} \right\| = \left| 0 - u\cos v, -u\sin v - 0, -u\sin^2 v - u\cos^2 v \right|$$

$$= \left| u\cos v, u\sin v, u(\sin^2 v + \cos^2 v) \right| = \left| u\cos v, u\sin v, u \right|$$

$$= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = u\sqrt{\cos^2 v + \sin^2 v + 1}$$

$$= u\sqrt{2}$$

HENCE

$$\text{AREA} = \int_{v=0}^{2\pi}\int_{u=0}^{1} 1 \, dS = \int_{v=0}^{2\pi}\int_{u=0}^{1} 1 \left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| du\, dv$$

$$= \int_{v=0}^{2\pi}\int_{u=0}^{1} u\sqrt{2} \, du\, dv = \int_{v=0}^{2\pi} \left[\frac{\sqrt{2}}{2}u\right]_{u=0}^{1} dv$$

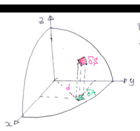
$$= \int_{v=0}^{2\pi} \frac{\sqrt{2}}{2} \, dv = \frac{\sqrt{2}}{2} \times 2\pi = \pi\sqrt{2}$$

Question 25

A thin uniform spherical shell has mass m and radius a.

Use surface integral projection techniques in x-y plane, to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



FIRSTLY THE MASS PER UNIT AREA OF THE SHELL IS

$$\boxed{\rho = \frac{m}{4\pi a^2}}$$

SURFACE HAS EQUATION

$$z = +\sqrt{a^2-x^2-y^2} = (a^2-x^2-y^2)^{\frac{1}{2}}$$

(TOP HALF)

$$\therefore f(x,y,z) = (a^2-x^2-y^2)^{\frac{1}{2}} - z$$

USING A "SHORTCUT" FOR THE SCALING FACTOR FROM dS TO $dy\,dx$

$$dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dy\,dx$$

$$dS = \sqrt{\left[-x(a^2-x^2-y^2)^{-\frac{1}{2}}\right]^2 + \left[-y(a^2-x^2-y^2)^{-\frac{1}{2}}\right]^2 + (-1)^2}\; dx\,dy$$

$$dS = \sqrt{\frac{x^2}{(a^2-x^2-y^2)} + \frac{y^2}{a^2-x^2-y^2} + 1}\; dx\,dy$$

$$dS = \sqrt{\frac{x^2+y^2+a^2-x^2-y^2}{a^2-x^2-y^2}}\; dx\,dy$$

$$dS = \sqrt{\frac{a^2}{a^2-x^2-y^2}}\; dx\,dy$$

$$\boxed{dS = \frac{a}{\sqrt{a^2-x^2-y^2}}\; dx\,dy}$$

MASS OF INFINITESIMAL AREA ELEMENT δS

$$\delta m = \rho\,\delta S$$

PROJECT ONTO THE xy PLANE

$$\delta m = \rho\left(\frac{a}{\sqrt{a^2-x^2-y^2}}\,\delta x\,\delta y\right)$$

MOMENT OF INERTIA OF "PROJECTED" MASS ELEMENT ABOUT THE z AXIS IS

$$\delta I = \delta m\, d^2$$

$$\delta I = \frac{a\rho}{\sqrt{a^2-x^2-y^2}} \times \left(\sqrt{x^2+y^2}\right)^2$$

$$\delta I = a\rho \times \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}}$$

$$\delta I = a \times \frac{m}{4\pi a^2} \times \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}}$$

$$\delta I = \frac{m}{4\pi a} \times \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}}$$

SUMMING UP & TAKING LIMITS OVER THE REGION $x^2+y^2=a^2$

$$\Rightarrow I = \iint_{\substack{R \\ (\text{circle } x^2+y^2=a^2)}} \frac{m}{4\pi a} \times \frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}}\, dx\,dy$$

SWITCH INTO PLANE POLARS

$r=a \quad 0 \leqslant \theta \leqslant 2\pi$

$$\Rightarrow I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{m}{4\pi a} \times \frac{r^2}{(a^2-r^2)^{\frac{1}{2}}}(r\,dr\,d\theta)$$

$$\Rightarrow I = \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{m}{4\pi a} \times \frac{r^3}{(a^2-r^2)^{\frac{1}{2}}}\, dr\,d\theta$$

$$\Rightarrow I = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{m}{4\pi a}\,\frac{r^3}{(a^2-r^2)^{\frac{1}{2}}}\, d\theta\,dr$$

$$\Rightarrow I = \int_{r=0}^{a} \frac{2m\pi}{4\pi a}\,\frac{r^3}{(a^2-r^2)^{\frac{1}{2}}}\, dr$$

$$\Rightarrow I = \frac{m}{2a}\int_0^a \frac{r^3}{(a^2-r^2)^{\frac{1}{2}}}\, dr$$

BY SUBSTITUTION $u^2 = a^2-r^2$

$2u\,du = -2r\,dr$

$dr = -\frac{u}{r}\,du$

r	u
0	a
a	0

$$\Rightarrow I = \frac{m}{2a}\int_a^0 \frac{r^3}{u}\left(\frac{-u}{r}\right) du$$

$$\Rightarrow I = \frac{m}{2a}\int_0^a r^2\, du$$

$$\Rightarrow I = \frac{m}{2a}\int_0^a a^2-u^2\, du$$

$$\Rightarrow I = \frac{m}{2a}\left[a^2u - \tfrac{1}{3}u^3\right]_0^a$$

$$\Rightarrow I = \frac{m}{2a}\left[\left(a^3-\tfrac{1}{3}a^3\right)-0\right]$$

$$\Rightarrow I = \frac{m}{2a} \times \frac{2}{3}a^3$$

$$\Rightarrow I = \frac{1}{3}ma^2$$

BUT THIS IS THE MOMENT OF INERTIA OF THE HEMISPHERE

$\therefore$ BY THE ADDITION RULE

$$I = \frac{2}{3}ma^2$$

FOR BOTH HEMISPHERES

Question 26

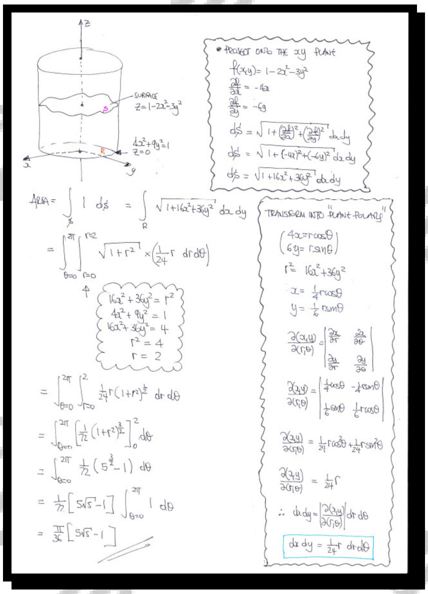
Find the area of the surface S which consists of the part of the surface with Cartesian equation

$$z = 1 - 2x^2 - 3y^2,$$

contained within the elliptic cylinder with Cartesian equation

$$4x^2 + 9y^2 = 1.$$

$$\boxed{\frac{\pi}{36}\left[5\sqrt{5} - 1\right]}$$



Question 27

The surface S is the hemisphere with Cartesian equation

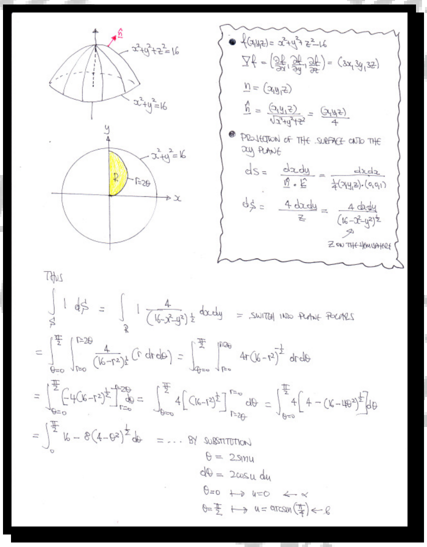
$$x^2 + y^2 + z^2 = 16, \ z \geq 0$$

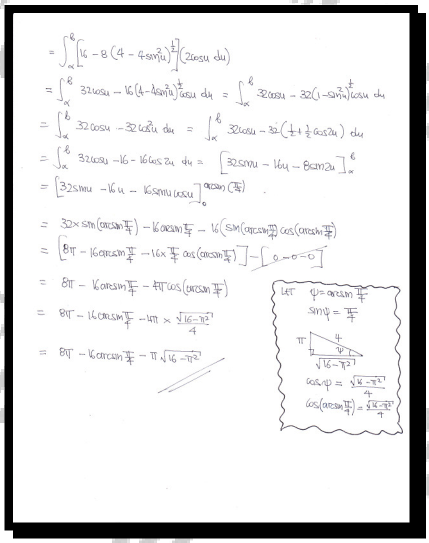
The projection of S onto the x-y plane is the area within the curve with polar equation

$$r = 2\theta, \ 0 \leq \theta \leq \frac{\pi}{2}.$$

Find, in exact form, the area of S.

$$\boxed{8\pi - \pi\sqrt{16-\pi^2} - 16\arcsin\frac{\pi}{4}}$$





Question 28

The surface S is the sphere with Cartesian equation

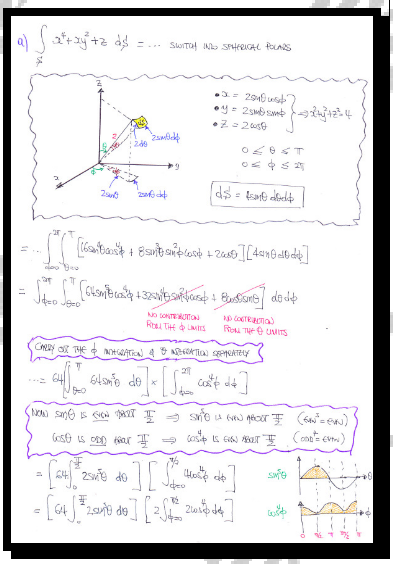
$$x^2 + y^2 + z^2 = 4$$

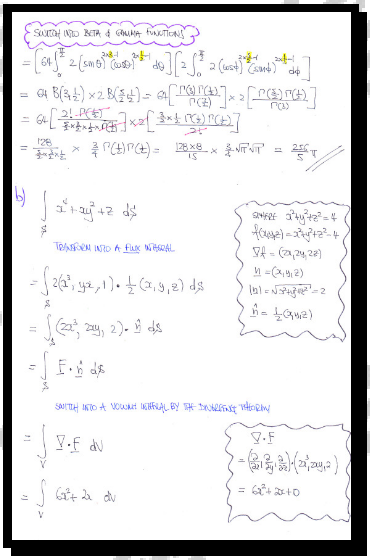
a) By using Spherical Polar coordinates, (r, θ, φ), evaluate by direct integration the following surface integral

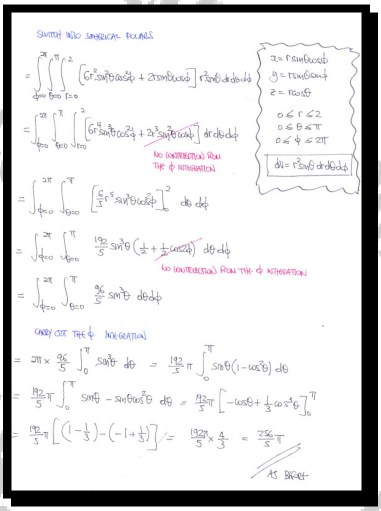
$$I = \oiint_S \left(x^4 + xy^2 + z\right)\, dS \, .$$

b) Verify the answer of part **(a)** by using the Divergence Theorem.

$$\boxed{\frac{256\pi}{5}}$$







Question 29

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta,\varphi) = (R + r\cos\theta)\cos\varphi, \qquad y(\theta,\varphi) = (R + r\cos\theta)\sin\varphi, \qquad z(\theta,\varphi) = r\sin\theta,$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$.

a) Find a general Cartesian equation for the surface of a torus.

A torus T has Cartesian equation

$$\left(4 - \sqrt{x^2 + y^2}\right)^2 = 1 - z^2.$$

b) Use a suitable parameterization of T to find its surface area.

$\boxed{z^2 + \left(R - \sqrt{x^2 + y^2}\right)^2 = r^2}$, $\boxed{\text{area} = (2\pi r)(2\pi R) = 16\pi^2}$

a)

$x(\theta,\phi) = (R + r\cos\theta)\cos\phi$

$y(\theta,\phi) = (R + r\cos\theta)\sin\phi$

$z(\theta,\phi) = r\sin\theta$

$0 \le \theta \le 2\pi$

$0 \le \phi \le 2\pi$

- $x^2 + y^2 = (R + r\cos\theta)^2\cos^2\phi + (R + r\cos\theta)\sin^2\phi$
  $= (R + r\cos\theta)^2\left[\cos^2\phi + \sin^2\phi\right]$
  $= (R + r\cos\theta)^2$

- NOW
  $\sqrt{x^2+y^2} = R + r\cos\theta$
  $-r\cos\theta = R - \sqrt{x^2+y^2}$
  $r^2\cos^2\theta = \left(R - \sqrt{x^2+y^2}\right)^2$

- ALSO
  $r\sin\theta = z$
  $r^2\sin^2\theta = z^2$

$r^2\cos^2\theta + r^2\sin^2\theta = z^2 + \left(R - \sqrt{x^2+y^2}\right)^2$

$r^2(\cos^2\theta + \sin^2\theta) = z^2 + \left(R - \sqrt{x^2+y^2}\right)^2$

$z^2 + \left(R - \sqrt{x^2+y^2}\right)^2 = r^2$

b)

- NOW $\left(4 - \sqrt{x^2+y^2}\right)^2 = 1 - z^2$
  $z^2 + \left(4 - \sqrt{x^2+y^2}\right)^2 = 1 \quad \leftarrow$ I.E $R = 4$, $r = 1$

- HENCE A PARAMETRIC REPRESENTATION FOR THIS TORUS WOULD BE

  $x = (4 + \cos\theta)\cos\phi$
  $y = (4 + \cos\theta)\sin\phi$
  $z = \sin\theta$
  $0 \le \theta \le 2\pi$
  $0 \le \phi \le 2\pi$

- NEXT $\underline{r}(\theta,\phi) = \left[(4+\cos\theta)\cos\phi, (4+\cos\theta)\sin\phi, \sin\theta\right]$
  $\underline{r}(\theta,\phi) = \left[4\cos\phi + \cos\theta\cos\phi, 4\sin\phi + \cos\theta\sin\phi, \sin\theta\right]$

$\frac{\partial \underline{r}}{\partial \theta} = \left[-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta\right]$

$\frac{\partial \underline{r}}{\partial \phi} = \left[-4\sin\phi - \cos\theta\sin\phi, 4\cos\phi + \cos\theta\cos\phi, 0\right]$

- NOW $\left|\frac{\partial \underline{r}}{\partial \theta} \wedge \frac{\partial \underline{r}}{\partial \phi}\right| = \left\|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & -\cos\theta \\ -4\sin\phi - \cos\theta\sin\phi & 4\cos\phi + \cos\theta\cos\phi & 0 \end{matrix}\right\|$

$= \left| 0 + 4\cos\phi\cos\theta + \cos^2\theta\cos\phi, 4\sin\phi\cos\theta + \cos^2\theta\sin\phi - 0, 4\sin\theta\cos^2\phi + \sin\theta\cos\theta\cos^2\phi + 4\sin\theta\sin^2\phi + \cos\theta\sin\theta\sin^2\phi \right|$

$= \left| 4\cos\theta\cos\phi + \cos^2\theta\cos\phi, 4\cos\theta\sin\phi + \cos^2\theta\sin\phi, 4\sin\theta(\cos^2\phi + \sin^2\phi) + \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) \right|$

$= \left| 4\cos\theta\cos\phi + \cos^2\theta\cos\phi, 4\cos\theta\sin\phi + \cos^2\theta\sin\phi, 4\sin\theta + \sin\theta\cos\theta \right|$

$= \left| (\cos\theta\cos\phi)(4+\cos\theta), (\cos\theta\sin\phi)(4+\cos\theta), \sin\theta(4+\cos\theta) \right|$

$= (4+\cos\theta)\left|\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta\right|$

$= (4+\cos\theta)\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}$

$= (4+\cos\theta)\sqrt{\cos^2\theta\left(\cancel{\cos^2\phi + \sin^2\phi}\right) + \sin^2\theta}$

$= 4 + \cos\theta$

- FINALLY

$\text{AREA} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} 1 \, dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} 1 \left|\frac{\partial \underline{r}}{\partial \theta} \wedge \frac{\partial \underline{r}}{\partial \phi}\right| d\theta \, d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} \left(4 + \cancel{\cos\theta}\right) d\theta \, d\phi$

NO CONTRIBUTION OVER THE LIMITS 0 to 2π

$= 4 \times 2\pi \times 2\pi$

$= 16\pi^2$

- NOTE THAT THE "STANDARD" FORMULA IS $(2\pi r)(2\pi R)$ WHICH HERE FOR $r=1$, $R=4$ YIELDS $(2\pi \times 1)(2\pi \times 4) = 16\pi^2$

Question 30

A spiral ramp is modelled by the surface S defined by the vector function

$$\mathbf{r}(u,v) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (u\cos v)\mathbf{i} + (u\sin v)\mathbf{j} + v\mathbf{k}\,,$$

where $0 \le u \le 1$, $0 \le v \le 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2+y^2}\; dS$$

$$\boxed{\pi\left[\sqrt{8}-1\right]}$$

$$\underline{r}(u,v) = [u\cos v, u\sin v, v] \qquad 0 \le u \le 1,\quad 0 \le v \le 3\pi$$

- FIRSTLY WE COMPUTE THE dS ELEMENT

$$\frac{\partial \underline{r}}{\partial u} = (\cos v, \sin v, 0)$$

$$\frac{\partial \underline{r}}{\partial v} = (-u\sin v, u\cos v, 1)$$

$$\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = [\sin v, -\cos v, u\cos^2 v + u\sin^2 v] = [\sin v, -\cos v, u]$$

$$\left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| = |\sin v, -\cos v, u| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\therefore \boxed{dS = \left|\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}\right| du\,dv}$$

$$dS = \sqrt{1+u^2}\; du\,dv$$

- $$\int_S \sqrt{x^2+y^2}\; dS = \int_{v=0}^{3\pi}\int_{u=0}^{1} \sqrt{u^2\cos^2 v + u^2\sin^2 v}\; \underbrace{\sqrt{1+u^2}\, du\,dv}_{dS}$$

$$= \int_{v=0}^{3\pi}\int_{u=0}^{1} u(1+u^2)^{\frac{1}{2}}\, du\,dv$$

$$= \left[\int_{v=0}^{3\pi} 1\; dv\right]\left[\int_0^1 u(1+u^2)^{\frac{1}{2}}\, du\right]$$

$$= \cancel{3}\pi \times \left[\tfrac{1}{\cancel{3}}(u^2+1)^{\frac{3}{2}}\right]_0^1$$

$$= \pi\left[2^{\frac{3}{2}} - 1\right]$$

$$= \pi\left[2\sqrt{2} - 1\right]$$

Question 31

The surface S is defined by the vector equation

$$\mathbf{F}(u,v) = \left[u\cos v, u\sin v, \frac{1}{u} \right]^T, \ u \neq 0.$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v = u^4, \ v = 2u^4,$$

and the straight lines with equations $u = 3^{\frac{1}{4}}$ and $u = 8^{\frac{1}{4}}$.

$$\boxed{\frac{19}{6}}$$

$$\boxed{\underline{F}(u,v) = \left[u\cos v, u\sin v, \frac{1}{u}\right] \qquad \begin{array}{c} u^4 \leqslant v \leqslant 2u^4 \\ 3^{\frac{1}{4}} \leqslant u \leqslant 8^{\frac{1}{4}} \end{array}}$$

<u>FIRSTLY</u> $\left.\begin{array}{l} \frac{\partial \underline{F}}{\partial u} = \left[\cos v, \sin v, -\frac{1}{u^2}\right] \\ \frac{\partial \underline{F}}{\partial v} = \left[-u\sin v, u\cos v, 0\right] \end{array}\right\} \implies$ FIND $\frac{\partial \underline{F}}{\partial u} \wedge \frac{\partial \underline{F}}{\partial v}$ NEXT

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & -\frac{1}{u^2} \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \left[\frac{1}{u}\cos v, \frac{1}{u}\sin v, u\cos^2 v + u\sin^2 v\right] = \left[\frac{1}{u}\cos v, \frac{1}{u}\sin v, u\right]$$

$$\left|\frac{\partial \underline{F}}{\partial u} \wedge \frac{\partial \underline{F}}{\partial v}\right| = \left|\frac{1}{u}\cos v, \frac{1}{u}\sin v, u\right| = \sqrt{\frac{1}{u^2}\cos^2 v + \frac{1}{u^2}\sin^2 v + u^2}$$

$$= \sqrt{\frac{1}{u^2} + u^2} = \sqrt{\frac{1+u^4}{u^2}} = \frac{1}{u}\sqrt{1+u^4}$$

$$d\underset{\sim}{S} = \left\| \frac{\partial \underline{F}}{\partial u} \wedge \frac{\partial \underline{F}}{\partial v} \right\| \, du \, dv$$

$$\boxed{d\underset{\sim}{S} = \frac{1}{u}\sqrt{1+u^4} \, du \, dv}$$

$$\underset{\sim}{S} = \int_{u=3^{\frac{1}{4}}}^{u=8^{\frac{1}{4}}} \int_{v=u^4}^{v=2u^4} 1 \, d\underset{\sim}{S} = \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \int_{v=u^4}^{v=2u^4} \frac{1}{u}(1+u^4)^{\frac{1}{2}} \, dv \, du$$

$$= \int_{u=3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} \left[\frac{v}{u}(1+u^4)^{\frac{1}{2}} \right]_{v=u^4}^{v=2u^4} du = \int_{u=3^{\frac{1}{4}}}^{u=8^{\frac{1}{4}}} 2u^3(1+u^4)^{\frac{1}{2}} - u^3(1+u^4)^{\frac{1}{2}} \, du$$

$$= \int_{u=3^{\frac{1}{4}}}^{u=8^{\frac{1}{4}}} u^3(1+u^4)^{\frac{1}{2}} \, du = \left[\frac{1}{6}(1+u^4)^{\frac{3}{2}} \right]_{3^{\frac{1}{4}}}^{8^{\frac{1}{4}}} = \frac{1}{6}\left[27 - 8\right]$$

$$= \frac{19}{6}$$

Question 32

The surface S is defined by the parametric equations

$$x = t\cosh\theta,\ \ y = t\sinh\theta,\ \ z = \tfrac{1}{2}\left(1-t^2\right),$$

where t and θ are real parameters such that $0 \le t \le 1$ and $0 \le \theta \le 1$.

Find, in exact form, the value of

$$\int_S xy \ dS .$$

$$\boxed{\frac{1}{30}\left[\frac{(\cosh 2+1)^{\frac{5}{2}}-1}{\cosh 2}+1-4\sqrt{2}\right] \approx 0.274397...}$$

$$\mathbf{r}(t,\theta) = \left(t\cosh\theta, t\sinh\theta, \tfrac{1}{2}-\tfrac{1}{2}t^2\right)$$

$$\frac{\partial \mathbf{r}}{\partial t} = (\cosh\theta, \sinh\theta, -t)$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = (t\sinh\theta, t\cosh\theta, 0)$$

THUS

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cosh\theta & \sinh\theta & -t \\ t\sinh\theta & t\cosh\theta & 0 \end{vmatrix} = \left[t^2\cosh\theta, -t^2\sinh\theta, t\cosh^2\theta - t\sinh^2\theta\right]$$

$$= \left[t^2\cosh\theta, -t^2\sinh\theta, t\right] \quad \text{SINCE } \cosh^2\theta - \sinh^2\theta \equiv 1$$

$$\therefore \left|t^2\cosh\theta, -t^2\sinh\theta, t\right| = \sqrt{t^4\cosh^2\theta + t^4\sinh^2\theta + t^2} = |t|\sqrt{t^2\cosh^2\theta + t^2\sinh^2\theta + 1}$$

$$= |t|\sqrt{t^2(\cosh^2\theta + \sinh^2\theta) + 1} = |t|\sqrt{t^2\cosh 2\theta + 1}$$

SINCE $\cosh^2\theta + \sinh^2\theta \equiv \cosh 2\theta$

$$\therefore dS = |t|\left(t^2\cosh 2\theta + 1\right)^{\frac{1}{2}}\, d\theta\, dt$$

THUS

$$\int_S xy \ dS = \int_{t=0}^{1}\int_{\theta=0}^{1} (t\cosh\theta)(t\sinh\theta)\,|t|\left(t^2\cosh 2\theta+1\right)^{\frac{1}{2}}\, d\theta\, dt$$

$$= \int_{t=0}^{1}\int_{\theta=0}^{1} t^3\cosh\theta\sinh\theta\left(t^2\cosh 2\theta+1\right)^{\frac{1}{2}}\, d\theta\, dt$$

$$= \int_{t=0}^{1}\int_{\theta=0}^{1} \tfrac{1}{2}t^3\sinh 2\theta\left(t^2\cosh 2\theta+1\right)^{\frac{1}{2}}\, d\theta\, dt$$

$$= \int_{t=0}^{1}\left[\tfrac{1}{6}t\left(t^2\cos 2\theta+1\right)^{\frac{3}{2}}\right]_{\theta=0}^{1} dt$$

$$= \int_{t=0}^{1} \tfrac{1}{6}t\left(t^2\cosh 2+1\right)^{\frac{3}{2}} - \tfrac{1}{6}t\left(t^2+1\right)^{\frac{3}{2}}\, dt$$

$$= \left[\frac{1}{30\cosh 2}\left(t^2\cosh 2+1\right)^{\frac{5}{2}} - \frac{1}{30}\left(t^2+1\right)^{\frac{5}{2}}\right]_0^1$$

$$= \left[\frac{1}{30\cosh 2}(\cosh 2+1)^{\frac{5}{2}} - \frac{1}{30}\times 2^{\frac{5}{2}}\right] - \left[\frac{1}{30\cosh 2} - \frac{1}{30}\right]$$

$$= \frac{1}{30}\left[\frac{1}{\cosh 2}(\cosh 2+1)^{\frac{5}{2}} - \frac{1}{\cosh 2} + 1 - 4\sqrt{2}\right]$$

$$= \frac{1}{30}\left[\frac{(\cosh 2+1)^{\frac{5}{2}}-1}{\cosh 2} + 1 - 4\sqrt{2}\right]$$