RESIDUES and INTEGRATION

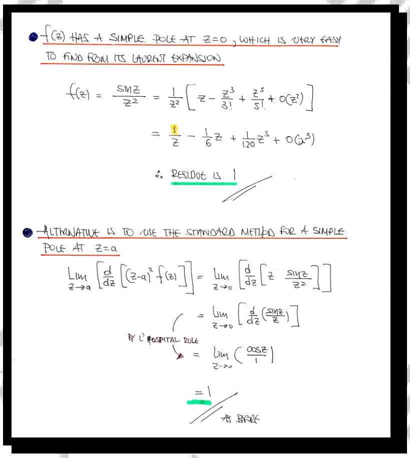
CALCULATIONS OF RESIDUES

Question 1

$$f(z) \equiv \frac{\sin z}{z^2}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=1}$

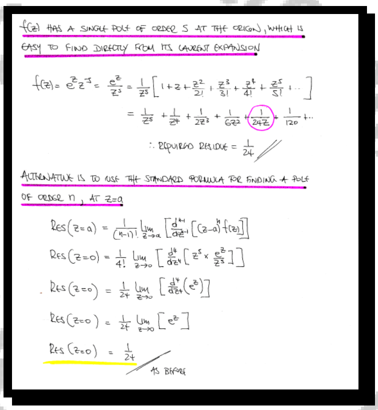


Question 2

$$f(z) \equiv e^z z^{-5}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{}$, $\boxed{res(z=0)=\frac{1}{24}}$

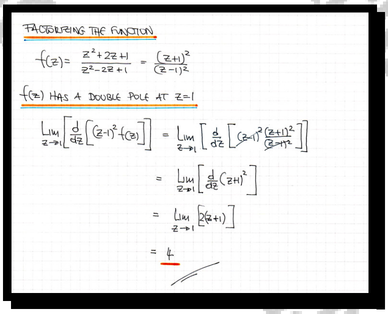


Question 3

$$f(z) \equiv \frac{z^2+2z+1}{z^2-2z+1}, \; z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$res(z=1)=4$

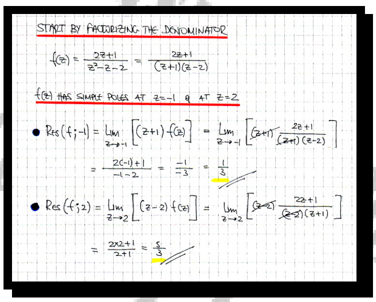


Question 4

$$f(z) \equiv \frac{2z+1}{z^2-z-2}, \; z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$res(z=2)=\frac{5}{3}$, $res(z=-1)=\frac{1}{3}$

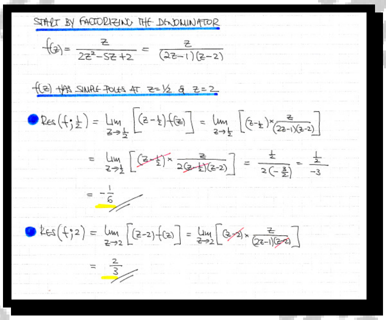


Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{res\left(z = \frac{1}{2}\right) = -\frac{1}{6}}$, $\boxed{res(z = 2) = \frac{2}{3}}$


START BY FACTORIZING THE DENOMINATOR

$f(z) = \frac{z}{2z^2-5z+2} = \frac{z}{(2z-1)(z-2)}$

$f(z)$ HAS SIMPLE POLES AT $z = \frac{1}{2}$ & $z = 2$

- $\text{Res}(f; \tfrac{1}{2}) = \lim_{z \to \frac{1}{2}} \left[(z-\tfrac{1}{2})f(z)\right] = \lim_{z \to \frac{1}{2}} \left[(z-\tfrac{1}{2}) \times \frac{z}{(2z-1)(z-2)}\right]$

  $= \lim_{z \to \frac{1}{2}} \left[(z-\tfrac{1}{2}) \times \frac{z}{2(z-\frac{1}{2})(z-2)}\right] = \frac{\frac{1}{2}}{2(-\frac{3}{2})} = \frac{\frac{1}{2}}{-3}$

  $= -\frac{1}{6}$

- $\text{Res}(f; 2) = \lim_{z \to 2} \left[(z-2)f(z)\right] = \lim_{z \to 2} \left[(z-2) \times \frac{z}{(2z-1)(z-2)}\right]$

  $= \frac{2}{3}$

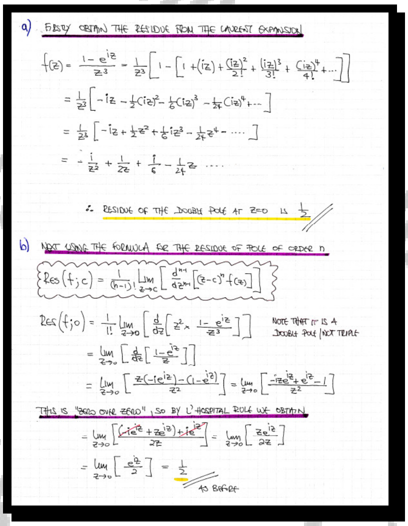
Question 6

$$f(z) \equiv \frac{1-e^{iz}}{z^3}, \quad z \in \mathbb{C}.$$

a) Find the first four terms in the Laurent expansion of $f(z)$ and hence state the residue of the pole of $f(z)$.

b) Determine the residue of the pole of $f(z)$ by an alternative method

$\boxed{res(z=0) = \frac{1}{2}}$


a) FIRSTLY OBTAIN THE RESIDUE FROM THE LAURENT EXPANSION

$$f(z) = \frac{1-e^{iz}}{z^3} = \frac{1}{z^3}\left[1-\left[1+(iz)+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\frac{(iz)^4}{4!}+\dots\right]\right]$$

$$= \frac{1}{z^3}\left[-iz-\frac{1}{2}(iz)^2-\frac{1}{6}(iz)^3-\frac{1}{24}(iz)^4+\dots\right]$$

$$= \frac{1}{z^3}\left[-iz+\frac{1}{2}z^2+\frac{1}{6}iz^3-\frac{1}{24}z^4-\dots\right]$$

$$= -\frac{i}{z^2}+\frac{1}{2z}+\frac{i}{6}-\frac{1}{24}z \dots$$

∴ RESIDUE OF THE DOUBLE POLE AT $z=0$ IS $\frac{1}{2}$

b) NEXT USING THE FORMULA FOR THE RESIDUE OF POLE OF ORDER n

$$\text{Res}(f;c) = \frac{1}{(n-1)!}\lim_{z\to c}\left[\frac{d^{n-1}}{dz^{n-1}}\left[(z-c)^n f(z)\right]\right]$$

$$\text{Res}(f;0) = \frac{1}{1!}\lim_{z\to 0}\left[\frac{d}{dz}\left[z^2 \times \frac{1-e^{iz}}{z^3}\right]\right]$$

NOTE THAT IT IS A DOUBLE POLE / NOT TRIPLE

$$= \lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{1-e^{iz}}{z}\right]\right]$$

$$= \lim_{z\to 0}\left[\frac{z(-ie^{iz})-(1-e^{iz})}{z^2}\right] = \lim_{z\to 0}\left[\frac{-ize^{iz}+e^{iz}-1}{z^2}\right]$$

THIS IS "ZERO OVER ZERO", SO BY L'HOSPITAL RULE WE OBTAIN

$$= \lim_{z\to 0}\left[\frac{(-ie^{iz}+ze^{iz})+ie^{iz}}{2z}\right] = \lim_{z\to 0}\left[\frac{ze^{iz}}{2z}\right]$$

$$= \lim_{z\to 0}\left[\frac{e^{iz}}{2}\right] = \frac{1}{2}$$

AS BEFORE

Question 7

$$f(z) \equiv \frac{z^2+4}{z^3+2z^2+2z}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

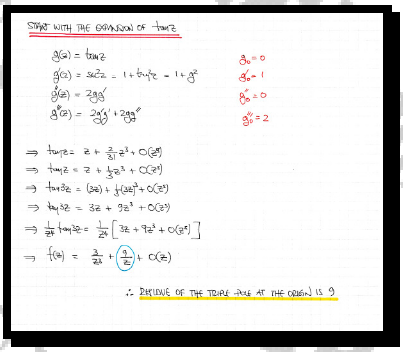
$\boxed{res(z=0)=2}$, $\boxed{res(z=-1+\mathrm{i})=\frac{1}{2}(-1+3\mathrm{i})}$, $\boxed{res(z=-1-\mathrm{i})=-\frac{1}{2}(1+3\mathrm{i})}$

Question 8

$$f(z) \equiv \frac{\tan 3z}{z^4}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$.

$\boxed{res(z=0)=9}$

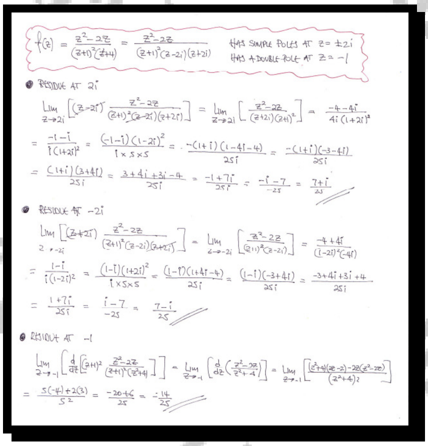


Question 9

$$f(z) \equiv \frac{z^2 - 2z}{(z^2+4)(z+1)^2}, \ z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res(z = 2\text{i}) = \frac{1}{25}(7+\text{i})}, \ \boxed{res(z = -2\text{i}) = \frac{1}{25}(7-\text{i})}, \ \boxed{res(z = -1) = -\frac{14}{25}}$$

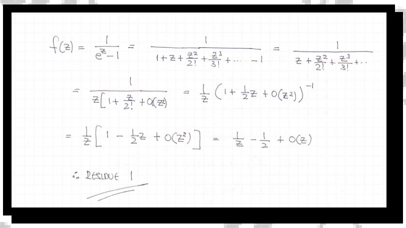


Question 10

$$f(z) \equiv \frac{1}{e^z - 1}, \ z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$, at the origin.

$$\boxed{res(z = 0) = 1}$$

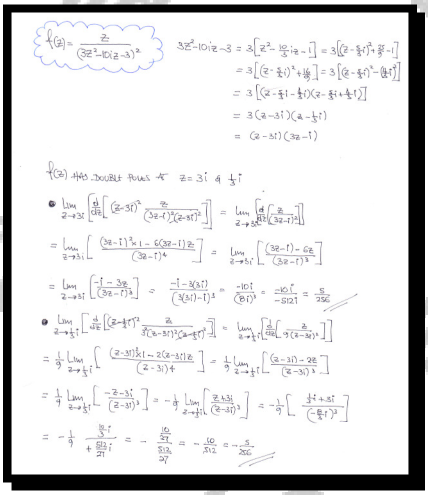


Question 11

$$f(z) \equiv \frac{z}{\left(3z^2 - 10\mathrm{i}\,z - 3\right)^2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$$\boxed{res\left(z = 3\mathrm{i}\right) = \frac{5}{256}}, \quad \boxed{res\left(z = \tfrac{1}{3}\mathrm{i}\right) = -\frac{5}{256}}$$

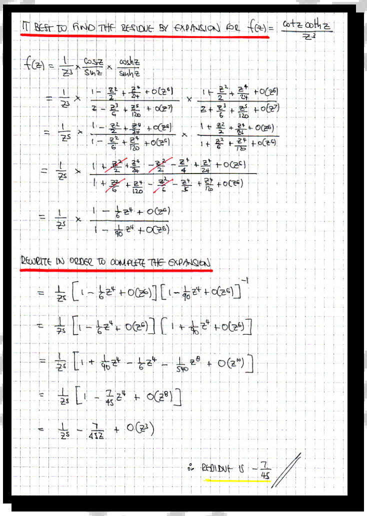


Question 12

$$f(z) \equiv \frac{\cot z \coth z}{z^3}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of $f(z)$ at $z = 0$.

$\boxed{}$, $\boxed{res(z=0) = -\frac{7}{45}}$



IT BEST TO FIND THE RESIDUE BY EXPANSION FOR $f(z) = \frac{\cot z \coth z}{z^3}$

$$f(z) = \frac{1}{z^3} \times \frac{\cos z}{\sin z} \times \frac{\cosh z}{\sinh z}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{z + \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} - \frac{z^2}{2} - \frac{z^4}{4} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^2}{6} - \frac{z^4}{36} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^5} \times \frac{1 - \frac{1}{6}z^4 + O(z^6)}{1 - \frac{1}{90}z^4 + O(z^6)}$$

REWRITE IN ORDER TO COMPLETE THE EXPANSION

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 - \frac{1}{90}z^4 + O(z^6)\right]^{-1}$$

$$= \frac{1}{z^5}\left[1 - \frac{1}{6}z^4 + O(z^6)\right]\left[1 + \frac{1}{90}z^4 + O(z^6)\right]$$

$$= \frac{1}{z^5}\left[1 + \frac{1}{90}z^4 - \frac{1}{6}z^4 - \frac{1}{540}z^8 + O(z^{10})\right]$$

$$= \frac{1}{z^5}\left[1 - \frac{7}{45}z^4 + O(z^8)\right]$$

$$= \frac{1}{z^5} - \frac{7}{45z} + O(z^3)$$

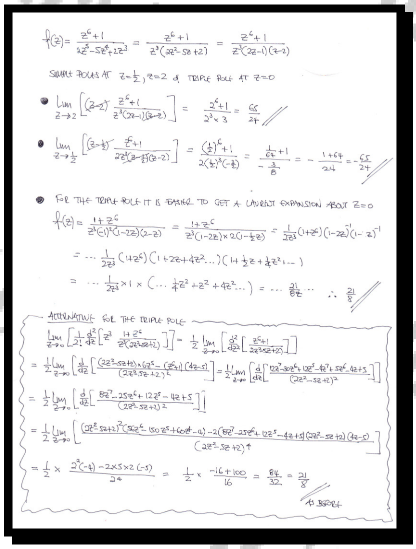
$\therefore$ RESIDUE IS $-\frac{7}{45}$

Question 13

$$f(z) \equiv \frac{z^6+1}{2z^5-5z^4+2z^3}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of $f(z)$.

$$\boxed{res\left(z=\frac{1}{2}\right)=-\frac{65}{24}}, \quad \boxed{res(z=2)=\frac{65}{24}}, \quad \boxed{res(z=0)=\frac{21}{8}}$$



$$f(z)=\frac{z^6+1}{2z^5-5z^4+2z^3}=\frac{z^6+1}{z^3(2z^2-5z+2)}=\frac{z^6+1}{z^3(2z-1)(z-2)}$$

SIMPLE POLES AT $z=\frac{1}{2}$, $z=2$ & TRIPLE POLE AT $z=0$

- $\lim_{z\to 2}\left[(z-2)\frac{z^6+1}{z^3(2z-1)(z-2)}\right]=\frac{2^6+1}{2^3\times 3}=\frac{65}{24}$
- $\lim_{z\to \frac{1}{2}}\left[(z-\tfrac{1}{2})\frac{z^6+1}{2z^3(z-\frac{1}{2})(z-2)}\right]=\frac{(\frac{1}{2})^6+1}{2(\frac{1}{2})^3(-\frac{3}{2})}=\frac{\frac{1}{64}+1}{-\frac{3}{8}}=-\frac{1+64}{24}=-\frac{65}{24}$
- FOR THE TRIPLE POLE IT IS EASIER TO GET A LAURENT EXPANSION ABOUT $z=0$

$$f(z)=\frac{1+z^6}{z^3(-1)^2(1-2z)(2-z)}=\frac{1+z^6}{z^3(1-2z)\times 2(1-\frac{1}{2}z)}=\frac{1}{2z^3}(1+z^6)(1-2z)^{-1}(1-\tfrac{1}{2}z)^{-1}$$

$$=\dots\frac{1}{2z^3}(1+z^6)(1+2z+4z^2\dots)(1+\tfrac{1}{2}z+\tfrac{1}{4}z^2+\dots)$$

$$=\dots\frac{1}{2z^3}\times 1\times(\dots\tfrac{1}{4}z^2+z^2+4z^2\dots)=\dots\frac{21}{8z}\dots \quad \therefore \frac{21}{8}$$

ALTERNATIVE FOR THE TRIPLE POLE

$$\lim_{z\to 0}\left[\frac{1}{2!}\frac{d^2}{dz^2}\left[z^3\frac{1+z^6}{z^3(2z^2-5z+2)}\right]\right]=\frac{1}{2}\lim_{z\to 0}\left[\frac{d^2}{dz^2}\left[\frac{z^6+1}{2z^2-5z+2}\right]\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{(2z^2-5z+2)\times 6z^5-(z^6+1)(4z-5)}{(2z^2-5z+2)^2}\right]\right]=\frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{12z^7-30z^6+12z^5-4z^7+5z^6-4z+5}{(2z^2-5z+2)^2}\right]\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\left[\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{(2z^2-5z+2)^2}\right]\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-2(8z^7-25z^6+12z^5-4z+5)(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4}\right]$$

$$=\frac{1}{2}\times\frac{2^2(-4)-2\times5\times2(-5)}{2^4}=\frac{1}{2}\times\frac{-16+100}{16}=\frac{84}{32}=\frac{21}{8}$$

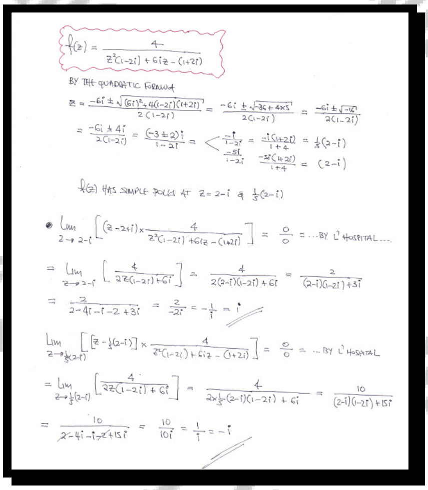
AS BEFORE

Question 14

$$f(z) \equiv \frac{4}{z^2(1-2\mathrm{i})+6z\mathrm{i}-(1+2\mathrm{i})}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of $f(z)$.

$\boxed{res(z=2-\mathrm{i})=\mathrm{i}}$, $\boxed{res\left(z=\frac{1}{5}(2-\mathrm{i})\right)=-\mathrm{i}}$



$f(z) = \frac{4}{z^2(1-2i)+6iz-(1+2i)}$

BY THE QUADRATIC FORMULA

$z = \frac{-6i \pm \sqrt{(6i)^2+4(1-2i)(1+2i)}}{2(1-2i)} = \frac{-6i \pm \sqrt{-36+4\times5}}{2(1-2i)} = \frac{-6i \pm \sqrt{-16}}{2(1-2i)}$

$= \frac{-6i \pm 4i}{2(1-2i)} = \frac{(-3\pm2)i}{1-2i} = \begin{cases} \frac{-i}{1-2i} = \frac{-i(1+2i)}{1+4} = \frac{1}{5}(2-i) \\ \frac{-5i}{1-2i} = \frac{-5i(1+2i)}{1+4} = (2-i) \end{cases}$

$f(z)$ HAS SIMPLE POLES AT $z=2-i$ & $\frac{1}{5}(2-i)$

- $\lim_{z\to 2-i}\left[(z-2+i)\times\frac{4}{z^2(1-2i)+6iz-(1+2i)}\right] = \frac{0}{0} = \ldots$ BY L'HOSPITAL...

$= \lim_{z\to 2-i}\left[\frac{4}{2z(1-2i)+6i}\right] = \frac{4}{2(2-i)(1-2i)+6i} = \frac{2}{(2-i)(1-2i)+3i}$

$= \frac{2}{2-4i-i-2+3i} = \frac{2}{-2i} = -\frac{1}{i} = i$

$\lim_{z\to\frac{1}{5}(2-i)}\left[\left[z-\frac{1}{5}(2-i)\right]\times\frac{4}{z^2(1-2i)+6iz-(1+2i)}\right] = \frac{0}{0} = \ldots$ BY L'HOSPITAL

$= \lim_{z\to\frac{1}{5}(2-i)}\left[\frac{4}{2z(1-2i)+6i}\right] = \frac{4}{2\times\frac{1}{5}(2-i)(1-2i)+6i} = \frac{10}{(2-i)(1-2i)+15i}$

$= \frac{10}{2-4i-i-2+15i} = \frac{10}{10i} = \frac{1}{i} = -i$

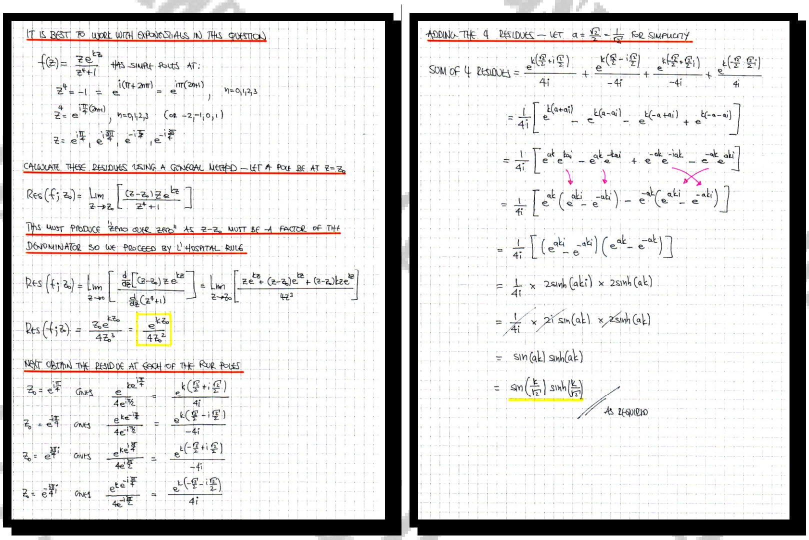
Question 15

$$f(z) \equiv \frac{z\,\mathrm{e}^{kz}}{z^4+1}, \; z \in \mathbb{C}, \; k \in \mathbb{R}, \; k > 0 .$$

Show that the sum of the residues of the four poles of $f(z)$, is

$$\sin\left(\frac{k}{\sqrt{2}}\right)\sinh\left(\frac{k}{\sqrt{2}}\right).$$

[ ], [proof]



REAL INTEGRALS

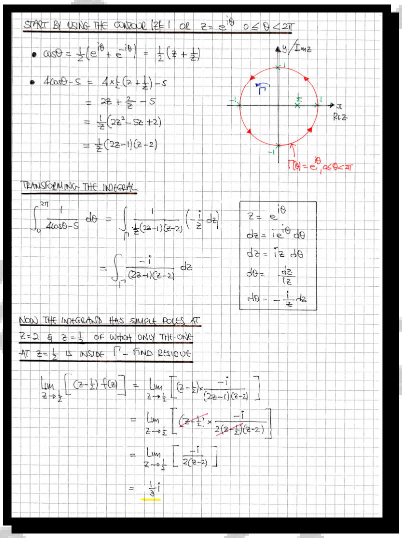
UNIT CIRCLE CONTOUR

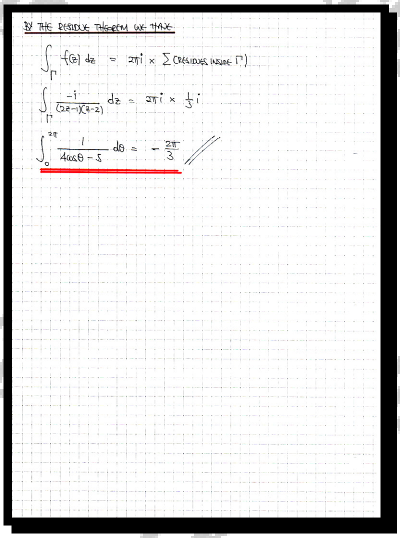
Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5}\ d\theta\,.$$

V, , $-\dfrac{2\pi}{3}$



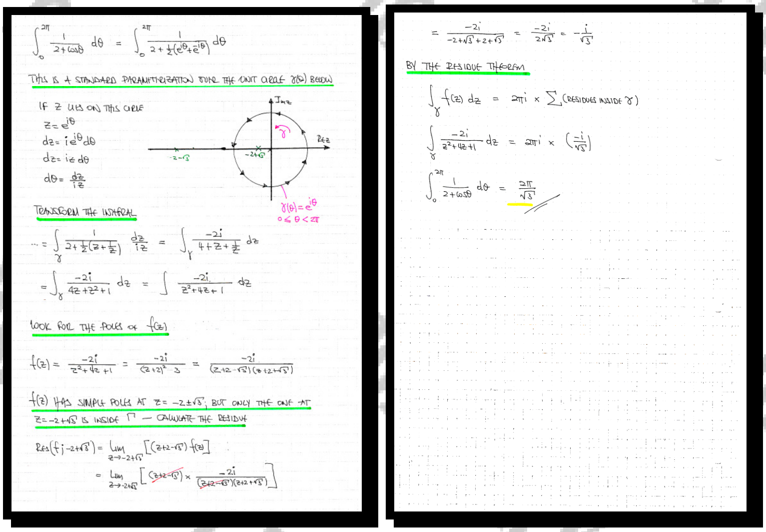


Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta .$$

[illegible] , $\boxed{\dfrac{2\pi}{\sqrt{3}}}$



$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_0^{2\pi} \frac{1}{2+\frac{1}{2}(e^{i\theta}+e^{-i\theta})}\, d\theta$$

THIS IS A STANDARD PARAMETRIZATION FOR THE UNIT CIRCLE $\gamma(\theta)$ BELOW

IF z LIES ON THIS CIRCLE

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$dz = iz\, d\theta$

$d\theta = \frac{dz}{iz}$

TRANSFORM THE INTEGRAL

$$\dots = \int_\gamma \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\,\frac{dz}{iz} = \int_\gamma \frac{-2i}{4+z+\frac{1}{z}}\, dz$$

$$= \int_\gamma \frac{-2i}{4z+z^2+1}\, dz = \int \frac{-2i}{z^2+4z+1}\, dz$$

LOOK FOR THE POLES OF $f(z)$

$$f(z) = \frac{-2i}{z^2+4z+1} = \frac{-2i}{(z+2)^2-3} = \frac{-2i}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$f(z)$ HAS SIMPLE POLES AT $z = -2\pm\sqrt{3}$; BUT ONLY THE ONE AT $z=-2+\sqrt{3}$ IS INSIDE Γ — CALCULATE THE RESIDUE

$$\text{Res}(f; -2+\sqrt{3}) = \lim_{z\to -2+\sqrt{3}} \left[(z+2-\sqrt{3})f(z)\right]$$

$$= \lim_{z\to -2+\sqrt{3}} \left[(z+2-\sqrt{3}) \times \frac{-2i}{(z+2-\sqrt{3})(z+2+\sqrt{3})}\right]$$

$$= \frac{-2i}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{-2i}{2\sqrt{3}} = -\frac{i}{\sqrt{3}}$$

BY THE RESIDUE THEOREM

$$\int_\gamma f(z)\, dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \gamma)$$

$$\int_\gamma \frac{-2i}{z^2+4z+1}\, dz = 2\pi i \times \left(\frac{-i}{\sqrt{3}}\right)$$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{3}}$$

Question 3

By integrating a suitable complex function over an appropriate contour find

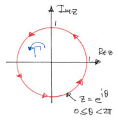
$$\int_0^{2\pi} \cos^6\theta \sin^6\theta \ d\theta .$$

$$\boxed{\dfrac{5\pi}{512}}$$

• USING THE CONTOUR $|z|=1$ OR $z=e^{i\theta}$, $0 \leq \theta < 2\pi$

THUS $z = e^{i\theta}$

$dz = ie^{i\theta}\,d\theta$



• HENCE THE INTEGRAL BECOMES

$$\int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta$$

$$= \int_0^{2\pi} (\cos\theta \sin\theta)^6 \, d\theta = \int_0^{2\pi} \left(\tfrac{1}{2}\sin 2\theta\right)^6 d\theta$$

$$= \frac{1}{64}\int_0^{2\pi} \sin^6 2\theta \, d\theta = \frac{1}{64}\int_0^{2\pi} \left[\frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right)\right]^6 d\theta$$

$$= \frac{1}{64}\left(\frac{1}{2i}\right)^6 \int_0^{2\pi} \left[\left(e^{i\theta}\right)^2 - \left(e^{-i\theta}\right)^2\right]^6 d\theta$$

$$= \frac{1}{64} \times \frac{1}{-64} \int_\Gamma \left(z^2 - \frac{1}{z^2}\right)^6 \left(\frac{dz}{iz}\right) \quad \leftarrow \boxed{d\theta = \frac{dz}{ie^{i\theta}}}$$

$$= \frac{i}{64^2}\int_\Gamma \frac{1}{z}\left(z^2 - \frac{1}{z^2}\right)^6 dz$$

• EXPAND BINOMIALLY

$$= \frac{i}{2^{12}}\int_\Gamma \frac{1}{z}\left[z^{12} - 6z^8 + 15z^4 - 20 + \frac{15}{z^4} - \frac{6}{z^8} + \frac{1}{z^{12}}\right] dz$$

$$= \frac{i}{2^{12}}\int_\Gamma z^{11} - 6z^7 + 15z^3 - \frac{20}{z} + \frac{15}{z^5} - \frac{6}{z^9} + \frac{1}{z^{13}} \, dz$$

• AS THE INTEGRAND IS IN SERIES FORM THE ONLY CONTRIBUTION IS FROM THE $\frac{1}{z}$ TERM IE THE RESIDUE IS -20

$$\Rightarrow \int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = 2\pi i \times \frac{i}{2^{12}} \times (-20)$$

$$\Rightarrow \int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = \frac{5\pi \times 2^3}{2^{12}} = \frac{5\pi}{2^9}$$

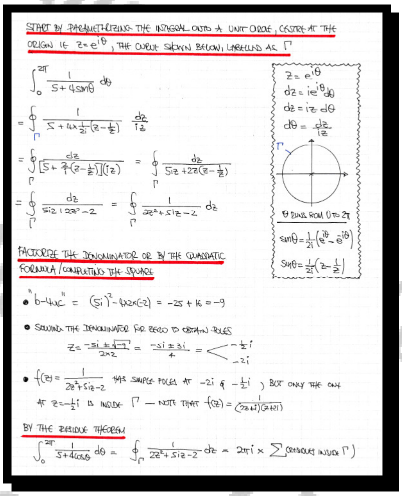
$$\Rightarrow \int_0^{2\pi} \cos^6\theta \sin^6\theta \, d\theta = \frac{5\pi}{512}$$

Question 4

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\, d\theta\,.$$

$\boxed{}$, $\boxed{\dfrac{2\pi}{3}}$



START BY PARAMETRIZING THE INTEGRAL ONTO A UNIT CIRCLE, CENTRE AT THE ORIGIN IE $z=e^{i\theta}$, THE CURVE SHOWN BELOW, LABELLED AS Γ

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\,d\theta$$

$$= \oint_\Gamma \frac{1}{5+4\times\frac{1}{2i}\left(z-\frac{1}{z}\right)}\,\frac{dz}{iz}$$

$$= \oint_\Gamma \frac{dz}{\left[5+\frac{2}{i}\left(z-\frac{1}{z}\right)\right](iz)} = \oint_\Gamma \frac{dz}{5iz+2z\left(z-\frac{1}{z}\right)}$$

$$= \oint_\Gamma \frac{dz}{5iz+2z^2-2} = \oint_\Gamma \frac{1}{2z^2+5iz-2}\,dz$$

$z=e^{i\theta}$
$dz=ie^{i\theta}d\theta$
$dz=iz\,d\theta$
$d\theta=\frac{dz}{iz}$

θ RUNS FROM 0 TO 2π

$\sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$

$\sin\theta=\frac{1}{2i}\left(z-\frac{1}{z}\right)$

FACTORIZE THE DENOMINATOR OR BY THE QUADRATIC FORMULA / COMPLETING THE SQUARE

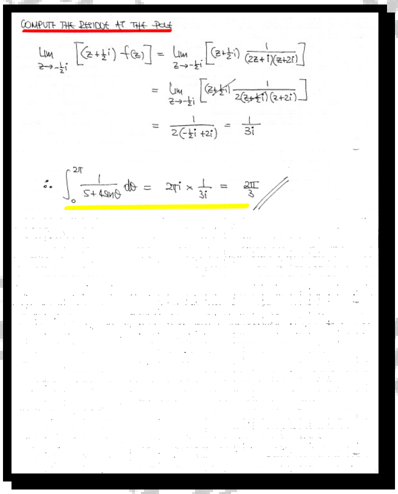
- "b^2-4ac" $=(5i)^2-4\times2\times(-2)=-25+16=-9$
- SOLVING THE DENOMINATOR FOR ZERO TO OBTAIN POLES

$$z=\frac{-5i\pm\sqrt{-9}}{2\times2}=\frac{-5i\pm3i}{4}= \begin{cases}-\frac{1}{2}i\\ -2i\end{cases}$$

- $f(z)=\frac{1}{2z^2+5iz-2}$ HAS SIMPLE POLES AT $-2i$ & $-\frac{1}{2}i$, BUT ONLY THE ONE AT $z=-\frac{1}{2}i$ IS INSIDE Γ — NOTE THAT $f(z)=\frac{1}{(2z+i)(z+2i)}$

BY THE RESIDUE THEOREM

$$\int_0^{2\pi}\frac{1}{5+4\cos\theta}\,d\theta = \oint_\Gamma \frac{1}{2z^2+5iz-2}\,dz = 2\pi i\times\sum(\text{RESIDUES INSIDE }\Gamma)$$



COMPUTE THE RESIDUE AT THE POLE

$$\lim_{z\to-\frac{1}{2}i}\left[\left(z+\tfrac{1}{2}i\right)f(z)\right] = \lim_{z\to-\frac{1}{2}i}\left[\left(z+\tfrac{1}{2}i\right)\frac{1}{(2z+i)(z+2i)}\right]$$

$$= \lim_{z\to-\frac{1}{2}i}\left[\left(z+\tfrac{1}{2}i\right)\frac{1}{2\left(z+\frac{1}{2}i\right)(z+2i)}\right]$$

$$= \frac{1}{2\left(-\frac{1}{2}i+2i\right)} = \frac{1}{3i}$$

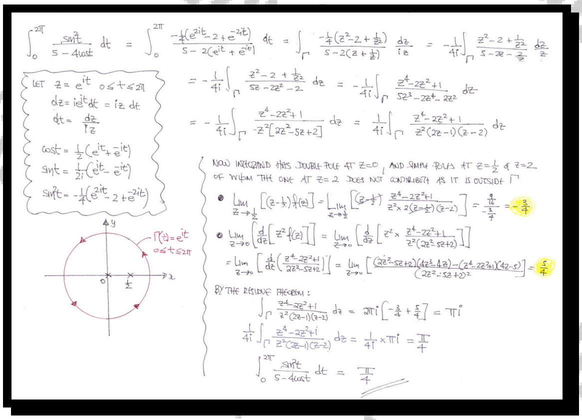
$$\therefore \int_0^{2\pi}\frac{1}{5+4\sin\theta}\,d\theta = 2\pi i\times\frac{1}{3i} = \frac{2\pi}{3}$$

Question 5

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{\sin^2 t}{5-4\cos t}\ dt \ .$$

$$\boxed{\frac{\pi}{4}}$$



$$\int_0^{2\pi} \frac{\sin^2 t}{5-4\cos t}\,dt = \int_0^{2\pi} \frac{-\frac{1}{4}(e^{2it}-2+e^{-2it})}{5-2(e^{it}+e^{-it})}\,dt = \int_\Gamma \frac{-\frac{1}{4}(z^2-2+\frac{1}{z^2})}{5-2(z+\frac{1}{z})}\,\frac{dz}{iz} = -\frac{1}{4i}\int_\Gamma \frac{z^2-2+\frac{1}{z^2}}{5-2z-\frac{2}{z}}\,\frac{dz}{z}$$

$$= -\frac{1}{4i}\int_\Gamma \frac{z^2-2+\frac{1}{z^2}}{5z-2z^2-2}\,dz = -\frac{1}{4i}\int_\Gamma \frac{z^4-2z^2+1}{5z^3-2z^4-2z^2}\,dz$$

$$= -\frac{1}{4i}\int_\Gamma \frac{z^4-2z^2+1}{-z^2[2z^2-5z+2]}\,dz = \frac{1}{4i}\int_\Gamma \frac{z^4-2z^2+1}{z^2(2z-1)(z-2)}\,dz$$

LET $z=e^{it}$ $0\le t\le 2\pi$

$dz = ie^{it}dt = iz\,dt$

$dt = \frac{dz}{iz}$

$\cos t = \frac{1}{2}(e^{it}+e^{-it})$

$\sin t = \frac{1}{2i}(e^{it}-e^{-it})$

$\sin^2 t = -\frac{1}{4}(e^{2it}-2+e^{-2it})$

NOW INTEGRAND HAS DOUBLE POLE AT $z=0$, AND SIMPLE POLES AT $z=\frac{1}{2}$ & $z=2$ OF WHOM THE ONE AT $z=2$ DOES NOT CONTRIBUTE AS IT IS OUTSIDE Γ

- $\lim_{z\to\frac{1}{2}}\left[(z-\tfrac{1}{2})f(z)\right] = \lim_{z\to\frac{1}{2}}\left[(z-\tfrac{1}{2})\frac{z^4-2z^2+1}{z^2\times 2(z-\frac{1}{2})(z-2)}\right] = \frac{\frac{9}{16}}{-\frac{3}{4}} = -\frac{3}{4}$
- $\lim_{z\to 0}\left(\frac{d}{dz}\left[z^2 f(z)\right]\right) = \lim_{z\to 0}\left[\frac{d}{dz}\left[z^2\times\frac{z^4-2z^2+1}{z^2(2z^2-5z+2)}\right]\right]$

$= \lim_{z\to 0}\left[\frac{d}{dz}\left(\frac{z^4-2z^2+1}{2z^2-5z+2}\right)\right] = \lim_{z\to 0}\left[\frac{(2z^2-5z+2)(4z^3-4z)-(z^4-2z^2+1)(4z-5)}{(2z^2-5z+2)^2}\right] = \frac{5}{4}$

BY THE RESIDUE THEOREM:

$$\int_\Gamma \frac{z^4-2z^2+1}{z^2(2z-1)(z-2)}\,dz = 2\pi i\left[-\frac{3}{4}+\frac{5}{4}\right] = \pi i$$

$$\frac{1}{4i}\int_\Gamma \frac{z^4-2z^2+1}{z^2(2z-1)(z-2)}\,dz = \frac{1}{4i}\times\pi i = \frac{\pi}{4}$$

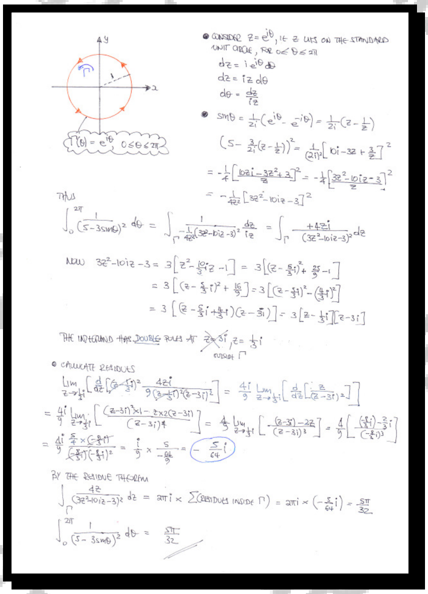
$$\int_0^{2\pi}\frac{\sin^2 t}{5-4\cos t}\,dt = \frac{\pi}{4}$$

Question 6

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}\, d\theta .$$

$$\boxed{\frac{5\pi}{32}}$$



• CONSIDER $z = e^{i\theta}$, IE z LIES ON THE STANDARD UNIT CIRCLE, FOR $0 \le \theta \le 2\pi$

$dz = ie^{i\theta}\,d\theta$

$dz = iz\,d\theta$

$d\theta = \frac{dz}{iz}$

• $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - \frac{1}{z})$

$(5 - \frac{3}{2i}(z-\frac{1}{z}))^2 = \frac{1}{(2i)^2}[10i - 3z + \frac{3}{z}]^2$

$= -\frac{1}{4}\left[\frac{10zi - 3z^2 + 3}{z}\right]^2 = -\frac{1}{4}\left[\frac{3z^2 - 10iz - 3}{z}\right]^2$

$= -\frac{1}{4z^2}[3z^2 - 10iz - 3]^2$

THUS

$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}\,d\theta = \int_\Gamma \frac{1}{-\frac{1}{4z^2}(3z^2-10iz-3)^2}\frac{dz}{iz} = \int_\Gamma \frac{+4zi}{(3z^2-10iz-3)^2}\,dz$

NOW $3z^2 - 10iz - 3 = 3[z^2 - \frac{10}{3}iz - 1] = 3[(z - \frac{5}{3}i)^2 + \frac{25}{9} - 1]$

$= 3[(z-\frac{5}{3}i)^2 + \frac{16}{9}] = 3[(z-\frac{5}{3}i)^2 - (\frac{4}{3}i)^2]$

$= 3[(z - \frac{5}{3}i + \frac{4}{3}i)(z - 3i)] = 3[z - \frac{1}{3}i][z-3i]$

THE INTEGRAND HAS DOUBLE POLES AT ~~$z = 3i$~~ (OUTSIDE Γ), $z = \frac{1}{3}i$

• CALCULATE RESIDUES

$\lim_{z\to\frac{1}{3}i}\left[\frac{d}{dz}\left[(z-\frac{1}{3}i)^2\frac{4zi}{9(z-\frac{1}{3}i)^2(z-3i)^2}\right]\right] = \frac{4i}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{d}{dz}\left[\frac{z}{(z-3i)^2}\right]\right]$

$= \frac{4i}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{(z-3i)^2\times1 - z\times2(z-3i)}{(z-3i)^4}\right] = \frac{4}{9}\lim_{z\to\frac{1}{3}i}\left[\frac{(z-3i)-2z}{(z-3i)^3}\right] = \frac{4}{9}\left[\frac{(-\frac{8}{3}i)-\frac{2}{3}i}{(-\frac{8}{3}i)^3}\right]$

$= \frac{4i}{9}\,\frac{\frac{5}{4}\times(-\frac{8}{3}i)}{(-\frac{8}{3}i)(-\frac{8}{3}i)^2} = \frac{i}{9}\times\frac{5}{-\frac{64}{9}} = -\frac{5}{64}i$

BY THE RESIDUE THEOREM

$\int_\Gamma \frac{4z}{(3z^2-10iz-3)^2}\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma) = 2\pi i \times (-\frac{5}{64}i) = \frac{5\pi}{32}$

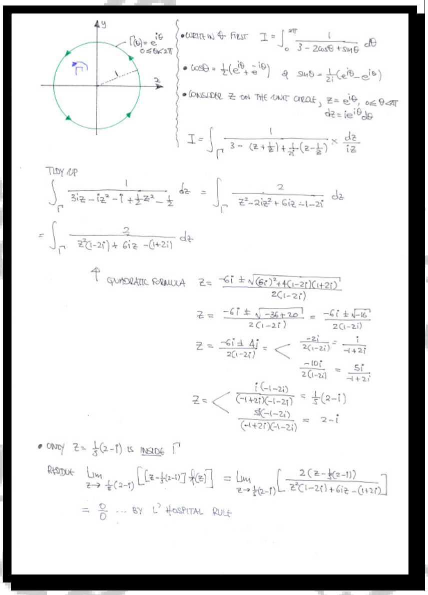
$\int_0^{2\pi}\frac{1}{(5-3\sin\theta)^2}\,d\theta = \frac{5\pi}{32}$

Question 7

$$I = \int_{0}^{2\pi} \frac{1}{3 - 2\cos x + \sin x}\, dx .$$

By integrating a suitable complex function over an appropriate contour find the exact value of I.

$\boxed{\pi}$



• WRITE IN θ FIRST $I = \int_0^{2\pi} \frac{1}{3 - 2\cos\theta + \sin\theta}\, d\theta$

• $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ & $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

• CONSIDER z ON THE UNIT CIRCLE, $z = e^{i\theta}$, $0 \le \theta < 2\pi$, $dz = ie^{i\theta}\, d\theta$

$$I = \int_{\Gamma} \frac{1}{3 - (z + \frac{1}{z}) + \frac{1}{2i}(z - \frac{1}{z})} \times \frac{dz}{iz}$$

TIDY UP

$$\int_{\Gamma} \frac{1}{3iz - iz^2 - i + \frac{1}{2}z^2 - \frac{1}{2}}\, dz = \int_{\Gamma} \frac{2}{z^2 - 2iz^2 + 6iz - 1 - 2i}\, dz$$

$$= \int_{\Gamma} \frac{2}{z^2(1-2i) + 6iz - (1+2i)}\, dz$$

↑ QUADRATIC FORMULA $z = \frac{-6i \pm \sqrt{(6i)^2 + 4(1-2i)(1+2i)}}{2(1-2i)}$

$$z = \frac{-6i \pm \sqrt{-36+20}}{2(1-2i)} = \frac{-6i \pm \sqrt{-16}}{2(1-2i)}$$

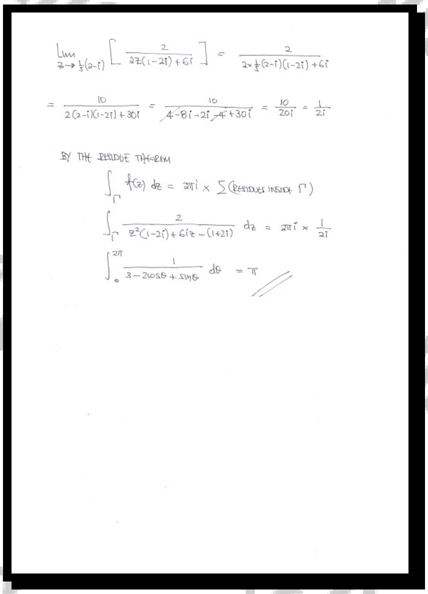
$$z = \frac{-6i \pm 4i}{2(1-2i)} = \begin{cases} \frac{-2i}{2(1-2i)} = \frac{i}{-1+2i} \\ \frac{-10i}{2(1-2i)} = \frac{5i}{-1+2i} \end{cases}$$

$$z = \begin{cases} \frac{i(-1-2i)}{(-1+2i)(-1-2i)} = \frac{1}{5}(2-i) \\ \frac{5i(-1-2i)}{(-1+2i)(-1-2i)} = 2-i \end{cases}$$

• ONLY $z = \frac{1}{5}(2-i)$ IS INSIDE Γ

RESIDUE $\lim_{z \to \frac{1}{5}(2-i)} \left[[z - \frac{1}{5}(2-i)] f(z) \right] = \lim_{z \to \frac{1}{5}(2-i)} \left[\frac{2(z - \frac{1}{5}(2-i))}{z^2(1-2i) + 6iz - (1+2i)} \right]$

$= \frac{0}{0}$ ... BY L'HOSPITAL RULE



$$\lim_{z \to \frac{1}{5}(2-i)} \left[\frac{2}{2z(1-2i) + 6i} \right] = \frac{2}{2 \times \frac{1}{5}(2-i)(1-2i) + 6i}$$

$$= \frac{10}{2(2-i)(1-2i) + 30i} = \frac{10}{4 - 8i - 2i - 4 + 30i} = \frac{10}{20i} = \frac{1}{2i}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z)\, dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{\Gamma} \frac{2}{z^2(1-2i) + 6iz - (1+2i)}\, dz = 2\pi i \times \frac{1}{2i}$$

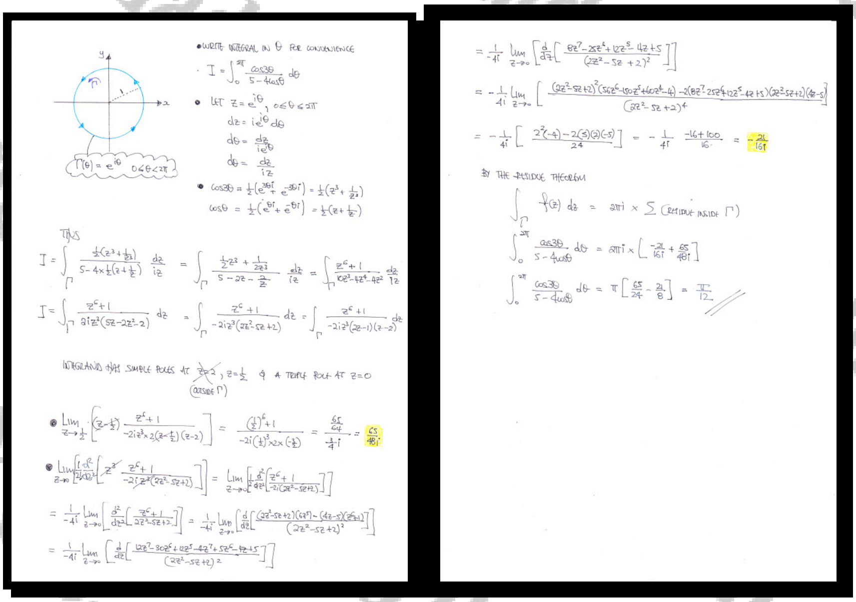
$$\int_0^{2\pi} \frac{1}{3 - 2\cos\theta + \sin\theta}\, d\theta = \pi$$

Question 8

$$I = \int_0^{2\pi} \frac{\cos 3x}{5 - 4\cos x} \, dx .$$

By integrating a suitable complex function over an appropriate contour find the exact value of I .

$$\boxed{\frac{\pi}{12}}$$



SEMI CIRCLE CONTOUR

Jordan's Lemma

Suppose that $f(z) \to 0$ uniformly, as $|z| \to \infty$, for $0 \le \arg z \le \pi$.

If $\alpha > 0$, then $\displaystyle\int_{\gamma_R} f(z) e^{i\alpha z}\, dz \to 0$ as $R \to \infty$, where $\gamma_R(\theta) = R e^{i\theta}$, for $0 \le \theta \le \pi$.

Proof

Given $\varepsilon > 0$ we may always pick R_0, so that if $R > R_0$, $|f(z)| < \varepsilon$, $\forall z \in \gamma_R$.

Thus

$$\left| \int_{\gamma_R} e^{i\alpha z} f(z)\, dz \right| = \left| \int_0^{\pi} e^{i\alpha R(\cos\theta + i\sin\theta)} f\left(R e^{i\theta}\right) i e^{i\theta} \; d\theta \right| =$$

$$\left| \int_0^{\pi} e^{i\alpha R\cos\theta} e^{-\alpha R\sin\theta} f\left(R e^{i\theta}\right) i e^{i\theta}\, d\theta \right| \le \int_0^{\pi} \left| e^{i\alpha R\cos\theta} e^{-\alpha R\sin\theta} f\left(R e^{i\theta}\right) i e^{i\theta} \right| \; d\theta =$$

$$\int_0^{\pi} \left| e^{i\alpha R\cos\theta} \right| \left| e^{-\alpha R\sin\theta} \right| \left| f\left(R e^{i\theta}\right) \right| |i| \left| e^{i\theta} \right| \; d\theta = \int_0^{\pi} e^{-\alpha R\sin\theta} \left| f\left(R e^{i\theta}\right) \right| \; d\theta \le$$

$$\varepsilon R \int_0^{\pi} e^{-\alpha R\sin\theta} \; d\theta = 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R\sin\theta} \; d\theta \qquad \left[\text{since } \sin\theta \text{ is even about } \frac{\pi}{2} \right]$$

Now by **Jordan's Inequality**

$$\boxed{\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1, \text{ if } 0 < \theta \le \frac{\pi}{2}}$$

$$\sin\theta \ge \frac{2\theta}{\pi} \;\Rightarrow e^{-\sin\theta} \le e^{\frac{2}{\pi}\theta}, \text{ if } 0 < \theta \le \frac{\pi}{2}$$

Hence

$$2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R\sin\theta} \; d\theta \le 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}\alpha R\theta} \; d\theta = 2\varepsilon R \left[-\frac{\pi}{2\alpha R} e^{-\frac{2}{\pi}\alpha R\theta} \right]_0^{\frac{\pi}{2}} =$$

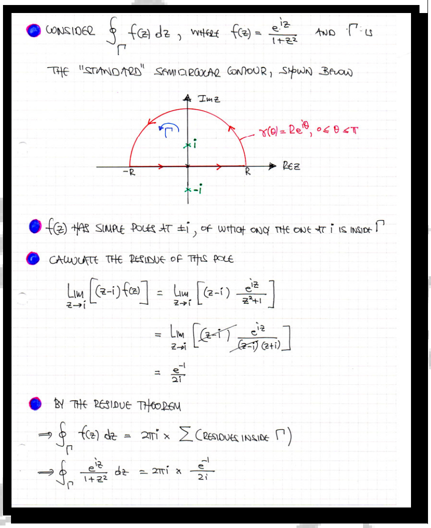
$$\frac{\varepsilon\pi}{\alpha} \left[e^{-\frac{2}{\pi}\alpha R\theta} \right]_{\frac{\pi}{2}}^{0} = \frac{\varepsilon\pi}{\alpha} \left[1 - e^{-\alpha R} \right] \to 0 \text{ since as } R \to \infty, \varepsilon \to 0 \; \square$$

Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx\,.$$

V , $\boxed{\frac{\pi}{e}}$



CONSIDER $\oint_{\Gamma} f(z)\,dz$, WHERE $f(z) = \frac{e^{iz}}{1+z^2}$ AND Γ IS THE "STANDARD" SEMICIRCULAR CONTOUR, SHOWN BELOW

$f(z)$ HAS SIMPLE POLES AT $\pm i$, OF WHICH ONLY THE ONE AT i IS INSIDE Γ

CALCULATE THE RESIDUE OF THIS POLE

$$\lim_{z\to i}\left[(z-i)f(z)\right] = \lim_{z\to i}\left[(z-i)\frac{e^{iz}}{z^2+1}\right]$$

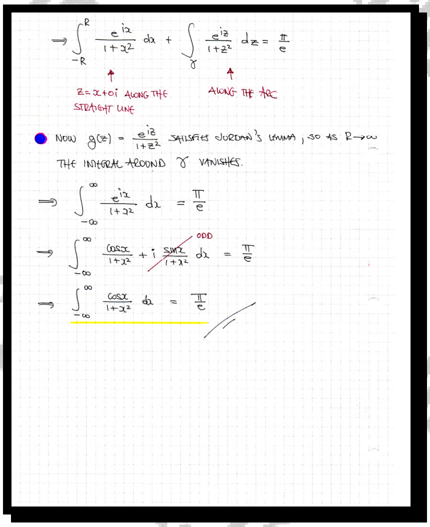
$$= \lim_{z\to i}\left[\cancel{(z-i)}\frac{e^{iz}}{\cancel{(z-i)}(z+i)}\right]$$

$$= \frac{e^{-1}}{2i}$$

BY THE RESIDUE THEOREM

$$\Rightarrow \oint_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \oint_{\Gamma} \frac{e^{iz}}{1+z^2}\,dz = 2\pi i \times \frac{e^{-1}}{2i}$$



$$\Rightarrow \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx + \int_{\gamma} \frac{e^{iz}}{1+z^2}\,dz = \frac{\pi}{e}$$

$z = x+0i$ ALONG THE STRAIGHT LINE; ALONG THE ARC

NOW $g(z) = \frac{e^{iz}}{1+z^2}$ SATISFIES JORDAN'S LEMMA, SO AS $R\to\infty$ THE INTEGRAL AROUND γ VANISHES.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} + i\,\cancel{\frac{\sin x}{1+x^2}}\,dx = \frac{\pi}{e} \quad \text{ODD}$$

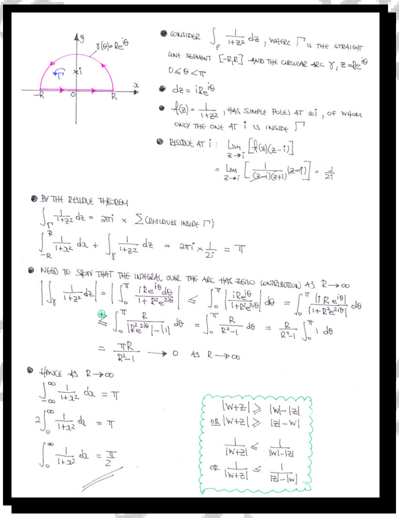
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{e}$$

Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^2}\ dx .$$

$$\boxed{\frac{\pi}{2}}$$



- CONSIDER $\int_{\Gamma} \frac{1}{1+z^2}\,dz$, WHERE Γ IS THE STRAIGHT LINE SEGMENT $[-R,R]$ AND THE CIRCULAR ARC γ, $z = Re^{i\theta}$ $0 \leq \theta < \pi$
- $dz = iRe^{i\theta}$
- $f(z) = \frac{1}{1+z^2}$, HAS SIMPLE POLES AT $\pm i$, OF WHOM ONLY THE ONE AT i IS INSIDE Γ
- RESIDUE AT i: $\lim_{z \to i}\left[f(z)(z-i)\right] = \lim_{z \to i}\left[\frac{1}{(z-i)(z+i)}(z-i)\right] = \frac{1}{2i}$
- BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{1}{1+z^2}\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{\gamma} \frac{1}{1+z^2}\,dz = 2\pi i \times \frac{1}{2i} = \pi$$

- NEED TO SHOW THAT THE INTEGRAL OVER THE ARC HAS ZERO CONTRIBUTION AS $R \to \infty$

$$\left|\int_{\gamma} \frac{1}{1+z^2}\,dz\right| = \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{1+R^2e^{2i\theta}}\right| \leq \int_0^{\pi} \left|\frac{iRe^{i\theta}}{1+R^2e^{2i\theta}}\right| d\theta = \int_0^{\pi} \frac{|iRe^{i\theta}|}{|1+R^2e^{2i\theta}|}\,d\theta$$

$$\overset{*}{\leq} \int_0^{\pi} \frac{R}{|R^2e^{2i\theta}|-|1|}\,d\theta = \int_0^{\pi} \frac{R}{R^2-1}\,d\theta = \frac{R}{R^2-1}\int_0^{\pi} 1\,d\theta$$

$$= \frac{\pi R}{R^2-1} \longrightarrow 0 \text{ AS } R \to \infty$$

- HENCE AS $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

$$2\int_0^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}$$

$|w+z| \geq |w|-|z|$
OR $|w+z| \geq |z|-|w|$

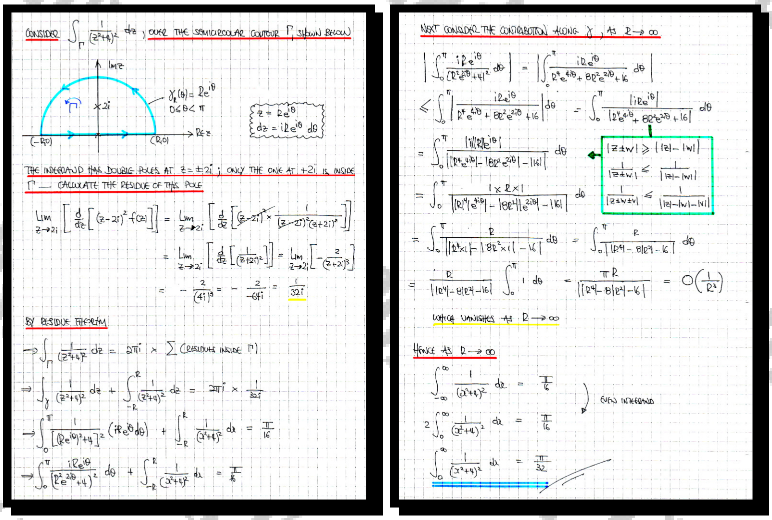
$\frac{1}{|w+z|} \leq \frac{1}{|w|-|z|}$
OR $\frac{1}{|w+z|} \leq \frac{1}{|z|-|w|}$

Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{\left(x^2+4\right)^2}\, dx .$$

$\boxed{}$, $\boxed{\frac{\pi}{32}}$



CONSIDER $\int_\Gamma \frac{1}{(z^2+4)^2}\,dz$, OVER THE SEMICIRCULAR CONTOUR Γ, SHOWN BELOW

$z = Re^{i\theta}$
$dz = iRe^{i\theta}\,d\theta$

THE INTEGRAND HAS DOUBLE POLES AT $z = \pm 2i$; ONLY THE ONE AT $+2i$ IS INSIDE Γ — CALCULATE THE RESIDUE OF THIS POLE

$$\lim_{z\to 2i}\left[\frac{d}{dz}\left[(z-2i)^2 f(z)\right]\right] = \lim_{z\to 2i}\left[\frac{d}{dz}\left[(z-2i)^2 \times \frac{1}{(z-2i)^2(z+2i)^2}\right]\right]$$

$$= \lim_{z\to 2i}\left[\frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]\right] = \lim_{z\to 2i}\left[-\frac{2}{(z+2i)^3}\right]$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{-64i} = \frac{1}{32i}$$

BY RESIDUE THEOREM

$$\Rightarrow \int_\Gamma \frac{1}{(z^2+4)^2}\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \int_\gamma \frac{1}{(z^2+4)^2}\,dz + \int_{-R}^{R} \frac{1}{(z^2+4)^2}\,dz = 2\pi i \times \frac{1}{32i}$$

$$\Rightarrow \int_0^{\pi} \frac{1}{\left[(Re^{i\theta})^2+4\right]^2}\left(iRe^{i\theta}\,d\theta\right) + \int_{-R}^{R} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

$$\Rightarrow \int_0^{\pi} \frac{iRe^{i\theta}}{\left(R^2e^{2i\theta}+4\right)^2}\,d\theta + \int_{-R}^{R} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

NEXT CONSIDER THE CONTRIBUTION ALONG γ, AS $R \to \infty$

$$\left|\int_0^{\pi} \frac{iRe^{i\theta}}{(R^2e^{2i\theta}+4)^2}\,d\theta\right| = \left|\int_0^{\pi} \frac{iRe^{i\theta}}{R^4e^{4i\theta}+8R^2e^{2i\theta}+16}\,d\theta\right|$$

$$\leqslant \int_0^{\pi}\left|\frac{iRe^{i\theta}}{R^4e^{4i\theta}+8R^2e^{2i\theta}+16}\right|d\theta = \int_0^{\pi} \frac{|iRe^{i\theta}|}{|R^4e^{4i\theta}+8R^2e^{2i\theta}+16|}\,d\theta$$

$$= \int_0^{\pi} \frac{|i||R||e^{i\theta}|}{\left||R^4e^{4i\theta}| - |8R^2e^{2i\theta}| - |16|\right|}\,d\theta$$

$|z \pm w| \geqslant \big||z| - |w|\big|$

$\frac{1}{|z \pm w|} \leqslant \frac{1}{\big||z|-|w|\big|}$

$\frac{1}{|z \pm w \pm v|} \leqslant \frac{1}{\big||z|-|w|-|v|\big|}$

$$= \int_0^{\pi} \frac{1 \times R \times 1}{\left||R|^4|e^{4i\theta}| - |8R^2||e^{2i\theta}| - |16|\right|}\,d\theta$$

$$= \int_0^{\pi} \frac{R}{\left||R^4 \times 1| - |8R^2 \times 1| - 16\right|}\,d\theta = \int_0^{\pi} \frac{R}{\left||R^4| - 8|R^2| - 16\right|}\,d\theta$$

$$= \frac{R}{\left||R^4| - 8|R^2| - 16\right|}\int_0^{\pi} 1\,d\theta = \frac{\pi R}{\left||R^4| - 8|R^2| - 16\right|} = O\left(\frac{1}{R^3}\right)$$

WHICH VANISHES AS $R \to \infty$

HENCE AS $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

EVEN INTEGRAND

$$2\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{16}$$

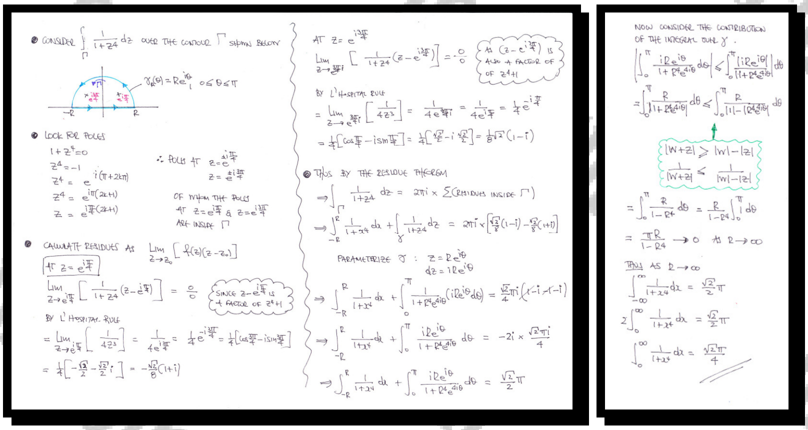
$$\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx = \frac{\pi}{32}$$

Question 4

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^4}\ dx\ .$$

$$\boxed{\frac{\pi\sqrt{2}}{4}}$$

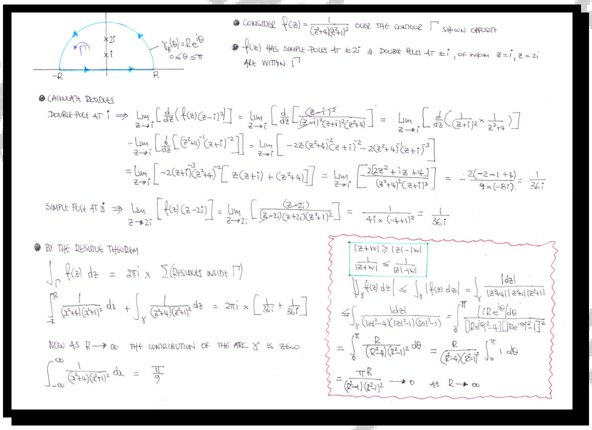


Question 5

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{\left(x^2+4\right)\left(x^2+1\right)^2}\ dx\ .$$

$$\boxed{\frac{\pi}{9}}$$

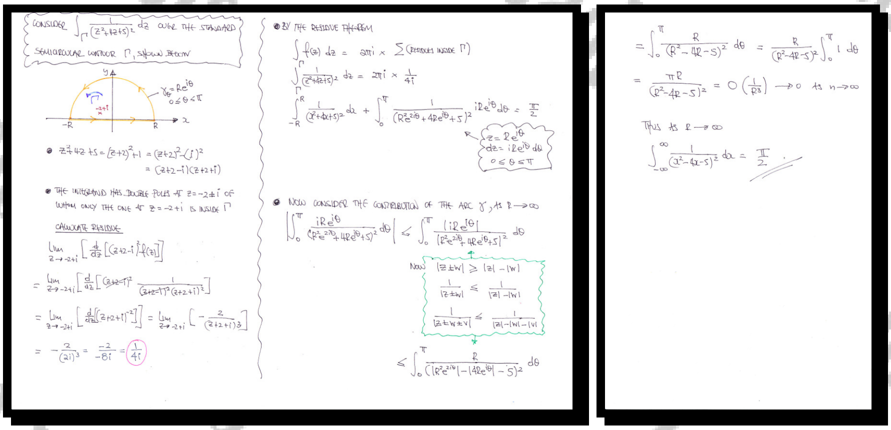


Question 6

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{\left(x^2+4x+5\right)^2}\ dx .$$

$$\boxed{\frac{\pi}{2}}$$

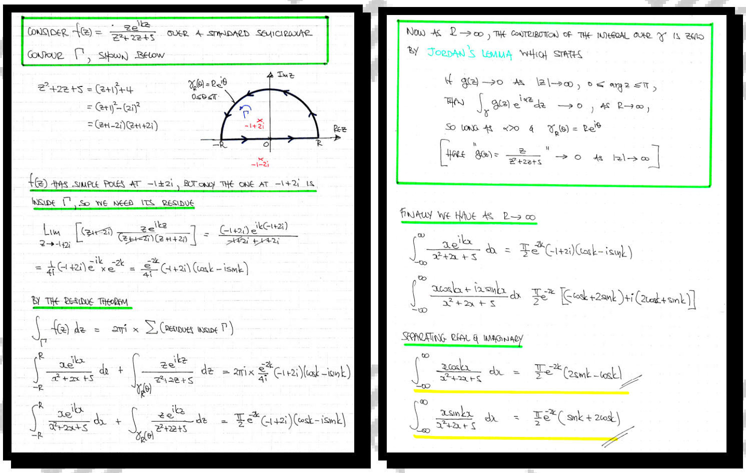


Question 7

Given that $k > 0$ find the exact value of

$$\int_{-\infty}^{\infty} \frac{x\cos kx}{x^2+2x+5}\,dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x\sin kx}{x^2+2x+5}\,dx.$$

$\boxed{}$, $\boxed{\frac{1}{2}\pi e^{-2k}\left(2\sin k - \cos k\right)}$, $\boxed{\frac{1}{2}\pi e^{-2k}\left(\sin k + 2\cos k\right)}$



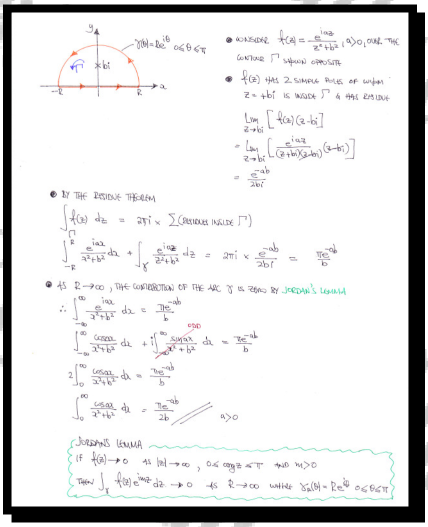
Question 8

By integrating a suitable complex function over an appropriate contour find

a) … $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\, dx$, $a > 0$.

b) … $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\, dx$, $a < 0$.

$\boxed{\dfrac{\pi e^{-ab}}{2b},\ a > 0}$, $\boxed{\dfrac{\pi e^{ab}}{2b},\ < 0}$



CONSIDER $f(z) = \dfrac{e^{iaz}}{z^2+b^2}$, $a>0$, OVER THE CONTOUR Γ SHOWN OPPOSITE

$f(z)$ HAS 2 SIMPLE POLES OF WHOM $z = +bi$ IS INSIDE Γ & HAS RESIDUE

$\lim_{z\to bi}\left[f(z)(z-bi)\right] = \lim_{z\to bi}\left[\dfrac{e^{iaz}}{(z+bi)(z-bi)}(z-bi)\right] = \dfrac{e^{-ab}}{2bi}$

BY THE RESIDUE THEOREM

$\int_\Gamma f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$

$\int_{-R}^{R} \dfrac{e^{iax}}{x^2+b^2}dx + \int_\gamma \dfrac{e^{iaz}}{z^2+b^2}dz = 2\pi i \times \dfrac{e^{-ab}}{2bi} = \dfrac{\pi e^{-ab}}{b}$

AS $R\to\infty$, THE CONTRIBUTION OF THE ARC γ IS ZERO BY JORDAN'S LEMMA

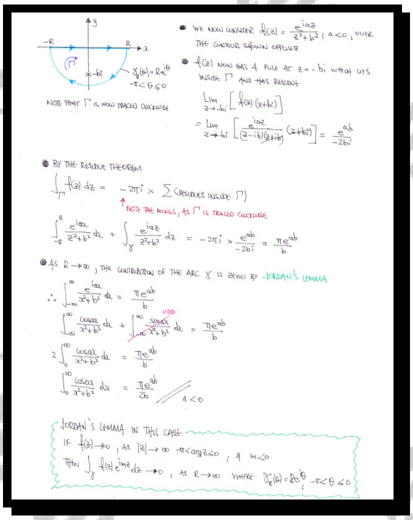
$\therefore \int_{-\infty}^{\infty} \dfrac{e^{iax}}{x^2+b^2}dx = \dfrac{\pi e^{-ab}}{b}$

$\int_{-\infty}^{\infty} \dfrac{\cos ax}{x^2+b^2}dx + i\int_{-\infty}^{\infty} \dfrac{\sin ax}{x^2+b^2}dx = \dfrac{\pi e^{-ab}}{b}$ (ODD)

$2\int_0^{\infty} \dfrac{\cos ax}{x^2+b^2}dx = \dfrac{\pi e^{-ab}}{b}$

$\int_0^{\infty} \dfrac{\cos ax}{x^2+b^2}dx = \dfrac{\pi e^{-ab}}{2b}$ // $a>0$

JORDANS LEMMA: IF $f(z)\to 0$ AS $|z|\to\infty$, $0\le \arg z\le\pi$ AND $m>0$ THEN $\int_\gamma f(z)e^{imz}dz \to 0$ AS $R\to\infty$ WHERE $\gamma_R(\theta) = Re^{i\theta}$ $0\le\theta\le\pi$



WE NOW CONSIDER $f(z) = \dfrac{e^{iaz}}{z^2+b^2}$, $a<0$, OVER THE CONTOUR SHOWN OPPOSITE

NOTE THAT Γ IS NOW TRACED CLOCKWISE

$f(z)$ NOW HAS A POLE AT $z=-bi$ WHICH LIES INSIDE Γ AND HAS RESIDUE

$\lim_{z\to -bi}\left[f(z)(z+bi)\right] = \lim_{z\to -bi}\left[\dfrac{e^{iaz}}{(z-ib)(z+ib)}(z+bi)\right] = \dfrac{e^{ab}}{-2bi}$

BY THE RESIDUE THEOREM

$\int_\Gamma f(z)\,dz = -2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$

NOTE THE MINUS, AS Γ IS TRACED CLOCKWISE

$\int_{-R}^{R} \dfrac{e^{iax}}{x^2+b^2}dx + \int_\gamma \dfrac{e^{iaz}}{z^2+b^2}dz = -2\pi i \times \dfrac{e^{ab}}{-2bi} = \dfrac{\pi e^{ab}}{b}$

AS $R\to\infty$, THE CONTRIBUTION OF THE ARC γ IS ZERO BY JORDAN'S LEMMA

$\therefore \int_{-\infty}^{\infty} \dfrac{e^{iax}}{x^2+b^2}dx = \dfrac{\pi e^{ab}}{b}$

$\int_{-\infty}^{\infty} \dfrac{\cos ax}{x^2+b^2}dx + \int_{-\infty}^{\infty} \dfrac{\sin ax}{x^2+b^2}dx = \dfrac{\pi e^{ab}}{b}$ (ODD)

$2\int_0^{\infty} \dfrac{\cos ax}{x^2+b^2}dx = \dfrac{\pi e^{ab}}{b}$

$\int_0^{\infty} \dfrac{\cos ax}{x^2+b^2}dx = \dfrac{\pi e^{ab}}{2b}$ // $a<0$

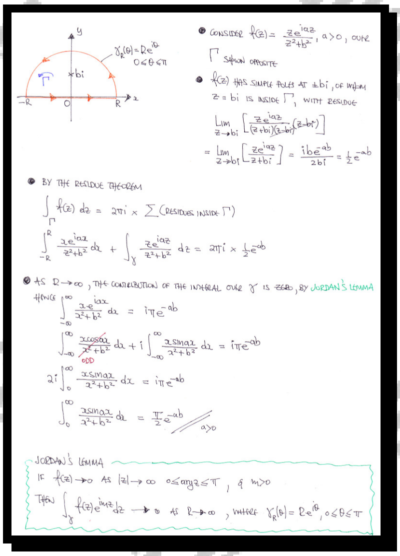
JORDAN'S LEMMA IN THIS CASE: IF $f(z)\to 0$, AS $|z|\to\infty$ $-\pi<\arg z\le 0$, & $m<0$ THEN $\int_\gamma f(z)e^{imz}dz \to 0$, AS $R\to\infty$ WHERE $\gamma_R(\theta) = Re^{i\theta}$ $-\pi<\theta\le 0$

Question 9

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\, dx, \ a>0 .$$

$$\boxed{\frac{1}{2}\pi\,\mathrm{e}^{-ab}}$$



• CONSIDER $f(z)=\frac{ze^{iaz}}{z^2+b^2}$, $a>0$, OVER Γ SHOWN OPPOSITE

• $f(z)$ HAS SIMPLE POLES AT $\pm bi$, OF WHICH $z=bi$ IS INSIDE Γ, WITH RESIDUE

$$\lim_{z\to bi}\left[\frac{ze^{iaz}}{(z+bi)(z-bi)}(z-bi)\right] = \lim_{z\to bi}\left[\frac{ze^{iaz}}{z+bi}\right] = \frac{ibe^{-ab}}{2bi} = \frac{1}{2}e^{-ab}$$

• BY THE RESIDUE THEOREM

$$\int_\Gamma f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{-R}^{R} \frac{xe^{iax}}{x^2+b^2}\,dx + \int_\gamma \frac{ze^{iaz}}{z^2+b^2}\,dz = 2\pi i \times \frac{1}{2}e^{-ab}$$

• AS $R\to\infty$, THE CONTRIBUTION OF THE INTEGRAL OVER γ IS ZERO, BY JORDAN'S LEMMA

HENCE $\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+b^2}\,dx = i\pi e^{-ab}$

$$\int_{-\infty}^{\infty} \frac{x\cos ax}{x^2+b^2}\,dx + i\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = i\pi e^{-ab}$$

ODD

$$2i\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = i\pi e^{-ab}$$

$$\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\,dx = \frac{\pi}{2}e^{-ab} \quad a>0$$

JORDAN'S LEMMA

IF $f(z)\to 0$ AS $|z|\to\infty$ $0\le \arg z\le\pi$, & $m>0$

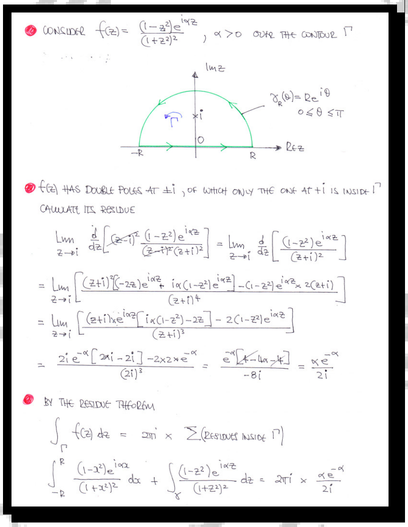
THEN $\int_\gamma f(z)e^{imz}\,dz \to 0$ AS $R\to\infty$, WHERE $\gamma_R(\theta)=Re^{i\theta}$, $0\le\theta\le\pi$

Question 10

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\left(1-x^2\right)\cos \alpha x}{\left(1+x^2\right)^2}\, dx\,,\ \ \alpha > 0\,.$$

$$\boxed{\frac{1}{2}\pi\alpha\,\mathrm{e}^{-\alpha}}$$



- CONSIDER $f(z) = \dfrac{(1-z^2)e^{i\alpha z}}{(1+z^2)^2}$, $\alpha > 0$ OVER THE CONTOUR Γ

- $f(z)$ HAS DOUBLE POLES AT $\pm i$, OF WHICH ONLY THE ONE AT $+i$ IS INSIDE Γ. CALCULATE ITS RESIDUE

$$\lim_{z\to i} \frac{d}{dz}\left[(z-i)^2 \frac{(1-z^2)e^{i\alpha z}}{(z-i)^2(z+i)^2}\right] = \lim_{z\to i} \frac{d}{dz}\left[\frac{(1-z^2)e^{i\alpha z}}{(z+i)^2}\right]$$

$$= \lim_{z\to i}\left[\frac{(z+i)^2\left[(-2z)e^{i\alpha z} + i\alpha(1-z^2)e^{i\alpha z}\right] - (1-z^2)e^{i\alpha z}\times 2(z+i)}{(z+i)^4}\right]$$

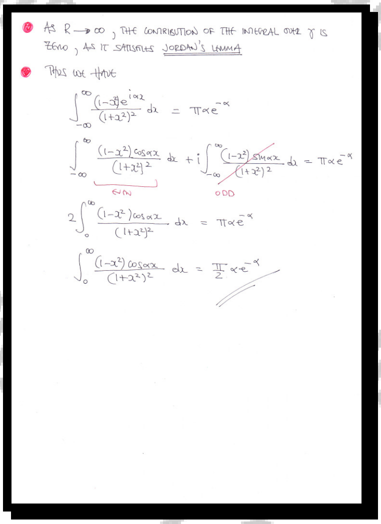
$$= \lim_{z\to i}\left[\frac{(z+i)\times e^{i\alpha z}\left[i\alpha(1-z^2) - 2z\right] - 2(1-z^2)e^{i\alpha z}}{(z+i)^3}\right]$$

$$= \frac{2i e^{-\alpha}\left[2\alpha i - 2i\right] - 2\times 2\times e^{-\alpha}}{(2i)^3} = \frac{e^{-\alpha}\left[4 - 4\alpha - 4\right]}{-8i} = \frac{\alpha e^{-\alpha}}{2i}$$

- BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z)\, dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$

$$\int_{-R}^{R} \frac{(1-x^2)e^{i\alpha x}}{(1+x^2)^2}\, dx + \int_{\gamma} \frac{(1-z^2)e^{i\alpha z}}{(1+z^2)^2}\, dz = 2\pi i \times \frac{\alpha e^{-\alpha}}{2i}$$



- AS $R \to \infty$, THE CONTRIBUTION OF THE INTEGRAL OVER γ IS ZERO, AS IT SATISFIES JORDAN'S LEMMA
- THUS WE HAVE

$$\int_{-\infty}^{\infty} \frac{(1-x^2)e^{i\alpha x}}{(1+x^2)^2}\, dx = \pi\alpha e^{-\alpha}$$

$$\underbrace{\int_{-\infty}^{\infty} \frac{(1-x^2)\cos\alpha x}{(1+x^2)^2}\, dx}_{\text{EVEN}} + i\underbrace{\int_{-\infty}^{\infty} \frac{(1-x^2)\sin\alpha x}{(1+x^2)^2}\, dx}_{\text{ODD}} = \pi\alpha e^{-\alpha}$$

$$2\int_0^{\infty} \frac{(1-x^2)\cos\alpha x}{(1+x^2)^2}\, dx = \pi\alpha e^{-\alpha}$$

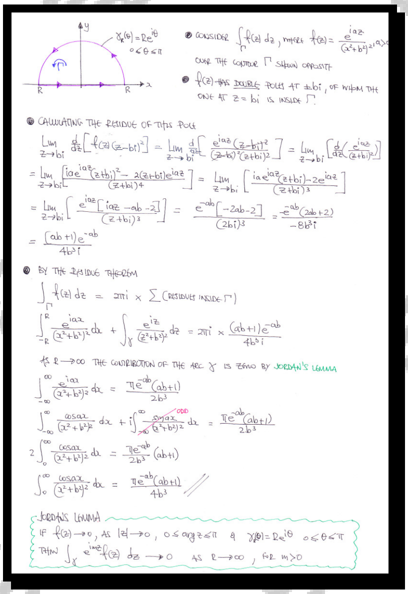
$$\int_0^{\infty} \frac{(1-x^2)\cos\alpha x}{(1+x^2)^2}\, dx = \frac{\pi}{2}\alpha e^{-\alpha}$$

Question 11

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{\cos ax}{\left(x^2+b^2\right)^2}\,dx\,,\ a>0\,.$$

$$\boxed{\frac{\pi e^{-ab}(ab+1)}{4b^3}}$$

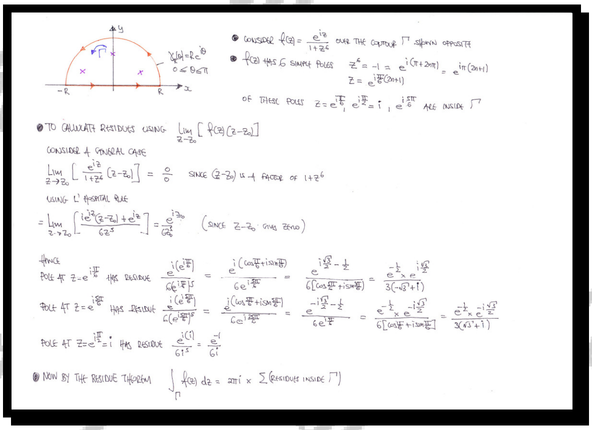


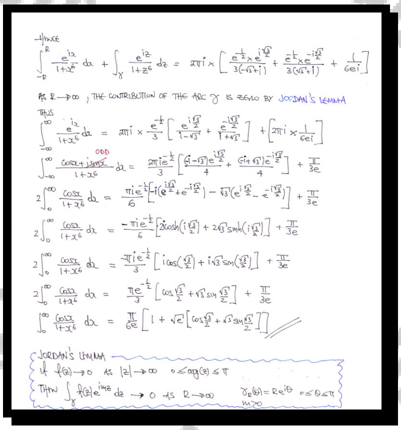
Question 12

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\cos x}{1+x^6}\ dx .$$

$$\boxed{\frac{\pi}{6\mathrm{e}}\left[1+\sqrt{\mathrm{e}}\left[\cos\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\right)\right]\right]}$$





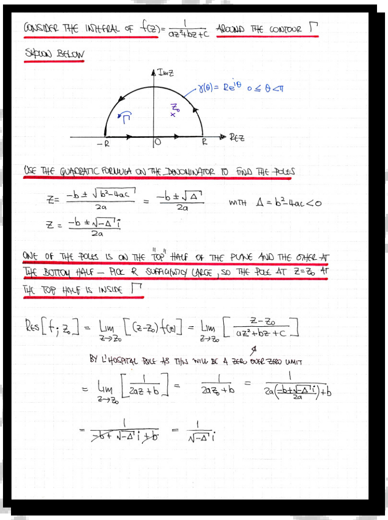
Question 13

By integrating a suitable complex function over an appropriate contour find an exact simplified value for

$$\int_{-\infty}^{\infty} \frac{1}{ax^2+bx+c}\,dx\,,$$

where a, b and c are real constants such that $a>0$ and $b^2-4ac<0$.

V, , $\boxed{\dfrac{2\pi}{\sqrt{4ac-b^2}}}$



CONSIDER THE INTEGRAL OF $f(z)=\frac{1}{az^2+bz+c}$ AROUND THE CONTOUR Γ SHOWN BELOW

USE THE QUADRATIC FORMULA ON THE DENOMINATOR TO FIND THE POLES

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{\Delta}}{2a}\qquad \text{WITH } \Delta=b^2-4ac<0$$

$$z=\frac{-b\pm\sqrt{-\Delta}\,i}{2a}$$

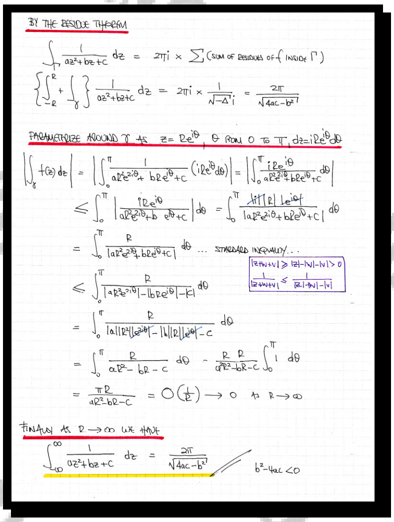
ONE OF THE POLES IS ON THE "TOP" HALF OF THE PLANE AND THE OTHER AT THE BOTTOM HALF — PICK R SUFFICIENTLY LARGE, SO THE POLE AT $z=z_0$ AT THE TOP HALF IS INSIDE Γ

$$\text{Res}\left[f;z_0\right]=\lim_{z\to z_0}\left[(z-z_0)f(z)\right]=\lim_{z\to z_0}\left[\frac{z-z_0}{az^2+bz+c}\right]$$

BY L'HOSPITAL RULE AS THIS WILL BE A ZERO OVER ZERO LIMIT

$$=\lim_{z\to z_0}\left[\frac{1}{2az+b}\right]=\frac{1}{2az_0+b}=\frac{1}{2a\left(\frac{-b+\sqrt{-\Delta}\,i}{2a}\right)+b}$$

$$=\frac{1}{-b+\sqrt{-\Delta}\,i+b}=\frac{1}{\sqrt{-\Delta}\,i}$$



BY THE RESIDUE THEOREM

$$\int_{\Gamma}\frac{1}{az^2+bz+c}\,dz=2\pi i\times\sum(\text{SUM OF RESIDUES OF } f \text{ INSIDE } \Gamma)$$

$$\left\{\int_{-R}^{R}+\int_{\gamma}\right\}\frac{1}{az^2+bz+c}\,dz=2\pi i\times\frac{1}{\sqrt{-\Delta}\,i}=\frac{2\pi}{\sqrt{4ac-b^2}}$$

PARAMETRIZE AROUND γ AS $z=Re^{i\theta}$, θ FROM 0 TO π, $dz=iRe^{i\theta}d\theta$

$$\left|\int_{\gamma}f(z)\,dz\right|=\left|\int_0^{\pi}\frac{1}{aR^2e^{2i\theta}+bRe^{i\theta}+c}\left(iRe^{i\theta}d\theta\right)\right|=\left|\int_0^{\pi}\frac{iRe^{i\theta}}{aR^2e^{2i\theta}+bRe^{i\theta}+c}\,d\theta\right|$$

$$\le\int_0^{\pi}\left|\frac{iRe^{i\theta}}{aR^2e^{2i\theta}+bRe^{i\theta}+c}\right|d\theta=\int_0^{\pi}\frac{|i||R||e^{i\theta}|}{|aR^2e^{2i\theta}+bRe^{i\theta}+c|}\,d\theta$$

$$=\int_0^{\pi}\frac{R}{|aR^2e^{2i\theta}+bRe^{i\theta}+c|}\,d\theta\quad\ldots\text{ STANDARD INEQUALITY}\ldots$$

$|z+w+v|\ge|z|-|w|-|v|>0$

$\frac{1}{|z+w+v|}\le\frac{1}{|z|-|w|-|v|}$

$$\le\int_0^{\pi}\frac{R}{|aR^2e^{2i\theta}|-|bRe^{i\theta}|-|c|}\,d\theta$$

$$=\int_0^{\pi}\frac{R}{|a||R^2||e^{2i\theta}|-|b||R||e^{i\theta}|-c}\,d\theta$$

$$=\int_0^{\pi}\frac{R}{aR^2-bR-c}\,d\theta=\frac{R}{aR^2-bR-c}\int_0^{\pi}1\,d\theta$$

$$=\frac{\pi R}{aR^2-bR-c}=O\left(\frac{1}{R}\right)\to 0 \text{ AS } R\to\infty$$

FINALLY AS $R\to\infty$ WE HAVE

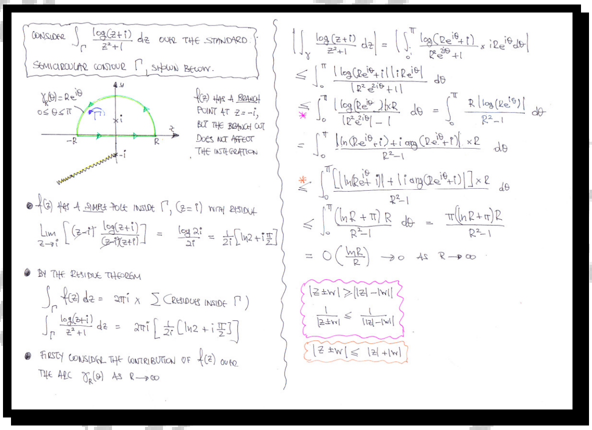
$$\int_{-\infty}^{\infty}\frac{1}{az^2+bz+c}\,dz=\frac{2\pi}{\sqrt{4ac-b^2}}\qquad b^2-4ac<0$$

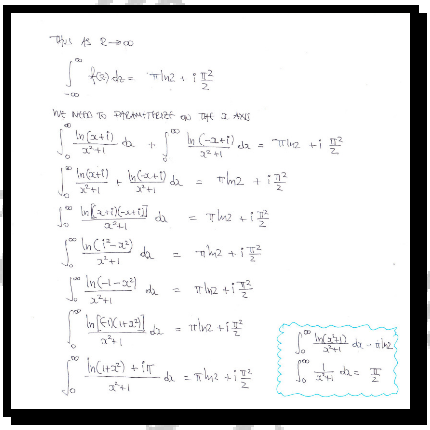
Question 14

$$I = \int_0^{\infty} \frac{\ln\left(x^2+1\right)}{x^2+1}\, dx .$$

By integrating $\dfrac{\ln(z+\mathrm{i})}{z^2+1}$ over a semicircular contour find the exact value of I.

$$\boxed{I = \pi \ln 2}$$





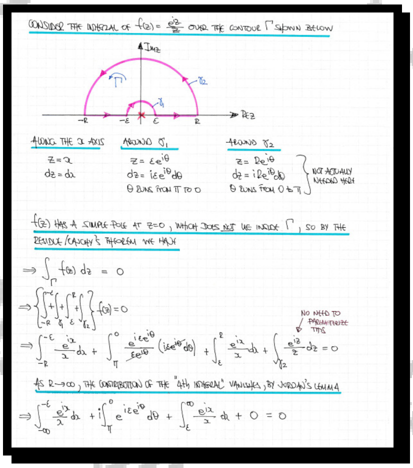
SEMI CIRCLE CONTOUR WITH HOLE

Question 1

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

V, MMR, proof



CONSIDER THE INTEGRAL OF $f(z) = \frac{e^{iz}}{z}$ OVER THE CONTOUR Γ SHOWN BELOW

ALONG THE x AXIS: $z = x$, $dz = dx$

AROUND γ_1: $z = \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta} d\theta$, θ RUNS FROM π TO 0

AROUND γ_2: $z = Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$, θ RUNS FROM 0 to π } NOT ACTUALLY NEEDED HERE

$f(z)$ HAS A SIMPLE POLE AT $z=0$, WHICH DOES NOT LIE INSIDE Γ, SO BY THE RESIDUE/CAUCHY'S THEOREM WE HAVE

$$\Rightarrow \int_{\Gamma} f(z)\, dz = 0$$

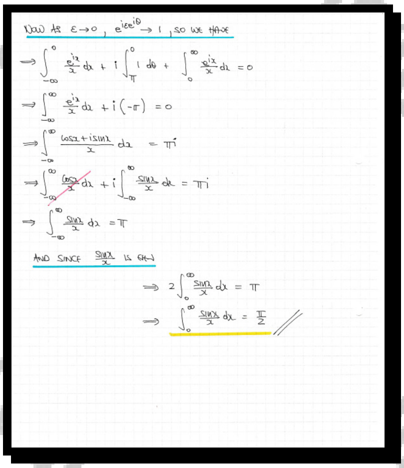
$$\Rightarrow \left\{ \int_{-R}^{-\varepsilon} + \int_{\gamma_1} + \int_{\varepsilon}^{R} + \int_{\gamma_2} \right\} f(z) = 0$$

NO NEED TO PARAMETERIZE THIS

$$\Rightarrow \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\pi}^{0} \frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon e^{i\theta}} (i\varepsilon e^{i\theta} d\theta) + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\, dx + \int_{\gamma_2} \frac{e^{iz}}{z}\, dz = 0$$

AS $R \to \infty$, THE CONTRIBUTION OF THE "4th INTEGRAL" VANISHES, BY JORDAN'S LEMMA

$$\Rightarrow \int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + i\int_{\pi}^{0} e^{i\varepsilon e^{i\theta}}\, d\theta + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx + 0 = 0$$



NOW AS $\varepsilon \to 0$, $e^{i\varepsilon e^{i\theta}} \to 1$, SO WE HAVE

$$\Rightarrow \int_{-\infty}^{0} \frac{e^{ix}}{x}\, dx + i\int_{\pi}^{0} 1\, d\theta + \int_{0}^{\infty} \frac{e^{ix}}{x}\, dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx + i(-\pi) = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\, dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

AND SINCE $\frac{\sin x}{x}$ IS EVEN

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \pi$$

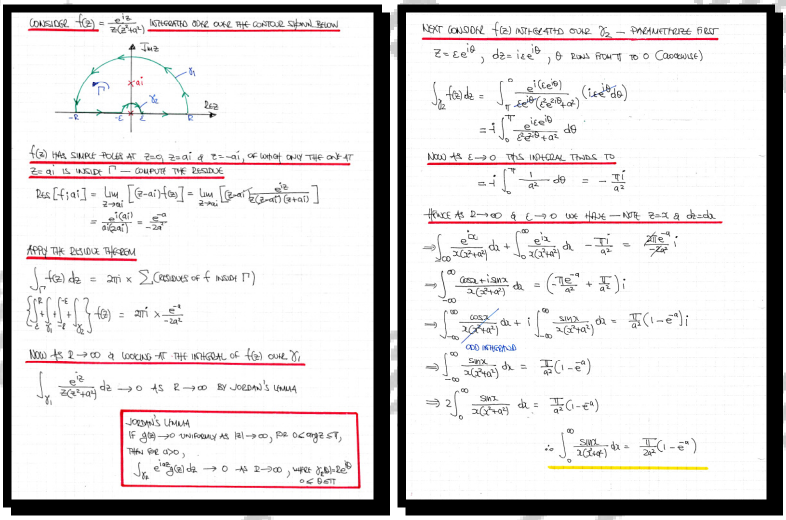
$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x\left(x^2+a^2\right)}\ dx = \frac{\pi}{2a^2}\left(1-\mathrm{e}^{-a}\right).$$

V, , proof



CONSIDER $f(z)=\frac{e^{iz}}{z(z^2+a^2)}$ INTEGRATED OVER OVER THE CONTOUR SHOWN BELOW

$f(z)$ HAS SIMPLE POLES AT $z=0$, $z=ai$ & $z=-ai$, OF WHICH ONLY THE ONE AT $z=ai$ IS INSIDE Γ — COMPUTE THE RESIDUE

$$\text{Res}[f;ai] = \lim_{z\to ai}\left[(z-ai)f(z)\right] = \lim_{z\to ai}\left[(z-ai)\frac{e^{iz}}{z(z-ai)(z+ai)}\right] = \frac{e^{i(ai)}}{ai(2ai)} = \frac{e^{-a}}{-2a^2}$$

APPLY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES OF } f \text{ INSIDE } \Gamma)$$

$$\left\{\int_{\varepsilon}^{R}+\int_{\gamma_1}+\int_{-R}^{-\varepsilon}+\int_{\gamma_2}\right\}f(z) = 2\pi i \times \frac{e^{-a}}{-2a^2}$$

NOW AS $R\to\infty$ & LOOKING AT THE INTEGRAL OF $f(z)$ OVER γ_1

$$\int_{\gamma_1}\frac{e^{iz}}{z(z^2+a^2)}\,dz \to 0 \text{ AS } R\to\infty \text{ BY JORDAN'S LEMMA}$$

JORDAN'S LEMMA
IF $g(z)\to 0$ UNIFORMLY AS $|z|\to\infty$, FOR $0 \le \arg z \le \pi$,
THEN FOR $a>0$,
$\int_{\gamma_R} e^{iaz}g(z)\,dz \to 0$ AS $R\to\infty$, WHERE $\gamma_R(\theta)=Re^{i\theta}$, $0\le\theta\le\pi$

NEXT CONSIDER $f(z)$ INTEGRATED OVER γ_2 — PARAMETRIZE FIRST

$z=\varepsilon e^{i\theta}$, $dz=i\varepsilon e^{i\theta}$, θ RUNS FROM π TO 0 (CLOCKWISE)

$$\int_{\gamma_2} f(z)\,dz = \int_{\pi}^{0}\frac{e^{i(\varepsilon e^{i\theta})}}{\varepsilon e^{i\theta}(\varepsilon^2e^{2i\theta}+a^2)}(i\varepsilon e^{i\theta}\,d\theta) = -i\int_0^{\pi}\frac{e^{i\varepsilon e^{i\theta}}}{\varepsilon^2e^{2i\theta}+a^2}\,d\theta$$

NOW AS $\varepsilon\to 0$ THIS INTEGRAL TENDS TO

$$= -i\int_0^{\pi}\frac{1}{a^2}\,d\theta = -\frac{\pi i}{a^2}$$

HENCE AS $R\to\infty$ & $\varepsilon\to 0$ WE HAVE — NOTE $z=x$ & $dz=dx$

$$\Rightarrow \int_{-\infty}^{0}\frac{e^{ix}}{x(x^2+a^2)}\,dx + \int_0^{\infty}\frac{e^{ix}}{x(x^2+a^2)}\,dx - \frac{\pi i}{a^2} = \frac{2\pi e^{-a}}{-2a^2}i$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{x(x^2+a^2)}\,dx = \left(-\frac{\pi e^{-a}}{a^2}+\frac{\pi}{a^2}\right)i$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x}{x(x^2+a^2)}\,dx + i\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-a})i$$

ODD INTEGRAND

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-a})$$

$$\Rightarrow 2\int_0^{\infty}\frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{a^2}(1-e^{-a})$$

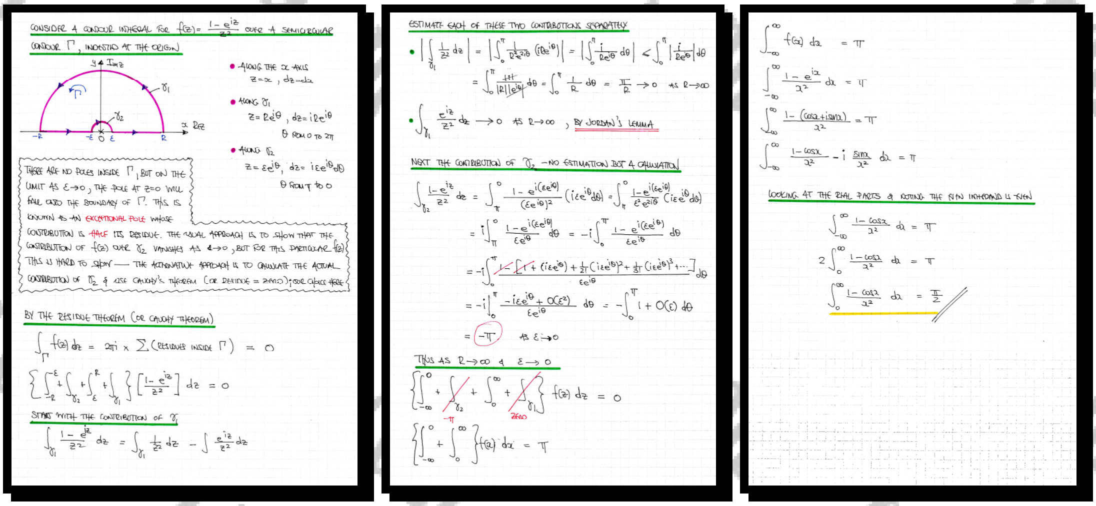
$$\therefore \int_0^{\infty}\frac{\sin x}{x(x^2+a^2)}\,dx = \frac{\pi}{2a^2}(1-e^{-a})$$

Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}.$$

[illegible], proof



Question 4

$$\int_0^{\infty} \frac{\ln x}{1+x^4}\ dx .$$

a) Find the value of the above improper integral, by integrating

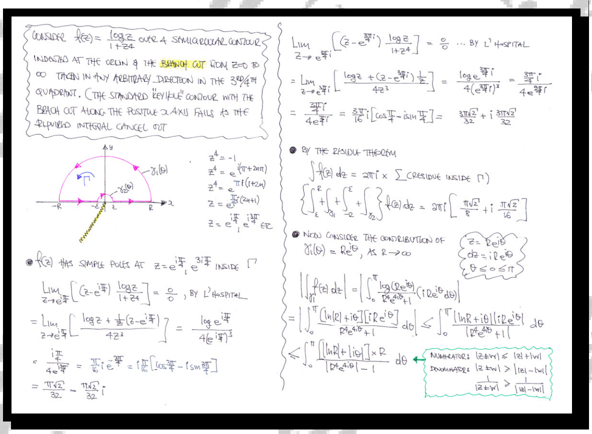
$$f(z) = \frac{\log z}{1+z^4}, \quad z \in \mathbb{C},$$

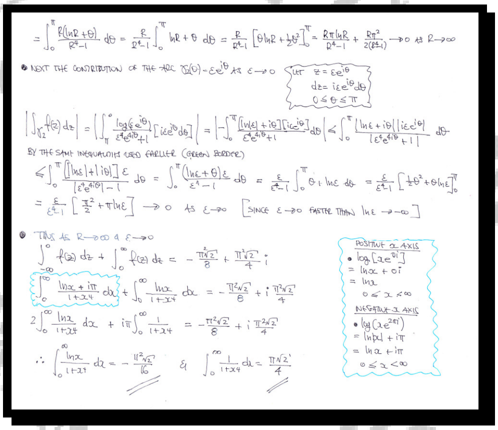
over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

b) State the value of

$$\int_0^{\infty} \frac{1}{1+x^4}\ dx .$$

$$\boxed{-\frac{\pi^2\sqrt{2}}{16}}, \quad \boxed{\frac{\pi\sqrt{2}}{4}}$$





Question 5

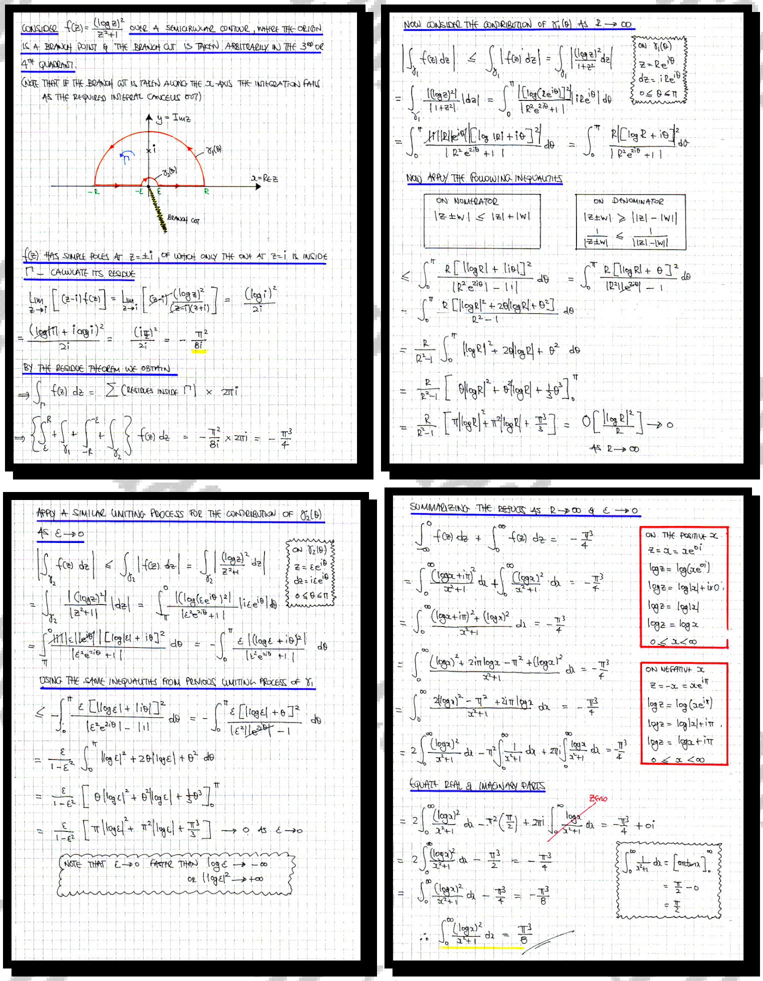
$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2}\, dx .$$

Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^2}, \quad z \in \mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

MM3-E , $\boxed{\frac{\pi^3}{8}}$



Question 6

$$\int_0^\infty \frac{(\ln x)^2}{1+x^4}\, dx .$$

a) Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^4}, \ z \in \mathbb{C},$$

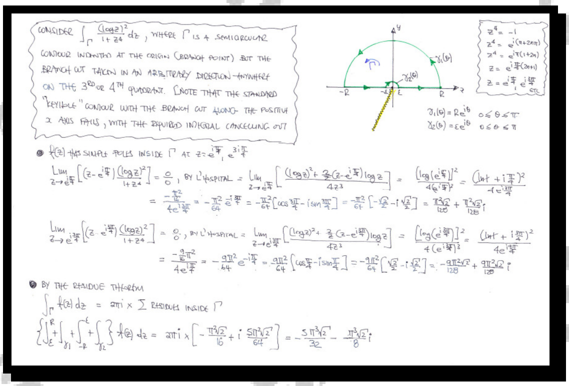
over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

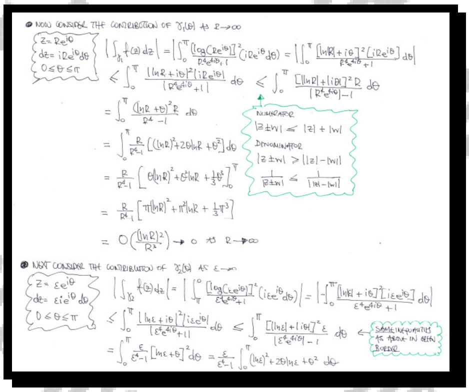
$$\left[\text{You may assume without proof that } \int_0^\infty \frac{1}{1+x^4}\, dx = \frac{\pi\sqrt{2}}{4} \right]$$

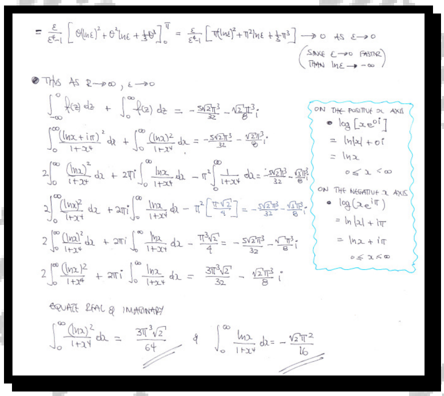
b) State the value of

$$\int_0^\infty \frac{\ln x}{1+x^4}\, dx .$$

$$\boxed{\frac{3\pi^3\sqrt{2}}{64}}, \ \boxed{-\frac{\pi^2\sqrt{2}}{16}}$$







KEYHOLE CONTOUR

(Branch Cuts)

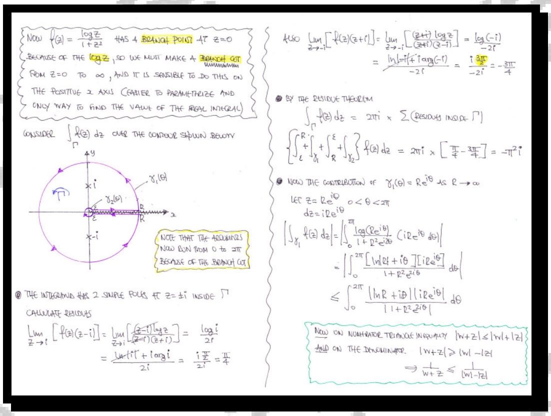
Question 1

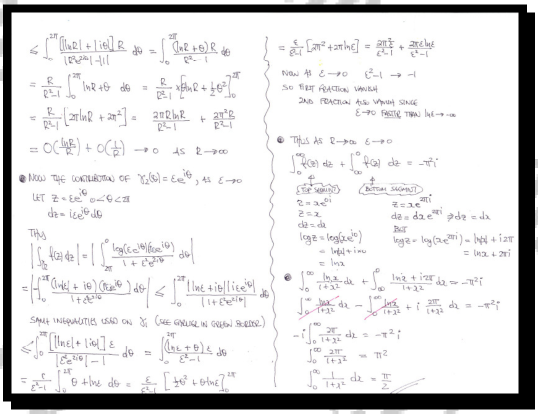
$$f(z)=\frac{\log z}{1+z^2},\ z\in\mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_0^\infty \frac{1}{1+x^2}\,dx=\frac{\pi}{2}.$$

proof



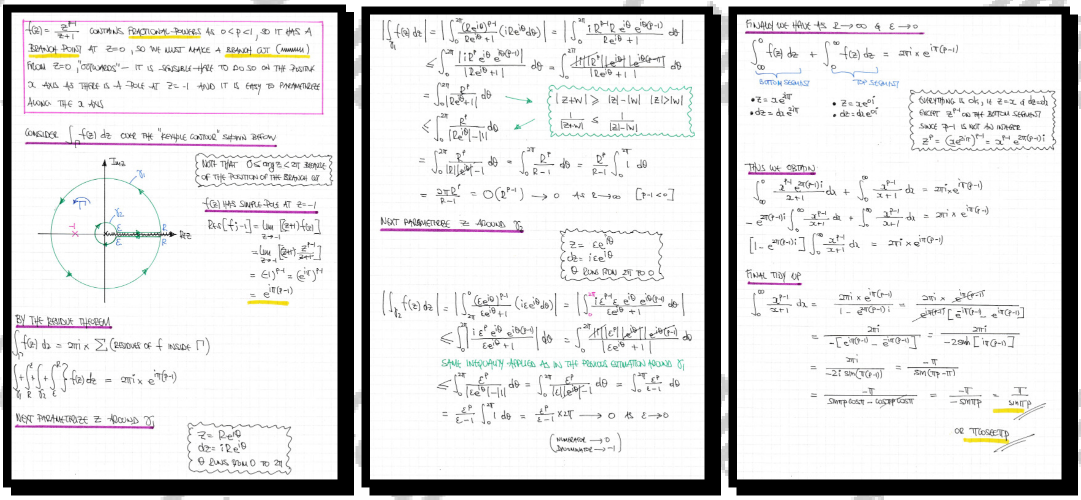


Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}\, dx = \pi \operatorname{cosec}(p\pi)\,, \ \ 0 < p < 1\,.$$

V, proof

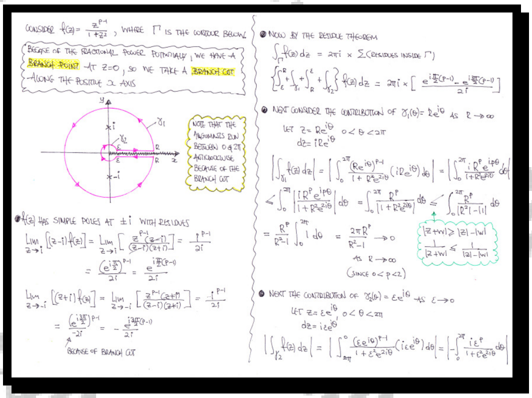


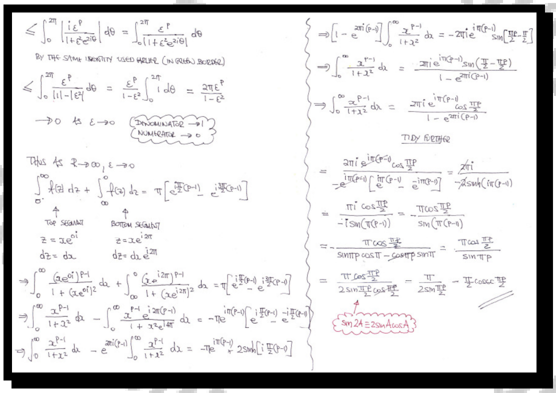
Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec}\left(\frac{p\pi}{2}\right), \quad 0 < p < 2.$$

proof





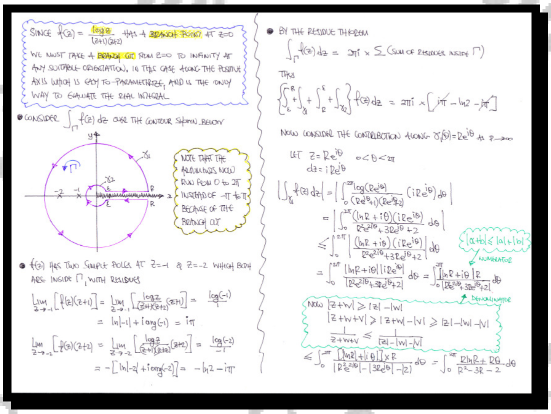
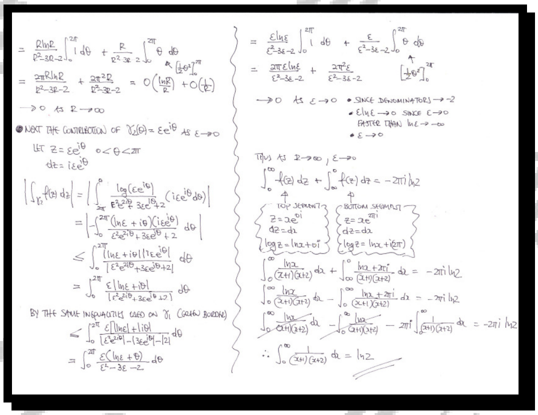
Question 4

$$f(z) = \frac{\log z}{(z+1)(z+2)}, \quad z \in \mathbb{C}.$$

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_0^\infty \frac{1}{(x+1)(x+2)}\,dx = \ln 2.$$

proof

Question 5

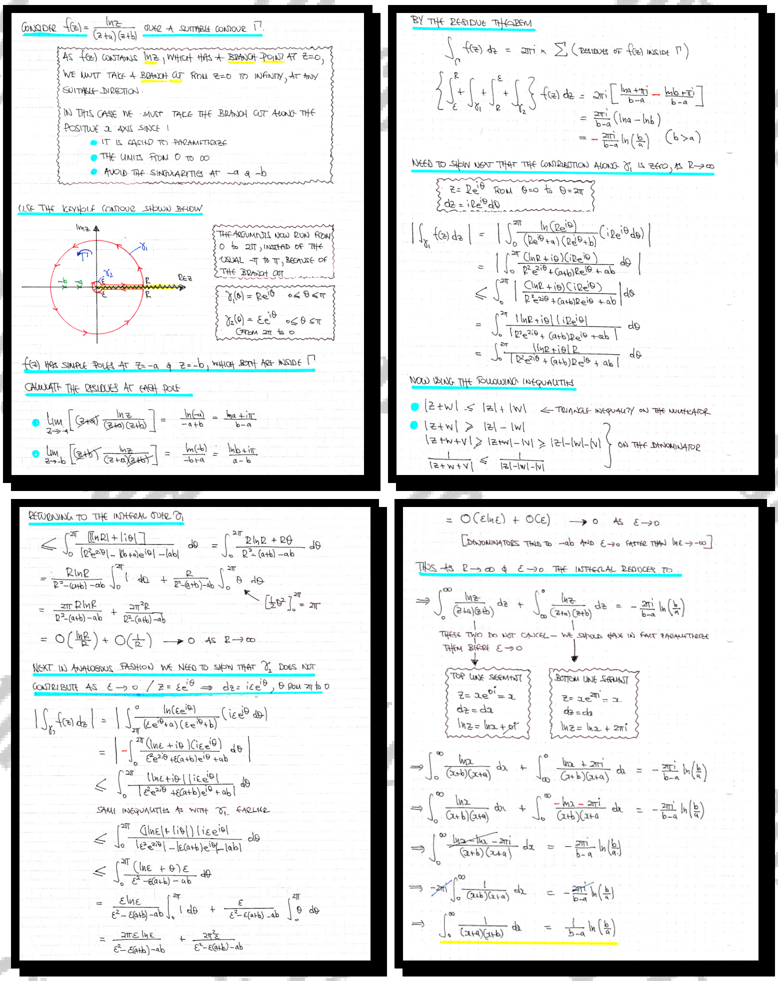
$$f(z) = \frac{\log z}{(z+a)(z+b)}, \quad z \in \mathbb{C},$$

where $a \in \mathbb{R}^{+}$, $b \in \mathbb{R}^{+}$ with $b > a$.

By integrating $f(z)$ over a suitable contour Γ, show that

$$\int_{0}^{\infty} \frac{1}{(x+a)(x+b)} \, dx = \frac{1}{b-a} \ln\left(\frac{b}{a}\right).$$

proof

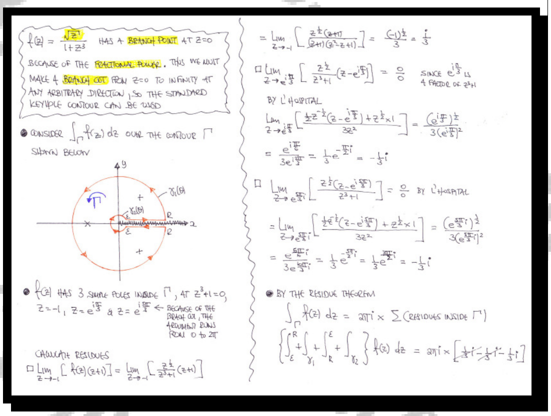


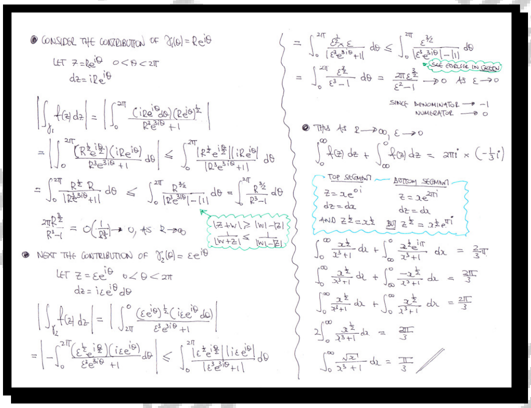
Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^3}\, dx = \frac{\pi}{3}.$$

proof





Question 7

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2}\, dx .$$

An attempt is made to find the value of the above improper integral, by integrating

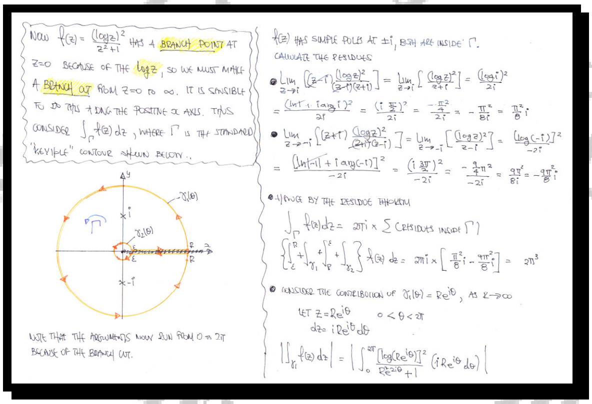
$$f(z) = \frac{(\ln z)^2}{1+z^2}, \ z \in \mathbb{C},$$

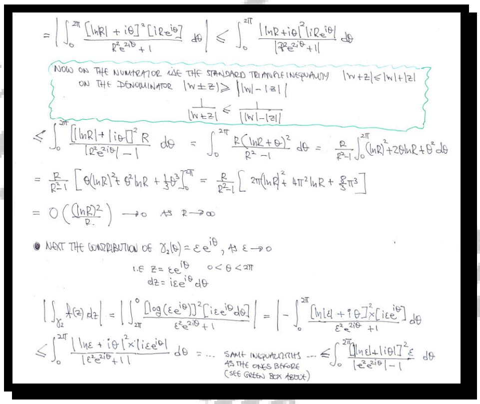
over the standard "keyhole" contour with a branch cut taken on the positive x axis.

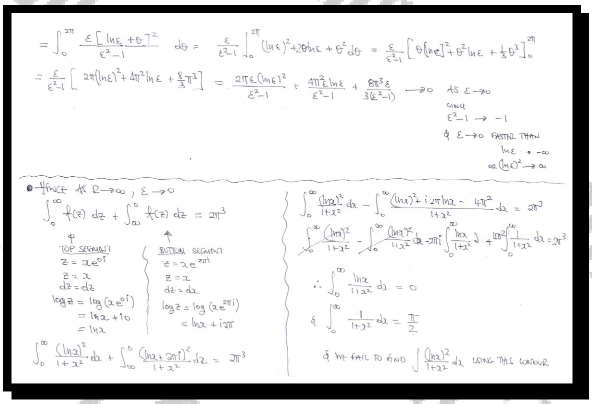
a) Show that such attempt fails.

b) Calculate the value of the two integrals that can be found during this attempt.

$$\boxed{\int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2}}, \quad \boxed{\int_0^{\infty} \frac{\ln x}{1+x^2}\, dx = 0}$$





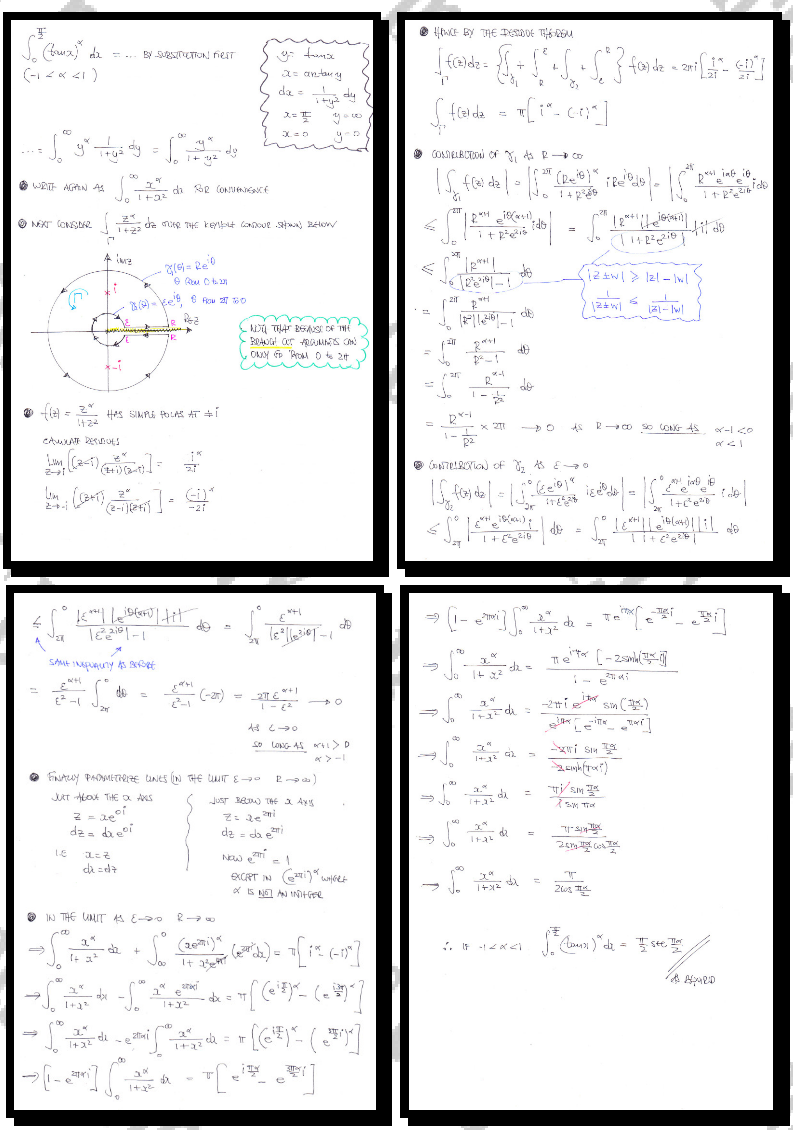


Question 8

Use a substitution followed by integration of a suitable complex function over an appropriate contour, to show that

$$\int_0^{\frac{1}{2}\pi} (\tan x)^{\alpha} \ dx = \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi\alpha\right), \ \ -1 < \alpha < 1 .$$

proof



SPECIAL CONTOURS

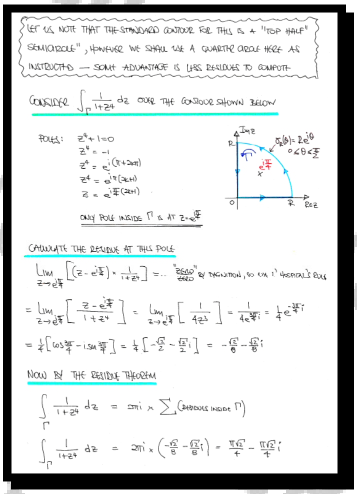
Question 1

Consider the contour Γ located in the first quadrant, defined as the boundary of a quarter circular sector of radius R, with centre at the origin O.

By integrating a suitable complex function over Γ show that

$$\int_0^{\infty} \frac{1}{1+x^4}\, dx = \frac{\pi\sqrt{2}}{4}.$$

$\boxed{}$, $\boxed{\frac{\pi\sqrt{2}}{4}}$



LET US NOTE THAT THE STANDARD CONTOUR FOR THIS IS A "TOP HALF" SEMICIRCLE", HOWEVER WE SHALL USE A QUARTER CIRCLE HERE AS INSTRUCTED — SOME ADVANTAGE IS LESS RESIDUES TO COMPUTE

CONSIDER $\int_\Gamma \frac{1}{1+z^4}\,dz$ OVER THE CONTOUR SHOWN BELOW

POLES: $z^4+1=0$
$z^4=-1$
$z^4=e^{i(\pi+2k\pi)}$
$z^4=e^{i\pi(2k+1)}$
$z=e^{i\frac{\pi}{4}(2k+1)}$

ONLY POLE INSIDE Γ IS AT $z=e^{i\frac{\pi}{4}}$

CALCULATE THE RESIDUE AT THIS POLE

$\lim_{z\to e^{i\frac{\pi}{4}}}\left[(z-e^{i\frac{\pi}{4}})\times\frac{1}{1+z^4}\right]=\dots$ "ZERO/ZERO" BY DEFINITION, SO USE L'HOSPITAL'S RULE

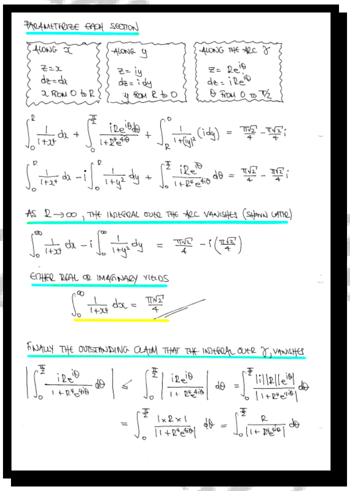
$=\lim_{z\to e^{i\frac{\pi}{4}}}\left[\frac{z-e^{i\frac{\pi}{4}}}{1+z^4}\right]=\lim_{z\to e^{i\frac{\pi}{4}}}\left[\frac{1}{4z^3}\right]=\frac{1}{4e^{\frac{3\pi}{4}i}}=\frac{1}{4}e^{-\frac{3\pi}{4}i}$

$=\frac{1}{4}\left[\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right]=\frac{1}{4}\left[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right]=-\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{8}i$

NOW BY THE RESIDUE THEOREM

$\int_\Gamma\frac{1}{1+z^4}\,dz=2\pi i\times\sum(\text{RESIDUES INSIDE }\Gamma)$

$\int_\Gamma\frac{1}{1+z^4}\,dz=2\pi i\times\left(-\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{8}i\right)=\frac{\pi\sqrt{2}}{4}-\frac{\pi\sqrt{2}}{4}i$



PARAMETRIZE EACH SECTION

ALONG x	ALONG y	ALONG THE ARC γ
$z=x$	$z=iy$	$z=Re^{i\theta}$
$dz=dx$	$dz=i\,dy$	$dz=iRe^{i\theta}$
x FROM 0 to R	y FROM R to 0	θ FROM 0 TO $\pi/2$

$\int_0^R\frac{1}{1+x^4}\,dx+\int_0^{\frac{\pi}{2}}\frac{iRe^{i\theta}\,d\theta}{1+R^4e^{4i\theta}}+\int_R^0\frac{1}{1+(iy)^4}(i\,dy)=\frac{\pi\sqrt{2}}{4}-\frac{\pi\sqrt{2}}{4}i$

$\int_0^R\frac{1}{1+x^4}\,dx-i\int_0^R\frac{1}{1+y^4}\,dy+\int_0^{\frac{\pi}{2}}\frac{iRe^{i\theta}}{1+R^4e^{4i\theta}}\,d\theta=\frac{\pi\sqrt{2}}{4}-\frac{\pi\sqrt{2}}{4}i$

AS $R\to\infty$, THE INTEGRAL OVER THE ARC VANISHES (SHOWN LATER)

$\int_0^\infty\frac{1}{1+x^4}\,dx-i\int_0^\infty\frac{1}{1+y^4}\,dy=\frac{\pi\sqrt{2}}{4}-i\left(\frac{\pi\sqrt{2}}{4}\right)$

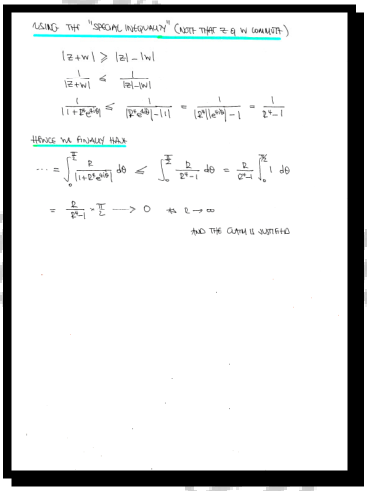
EITHER REAL OR IMAGINARY YIELDS

$\int_0^\infty\frac{1}{1+x^4}\,dx=\frac{\pi\sqrt{2}}{4}$

FINALLY THE OUTSTANDING CLAIM THAT THE INTEGRAL OVER γ, VANISHES

$\left|\int_0^{\frac{\pi}{2}}\frac{iRe^{i\theta}}{1+R^4e^{4i\theta}}\,d\theta\right|\leqslant\int_0^{\frac{\pi}{2}}\left|\frac{iRe^{i\theta}}{1+R^4e^{4i\theta}}\right|d\theta=\int_0^{\frac{\pi}{2}}\frac{|i||R||e^{i\theta}|}{|1+R^4e^{4i\theta}|}\,d\theta$

$=\int_0^{\frac{\pi}{2}}\frac{1\times R\times1}{|1+R^4e^{4i\theta}|}\,d\theta=\int_0^{\frac{\pi}{2}}\frac{R}{|1+R^4e^{4i\theta}|}\,d\theta$



USING THE "SPECIAL INEQUALITY" (NOTE THAT z & w COMMUTE)

$|z+w|\geqslant|z|-|w|$

$\frac{1}{|z+w|}\leqslant\frac{1}{|z|-|w|}$

$\frac{1}{|1+R^4e^{4i\theta}|}\leqslant\frac{1}{|R^4e^{4i\theta}|-|1|}=\frac{1}{|R^4||e^{4i\theta}|-1}=\frac{1}{R^4-1}$

HENCE WE FINALLY HAVE

$\dots=\int_0^{\frac{\pi}{2}}\frac{R}{|1+R^4e^{4i\theta}|}\,d\theta\leqslant\int_0^{\frac{\pi}{2}}\frac{R}{R^4-1}\,d\theta=\frac{R}{R^4-1}\int_0^{\pi/2}1\,d\theta$

$=\frac{R}{R^4-1}\times\frac{\pi}{2}\longrightarrow0$ AS $R\to\infty$

AND THE CLAIM IS JUSTIFIED

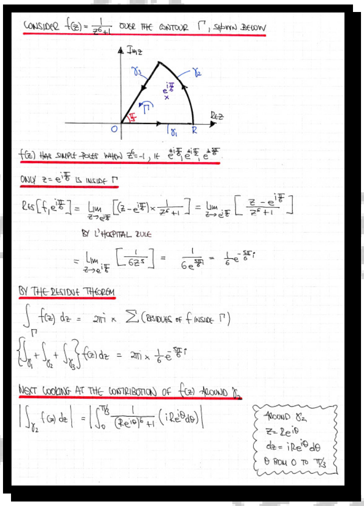
Question 2

By integrating a suitable complex function over a contour defined as the outline of a circular sector subtending an angle of $\frac{1}{3}\pi$ at the origin, find an exact value for

$$\int_0^{\infty} \frac{1}{1+x^6}\, dx \, .$$

No credit will be given for integration over alternative contours.

V, , $\boxed{\frac{\pi}{3}}$



CONSIDER $f(z) = \frac{1}{z^6+1}$ OVER THE CONTOUR Γ, SHOWN BELOW

$f(z)$ HAS SIMPLE POLES WHEN $z^6=-1$, IE $e^{\pm i\frac{\pi}{6}}, e^{\pm i\frac{\pi}{2}}, e^{\pm\frac{5\pi}{6}}$

ONLY $z = e^{i\frac{\pi}{6}}$ IS INSIDE Γ

$\text{Res}[f, e^{i\frac{\pi}{6}}] = \lim_{z \to e^{i\frac{\pi}{6}}} \left[(z - e^{i\frac{\pi}{6}}) \times \frac{1}{z^6+1} \right] = \lim_{z \to e^{i\frac{\pi}{6}}} \left[\frac{z - e^{i\frac{\pi}{6}}}{z^6+1} \right]$

BY L'HOSPITAL RULE

$= \lim_{z \to e^{i\frac{\pi}{6}}} \left[\frac{1}{6z^5} \right] = \frac{1}{6e^{\frac{5\pi}{6}i}} = \frac{1}{6}e^{-\frac{5\pi}{6}i}$

BY THE RESIDUE THEOREM

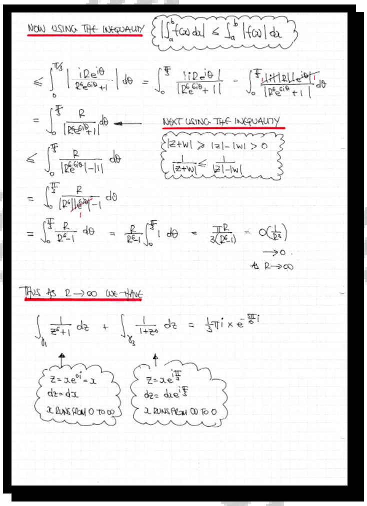
$\int_{\Gamma} f(z)\, dz = 2\pi i \times \sum(\text{RESIDUES OF } f \text{ INSIDE } \Gamma)$

$\left\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right\} f(z)\, dz = 2\pi i \times \frac{1}{6}e^{-\frac{5\pi}{6}i}$

NEXT LOOKING AT THE CONTRIBUTION OF $f(z)$ AROUND γ_2

$\left| \int_{\gamma_2} f(z)\, dz \right| = \left| \int_0^{\pi/3} \frac{1}{(Re^{i\theta})^6+1} (iRe^{i\theta} d\theta) \right|$

AROUND γ_2
$z = Re^{i\theta}$
$dz = iRe^{i\theta} d\theta$
θ FROM 0 TO $\pi/3$



NOW USING THE INEQUALITY $\left| \int_a^b f(x)\, dx \right| \leqslant \int_a^b |f(x)|\, dx$

$\leqslant \int_0^{\pi/3} \left| \frac{iRe^{i\theta}}{R^6e^{6i\theta}+1} \right| d\theta = \int_0^{\frac{\pi}{3}} \frac{|iRe^{i\theta}|}{|R^6e^{6i\theta}+1|} = \int_0^{\frac{\pi}{3}} \frac{|i||R||e^{i\theta}|}{|R^6e^{6i\theta}+1|} d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{R}{|R^6e^{6i\theta}+1|} d\theta$

NEXT USING THE INEQUALITY
$|z+w| \geqslant |z|-|w| > 0$
$\frac{1}{|z+w|} \leqslant \frac{1}{|z|-|w|}$

$\leqslant \int_0^{\frac{\pi}{3}} \frac{R}{|R^6e^{6i\theta}|-|1|} d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{R}{|R^6||e^{6i\theta}|-1} d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{R}{R^6-1} d\theta = \frac{R}{R^6-1}\int_0^{\frac{\pi}{3}} 1\, d\theta = \frac{\pi R}{3(R^6-1)} = O\left(\frac{1}{R^5}\right)$

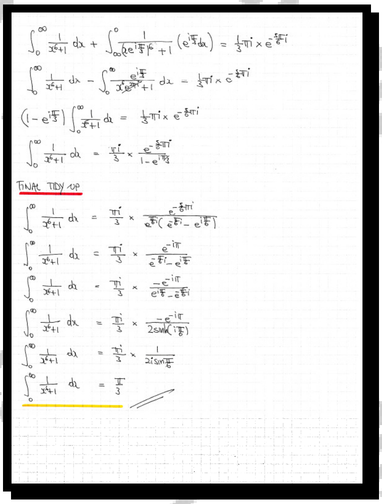
$\to 0$ AS $R \to \infty$

THUS AS $R \to \infty$ WE HAVE

$\int_{\gamma_1} \frac{1}{z^6+1} dz + \int_{\gamma_3} \frac{1}{1+z^6} dz = \frac{1}{3}\pi i \times e^{-\frac{5\pi}{6}i}$

$z = xe^{0i} = x$
$dz = dx$
x RUNS FROM 0 TO ∞

$z = xe^{i\frac{\pi}{3}}$
$dz = dx\, e^{i\frac{\pi}{3}}$
x RUNS FROM ∞ TO 0



$\int_0^{\infty} \frac{1}{x^6+1} dx + \int_{\infty}^{0} \frac{1}{(xe^{i\frac{\pi}{3}})^6+1} (e^{i\frac{\pi}{3}} dx) = \frac{1}{3}\pi i \times e^{-\frac{5\pi}{6}i}$

$\int_0^{\infty} \frac{1}{x^6+1} dx - \int_0^{\infty} \frac{e^{i\frac{\pi}{3}}}{x^6e^{2\pi i}+1} dx = \frac{1}{3}\pi i \times e^{-\frac{5}{6}\pi i}$

$\left(1 - e^{i\frac{\pi}{3}}\right) \int_0^{\infty} \frac{1}{x^6+1} dx = \frac{1}{3}\pi i \times e^{-\frac{5}{6}\pi i}$

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{e^{-\frac{5}{6}\pi i}}{1-e^{i\pi/3}}$

FINAL TIDY UP

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{e^{-\frac{5}{6}\pi i}}{e^{\frac{\pi}{6}i}(e^{-\frac{\pi}{6}i} - e^{i\frac{\pi}{6}})}$

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{e^{-i\pi}}{e^{-\frac{\pi}{6}i} - e^{i\frac{\pi}{6}}}$

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{-e^{-i\pi}}{e^{i\frac{\pi}{6}} - e^{-\frac{\pi}{6}i}}$

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{-e^{-i\pi}}{2\sinh(i\frac{\pi}{6})}$

$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi i}{3} \times \frac{1}{2i\sin\frac{\pi}{6}}$

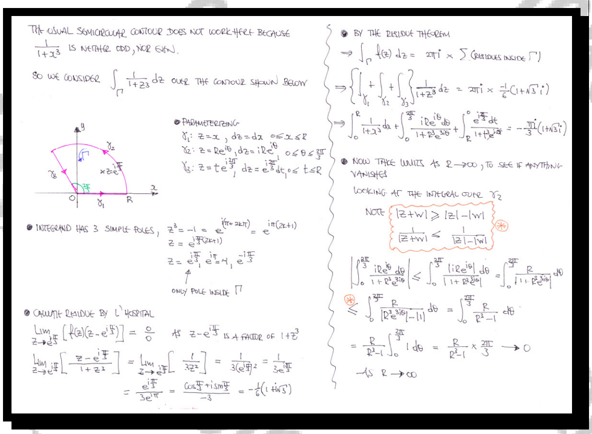
$\int_0^{\infty} \frac{1}{x^6+1} dx = \frac{\pi}{3}$

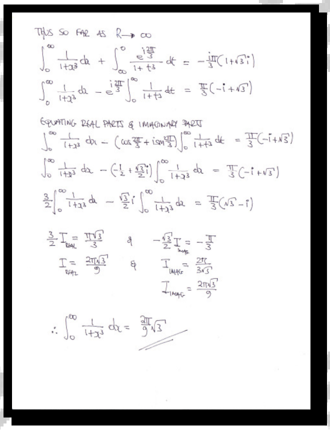
Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^\infty \frac{1}{1+x^3}\ dx .$$

$$\boxed{\frac{2\pi\sqrt{3}}{9}}$$



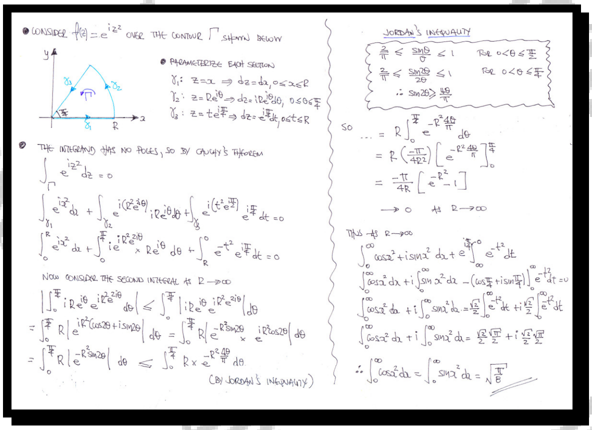


Question 4

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \cos\left(x^2\right)\ dx = \int_0^{\infty} \sin\left(x^2\right)\ dx = \sqrt{\frac{\pi}{8}}.$$

proof

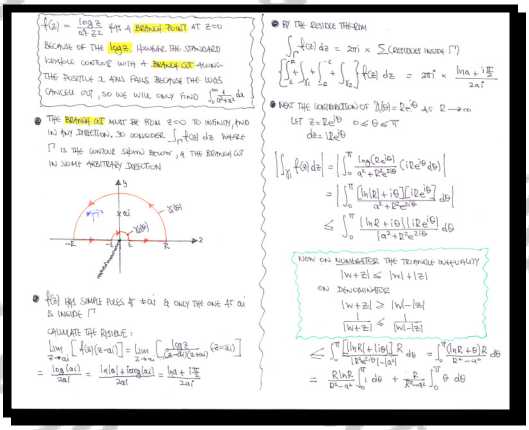


Question 5

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\ln x}{a^2+x^2}\, dx = \frac{\pi \ln a}{2a}.$$

proof



$f(z) = \frac{\log z}{a^2+z^2}$ HAS A BRANCH POINT AT $z=0$ BECAUSE OF THE $\log z$. HOWEVER THE STANDARD KEYHOLE CONTOUR WITH A BRANCH CUT ALONG THE POSITIVE x AXIS FAILS BECAUSE THE LOGS CANCEL OUT, SO WE WILL ONLY FIND $\int_0^{\infty} \frac{1}{a^2+x^2}\,dx$

- THE BRANCH CUT MUST BE FROM $z=0$ TO INFINITY, AND IN ANY DIRECTION. SO CONSIDER $\int_{\Gamma} f(z)\,dz$ WHERE Γ IS THE CONTOUR SHOWN BELOW, & THE BRANCH CUT IN SOME ARBITRARY DIRECTION

- $f(z)$ HAS SIMPLE POLES AT $\pm ai$ & ONLY THE ONE AT ai IS INSIDE Γ

CALCULATE THE RESIDUE:

$$\lim_{z \to ai}\left[f(z)(z-ai)\right] = \lim_{z\to ai}\left[\frac{\log z}{(z-ai)(z+ai)}(z-ai)\right]$$
$$= \frac{\log(ai)}{2ai} = \frac{\ln|a| + i\arg(ai)}{2ai} = \frac{\ln a + i\frac{\pi}{2}}{2ai}$$

- BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z)\,dz = 2\pi i \times \sum(\text{RESIDUES INSIDE } \Gamma)$$
$$\left\{\int_{\varepsilon}^{R} + \int_{\gamma_1} + \int_{-R}^{-\varepsilon} + \int_{\gamma_2}\right\} f(z)\,dz = 2\pi i \times \frac{\ln a + i\frac{\pi}{2}}{2ai}$$

- NEXT THE CONTRIBUTION OF $\gamma_1(\theta) = Re^{i\theta}$ AS $R \to \infty$

LET $z = Re^{i\theta}$ $\quad 0 \leqslant \theta \leqslant \pi$

$dz = iRe^{i\theta}$

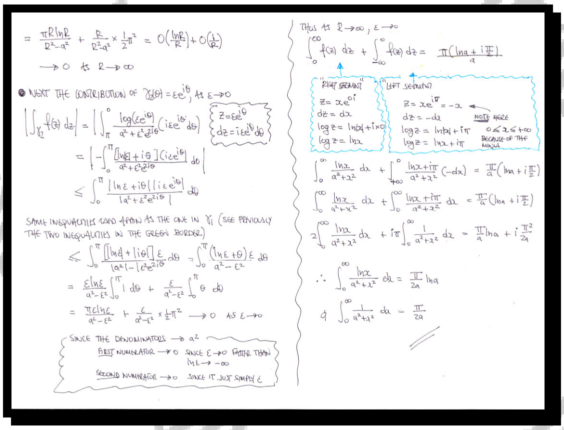
$$\left|\int_{\gamma_1} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{\log(Re^{i\theta})}{a^2+R^2e^{2i\theta}}(iRe^{i\theta}\,d\theta)\right|$$
$$= \left|\int_0^{\pi} \frac{[\ln|R| + i\theta][iRe^{i\theta}]}{a^2+R^2e^{2i\theta}}\,d\theta\right|$$
$$\leqslant \int_0^{\pi} \frac{|\ln R + i\theta||iRe^{i\theta}|}{|a^2+R^2e^{2i\theta}|}\,d\theta$$

NOW ON NUMERATOR THE TRIANGLE INEQUALITY $|w+z| \leqslant |w| + |z|$

ON DENOMINATOR $|w+z| \geqslant |w| - |z|$

$$\frac{1}{|w+z|} \leqslant \frac{1}{|w|-|z|}$$

$$\leqslant \int_0^{\pi} \frac{[|\ln R| + |i\theta|]R}{|R^2e^{2i\theta}| - |a^2|}\,d\theta = \int_0^{\pi} \frac{(\ln R + \theta)R}{R^2-a^2}\,d\theta$$
$$= \frac{R\ln R}{R^2-a^2}\int_0^{\pi} 1\,d\theta + \frac{R}{R^2-a^2}\int_0^{\pi}\theta\,d\theta$$



$$= \frac{\pi R\ln R}{R^2-a^2} + \frac{R}{R^2-a^2}\times\frac{1}{2}\pi^2 = O\left(\frac{\ln R}{R}\right) + O\left(\frac{1}{R}\right)$$
$\to 0$ AS $R \to \infty$

- NEXT THE CONTRIBUTION OF $\gamma_2(\theta) = \varepsilon e^{i\theta}$, AS $\varepsilon \to 0$

$z = \varepsilon e^{i\theta}$, $dz = i\varepsilon e^{i\theta}\,d\theta$

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_{\pi}^{0} \frac{\log(\varepsilon e^{i\theta})}{a^2+\varepsilon^2e^{2i\theta}}(i\varepsilon e^{i\theta}\,d\theta)\right|$$
$$= \left|-\int_0^{\pi} \frac{[\ln|\varepsilon| + i\theta](i\varepsilon e^{i\theta})}{a^2+\varepsilon^2e^{2i\theta}}\,d\theta\right|$$
$$\leqslant \int_0^{\pi} \frac{|\ln\varepsilon + i\theta||i\varepsilon e^{i\theta}|}{|a^2+\varepsilon^2e^{2i\theta}|}\,d\theta$$

SAME INEQUALITIES USED AGAIN AS THE ONE IN γ_1 (SEE PREVIOUSLY THE TWO INEQUALITIES IN THE GREEN BORDER)

$$\leqslant \int_0^{\pi} \frac{[|\ln\varepsilon| + |i\theta|]\varepsilon}{|a^2| - |\varepsilon^2e^{2i\theta}|}\,d\theta = \int_0^{\pi} \frac{(\ln\varepsilon + \theta)\varepsilon}{a^2-\varepsilon^2}\,d\theta$$
$$= \frac{\varepsilon\ln\varepsilon}{a^2-\varepsilon^2}\int_0^{\pi} 1\,d\theta + \frac{\varepsilon}{a^2-\varepsilon^2}\int_0^{\pi}\theta\,d\theta$$
$$= \frac{\pi\varepsilon\ln\varepsilon}{a^2-\varepsilon^2} + \frac{\varepsilon}{a^2-\varepsilon^2}\times\frac{1}{2}\pi^2 \to 0 \text{ AS } \varepsilon \to 0$$

SINCE THE DENOMINATORS $\to a^2$

FIRST NUMERATOR $\to 0$ SINCE $\varepsilon \to 0$ FASTER THAN $\ln\varepsilon \to -\infty$

SECOND NUMERATOR $\to 0$ SINCE IT JUST SIMPLY ε

THUS AS $R \to \infty$, $\varepsilon \to 0$

$$\int_0^{\infty} f(z)\,dz + \int_{-\infty}^{0} f(z)\,dz = \frac{\pi(\ln a + i\frac{\pi}{2})}{a}$$

"RIGHT SEGMENT": $z = xe^{0i}$, $dz = dx$, $\log z = \ln|x| + i\times 0$, $\log z = \ln x$

"LEFT SEGMENT": $z = xe^{i\pi} = -x$, $dz = -dx$, $\log z = \ln|x| + i\pi$, $\log z = \ln x + i\pi$ — NOTE HERE $0 \leqslant x \leqslant +\infty$ BECAUSE OF THE MINUS

$$\int_0^{\infty} \frac{\ln x}{a^2+x^2}\,dx + \int_{+\infty}^{0} \frac{\ln x + i\pi}{a^2+x^2}(-dx) = \frac{\pi}{a}\left(\ln a + i\frac{\pi}{2}\right)$$
$$\int_0^{\infty} \frac{\ln x}{a^2+x^2}\,dx + \int_0^{\infty} \frac{\ln x + i\pi}{a^2+x^2}\,dx = \frac{\pi}{a}\left(\ln a + i\frac{\pi}{2}\right)$$
$$2\int_0^{\infty} \frac{\ln x}{a^2+x^2}\,dx + i\pi\int_0^{\infty} \frac{1}{a^2+x^2}\,dx = \frac{\pi}{a}\ln a + i\frac{\pi^2}{2a}$$

$$\therefore \int_0^{\infty} \frac{\ln x}{a^2+x^2}\,dx = \frac{\pi}{2a}\ln a$$

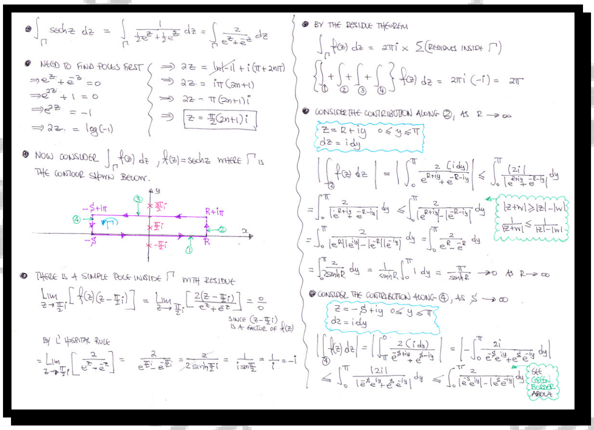
$$\text{\&} \int_0^{\infty} \frac{1}{a^2+x^2}\,dx = \frac{\pi}{2a}$$

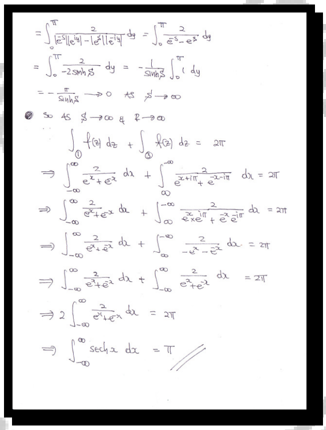
Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_{-\infty}^{\infty} \operatorname{sech} x \;\, dx = \pi .$$

proof





Question 7

It is required to evaluate the integral

$$\int_0^{\infty} \mathrm{e}^{-x^2} \cos x \ dx .$$

a) Show that the above integral can be written as

$$\frac{1}{2}\mathrm{e}^{-\frac{1}{4}} \int_{-\infty}^{\infty} \mathrm{e}^{-\left(x+\frac{1}{2}\mathrm{i}\right)^2} \ dx$$

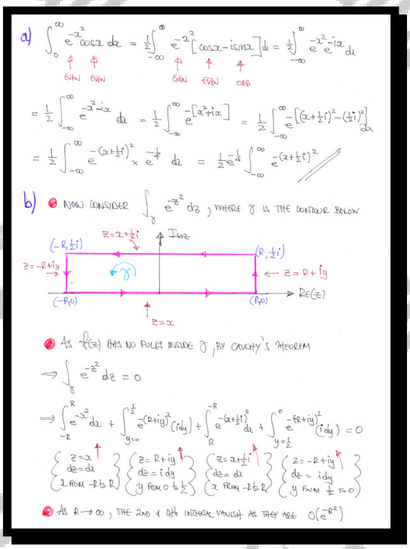
b) By integrating the complex function $f(z)=\mathrm{e}^{-z^2}$, over a rectangular contour with vertices at $(-R,0)$, $(R,0)$, $\left(R,\frac{1}{2}\mathrm{i}\right)$ and $\left(-R,\frac{1}{2}\mathrm{i}\right)$, show that

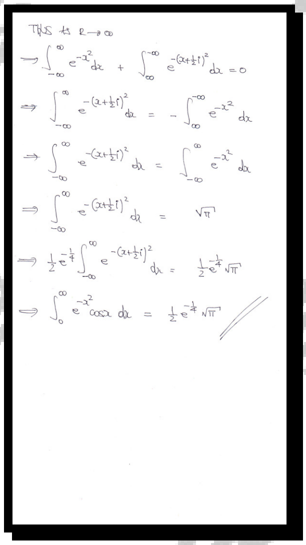
$$\int_0^{\infty} \mathrm{e}^{-x^2} \cos x \ dx = \frac{1}{2}\mathrm{e}^{-\frac{1}{4}}\sqrt{\pi} .$$

You may assume without proof that

$$\int_0^{\infty} \mathrm{e}^{-x^2} \ dx \ = \frac{1}{2}\sqrt{\pi} .$$

proof





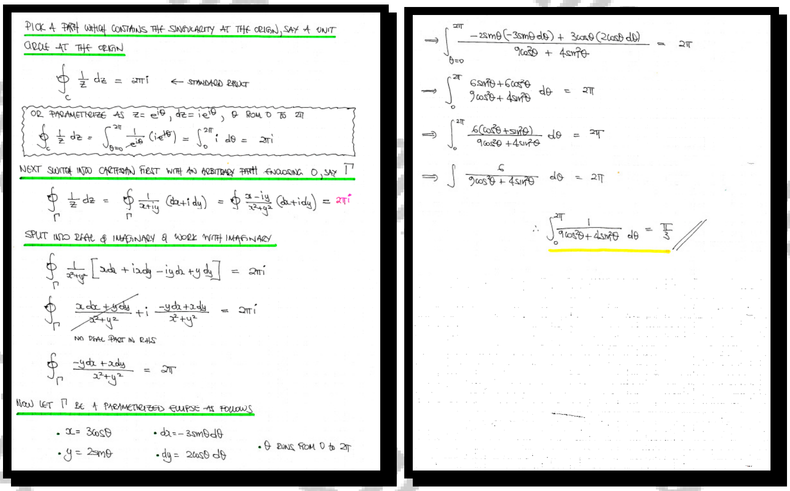
Question 8

$$f(z) \equiv \frac{1}{z}, \ z \in \mathbb{C}, \ \ z \neq 0 .$$

By considering the integral of $f(z)$ over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{9\cos^2\theta + 4\sin^2\theta} \, d\theta = \frac{\pi}{3} .$$

[ ], [proof]



PICK A PATH WHICH CONTAINS THE SINGULARITY AT THE ORIGIN, SAY A UNIT CIRCLE AT THE ORIGIN

$\oint_C \frac{1}{z}\,dz = 2\pi i$ ← STANDARD RESULT

OR PARAMETRIZE AS $z = e^{i\theta}$, $dz = ie^{i\theta}$, θ FROM 0 TO 2π

$\oint_C \frac{1}{z}\,dz = \int_{\theta=0}^{2\pi} \frac{1}{e^{i\theta}}(ie^{i\theta}) = \int_0^{2\pi} i\,d\theta = 2\pi i$

NEXT SWITCH INTO CARTESIAN FIRST WITH AN ARBITRARY PATH ENCLOSING O, SAY Γ

$\oint_\Gamma \frac{1}{z}\,dz = \oint_\Gamma \frac{1}{x+iy}(dx+i\,dy) = \oint \frac{x-iy}{x^2+y^2}(dx+i\,dy) = 2\pi i$

SPLIT INTO REAL & IMAGINARY & WORK WITH IMAGINARY

$\oint_\Gamma \frac{1}{x^2+y^2}\left[x\,dx + ix\,dy - iy\,dx + y\,dy\right] = 2\pi i$

$\oint_\Gamma \frac{x\,dx+y\,dy}{x^2+y^2} + i\,\frac{-y\,dx+x\,dy}{x^2+y^2} = 2\pi i$

NO REAL PART IN RHS

$\oint_\Gamma \frac{-y\,dx+x\,dy}{x^2+y^2} = 2\pi$

NOW LET Γ BE A PARAMETRIZED ELLIPSE AS FOLLOWS

- $x = 3\cos\theta$
- $y = 2\sin\theta$
- $dx = -3\sin\theta\,d\theta$
- $dy = 2\cos\theta\,d\theta$
- θ RUNS FROM 0 to 2π

$\Longrightarrow \int_{\theta=0}^{2\pi} \frac{-2\sin\theta(-3\sin\theta\,d\theta) + 3\cos\theta(2\cos\theta\,d\theta)}{9\cos^2\theta + 4\sin^2\theta} = 2\pi$

$\Longrightarrow \int_0^{2\pi} \frac{6\sin^2\theta + 6\cos^2\theta}{9\cos^2\theta + 4\sin^2\theta}\,d\theta = 2\pi$

$\Longrightarrow \int_0^{2\pi} \frac{6(\cos^2\theta + \sin^2\theta)}{9\cos^2\theta + 4\sin^2\theta}\,d\theta = 2\pi$

$\Longrightarrow \int \frac{6}{9\cos^2\theta + 4\sin^2\theta}\,d\theta = 2\pi$

$\therefore \int_0^{2\pi} \frac{1}{9\cos^2\theta + 4\sin^2\theta}\,d\theta = \frac{\pi}{3}$

Question 9

The complex number $z = c + a\cos\theta + \mathrm{i}b\sin\theta$, $0 \le \theta < 2\pi$, traces a closed contour C, where a, b and c are positive real numbers with $a > c$.

By considering

$$\oint_C \frac{1}{z}\, dz\,,$$

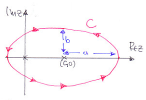
show that

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(c + a\cos\theta)^2 + (b\sin\theta)^2}\, d\theta = \frac{2\pi}{b}.$$

proof

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{b}\,, \quad a,b,c \in \mathbb{R},\ a > c,\ a,b,c > 0$$

- LET $z = c + a\cos\theta + ib\sin\theta$ $\quad \theta \in [0, 2\pi]$, $\quad a > c$



$dz = -a\sin\theta\, d\theta + ib\cos\theta\, d\theta$

$dz = (-a\sin\theta + ib\cos\theta)\, d\theta$

- NOW CONSIDERING THE INTEGRAL GIVEN

$$\Longrightarrow \oint_C \frac{1}{z}\, dz = 2\pi i \quad \text{(ORIGIN IS INSIDE C AS AS } a > c\text{)}$$

$$\Longrightarrow \int_{\theta=0}^{2\pi} \frac{1}{[c + a\cos\theta] + i[b\sin\theta]} (-a\sin\theta + ib\cos\theta)\, d\theta = 2\pi i$$

$$\Longrightarrow \int_0^{2\pi} \frac{[-a\sin\theta + ib\cos\theta][(c + a\cos\theta) - ib\sin\theta]}{[(c + a\cos\theta) + ib\sin\theta][(c + a\cos\theta) - ib\sin\theta]}\, d\theta = 2\pi i$$

$$\Longrightarrow \int_0^{2\pi} \frac{(-a\sin\theta + ib\cos\theta)[(c + a\cos\theta) - ib\sin\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta}\, d\theta = 2\pi i$$

$$\Longrightarrow \int_0^{2\pi} \frac{[-a(c + a\cos\theta)\sin\theta + b^2\cos\theta\sin\theta] + i[bc\cos\theta + ab\cos^2\theta + ab\sin^2\theta]}{(c + a\cos\theta)^2 + b^2\sin^2\theta}\, d\theta = 2\pi i$$

- NOW SEPARATING THE INTEGRAL INTO REAL & IMAGINARY

$$\int_0^{2\pi} \frac{(b^2 - a^2)\cos\theta\sin\theta - ac\sin\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\, d\theta + i\int_0^{2\pi} \frac{ab + bc\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\, d\theta = 2\pi i$$

- THE REAL INTEGRAL YIELDS ZERO

LOOKING AT THE IMAGINARY INTEGRAL

$$ib\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\, d\theta = 2\pi i$$

$$\int_0^{2\pi} \frac{a + c\cos\theta}{(a\cos\theta + c)^2 + b^2\sin^2\theta}\, d\theta = \frac{2\pi}{b}$$