# SPECIAL FUNCTIONS

# CHEBYSHEV POLYNOMIALS

## Question 1

Chebyshev's differential equation is given below.

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0, \quad n=0, 1, 2, 3, 4, \ldots$$

**a)** Use the substitution $x=\cos t$, to show that a general solution for Chebyshev's differential equation is

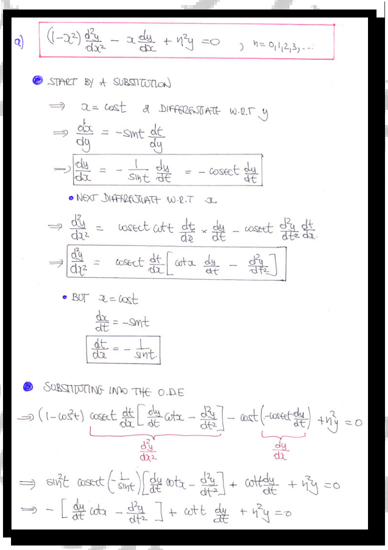
$$y=A\,\mathrm{T}_n(x)+B\,\mathrm{U}_n(x),\ |x|<1,$$

with $\mathrm{T}_n(x)=\cos\left[n\arccos(x)\right]$, $\mathrm{U}_n(x)=\sin\left[n\arccos(x)\right]$, and $A$ and $B$ are arbitrary constants.

**b)** Show further that $\mathrm{T}_n(x)$ can be written as

$$\mathrm{T}_n(x)=\frac{1}{2}\left[\left(x+\mathrm{i}\sqrt{1-x^2}\right)+\left(x-\mathrm{i}\sqrt{1-x^2}\right)\right].$$

proof


a) $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0$, $n=0,1,2,3,\ldots$

• START BY A SUBSTITUTION

$\Rightarrow x=\cos t$ & DIFFERENTIATE W.R.T $y$

$\Rightarrow \frac{dx}{dy}=-\sin t\frac{dt}{dy}$

$\Rightarrow \frac{dy}{dx}=-\frac{1}{\sin t}\frac{dy}{dt}=-\operatorname{cosec}t\frac{dy}{dt}$

• NEXT DIFFERENTIATE W.R.T $x$

$\Rightarrow \frac{d^2y}{dx^2}=\operatorname{cosec}t\cot t\frac{dt}{dx}\times\frac{dy}{dt}-\operatorname{cosec}t\frac{d^2y}{dt^2}\frac{dt}{dx}$

$\Rightarrow \frac{d^2y}{dx^2}=\operatorname{cosec}t\frac{dt}{dx}\left[\cot x\frac{dy}{dt}-\frac{d^2y}{dt^2}\right]$

• BUT $x=\cos t$

$\frac{dx}{dt}=-\sin t$

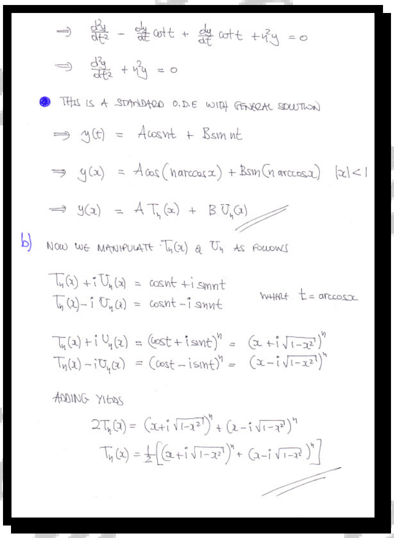
$\frac{dt}{dx}=-\frac{1}{\sin t}$

• SUBSTITUTING INTO THE O.D.E

$\Rightarrow (1-\cos^2t)\operatorname{cosec}t\frac{dt}{dx}\left[\frac{dy}{dt}\cot x-\frac{d^2y}{dt^2}\right]-\cos t\left(-\operatorname{cosec}t\frac{dy}{dt}\right)+n^2y=0$

$\Rightarrow \sin^2t\operatorname{cosec}t\left(-\frac{1}{\sin t}\right)\left[\frac{dy}{dt}\cot x-\frac{d^2y}{dt^2}\right]+\cot t\frac{dy}{dt}+n^2y=0$

$\Rightarrow -\left[\frac{dy}{dt}\cot x-\frac{d^2y}{dt^2}\right]+\cot t\frac{dy}{dt}+n^2y=0$


$\Rightarrow \frac{d^2y}{dt^2}-\frac{dy}{dt}\cot t+\frac{dy}{dt}\cot t+n^2y=0$

$\Rightarrow \frac{d^2y}{dt^2}+n^2y=0$

• THIS IS A STANDARD O.D.E WITH GENERAL SOLUTION

$\Rightarrow y(t)=A\cos nt+B\sin nt$

$\Rightarrow y(x)=A\cos(n\arccos x)+B\sin(n\arccos x)\quad |x|<1$

$\Rightarrow y(x)=A\,T_n(x)+B\,U_n(x)$

b) NOW WE MANIPULATE $T_n(x)$ & $U_n$ AS FOLLOWS

$T_n(x)+iU_n(x)=\cos nt+i\sin nt$

$T_n(x)-iU_n(x)=\cos nt-i\sin nt$ WHERE $t=\arccos x$

$T_n(x)+iU_n(x)=(\cos t+i\sin t)^n=(x+i\sqrt{1-x^2})^n$

$T_n(x)-iU_n(x)=(\cos t-i\sin t)^n=(x-i\sqrt{1-x^2})^n$

ADDING YIELDS

$2T_n(x)=(x+i\sqrt{1-x^2})^n+(x-i\sqrt{1-x^2})^n$

$T_n(x)=\frac{1}{2}\left[(x+i\sqrt{1-x^2})^n+(x-i\sqrt{1-x^2})^n\right]$

**Question 2**

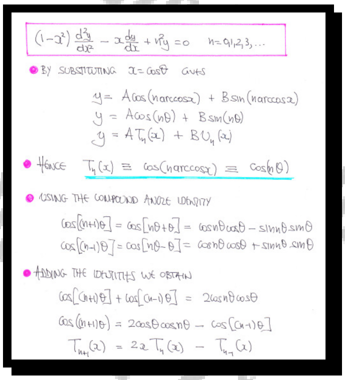
Chebyshev's polynomials of the first kind $\mathrm{T}_n(x)$ are defined as

$$\mathrm{T}_n(x) = \cos\left[n \arccos(x)\right], \quad n = 0, 1, 2, 3, 4, \ldots$$

By writing $\theta = \arccos(x)$ and considering the compound angle trigonometric identities for $\cos\left[(n \pm 1)\theta\right]$, obtain the recurrence relation

$$\mathrm{T}_{n+1}(x) = 2x\mathrm{T}_n(x) - \mathrm{T}_{n-1}(x), \quad |x| < 1.$$

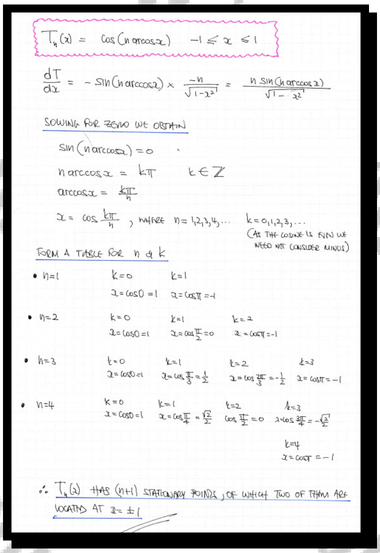
proof



**Question 3**

Investigate the stationary points of the Chebyshev polynomials of the first kind $T_n(x)$.

$\boxed{n+1 \text{ stationary points, two them at } x = \pm 1}$



$T_n(x) = \cos(n \arccos x) \quad -1 \leqslant x \leqslant 1$

$\frac{dT}{dx} = -\sin(n \arccos x) \times \frac{-n}{\sqrt{1-x^2}} = \frac{n \sin(n \arccos x)}{\sqrt{1-x^2}}$

SOLVING FOR ZERO WE OBTAIN

$\sin(n \arccos x) = 0$

$n \arccos x = k\pi \quad k \in \mathbb{Z}$

$\arccos x = \frac{k\pi}{n}$

$x = \cos \frac{k\pi}{n}$, WHERE $n = 1, 2, 3, 4, \ldots$ $k = 0, 1, 2, 3, \ldots$ (AS THE COSINE IS EVEN WE NEED NOT CONSIDER MINUS)

FORM A TABLE FOR $n$ & $k$

- $n=1$: $k=0$, $x = \cos 0 = 1$; $k=1$, $x = \cos \pi = -1$
- $n=2$: $k=0$, $x = \cos 0 = 1$; $k=1$, $x = \cos \frac{\pi}{2} = 0$; $k=2$, $x = \cos \pi = -1$
- $n=3$: $k=0$, $x = \cos 0 = 1$; $k=1$, $x = \cos \frac{\pi}{3} = \frac{1}{2}$; $k=2$, $x = \cos \frac{2\pi}{3} = -\frac{1}{2}$; $k=3$, $x = \cos \pi = -1$
- $n=4$: $k=0$, $x = \cos 0 = 1$; $k=1$, $x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$; $k=2$, $\cos \frac{\pi}{2} = 0$; $k=3$, $x = \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$; $k=4$, $x = \cos \pi = -1$

$\therefore$ $T_n(x)$ HAS $(n+1)$ STATIONARY POINTS, OF WHICH TWO OF THEM ARE LOCATED AT $x = \pm 1$

**Question 4**

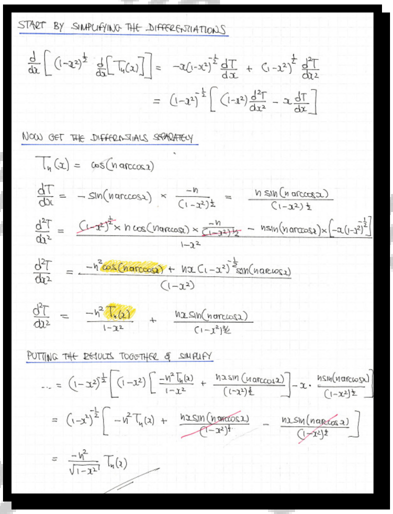
The Chebyshev polynomials of the first kind $T_n(x)$ are defined as

$$T_n(x) = \cos(n \arccos x), \ -1 \le x \le 1, \ n \in \mathbb{N}.$$

Show that

$$\frac{d}{dx}\left[\left(1-x^2\right)^{\frac{1}{2}} \frac{d}{dx}\left[T_n(x)\right]\right] = \frac{-n^2 \, T_n(x)}{\sqrt{1-x^2}}.$$

proof



**Question 5**

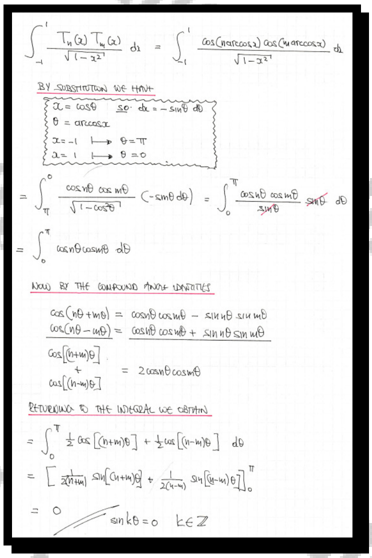
The Chebyshev polynomials of the first kind $T_n(x)$ are defined as

$$T_n(x) = \cos(n \arccos x), \ -1 \le x \le 1, \ n \in \mathbb{N}.$$

Show that

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}} \, dx = 0.$$

proof



$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos(n\arccos x)\cos(m\arccos x)}{\sqrt{1-x^2}}\,dx$$

BY SUBSTITUTION WE HAVE

$x = \cos\theta$ so $dx = -\sin\theta\,d\theta$
$\theta = \arccos x$
$x=-1 \mapsto \theta = \pi$
$x = 1 \mapsto \theta = 0$

$$= \int_{\pi}^{0} \frac{\cos n\theta \cos m\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta\,d\theta) = \int_{0}^{\pi} \frac{\cos n\theta \cos m\theta}{\sin\theta}\sin\theta\,d\theta$$

$$= \int_{0}^{\pi} \cos n\theta \cos m\theta \, d\theta$$

NOW BY THE COMPOUND ANGLE IDENTITIES

$$\cos(n\theta + m\theta) = \cos n\theta \cos m\theta - \sin n\theta \sin m\theta$$
$$\cos(n\theta - m\theta) = \cos n\theta \cos m\theta + \sin n\theta \sin m\theta$$
$$\cos[(n+m)\theta] + \cos[(n-m)\theta] = 2\cos n\theta \cos m\theta$$

RETURNING TO THE INTEGRAL WE OBTAIN

$$= \int_0^{\pi} \tfrac{1}{2}\cos[(n+m)\theta] + \tfrac{1}{2}\cos[(n-m)\theta]\,d\theta$$

$$= \left[\frac{1}{2(n+m)}\sin[(n+m)\theta] + \frac{1}{2(n-m)}\sin[(n-m)\theta]\right]_0^{\pi}$$

$$= 0$$

$\sin k\theta = 0 \quad k \in \mathbb{Z}$

## Question 6

Chebyshev's differential equation is given below.

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0\,, \qquad n = 0, 1, 2, 3, 4, \ldots$$

**a)** By using the substitution $x = \cosh t$, show that a general solution for Chebyshev's differential equation is

$$y = A\,\mathrm{T}_n(x) + B\,\mathrm{U}_n(x)\,, \ |x| > 1\,,$$

with $\mathrm{T}_n(x) = \cosh\left[n\,\text{arcosh}(x)\right]$, $\mathrm{U}_n(x) = \sinh\left[n\,\text{arcosh}(x)\right]$, and $A$ and $B$ are arbitrary constants.

By using the substitution $x = \cos t$, Chebyshev's differential equation has general solution

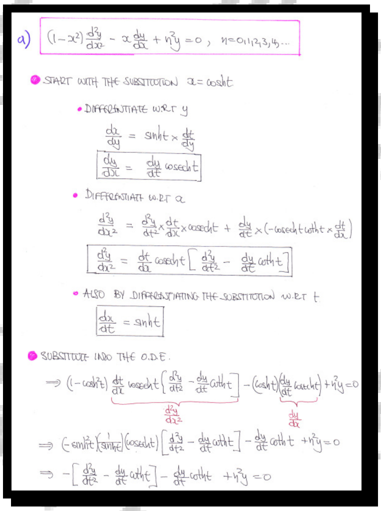
$$y = A\,\mathrm{T}_n(x) + B\,\mathrm{U}_n(x)\,, \ |x| < 1\,,$$

with $\mathrm{T}_n(x) = \cos\left[n\arccos(x)\right]$, $\mathrm{U}_n(x) = \sin\left[n\arccos(x)\right]$, and as above $A$ and $B$ are arbitrary constants.

**b)** Show that $\mathrm{T}_n(x)$, for both the general solutions obtained via the substitutions $x = \cos t$ and $x = \cosh t$, can be written as

$$\mathrm{T}_n(x) = \frac{1}{2}\left[\left(x + \mathrm{i}\sqrt{1-x^2}\right) + \left(x - \mathrm{i}\sqrt{1-x^2}\right)\right].$$

proof



a) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$, $n = 0,1,2,3,4\ldots$

START WITH THE SUBSTITUTION $x = \cosh t$

- DIFFERENTIATE W.R.T $y$

$\frac{dx}{dy} = \sinh t \times \frac{dt}{dy}$

$\frac{dy}{dx} = \frac{dy}{dt}\operatorname{cosech} t$

- DIFFERENTIATE W.R.T $x$

$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\times\frac{dt}{dx}\times\operatorname{cosech} t + \frac{dy}{dt}\times\left(-\operatorname{cosech} t\coth t\times\frac{dt}{dx}\right)$

$\frac{d^2y}{dx^2} = \frac{dt}{dx}\operatorname{cosech} t\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\coth t\right]$

- ALSO BY DIFFERENTIATING THE SUBSTITUTION W.R.T $t$

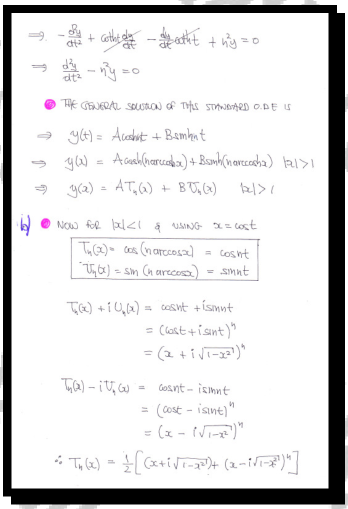
$\frac{dx}{dt} = \sinh t$

SUBSTITUTE INTO THE O.D.E.

$\Rightarrow (1-\cosh^2 t)\frac{dt}{dx}\operatorname{cosech} t\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\coth t\right] - (\cosh t)\left(\frac{dy}{dt}\operatorname{cosech} t\right) + n^2y = 0$

$\Rightarrow (-\sinh^2 t)\left(\frac{1}{\sinh t}\right)(\operatorname{cosech} t)\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\coth t\right] - \frac{dy}{dt}\coth t + n^2y = 0$

$\Rightarrow -\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\coth t\right] - \frac{dy}{dt}\coth t + n^2y = 0$



$\Rightarrow -\frac{d^2y}{dt^2} + \coth t\frac{dy}{dt} - \frac{dy}{dt}\coth t + n^2y = 0$

$\Rightarrow \frac{d^2y}{dt^2} - n^2y = 0$

THE GENERAL SOLUTION OF THIS STANDARD O.D.E IS

$\Rightarrow y(t) = A\cosh nt + B\sinh nt$

$\Rightarrow y(x) = A\cosh(n\,\text{arcosh}\,x) + B\sinh(n\,\text{arcosh}\,x)$ $|x|>1$

$\Rightarrow y(x) = A\mathrm{T}_n(x) + B\mathrm{U}_n(x)$ $|x|>1$

b) NOW FOR $|x|<1$ & USING $x = \cos t$

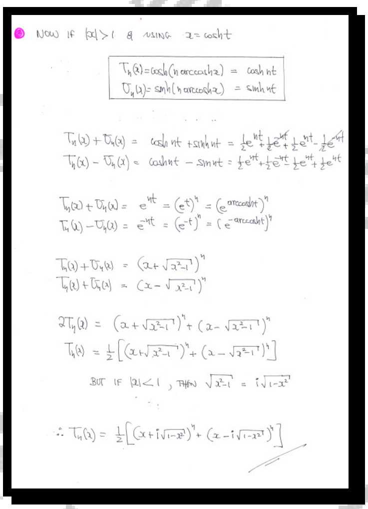
$\mathrm{T}_n(x) = \cos(n\arccos x) = \cos nt$

$\mathrm{U}_n(x) = \sin(n\arccos x) = \sin nt$

$\mathrm{T}_n(x) + i\,\mathrm{U}_n(x) = \cos nt + i\sin nt = (\cos t + i\sin t)^n = (x + i\sqrt{1-x^2})^n$

$\mathrm{T}_n(x) - i\,\mathrm{U}_n(x) = \cos nt - i\sin nt = (\cos t - i\sin t)^n = (x - i\sqrt{1-x^2})^n$

$\therefore \mathrm{T}_n(x) = \frac{1}{2}\left[(x+i\sqrt{1-x^2}) + (x-i\sqrt{1-x^2})^n\right]$



NOW IF $|x|>1$ & USING $x = \cosh t$

$\mathrm{T}_n(x) = \cosh(n\,\text{arcosh}\,x) = \cosh nt$

$\mathrm{U}_n(x) = \sinh(n\,\text{arcosh}\,x) = \sinh nt$

$\mathrm{T}_n(x) + \mathrm{U}_n(x) = \cosh nt + \sinh nt = \frac{1}{2}e^{nt} + \frac{1}{2}e^{-nt} + \frac{1}{2}e^{nt} - \frac{1}{2}e^{-nt}$

$\mathrm{T}_n(x) - \mathrm{U}_n(x) = \cosh nt - \sinh nt = \frac{1}{2}e^{nt} + \frac{1}{2}e^{-nt} - \frac{1}{2}e^{nt} + \frac{1}{2}e^{-nt}$

$\mathrm{T}_n(x) + \mathrm{U}_n(x) = e^{nt} = (e^t)^n = (e^{\text{arcosh}\,t})^n$

$\mathrm{T}_n(x) - \mathrm{U}_n(x) = e^{-nt} = (e^{-t})^n = (e^{-\text{arcosh}\,t})^n$

$\mathrm{T}_n(x) + \mathrm{U}_n(x) = (x + \sqrt{x^2-1})^n$

$\mathrm{T}_n(x) + \mathrm{U}_n(x) = (x - \sqrt{x^2-1})^n$

$2\mathrm{T}_n(x) = (x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n$

$\mathrm{T}_n(x) = \frac{1}{2}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right]$

BUT IF $|x|<1$, THEN $\sqrt{x^2-1} = i\sqrt{1-x^2}$

$\therefore \mathrm{T}_n(x) = \frac{1}{2}\left[(x+i\sqrt{1-x^2})^n + (x-i\sqrt{1-x^2})^n\right]$

## Question 7

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0\,,\;\; n=0,1,2,3,\ldots$$

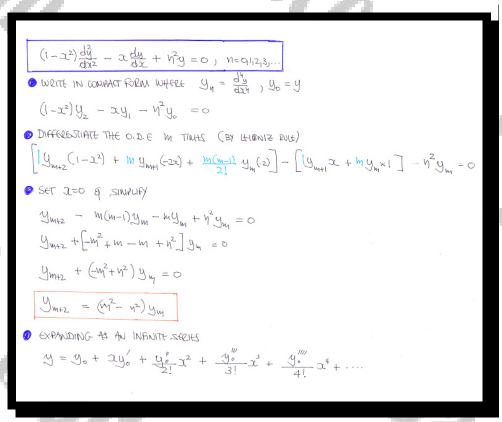
Find a series solution for Chebyshev's equation, by using the Leibniz method

$$y=$$

$$A\left[x-\frac{\left(1-n^2\right)}{3!}x^3-\frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5-\frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7-\ldots\right]$$

$$+$$

$$B\left[1-\frac{n^2}{2!}x^2-\frac{n^2\left(4-n^2\right)}{4!}x^4-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6-\frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8-\ldots\right]$$



$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0,\quad n=0,1,2,3,\ldots$$

WRITE IN COMPACT FORM WHERE $y_n=\frac{d^ny}{dx^n}$, $y_0=y$

$$(1-x^2)y_2-xy_1-n^2y_0=0$$

DIFFERENTIATE THE O.D.E $m$ TIMES (BY LEIBNIZ RULE)

$$\left[1y_{m+2}(1-x^2)+m\,y_{m+1}(-2x)+\frac{m(m-1)}{2!}y_m(-2)\right]-\left[1y_{m+1}x+m\,y_m\times 1\right]-n^2y_m=0$$

SET $x=0$ & SIMPLIFY

$$y_{m+2}-m(m-1)y_m-my_m+n^2y_m=0$$

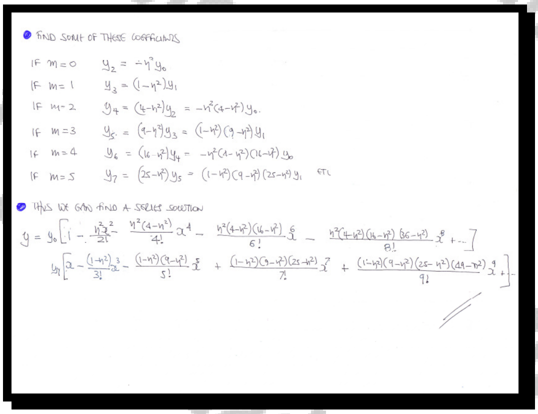
$$y_{m+2}+\left[-m^2+m-m+n^2\right]y_m=0$$

$$y_{m+2}+(-m^2+n^2)y_m=0$$

$$y_{m+2}=(m^2-n^2)y_m$$

EXPANDING AS AN INFINITE SERIES

$$y=y_0+xy_0'+\frac{y_0''}{2!}x^2+\frac{y_0'''}{3!}x^3+\frac{y_0''''}{4!}x^4+\ldots$$



FIND SOME OF THESE COEFFICIENTS

IF $m=0$ $\quad y_2=-n^2y_0$

IF $m=1$ $\quad y_3=(1-n^2)y_1$

IF $m=2$ $\quad y_4=(4-n^2)y_2=-n^2(4-n^2)y_0$

IF $m=3$ $\quad y_5=(9-n^2)y_3=(1-n^2)(9-n^2)y_1$

IF $m=4$ $\quad y_6=(16-n^2)y_4=-n^2(4-n^2)(16-n^2)y_0$

IF $m=5$ $\quad y_7=(25-n^2)y_5=(1-n^2)(9-n^2)(25-n^2)y_1$ ETC

THUS WE CAN FIND A SERIES SOLUTION

$$y=y_0\left[1-\frac{n^2x^2}{2!}-\frac{n^2(4-n^2)}{4!}x^4-\frac{n^2(4-n^2)(16-n^2)}{6!}x^6-\frac{n^2(4-n^2)(16-n^2)(36-n^2)}{8!}x^8+\ldots\right]$$

$$y_1\left[x-\frac{(1-n^2)}{3!}x^3-\frac{(1-n^2)(9-n^2)}{5!}x^5+\frac{(1-n^2)(9-n^2)(25-n^2)}{7!}x^7+\frac{(1-n^2)(9-n^2)(25-n^2)(49-n^2)}{9!}x^9+\ldots\right]$$

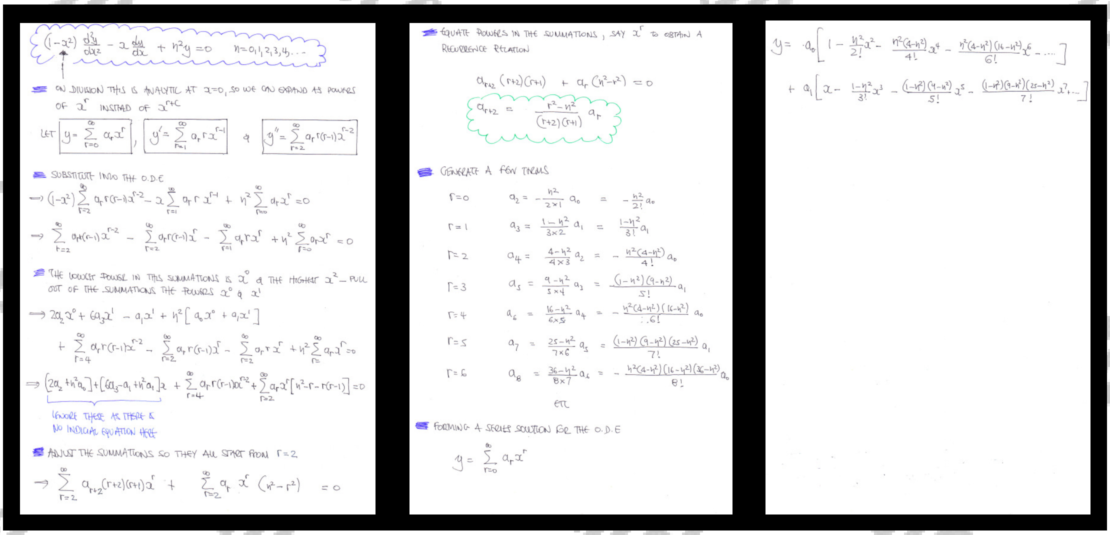
## Question 8

Chebyshev's equation is shown below

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0, \quad n = 0, 1, 2, 3, \ldots$$

Find a series solution for Chebyshev's equation, by using the Frobenius method.

$$y = A\left[x - \frac{\left(1-n^2\right)}{3!}x^3 - \frac{\left(1-n^2\right)\left(9-n^2\right)}{5!}x^5 - \frac{\left(1-n^2\right)\left(9-n^2\right)\left(25-n^2\right)}{7!}x^7 - \ldots\right]$$
$$+$$
$$B\left[1 - \frac{n^2}{2!}x^2 - \frac{n^2\left(4-n^2\right)}{4!}x^4 - \frac{n^2\left(4-n^2\right)\left(16-n^2\right)}{6!}x^6 - \frac{n^2\left(4-n^2\right)\left(16-n^2\right)\left(36-n^2\right)}{8!}x^8 - \ldots\right]$$



$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0 \qquad n = 0,1,2,3,4,\ldots$$

ON DIVISION THIS IS ANALYTIC AT $x=0$, SO WE CAN EXPAND AS POWERS OF $x^r$ INSTEAD OF $x^{r+c}$

LET $y = \sum_{r=0}^{\infty} a_r x^r$, $y' = \sum_{r=1}^{\infty} a_r r x^{r-1}$ & $y'' = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow (1-x^2)\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - x\sum_{r=1}^{\infty} a_r r x^{r-1} + n^2\sum_{r=0}^{\infty} a_r x^r = 0$$

$$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=1}^{\infty} a_r r x^r + n^2\sum_{r=0}^{\infty} a_r x^r = 0$$

THE LOWEST POWER IN THE SUMMATIONS IS $x^0$ & THE HIGHEST $x^2$ – PULL OUT OF THE SUMMATIONS THE POWERS $x^0$ & $x^1$

$$\Rightarrow 2a_2x^0 + 6a_3x^1 - a_1x^1 + n^2\left[a_0x^0 + a_1x^1\right]$$
$$+ \sum_{r=4}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=2}^{\infty} a_r r(r-1)x^r - \sum_{r=2}^{\infty} a_r r x^r + n^2\sum_{r=2}^{\infty} a_r x^r = 0$$

$$\Rightarrow \left[2a_2 + n^2a_0\right] + \left[6a_3 - a_1 + n^2a_1\right]x + \sum_{r=4}^{\infty} a_r r(r-1)x^{r-2} + \sum_{r=2}^{\infty} a_r x^r\left[n^2 - r - r(r-1)\right] = 0$$

IGNORE THESE AS THERE IS NO INDICIAL EQUATION HERE

ADJUST THE SUMMATIONS SO THEY ALL START FROM $r=2$

$$\Rightarrow \sum_{r=2}^{\infty} a_{r+2}(r+2)(r+1)x^r + \sum_{r=2}^{\infty} a_r x^r\left(n^2 - r^2\right) = 0$$

EQUATE POWERS IN THE SUMMATIONS, SAY $x^r$ TO OBTAIN A RECURRENCE RELATION

$$a_{r+2}(r+2)(r+1) + a_r(n^2 - r^2) = 0$$

$$a_{r+2} = \frac{r^2 - n^2}{(r+2)(r+1)}a_r$$

GENERATE A FEW TERMS

$r=0 \qquad a_2 = -\frac{n^2}{2\times1}a_0 = -\frac{n^2}{2!}a_0$

$r=1 \qquad a_3 = \frac{1-n^2}{3\times2}a_1 = \frac{1-n^2}{3!}a_1$

$r=2 \qquad a_4 = \frac{4-n^2}{4\times3}a_2 = -\frac{n^2(4-n^2)}{4!}a_0$

$r=3 \qquad a_5 = \frac{9-n^2}{5\times4}a_3 = \frac{(1-n^2)(9-n^2)}{5!}a_1$

$r=4 \qquad a_6 = \frac{16-n^2}{6\times5}a_4 = -\frac{n^2(4-n^2)(16-n^2)}{6!}a_0$

$r=5 \qquad a_7 = \frac{25-n^2}{7\times6}a_5 = \frac{(1-n^2)(9-n^2)(25-n^2)}{7!}a_1$

$r=6 \qquad a_8 = \frac{36-n^2}{8\times7}a_6 = -\frac{n^2(4-n^2)(16-n^2)(36-n^2)}{8!}a_0$

ETC

FORMING A SERIES SOLUTION FOR THE O.D.E

$$y = \sum_{r=0}^{\infty} a_r x^r$$

$$y = a_0\left[1 - \frac{n^2}{2!}x^2 - \frac{n^2(4-n^2)}{4!}x^4 - \frac{n^2(4-n^2)(16-n^2)}{6!}x^6 - \ldots\right]$$
$$+ a_1\left[x - \frac{1-n^2}{3!}x^3 - \frac{(1-n^2)(4-n^2)}{5!}x^5 - \frac{(1-n^2)(9-n^2)(25-n^2)}{7!}x^7 + \ldots\right]$$

# LAMBERT FUNCTIONS

## Question 1

The Lambert $W$ function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(xe^{x}\right) \equiv x,$$

and hence if $xe^{x} = y$ then $x = W(y)$.

For example

$-xe^{-x} = 2 \;\Rightarrow\; -x = W(2)$, $\quad (x+\pi)e^{x+\pi} = \frac{1}{2} \;\Rightarrow\; x+\pi = W\left(\frac{1}{2}\right)$ and so on.

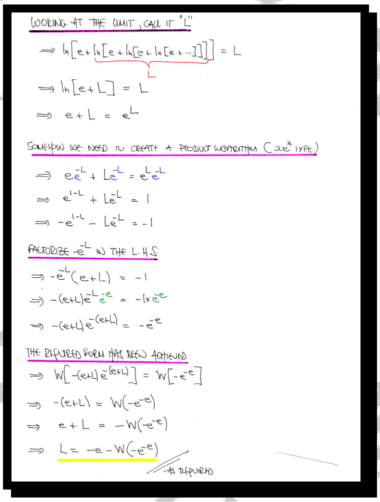
Use this result to show that the limit of

$$\ln\left(e+\ln\left(e+\ln\left(e+\ln\left(e+\ldots\right)\right)\right)\right)$$

is given by

$$-e-W\left[-e^{-e}\right].$$

[ ] , proof



**Question 2**

The Lambert $W$ function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(xe^{x}\right) \equiv x, \quad x \in \left[-e^{-1}, \infty\right)$$

and hence if $xe^{x} = y$ then $x = W(y)$.

For example

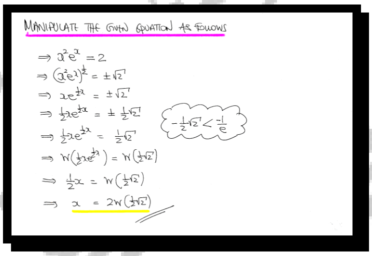
$$-xe^{-x} = 2 \;\Rightarrow\; -x = W(2), \quad (x+\pi)e^{x+\pi} = \tfrac{1}{2} \;\Rightarrow\; x+\pi = W\left(\tfrac{1}{2}\right) \text{ and so on.}$$

Use this result to find the exact solution of

$$x^{2}e^{x} = 2.$$

Give the answer in the form $x = \lambda W(\mu)$, where $\lambda$ and $\mu$ are constants.

$\boxed{\phantom{xx}}$, $\boxed{x = 2W\left(\tfrac{1}{2}\sqrt{2}\right)}$



**Question 3**

The Lambert $W$ function, also called the omega function or product logarithm, is a multivalued function which has the property

$$W\left(x\mathrm{e}^{x}\right) \equiv x, \quad x \in \left[-\mathrm{e}^{-1}, \infty\right)$$

and hence if $x\mathrm{e}^{x} = y$ then $x = W(y)$.

For example

$$-x\mathrm{e}^{-x} = 2 \;\Rightarrow\; -x = W(2), \quad (x+\pi)\mathrm{e}^{x+\pi} = \tfrac{1}{2} \;\Rightarrow\; x+\pi = W\left(\tfrac{1}{2}\right) \text{ and so on.}$$

Use this result to find the exact solution of

$$x + \mathrm{e}^{x} = 2.$$

Give the answer in the form $x = k - W\left(\mathrm{e}^{k}\right)$, where $k$ is a constant.

$\boxed{\phantom{x}}$, $\boxed{x = 2 - W\left(\mathrm{e}^{2}\right)}$