

Created by T. Madas

RESIDUES and APPLICATIONS in SERIES SUMMATION

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The Residue Theorem can often be used to sum various types of series.

The following results are valid under some restrictions on $f(z)$, which more often than not are satisfied when the series converges.

$$\sum_{r=-\infty}^{\infty} f(r)$$

use $\oint_{\Gamma_n} f(z) \pi \cot \pi z \, dz$, where Γ_n is the square with vertices at $(n + \frac{1}{2})(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} (-1)^r f(r)$$

use $\oint_{\Gamma_n} f(z) \pi \operatorname{cosec} \pi z \, dz$, where Γ_n is the square with vertices at $(n + \frac{1}{2})(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} f\left(\frac{2r+1}{2}\right)$$

use $\oint_{\Gamma_n} f(z) \pi \tan \pi z \, dz$, where Γ_n is the square with vertices at $n(\pm 1 \pm i)$

$$\sum_{r=-\infty}^{\infty} (-1)^r f\left(\frac{2r+1}{2}\right)$$

use $\oint_{\Gamma_n} f(z) \pi \sec \pi z \, dz$, where Γ_n is the square with vertices at $n(\pm 1 \pm i)$

Question 1

$$f(z) = \frac{\pi \cot \pi z}{(a+z)^2}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(a+r)^2} = \pi^2 \operatorname{cosec}^2(\pi a), \quad a \notin \mathbb{Z}.$$

proof

CONSIDER $\int_{\Gamma} \frac{\pi \cot \pi z}{(a+z)^2} dz$ WITH THE SQUARE OUTLINE WITH VERTICES $(n+\frac{1}{2})\pi i, (n+\frac{1}{2})\pi i + 2\pi, (n+\frac{1}{2})\pi i + 2\pi + i, (n+\frac{1}{2})\pi i + i$

THE SQUARE HAS A DOUBLE POLE AT $z = -a$, AND SIMPLE POLES AT $z = \dots, -n, \dots, -n+1, 1, 2, \dots, n$ WHERE $z = n+1/2 + ik$, BECAUSE OF $\cot z = \frac{\cos z}{\sin z}$

CAUSAL RESIDUES

$$\lim_{z \rightarrow a} \left[\frac{1}{z-a} \left(\frac{\pi \cot \pi z}{(a+z)^2} \right) \right] = \lim_{z \rightarrow a} \left[\frac{\pi \cot \pi z}{2(a+z)} \right]$$

$$\lim_{z \rightarrow a} \left[\frac{\pi \cot \pi z}{2(a+z)} \right] = \lim_{z \rightarrow a} \left[\frac{\pi \cot \pi z}{2(a+z)} \right] = \frac{\pi \cot \pi a}{2(a+a)} = \frac{\pi \cot \pi a}{4a}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \sum (\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{\Gamma} \frac{\pi \cot \pi z}{(a+z)^2} dz = 2\pi i \left[\frac{\pi \cot \pi a}{4a} + \sum_{k=-n}^n \frac{1}{(a+k)^2} \right]$$

THE $\int_{\Gamma} f(z) dz$ IS BOUNDED ON Γ (i.e. $|\int_{\Gamma} f(z) dz| \leq M$) (PROOF ATTACHED AT THE END)

$|z| > n + \frac{1}{2}$ ON Γ

HERE

$$\left| \int_{\Gamma} f(z) dz \right| = \left| \int_{\Gamma} \frac{\pi \cot \pi z}{(a+z)^2} dz \right| \leq \int_{\Gamma} \frac{\pi |\cot \pi z|}{|a+z|^2} |dz|$$

$$\leq \int_{\Gamma} \frac{\pi |\cot \pi z|}{|z|^2} |dz| \leq \int_{\Gamma} \frac{\pi}{|z|^2} |dz|$$

$$= \frac{\pi M \times 8(n+\frac{1}{2})}{(n+\frac{1}{2})^2 - 2\pi(n+\frac{1}{2}) - a^2} \rightarrow 0 \text{ as } n \rightarrow \infty$$

RETURNING TO THE INTEGRAL AS $n \rightarrow \infty$ $0 = 2\pi i \left[\frac{\pi \cot \pi a}{4a} + \sum_{k=-\infty}^{\infty} \frac{1}{(a+k)^2} \right]$

INSTEAD OF THE 2PI AND REARRANGE TO GET $\sum_{k=-\infty}^{\infty} \frac{1}{(a+k)^2} = \pi^2 \operatorname{cosec}^2(\pi a)$

PROOF THAT $\cot \pi z$ IS BOUNDED ON THE SQUARE OUTLINE WITH VERTICES $(n+\frac{1}{2})\pi i, (n+\frac{1}{2})\pi i + 2\pi, \dots$

- $\left| \cot \pi z \right| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{i\pi z}| - |e^{-i\pi z}|}$

DO THE HORIZONTAL SIDES OF THE OUTLINE $z = x + i(n+\frac{1}{2})$, $-\pi n < x < \pi(n+\frac{1}{2})$

$$|e^{i\pi z}| = |e^{i\pi(x+i(n+\frac{1}{2}))}| = |e^{i\pi x} e^{-\pi(n+\frac{1}{2})}| = e^{-\pi(n+\frac{1}{2})}$$

$$|e^{-i\pi z}| = |e^{-i\pi(x+i(n+\frac{1}{2}))}| = |e^{-i\pi x} e^{\pi(n+\frac{1}{2})}| = e^{\pi(n+\frac{1}{2})}$$

$$\therefore \left| \cot \pi z \right| \leq \frac{e^{-\pi(n+\frac{1}{2})} + e^{\pi(n+\frac{1}{2})}}{e^{-\pi(n+\frac{1}{2})} - e^{\pi(n+\frac{1}{2})}} = \frac{2 \cosh(\pi(n+\frac{1}{2}))}{2 \sinh(\pi(n+\frac{1}{2}))} = \coth(\pi(n+\frac{1}{2})) < 1 \text{ FOR ALL } (n \in \mathbb{Z})$$

THE SAME IS TRUE FOR THE OTHER HORIZONTAL SIDES.

- NOW ON THE VERTICAL SIDES OF THE OUTLINE $z = x + i(n+\frac{1}{2}) + iy$, $-\pi n < x < \pi(n+\frac{1}{2})$

$$\left| \cot \pi z \right| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{e^{i\pi(x+i(n+\frac{1}{2})+iy)} + e^{-i\pi(x+i(n+\frac{1}{2})+iy)}}{e^{i\pi(x+i(n+\frac{1}{2})+iy)} - e^{-i\pi(x+i(n+\frac{1}{2})+iy)}} \right|$$

$$= \left| \frac{e^{i\pi x} e^{-\pi(n+\frac{1}{2})} e^{-\pi y} + e^{-i\pi x} e^{\pi(n+\frac{1}{2})} e^{-\pi y}}{e^{i\pi x} e^{-\pi(n+\frac{1}{2})} e^{-\pi y} - e^{-i\pi x} e^{\pi(n+\frac{1}{2})} e^{-\pi y}} \right| = \left| \frac{e^{-\pi y} (e^{i\pi x} e^{-\pi(n+\frac{1}{2})} + e^{-i\pi x} e^{\pi(n+\frac{1}{2})})}{e^{-\pi y} (e^{i\pi x} e^{-\pi(n+\frac{1}{2})} - e^{-i\pi x} e^{\pi(n+\frac{1}{2})})} \right| < 1$$

$\therefore \cot \pi z$ IS BOUNDED ON THE OUTLINE

Question 2

$$f(z) = \frac{\pi \cot \pi z}{(3z+1)(2z+1)}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=-\infty}^{\infty} \frac{1}{(3r+1)(2r+1)} = \frac{\pi}{\sqrt{3}}.$$

 , proof

SOLUTION $\int_{\Gamma} \frac{\pi \cot \pi z}{(3z+1)(2z+1)} dz$ over the square contour with vertices at $(-n+\frac{1}{2})(1+i)$, $(n+\frac{1}{2})(1+i)$, $(n+\frac{1}{2})(-1-i)$, $(-n+\frac{1}{2})(-1-i)$.

CONTOUR CHOICE

- $\lim_{z \rightarrow \infty} f(z) = 0$ (check)
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RESIDUE THEORY

By the residue theorem we know that

$$\int_{\Gamma} f(z) dz = 2\pi i \sum \text{Residues inside } \Gamma$$

IT CAN BE SHOWN THAT $\cot \pi z$ IS BOUNDED ON Γ , i.e. $|\cot \pi z| < M$ FOR SOME $M < \infty$. THIS IS SHOWN AS A SEPARATE PROOF AT THE END OF THIS QUESTION.

THEY USE THE FOLLOWING

$$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

BOUNDING THE INTEGRAL

$$\int_{\Gamma} \frac{\pi \cot \pi z}{(3z+1)(2z+1)} dz \leq \int_{\Gamma} \frac{\pi M}{|3z+1||2z+1|} |dz|$$

BOUNDING THE INTEGRAL

$$O = 2\pi i \left[\frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{(3n+1)(2n+1)} \right]$$

BOUNDING THE INTEGRAL

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(2n+1)} = \frac{\pi}{\sqrt{3}}$$

BOUNDING THE INTEGRAL

ON THE "VERTICAL" SIDES OF THE CONTOUR, $z = \pm(n+\frac{1}{2}) + iy$, $-(n+\frac{1}{2}) < n+\frac{1}{2}$

- $|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$
- $|\cot \pi z| = \dots$ BY APPROXIMATELY EQUALS $\dots = e^{2\pi y}$

BOUNDING THE INTEGRAL

ON THE "VERTICAL" SIDES OF THE CONTOUR, $z = \pm(n+\frac{1}{2}) + iy$ where $-(n+\frac{1}{2}) < n+\frac{1}{2} < n+\frac{1}{2}$ PROCEED AS BEFORE

$$\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{\pi \cot \pi z}{(3z+1)(2z+1)} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{e^{3\pi z} + 1} \frac{1}{e^{2\pi z} + 1} dz$$

BOUNDING THE INTEGRAL

$$\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{\pi \cot \pi z}{(3z+1)(2z+1)} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{e^{3\pi z} + 1} \frac{1}{e^{2\pi z} + 1} dz$$

BOUNDING THE INTEGRAL

$$\int_{\Gamma} f(z) dz = \int_{\Gamma} \frac{\pi \cot \pi z}{(3z+1)(2z+1)} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{e^{3\pi z} + 1} \frac{1}{e^{2\pi z} + 1} dz$$

Question 3

$$f(z) = \frac{\pi \cot \pi z}{4z^2 - 1}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{1}{4r^2 - 1} = \frac{1}{2}.$$

proof

Handwritten mathematical proof for Question 3. The left page shows the contour integral setup with a square contour Γ in the complex plane, vertices at $(n+1/2) \pm i\epsilon$ and $(n+1/2) \mp i\epsilon$. It details the residue calculation at $z=0$ and $z=\pm 1/2$, and the limit process as $\epsilon \rightarrow 0$. The right page shows the residue calculation at $z=0$ using the Laurent series of $\cot \pi z$, and the limit process as $n \rightarrow \infty$ to show the integral over the contour vanishes, leading to the final result.

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Question 4

$$f(z) = \frac{\pi \cot \pi z}{z^2}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

proof

CONSIDER $\int_{\Gamma} \frac{\pi \cot \pi z}{z^2} dz$ where Γ is the usual SQUARE contour with vertices at $(\pm 1/2) \pm i n$, $n \in \mathbb{N}$.

(i) WE SHOWN EARLIER THAT $\cot \pi z$ HAS SIMPLE POLES AT $z = k$, $k \in \mathbb{Z}$ WITH RESIDUES OF 1. ALL ARE WITHIN THE SQUARE OF THE $\frac{1}{2}$ CENTERED ON THE REAL AXIS.

RESIDUES:

- $\lim_{z \rightarrow k} (z-k) \cot \pi z = \lim_{z \rightarrow k} \frac{z-k}{\sin \pi z} = \frac{1}{\pi \cos \pi k} = \frac{1}{\pi} (-1)^k$
- $\lim_{z \rightarrow k} \frac{z-k}{\cos \pi z} = \frac{1}{-\pi \sin \pi k} = \frac{1}{-\pi} (-1)^k$
- $\lim_{z \rightarrow k} \frac{z-k}{\sin \pi z} = \frac{1}{\pi \cos \pi k} = \frac{1}{\pi} (-1)^k$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{\pi \cot \pi z}{z^2} dz = 2\pi i \sum_{k=-n}^n \text{Res}_{z=k} \left[\frac{\pi \cot \pi z}{z^2} \right]$$

NOW IT CAN BE SHOWN THAT $\cot \pi z$ IS BOUNDED ON Γ , i.e. $|\cot \pi z| \leq M$ FOR SOME $M \in \mathbb{R}$.

ALSO $|z| \geq n - \frac{1}{2} \forall z \in \Gamma$

THIS $\int_{\Gamma} \frac{\pi \cot \pi z}{z^2} dz = \int_{\Gamma} \frac{\pi \cot \pi z}{z^2} dz < \int_{\Gamma} \frac{M}{|z|^2} |dz| < \int_{\Gamma} \frac{M}{(n-1/2)^2} |dz| = \frac{M}{(n-1/2)^2} \cdot 4(n) = \frac{4Mn}{(n-1/2)^2} \rightarrow 0$ AS $n \rightarrow \infty$

RETURNING TO THE ORIGINAL AS $n \rightarrow \infty$

$$0 = 2\pi i \left[-\frac{1}{3} + \sum_{k=1}^n \frac{1}{k^2} + \sum_{k=1}^n \frac{1}{k^2} \right]$$

DIVIDE BY $2\pi i$ AND REARRANGE

$$\sum_{k=1}^n \frac{1}{k^2} + \sum_{k=1}^n \frac{1}{k^2} = \frac{1}{3}$$

$$\Rightarrow \left[\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \right] + \left[\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \right] = \frac{1}{3}$$

$$\Rightarrow \sum_{k=1}^n \frac{1}{k^2} = \frac{1}{6}$$

PROOF THAT $\cot \pi z$ IS BOUNDED ON THE SQUARE contour WITH VERTICES $(\pm 1/2) \pm i n$, $n=1,2,3,4, \dots$

$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \frac{1}{2} \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{i\pi z}| - |e^{-i\pi z}|}$

DO THE HORIZONTAL SIDES OF THE contour $z = x + i(\pm n)$, $-1/2 \leq x \leq 1/2$

$|e^{i\pi z}| = |e^{i\pi(x+in)}| = |e^{i\pi x} e^{-\pi n}| = e^{-\pi n}$

$|e^{-i\pi z}| = |e^{-i\pi(x-in)}| = |e^{-i\pi x} e^{-\pi n}| = e^{-\pi n}$

$\therefore |\cot \pi z| \leq \dots = \frac{e^{-\pi n} + e^{-\pi n}}{e^{-\pi n} - e^{-\pi n}} = \frac{2e^{-\pi n}}{0} \rightarrow \infty$ (WRONG!)

NOW ON THE VERTICAL SIDES OF THE contour $z = (\pm 1/2) + iy$, $-n \leq y \leq n$

$|\cot \pi z| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \frac{1}{2} \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \frac{1}{2} \left| \frac{e^{i\pi(\pm 1/2 + iy)} + e^{-i\pi(\pm 1/2 + iy)}}{e^{i\pi(\pm 1/2 + iy)} - e^{-i\pi(\pm 1/2 + iy)}} \right|$

$\therefore \cot \pi z$ IS BOUNDED ON THE contour

Question 5

$$f(z) = \frac{\pi \cot \pi z}{z^4}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90}.$$

proof

CONSIDER $\int_{\Gamma} \frac{\pi \cot \pi z}{z^4} dz$ where Γ is the usual square contour with vertices at $(n+\frac{1}{2})\pi i, (-n+\frac{1}{2})\pi i, (-n+\frac{1}{2})\pi i, (n+\frac{1}{2})\pi i$.

TO EVALUATE THE RESIDUE OF A Pole of order 5 is probably better to get the Laurent expansion

ALSO OBSERVE THAT $\cot \pi z$ IS BOUNDED ON Γ_n IF $|z| \leq n$ (IT IS SIMILAR IF THE VERT. SIDES)

ALSO OBSERVE THAT $|z| \leq n+1/2$ $\forall z \in \Gamma_n$

THE INTEGRAL OVER THE VERTICAL SIDES IS $O(1/n^3) \rightarrow 0$ AS $n \rightarrow \infty$

RETURNING TO THE INTEGRAL AS $n \rightarrow \infty$

BY THE RESIDUE THEOREM

THE INTEGRAL IS $2\pi i \times \sum (\text{Residues inside } \Gamma_n)$

PROOF THAT $\cot \pi z$ IS BOUNDED ON THE SQUARE CONTOUR WITH VERTICES $(n+\frac{1}{2})\pi i, (-n+\frac{1}{2})\pi i, (-n+\frac{1}{2})\pi i, (n+\frac{1}{2})\pi i$

DO THE HORIZONTAL SIDES OF THE CONTOUR $z = x \pm i(n+\frac{1}{2})$, $-n+\frac{1}{2} < x < n+\frac{1}{2}$

THE BOUND IS INCREASING AS n INCREASES TO INFINITY

NOW ON THE VERTICAL SIDES OF THE CONTOUR $z = (n+\frac{1}{2})\pi i + iy$, $-n+\frac{1}{2} \leq y \leq n+\frac{1}{2}$

$\cot \pi z$ IS BOUNDED ON THE CONTOUR (FOR A SUFFICIENTLY LARGE n)

Question 6

$$f(z) = \frac{\pi \cot \pi z}{(z^2 + 1)^2}, z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{1}{(r^2 + 1)^2} = \frac{1}{4} \pi^2 \operatorname{cosech}^2 \pi + \frac{1}{4} \pi \coth \pi - \frac{1}{2}.$$

, proof

CONSIDER $\int_{\Gamma} \frac{f(z)}{dz}$ OVER THE USUAL SQUARE CONTOUR WITH VERTICES $(-N+1/2, N+1/2)$

(a) HAS SAME POLES AT EVERY SQUARE COUSE ON THE x-AXIS (DATE = $\frac{2\pi i n}{2\pi}$)
 (b) HAS DOUBLE POLES AT $\pm i$ ($\frac{2\pi i n}{2\pi}$)

CALCULATE RESIDUES

- $\lim_{z \rightarrow 0} \left[(z-0) f(z) \right] = \lim_{z \rightarrow 0} \left[\frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] = \lim_{z \rightarrow 0} \left[\frac{\pi \frac{\cos \pi z}{\sin \pi z}}{(z^2 + 1)^2} \right] = \frac{\pi \cos \pi z}{(z^2 + 1)^2} \Big|_{z=0} = \frac{\pi \cdot 1}{1} = \pi$
 THIS IS ZERO FOR THE OTHER TWO, SO BY L'HOPITAL'S RULE
- $\lim_{z \rightarrow i} \left[\frac{d}{dz} (z-i)^2 f(z) \right] = \lim_{z \rightarrow i} \left[\frac{d}{dz} \left[\frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] \right]$
 $= \lim_{z \rightarrow i} \left[\frac{\pi \cot \pi z \cdot (-2(z-i))}{(z^2 + 1)^3} + \frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] = \frac{\pi \cot \pi i \cdot (-2(i-i))}{(i^2 + 1)^3} + \frac{\pi \cot \pi i}{(i^2 + 1)^2} = 0 + \frac{\pi \cot \pi i}{0} = \frac{\pi \cot \pi i}{0}$
- $\lim_{z \rightarrow -i} \left[\frac{d}{dz} (z+i)^2 f(z) \right] = \lim_{z \rightarrow -i} \left[\frac{d}{dz} \left[\frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] \right]$
 $= \lim_{z \rightarrow -i} \left[\frac{\pi \cot \pi z \cdot (-2(z+i))}{(z^2 + 1)^3} + \frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] = \frac{\pi \cot \pi (-i) \cdot (-2(-i+i))}{((-i)^2 + 1)^3} + \frac{\pi \cot \pi (-i)}{((-i)^2 + 1)^2} = 0 + \frac{\pi \cot \pi (-i)}{0} = \frac{\pi \cot \pi (-i)}{0}$

$\lim_{z \rightarrow 1} \left[\frac{d}{dz} (z-1) f(z) \right] = \lim_{z \rightarrow 1} \left[\frac{d}{dz} \left[\frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] \right]$
 $= \lim_{z \rightarrow 1} \left[\frac{\pi \cot \pi z \cdot (-2(z-1))}{(z^2 + 1)^3} + \frac{\pi \cot \pi z}{(z^2 + 1)^2} \right] = \frac{\pi \cot \pi \cdot (-2(1-1))}{(1+1)^3} + \frac{\pi \cot \pi}{(1+1)^2} = 0 + \frac{\pi \cot \pi}{4}$

BY THE RESIDUE-THEOREM NOW

$\int_{\Gamma} f(z) dz = 2\pi i \sum (\text{RESIDUES INSIDE } \Gamma)$

$\int_{\Gamma} \frac{\pi \cot \pi z}{(z^2 + 1)^2} dz = 2\pi i \left[\pi \cot \pi z + \frac{1}{2} \cot \pi z + \frac{1}{2} \cot \pi z \right]$

IT CAN BE SHOWN THAT DATE IS BOUNDED ON Γ , I.E. $|\cot \pi z| \leq M$ FOR SOME $M \in \mathbb{R}$ — THIS IS SHOWN AT THE VERY END OF THE QUESTION, AS A SEPARATE PROOF

WE ALSO HAVE THE FOLLOWING:

$|z| > \frac{1}{2}$
 $\frac{1}{|z|} < \frac{1}{1/2} = 2$

$\left| \int_{\Gamma} \frac{\pi \cot \pi z}{(z^2 + 1)^2} dz \right| \leq \int_{\Gamma} \left| \frac{\pi \cot \pi z}{(z^2 + 1)^2} \right| |dz| = \int_{\Gamma} \frac{\pi M}{(|z|^2 + 1)^2} |dz|$

$\frac{1}{(|z|^2 + 1)^2} \leq \frac{1}{(1/4 + 1)^2} = \frac{1}{5/4} = \frac{4}{5}$

$\int_{\Gamma} \frac{\pi M}{(|z|^2 + 1)^2} |dz| \leq \frac{\pi M}{4} \int_{\Gamma} |dz|$

LENGTH OF THE COUSE Γ IS $4(N+1/2)$

$\leq \frac{\pi M}{4} \cdot 4(N+1/2) = \pi M(N+1/2) = O(N)$

RETURNING TO THE INTEGRAL AS $N \rightarrow \infty$

$\rightarrow 0 = 2\pi i \left[\pi \cot \pi z + \frac{1}{2} \cot \pi z + \frac{1}{2} \cot \pi z \right]$

$\rightarrow \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^2} = \frac{1}{4} \pi^2 \operatorname{cosech}^2 \pi + \frac{1}{4} \pi \coth \pi$

$\therefore \sum_{n=1}^{\infty} \frac{1}{(n^2 + 1)^2} = \frac{1}{4} \pi^2 \operatorname{cosech}^2 \pi + \frac{1}{4} \pi \coth \pi - \frac{1}{2}$

PROOF THAT DATE IS BOUNDED ON THE SQUARE COUSUR WITH VERTICES AT $(n+1/2, n+1/2)$, $n=1, 2, 3, \dots$

$\left| \cot \pi z \right| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$ ($|1|=1$)

$\frac{|e^{i\pi z} + e^{-i\pi z}|}{|e^{i\pi z} - e^{-i\pi z}|} \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{|e^{i\pi z} - e^{-i\pi z}|}$

ON THE HORIZONTAL SIDES OF THE COUSUR, $z = x + iy$, $0 < x < n+1/2$, $y = n+1/2$

- $|e^{i\pi z}| = |e^{i\pi(x+iy)}| = |e^{i\pi x} e^{-\pi y}| = e^{-\pi y}$
- $|e^{-i\pi z}| = \dots$ ANALOGOUS REASONING

$\therefore \left| \cot \pi z \right| \leq \frac{e^{-\pi y} + e^{\pi y}}{|e^{i\pi z} - e^{-i\pi z}|} = \frac{2 \cosh \pi y}{|2i \sinh \pi y|} = \frac{\cosh \pi y}{\sinh \pi y} = \coth \pi y$

AS $y = n+1/2$ IS A DECREASING FUNCTION

$\coth \pi y > \coth \pi > \coth \pi$

SO THIS YET, ONLY π IS MAX

NEXT ON THE VERTICAL SIDES OF THE COUSUR $z = x + iy$, $x = n+1/2$, $0 \leq y \leq n+1/2$

$\left| \cot \pi z \right| = \left| \frac{\cos \pi z}{\sin \pi z} \right| = \left| \frac{e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}}{e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}} \right|$

$\frac{e^{i\pi z} + 1}{e^{i\pi z} - 1} = \frac{e^{i\pi(x+iy)} + 1}{e^{i\pi(x+iy)} - 1}$

$= \frac{e^{i\pi x} e^{-\pi y} + 1}{e^{i\pi x} e^{-\pi y} - 1}$

$= \frac{e^{i\pi(x+iy)} + 1}{e^{i\pi(x+iy)} - 1}$

$= \frac{e^{i\pi x} e^{-\pi y} + 1}{e^{i\pi x} e^{-\pi y} - 1}$

$= \frac{e^{i\pi x} e^{-\pi y} + 1}{e^{i\pi x} e^{-\pi y} - 1}$

$= \left| \frac{e^{i\pi x} e^{-\pi y} + 1}{e^{i\pi x} e^{-\pi y} - 1} \right|$

$= \left| \frac{e^{-\pi y} + e^{i\pi x}}{e^{-\pi y} - e^{i\pi x}} \right|$

$= \left| \frac{e^{-\pi y} + 1}{e^{-\pi y} - 1} \right|$

$\therefore \cot \pi z$ IS BOUNDED ON THE COUSUR

"HORIZONTAL" BY $\coth \pi$ } IF M (AND m) IS THE SUPRENUM OF THESE TWO "VERTICAL" BY 1 }
 M IS SUP (WHATEVER)

Question 7

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(a+r)^2} = \pi^2 \operatorname{cosec}(\pi a) \cot(\pi a), \quad a \notin \mathbb{Z}.$$

proof

Left Column Notes:

- Since simple poles at $z = n + \frac{1}{2}$, we consider the integral of $f(z) = \frac{\pi \operatorname{cosec} \pi z}{(a+z)^2}$ over the contour Γ for $n \in \mathbb{N}$, i.e. the square contour with vertices at $(n+\frac{1}{2})(1+i)$, $(n+\frac{1}{2})(1-i)$, $(n+\frac{1}{2})(-1-i)$, $(n+\frac{1}{2})(-1+i)$.
- Calculate residues: $\lim_{z \rightarrow -a} (z+a)^2 f(z) = \lim_{z \rightarrow -a} \left[\frac{\pi \operatorname{cosec} \pi z}{(z+a)^2} \right] = \frac{\pi \operatorname{cosec}(\pi(-a))}{(-a)^2} = -\pi^2 \operatorname{cosec}(\pi a) \cot(\pi a)$.
- Next the simple poles at $z = n + \frac{1}{2}$, say n : $\lim_{z \rightarrow n+\frac{1}{2}} (z - n - \frac{1}{2}) f(z) = \lim_{z \rightarrow n+\frac{1}{2}} \left[\frac{\pi \operatorname{cosec} \pi z}{(z+a)^2} \right] = \frac{\pi \operatorname{cosec}(\pi(n+\frac{1}{2}))}{(n+\frac{1}{2}+a)^2} = \frac{\pi (-1)^n}{(n+\frac{1}{2}+a)^2}$.
- By the Residue Theorem: $\int_{\Gamma} f(z) dz = 2\pi i \times \sum \text{Residues inside } \Gamma$.

Right Column Notes:

- Now $\operatorname{cosec} \pi z$ is bounded on Γ , i.e. $|\operatorname{cosec} \pi z| \leq M$.
- $z > \frac{1}{2}$ on Γ : $\frac{1}{2} \leq z \leq n + \frac{1}{2}$. $|\operatorname{cosec} \pi z| \leq \frac{1}{|\sin \pi z|} \leq \frac{1}{|\sin \pi \frac{1}{2}|} = 1$.
- As $n \rightarrow \infty$, the integral over the top and bottom sides of the contour goes to zero.
- Final result: $\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(a+r)^2} = \pi^2 \operatorname{cosec}(\pi a) \cot(\pi a)$.

PROOF THAT COSECANT IS BOUNDED ON THE SQUARE CONTOUR WITH VERTICES AT $(n+\frac{1}{2})(1+i)$, $(n+\frac{1}{2})(1-i)$, ...

- $|\operatorname{cosec} \pi z| = \left| \frac{1}{\sin \pi z} \right| = \left| \frac{1}{\frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})} \right| = \frac{2}{|e^{i\pi z} - e^{-i\pi z}|} \leq \frac{2}{|e^{i\pi z}| - |e^{-i\pi z}|}$.
- Now on the horizontal segments $z = x + iy$, $-y < y < y + \frac{1}{2}$: $|e^{i\pi z}| = |e^{i\pi x - \pi y}| = e^{-\pi y} |e^{i\pi x}| = e^{-\pi y}$. $|e^{-i\pi z}| = |e^{-i\pi x + \pi y}| = e^{\pi y} |e^{-i\pi x}| = e^{\pi y}$.
- $\therefore |\operatorname{cosec} \pi z| \leq \frac{2}{e^{-\pi y} - e^{\pi y}} = \frac{2}{2 \cosh(\pi y)} = \frac{1}{\cosh(\pi y)} \leq 1$.
- On the vertical segments $z = (n+\frac{1}{2}) + iy$, $-y < y < y + \frac{1}{2}$: $|e^{i\pi z}| = |e^{i\pi(n+\frac{1}{2}) - \pi y}| = e^{-\pi y} |e^{i\pi(n+\frac{1}{2})}| = e^{-\pi y}$. $|e^{-i\pi z}| = |e^{-i\pi(n+\frac{1}{2}) + \pi y}| = e^{\pi y} |e^{-i\pi(n+\frac{1}{2})}| = e^{\pi y}$.
- $\therefore |\operatorname{cosec} \pi z| \leq \frac{2}{e^{-\pi y} - e^{\pi y}} = \frac{1}{\cosh(\pi y)} \leq 1$.

Question 8

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{(2z+1)(3z+1)}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)(3r+1)} = \frac{\pi}{3}(2\sqrt{3}-3).$$

 , proof

The handwritten solution is divided into several sections:

- Problem Statement:** As this special function is given we consider the integral of $f(z) = \frac{\pi \operatorname{cosec} \pi z}{(2z+1)(3z+1)}$ around the square contour Γ , with vertices $(n+\frac{1}{2})(-i)$, $(n+\frac{1}{2})$, $(n+\frac{1}{2})i$, and $(n+\frac{1}{2})(-i)$.
- Contour Diagram:** A diagram of a square contour Γ in the complex plane. The vertices are labeled as $(n+\frac{1}{2})(-i)$, $(n+\frac{1}{2})$, $(n+\frac{1}{2})i$, and $(n+\frac{1}{2})(-i)$. The real axis is marked with $z = (n+\frac{1}{2})$ and $z = (n+\frac{1}{2}) - i$. The imaginary axis is marked with $z = (n+\frac{1}{2})i$ and $z = (n+\frac{1}{2})(-i)$.
- Residue Calculations:**
 - At $z = -\frac{1}{2}$: $\operatorname{Res}(f, -\frac{1}{2}) = \lim_{z \rightarrow -\frac{1}{2}} (z + \frac{1}{2}) f(z) = \frac{\pi \operatorname{cosec} \pi z}{3z+1} \Big|_{z=-\frac{1}{2}} = \frac{\pi \operatorname{cosec}(-\frac{\pi}{2})}{-\frac{1}{2}}$
 - At $z = -\frac{1}{3}$: $\operatorname{Res}(f, -\frac{1}{3}) = \lim_{z \rightarrow -\frac{1}{3}} (z + \frac{1}{3}) f(z) = \frac{\pi \operatorname{cosec} \pi z}{2z+1} \Big|_{z=-\frac{1}{3}} = \frac{\pi \operatorname{cosec}(-\frac{\pi}{3})}{-\frac{1}{3}}$
- Residue Theorem:** $\int_{\Gamma} f(z) dz = 2\pi i \sum \operatorname{Residues} = 2\pi i \left(\frac{\pi \operatorname{cosec}(-\frac{\pi}{2})}{-\frac{1}{2}} + \frac{\pi \operatorname{cosec}(-\frac{\pi}{3})}{-\frac{1}{3}} \right)$
- Final Result:** $\sum_{r=-\infty}^{\infty} \frac{(-1)^r}{(2r+1)(3r+1)} = \frac{\pi}{3}(2\sqrt{3}-3)$

Question 9

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{4r^2 - 1} = \frac{1}{4}(2 - \pi).$$

proof

Handwritten notes detailing the proof. The notes include a diagram of a square contour in the complex plane with vertices at $(n+\frac{1}{2}) \pm i\epsilon$ and $(n+\frac{1}{2}) \mp i\epsilon$. It details the residue calculation for poles at $z = \frac{1}{2n}$ and $z = \frac{1}{2n+1}$, and uses the residue theorem to relate the integral to the sum of residues. The final result is derived as $\frac{1}{4}(2 - \pi)$.

Handwritten notes showing a detailed proof of the bound on the cosecant function. It includes a diagram of a square contour in the complex plane and a table of residues for the function $f(z) = \frac{\pi \operatorname{cosec} \pi z}{4z^2 - 1}$.

Location	Residue
$z = \frac{1}{2n}$	$\frac{(-1)^n}{2n}$
$z = \frac{1}{2n+1}$	$\frac{(-1)^{n+1}}{2n+1}$

Question 10

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^2}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{r^2} = -\frac{1}{12} \pi^2.$$

proof

The handwritten notes are divided into three sections:

- Left Section:** Discusses the contour integral $\int_{\Gamma} \frac{\pi \operatorname{cosec} \pi z}{z^2} dz$ where Γ is a square contour with vertices at $(n+\frac{1}{2})\pi i$, $(n+\frac{1}{2})\pi i + 2\pi i$, $(n+\frac{1}{2})\pi i - 2\pi i$, and $(n+\frac{1}{2})\pi i - 2\pi i - 2\pi i$. It notes that the contour encloses poles of $\operatorname{cosec} \pi z$ at $z = n\pi$ and $z = (n+1)\pi$. The residue at $z = n\pi$ is $\frac{1}{\pi} (-1)^n$. The integral is then related to the sum $\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(n+k\pi)^2}$.
- Middle Section:** Shows the limit $\lim_{n \rightarrow \infty} \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{\pi \operatorname{cosec} \pi z}{z^2} dz \right] = \lim_{n \rightarrow \infty} \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{\pi \operatorname{cosec} \pi z}{z^2} dz \right] = \frac{1}{2\pi i} \int_{\Gamma} \frac{\pi \operatorname{cosec} \pi z}{z^2} dz$. It uses the residue theorem to equate this to the sum of residues inside the contour, which is $\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(n+k\pi)^2}$.
- Right Section:** Proves that $\operatorname{cosec} \pi z$ is bounded on the contour Γ . It shows that $|\operatorname{cosec} \pi z| \leq M$ for some M on Γ . It also shows that the integral $\int_{\Gamma} \frac{\pi \operatorname{cosec} \pi z}{z^2} dz \rightarrow 0$ as $n \rightarrow \infty$. This leads to the equation $\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(n+k\pi)^2} = 0$.

The handwritten notes are divided into two sections:

- Top Section:** Proves that $\operatorname{cosec} \pi z$ is bounded on the square contour Γ with vertices at $(n+\frac{1}{2})\pi i$, $(n+\frac{1}{2})\pi i + 2\pi i$, $(n+\frac{1}{2})\pi i - 2\pi i$, and $(n+\frac{1}{2})\pi i - 2\pi i - 2\pi i$. It shows that $|\operatorname{cosec} \pi z| \leq \frac{2}{|2\pi i - 2\pi i|} = 1$ on the contour.
- Bottom Section:** Shows that $\operatorname{cosec} \pi z$ is bounded on the contour Γ . It shows that $|\operatorname{cosec} \pi z| \leq \frac{2}{|2\pi i - 2\pi i|} = 1$ on the contour.

Question 11

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{z^4}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^4} = \frac{7\pi^4}{720}.$$

proof

The handwritten proof is divided into several sections:

- Top Left:** A diagram of a square contour Γ in the complex plane with vertices at $(-n+1/2), (n+1/2), (n+1/2), (-n+1/2)$ on the real axis. The contour is traversed counter-clockwise. Poles are marked at $z=0$ and $z=1/n$.
- Top Middle:** A calculation of the residue at $z=0$. It uses the Laurent expansion of $\operatorname{cosec} \pi z$ around $z=0$.

$$\operatorname{cosec} \pi z = \frac{1}{\pi z} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} (\pi z)^{2k-1}$$
 The residue at $z=0$ is identified as $\frac{1}{\pi}$.
- Top Right:** A diagram showing the contour Γ and the location of poles at $z = \frac{1}{n}$. It notes that $\operatorname{cosec} \pi z$ is bounded on Γ for $n \in \mathbb{N}$.
- Middle Left:** A calculation of the residue at $z = \frac{1}{n}$. It uses the expansion $\operatorname{cosec} \pi z = \frac{1}{\pi(z - 1/n)} + O(1)$ and finds the residue to be $(-1)^{n+1} \pi$.
- Middle Right:** An application of the residue theorem:

$$\int_{\Gamma} f(z) dz = 2\pi i \left(\frac{1}{\pi} + (-1)^{n+1} \pi \right) = 2(-1)^{n+1} (1 - n^2)$$
 It then shows that the integral over the contour goes to zero as $n \rightarrow \infty$.
- Bottom:** A detailed proof that $\operatorname{cosec} \pi z$ is bounded on the contour. It shows that on the horizontal segments, $|\operatorname{cosec} \pi z| \leq \frac{1}{|\sin \pi x|}$, and on the vertical segments, $|\operatorname{cosec} \pi z| \leq \frac{1}{|\cos \pi y|}$. Both are bounded by $\frac{1}{\sin \frac{\pi}{2n}}$.

Question 12

$$f(z) = \frac{\pi \tan \pi z}{z^4}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^4} = \frac{\pi^4}{96}.$$

 , proof

As the things go up in the denominator, we consider the interval of $f(z) = \frac{\pi \tan \pi z}{z^4}$ over the square contour Γ , with vertices at the points $n(i+1), n \in \mathbb{Z}$.

$f(z)$ has simple poles at the "half integer values" because of $\tan z = \frac{\sin z}{\cos z}$.

$f(z)$ has a triple pole at the origin because of the z^4 (denominator term here).

CAUTION! RESIDUES

- $\lim_{z \rightarrow n(i+1)} [z - n(i+1)] f(z) = \lim_{z \rightarrow n(i+1)} \left[\frac{\pi \cos \pi z}{\sin \pi z} \right]_{z=n(i+1)}$
- $\lim_{z \rightarrow n(i+1)} \left[\frac{\pi \cos \pi z}{\sin \pi z} \right]_{z=n(i+1)}$... THE INDEFINITE FORM "0/0" MORE THAN ONE TIME SO BY L'HOPITAL'S RULE
- $\lim_{z \rightarrow n(i+1)} \left[\frac{\pi \cos \pi z + \pi^2 \sin \pi z}{-\pi \cos \pi z} \right]_{z=n(i+1)} = \frac{\pi \cos \pi n(i+1) + \pi^2 \sin \pi n(i+1)}{-\pi \cos \pi n(i+1)} = \frac{1}{\cos \pi n(i+1)}$

TO FIND THE RESIDUE OF THE TRIPLE POLE AT THE ORIGIN IS DEFINED GET IT FROM THE LARGEST EXPANSION OF $f(z)$

- $g(z) = \tan z$
- $g'(z) = \sec^2 z = 1 + \tan^2 z = 1 + g^2$
- $g''(z) = 2g g'$
- $g'''(z) = 2g'(g')^2 + 2g g''$

where $z = \frac{1}{2} \pi + o(z)$
 where $z = \frac{1}{2} \pi + o(z)$

$f(z) = \frac{\pi \tan \pi z}{z^4} = \frac{\pi}{z^4} \left[\tan z + \frac{1}{3} \tan^3 z + o(\tan^5 z) \right]$
 $\therefore \text{Res}(f, 0) = \frac{\pi}{3}$

NOW BY THE RESIDUE THEOREM

$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz = 2\pi i \left[-\frac{\pi^2}{3} \left(\frac{1}{2} \right)^4 + \frac{\pi^2}{3} \right]$

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz = 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right]$

IT WOULD BEWILD AT THE END OF THE QUESTION THAT $f(z)$ IS BOUNDED ON Γ , LE TALKING TO M FOR SOME REASON.

TRICKY USE THIS:

$|z| > n$ (one dimension)

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz < \int_{\Gamma} \frac{\pi |\tan \pi z|}{|z|^4} |dz| < \int_{\Gamma} \frac{\pi e^{-\cos \pi z}}{|z|^4} |dz|$

$= \frac{\pi}{n^4} \int_{\Gamma} |dz| = \frac{\pi}{n^4} \times 4n = O\left(\frac{1}{n^3}\right) \rightarrow 0$ as $n \rightarrow \infty$

TRICKY RELATIONS TO THE ORIGINAL AS $n \rightarrow \infty$

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz = 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right]$

$0 = 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right] - 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right]$ (CHANGE SIGN) (MAYBE IS R)

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz = 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right]$

$\int_{\Gamma} \frac{\pi \tan \pi z}{z^4} dz = 2\pi i \left[\frac{\pi^2}{3} - \frac{\pi^2}{24} \right]$

AS REQUIRED

PROOF THAT $f(z)$ IS BOUNDED ON THE SQUARE OUTSIDE Γ , WITH VERTICES AT $n(i+1), n=1,2,3,4, \dots$

$|f(z)| = \left| \frac{\pi \tan \pi z}{z^4} \right| = \frac{\pi |e^{i\pi z} - e^{-i\pi z}|}{|z|^4 |e^{i\pi z} + e^{-i\pi z}|}$

NUMERATOR
 $|e^{i\pi z} - e^{-i\pi z}| = |e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}| = |e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}|$

NUMERATOR
 $|e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}| \leq |e^{i\pi x} e^{-\pi y}| + |e^{-i\pi x} e^{\pi y}| = e^{-\pi y} + e^{\pi y}$

DENOMINATOR
 $|e^{i\pi z} + e^{-i\pi z}| = |e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}| = |e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}|$

$|e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}| \geq |e^{-\pi y} - e^{\pi y}| = e^{-\pi y} - e^{\pi y}$

ON THE "HORIZONTAL" SECTIONS OF THE OUTSIDE, $z = x+iy, -x < x < n$

- $|e^{i\pi z}| = |e^{i\pi(x+iy)}| = |e^{i\pi x} e^{-\pi y}| = e^{-\pi y}$
- $|e^{-i\pi z}| = |e^{-i\pi(x+iy)}| = |e^{-i\pi x} e^{\pi y}| = e^{\pi y}$

$\frac{|e^{i\pi z} - e^{-i\pi z}|}{|e^{i\pi z} + e^{-i\pi z}|} = \frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}} = \frac{2 \cosh(\pi y)}{2 \sinh(\pi y)} = \coth(\pi y)$

$= \frac{2 \cosh(\pi y)}{2 \sinh(\pi y)} = \coth(\pi y) < \coth(\pi)$

AS $y = \cosh x$ IS A DECREASING FUNCTION THE GREATEST VALUE OF $\coth(\pi y)$ OCCURS WHEN $y=1$, IS $\coth(\pi)$

TRICKY USE THIS:

ON THE "VERTICAL" SECTIONS OF THE OUTSIDE, $z = n+iy, -x < x < n$

$|f(z)| = \left| \frac{\pi \tan \pi z}{z^4} \right| = \frac{\pi |e^{i\pi z} - e^{-i\pi z}|}{|z|^4 |e^{i\pi z} + e^{-i\pi z}|}$

MULTIPLY TOP AND BOTTOM BY $e^{i\pi z}$

$= \frac{\pi |e^{2i\pi z} - 1|}{|z|^4 |e^{2i\pi z} + 1|} = \frac{\pi |e^{2i\pi(n+iy)} - 1|}{|z|^4 |e^{2i\pi(n+iy)} + 1|}$

$= \frac{\pi |e^{2i\pi n} e^{-2\pi y} - 1|}{|z|^4 |e^{2i\pi n} e^{-2\pi y} + 1|} = \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|}$

$= \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|} = \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|}$

$= \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|} = \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|}$

$= \frac{\pi |e^{-2\pi y} - 1|}{|z|^4 |e^{-2\pi y} + 1|} < 1$

"DEPENDENT" $f(z)$ IS BOUNDED BY $\coth(\pi)$

"INDEPENDENT" $f(z)$ IS BOUNDED BY 1 $\implies M$ (SOME VALUE) IS THE SUPRENUM OF THESE TWO

$\therefore f(z)$ IS BOUNDED ON THE OUTSIDE

Question 13

$$f(z) = \frac{\pi \sec \pi z}{z^3}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ , show that

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^3} = \frac{\pi^3}{32}.$$

 , proof

As the terms go up in this in the denominator, the terms alternate we consider the integral of $f(z) = \frac{\pi \sec \pi z}{z^3}$, over the square contour Γ with vertices at $\pm(n+1/2) \pm i\epsilon$.

Now on the square poles at half integer values because of the secant $\frac{1}{\cos \pi z}$.
 (i) has a zero pole at the origin because of the z^3 .
 Calculate residues at these points.

• To get the residue of the three pole at the origin we use the Laurent expansion
 $f(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3} \left[\frac{1}{\cos \pi z} \right] = \frac{\pi}{z^3} \left[1 - \frac{\pi^2 z^2}{6} + O(z^4) \right]^{-1}$
 $= \frac{\pi}{z^3} \left[1 + \frac{\pi^2 z^2}{6} + O(z^4) \right] = \dots + \frac{\pi^3}{24} z + \dots$ (Residue $\frac{\pi^3}{24}$)

Now on the residue theorem we have
 $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{Residues of } f \text{ inside } \Gamma)$
 $\int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^n \frac{\pi \sec \pi k}{(2k+1)^3} \right]$

It will be shown at the end of the question that the contour integral goes to 0 as $n \rightarrow \infty$, i.e. $\lim_{n \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$ using the fact we think $|z| \geq n$ and $\frac{1}{|z|} \leq \frac{1}{n}$.

$\left| \int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz \right| \leq \int_{\Gamma} \frac{|\pi \sec \pi z|}{|z|^3} |dz| \leq \int_{\Gamma} \frac{\pi M}{n^3} |dz|$
 $= \frac{\pi M}{n^3} \times 4n = \frac{4\pi M}{n^2} \rightarrow 0$ as $n \rightarrow \infty$

Finally returning to the residue theorem as $n \rightarrow \infty$
 $\Rightarrow \int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^{\infty} \frac{\pi \sec \pi k}{(2k+1)^3} \right]$
 $\Rightarrow 0 = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^{\infty} \frac{\pi \sec \pi k}{(2k+1)^3} \right]$

• To get the residue of the three pole at the origin we use the Laurent expansion
 $f(z) = \frac{\pi \sec \pi z}{z^3} = \frac{\pi}{z^3} \left[\frac{1}{\cos \pi z} \right] = \frac{\pi}{z^3} \left[1 - \frac{\pi^2 z^2}{6} + O(z^4) \right]^{-1}$
 $= \frac{\pi}{z^3} \left[1 + \frac{\pi^2 z^2}{6} + O(z^4) \right] = \dots + \frac{\pi^3}{24} z + \dots$ (Residue $\frac{\pi^3}{24}$)

Now on the residue theorem we have
 $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{Residues of } f \text{ inside } \Gamma)$
 $\int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^n \frac{\pi \sec \pi k}{(2k+1)^3} \right]$

It will be shown at the end of the question that the contour integral goes to 0 as $n \rightarrow \infty$, i.e. $\lim_{n \rightarrow \infty} \int_{\Gamma} f(z) dz = 0$ using the fact we think $|z| \geq n$ and $\frac{1}{|z|} \leq \frac{1}{n}$.

$\left| \int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz \right| \leq \int_{\Gamma} \frac{|\pi \sec \pi z|}{|z|^3} |dz| \leq \int_{\Gamma} \frac{\pi M}{n^3} |dz|$
 $= \frac{\pi M}{n^3} \times 4n = \frac{4\pi M}{n^2} \rightarrow 0$ as $n \rightarrow \infty$

Finally returning to the residue theorem as $n \rightarrow \infty$
 $\Rightarrow \int_{\Gamma} \frac{\pi \sec \pi z}{z^3} dz = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^{\infty} \frac{\pi \sec \pi k}{(2k+1)^3} \right]$
 $\Rightarrow 0 = 2\pi i \left[\frac{\pi^3}{24} + \sum_{k=1}^{\infty} \frac{\pi \sec \pi k}{(2k+1)^3} \right]$

$\Rightarrow \sum_{k=1}^{\infty} \frac{\pi \sec \pi k}{(2k+1)^3} = -\frac{\pi^3}{24}$
 $\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k \pi}{(2k+1)^3} = -\frac{\pi^3}{24}$
 $\Rightarrow \dots - \frac{\pi}{24} + \frac{\pi}{16} - \frac{\pi}{8} + 1 + \frac{\pi}{8} - \frac{\pi}{16} + \frac{\pi}{24} + \dots = \frac{\pi^3}{16}$
 $\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k \pi}{(2k+1)^3} = \frac{\pi^3}{32}$ or $\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}$

Proof that sec πz is bounded on the square contour Γ with vertices at $n \pm i\epsilon$, $n+1 \pm i\epsilon$.

$|\sec \pi z| = \left| \frac{1}{\cos \pi z} \right| = \left| \frac{1}{\frac{1}{2}(e^{i\pi z} + e^{-i\pi z})} \right| = \frac{2}{|e^{i\pi z} + e^{-i\pi z}|} \leq \frac{2}{|e^{i\pi z}| - |e^{-i\pi z}|}$

Now on the "horizontal" sections of the contour, $z = x \pm i\epsilon$, $-\epsilon < \epsilon < \epsilon$

- $|e^{i\pi z}| = |e^{i\pi(x \pm i\epsilon)}| = |e^{i\pi x} e^{\mp \pi \epsilon}| = e^{\mp \pi \epsilon}$
- $|e^{-i\pi z}| = |e^{-i\pi(x \pm i\epsilon)}| = |e^{-i\pi x} e^{\pm \pi \epsilon}| = e^{\pm \pi \epsilon}$

$\therefore \frac{2}{|e^{i\pi z}| - |e^{-i\pi z}|} = \frac{2}{|e^{\mp \pi \epsilon} - e^{\pm \pi \epsilon}|} = \frac{2}{2\sinh(\pi \epsilon)} = \frac{1}{\sinh(\pi \epsilon)}$

$\therefore \cosh(\pi \epsilon) \leq \cosh(\pi \epsilon)$ (L.H.S. \geq R.H.S.)

Now on the "vertical" sections of the contour, $z = \pm n + iy$, $-\epsilon < y < \epsilon$

$|\sec \pi z| = \left| \frac{1}{\cos \pi z} \right| = \frac{1}{\left| \frac{1}{2}(e^{i\pi z} + e^{-i\pi z}) \right|} = \frac{2}{|e^{i\pi z} + e^{-i\pi z}|}$
 $= \frac{2}{|e^{i\pi(\pm n + iy)} + e^{-i\pi(\pm n + iy)}|} = \frac{2}{|e^{\pm i\pi n} e^{\mp \pi y} + e^{\mp i\pi n} e^{\pm \pi y}|}$

n	1	2	3
$e^{i\pi n}$	$e^{i\pi} = -1$	$e^{2i\pi} = 1$	$e^{3i\pi} = -1$
$e^{-i\pi n}$	$e^{-i\pi} = -1$	$e^{-2i\pi} = 1$	$e^{-3i\pi} = -1$

$\therefore \frac{2}{|(-1)e^{\mp \pi y} + (-1)e^{\pm \pi y}|} = \frac{2}{|e^{\mp \pi y} + e^{\pm \pi y}|} = \frac{2}{2\cosh(\pi y)}$
 $= \frac{1}{\cosh(\pi y)} \leq 1$

"Horizontally", sec πz is bounded by $\cosh(\pi \epsilon)$ \Rightarrow M (max outside) is the maximum of these two values
 "Vertically", sec πz is bounded by 1

$\therefore \sec \pi z$ is bounded on Γ