

Created by T. Madas

# **RESIDUES and INTEGRATION**

Created by T. Madas

Created by T. Madas

# CALCULATIONS OF RESIDUES

Created by T. Madas

## Question 1

$$f(z) \equiv \frac{\sin z}{z^2}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$ .

$$\boxed{\phantom{0}}, \quad \boxed{res(z=0)=1}$$

•  $f(z)$  HAS A SIMPLE POLE AT  $z=0$ , WHICH IS EASY TO FIND FROM ITS LAURENT EXPANSION

$$f(z) = \frac{\sin z}{z^2} = \frac{1}{z^2} \left[ z - \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7) \right]$$

$$= \frac{1}{z} - \frac{1}{6}z + \frac{1}{120}z^3 + O(z^5)$$

∴ RESIDUE IS 1

• ALTERNATIVE IS TO USE THE STANDARD METHOD FOR A SIMPLE POLE AT  $z=a$

$$\lim_{z \rightarrow 0} \left[ \frac{d}{dz} (z-0)^2 f(z) \right] = \lim_{z \rightarrow 0} \left[ \frac{d}{dz} \left[ z \frac{\sin z}{z^2} \right] \right]$$

$$= \lim_{z \rightarrow 0} \left[ \frac{d}{dz} \left( \frac{\sin z}{z} \right) \right]$$

BY L'HOPITAL'S RULE

$$= \lim_{z \rightarrow 0} \left( \frac{\cos z}{1} \right)$$

$$= 1$$

AS BEFORE

## Question 2

$$f(z) \equiv e^z z^{-5}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$ .

$$\boxed{\phantom{0}}, \quad \boxed{res(z=0) = \frac{1}{24}}$$

$f(z)$  HAS A SINGLE POLE OF ORDER 5 AT THE ORIGIN, WHICH IS EASY TO FIND DIRECTLY FROM ITS LAURENT EXPANSION

$$f(z) = e^z z^{-5} = \frac{e^z}{z^5} = \frac{1}{z^5} \left[ 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \dots \right]$$

$$= \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{2z^3} + \frac{1}{6z^2} + \frac{1}{24z} + \frac{1}{120} + \dots$$

∴ REQUIRED RESIDUE =  $\frac{1}{24}$

ALTERNATIVE IS TO USE THE STANDARD FORMULA FOR FINDING A POLE OF ORDER  $n$ , AT  $z=a$

$$res(z=0) = \frac{1}{(n-1)!} \lim_{z \rightarrow 0} \left[ \frac{d^{n-1}}{dz^{n-1}} (z-0)^n f(z) \right]$$

$$res(z=0) = \frac{1}{4!} \lim_{z \rightarrow 0} \left[ \frac{d^4}{dz^4} \left( z^5 \times \frac{e^z}{z^5} \right) \right]$$

$$= \frac{1}{24} \lim_{z \rightarrow 0} \left[ \frac{d^4}{dz^4} (e^z) \right]$$

$$res(z=0) = \frac{1}{24} \lim_{z \rightarrow 0} [e^z]$$

$$res(z=0) = \frac{1}{24}$$

AS BEFORE

## Question 3

$$f(z) \equiv \frac{z^2 + 2z + 1}{z^2 - 2z + 1}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$ .

$$\boxed{\phantom{0}}, \quad \boxed{res(z=1)=4}$$

FACTORIZING THE FUNCTION

$$f(z) = \frac{z^2 + 2z + 1}{z^2 - 2z + 1} = \frac{(z+1)^2}{(z-1)^2}$$

$f(z)$  HAS A DOUBLE POLE AT  $z=1$

$$\begin{aligned} \lim_{z \rightarrow 1} \left[ \frac{d}{dz} \left[ (z-1)^2 f(z) \right] \right] &= \lim_{z \rightarrow 1} \left[ \frac{d}{dz} \left[ (z-1)^2 \frac{(z+1)^2}{(z-1)^2} \right] \right] \\ &= \lim_{z \rightarrow 1} \left[ \frac{d}{dz} (z+1)^2 \right] \\ &= \lim_{z \rightarrow 1} [2(z+1)] \\ &= \underline{\underline{4}} \end{aligned}$$

## Question 4

$$f(z) \equiv \frac{2z+1}{z^2 - z - 2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of  $f(z)$ .

$$\boxed{\phantom{0}}, \quad \boxed{res(z=2)=\frac{5}{3}}, \quad \boxed{res(z=-1)=\frac{1}{3}}$$

START BY FACTORIZING THE DENOMINATOR

$$f(z) = \frac{2z+1}{z^2 - z - 2} = \frac{2z+1}{(z+1)(z-2)}$$

$f(z)$  HAS SIMPLE POLES AT  $z=-1$  & AT  $z=2$

- $\bullet \text{ Res}(f; -1) = \lim_{z \rightarrow -1} \left[ (z+1) f(z) \right] = \lim_{z \rightarrow -1} \left[ (z+1) \frac{2z+1}{(z+1)(z-2)} \right]$ 

$$= \frac{2(-1)+1}{-1-2} = \frac{-1}{-3} = \underline{\underline{\frac{1}{3}}}$$
- $\bullet \text{ Res}(f; 2) = \lim_{z \rightarrow 2} \left[ (z-2) f(z) \right] = \lim_{z \rightarrow 2} \left[ (z-2) \frac{2z+1}{(z+1)(z-2)} \right]$ 

$$= \frac{2(2)+1}{2+1} = \underline{\underline{\frac{5}{3}}}$$



## Question 5

$$f(z) \equiv \frac{z}{2z^2 - 5z + 2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of  $f(z)$ .

$$\boxed{\phantom{0}}, \quad \text{res}\left(z = \frac{1}{2}\right) = -\frac{1}{6}, \quad \text{res}(z = 2) = \frac{2}{3}$$

START BY FACTORISING THE DENOMINATOR

$$f(z) = \frac{z}{2z^2 - 5z + 2} = \frac{z}{(2z-1)(z-2)}$$

$f(z)$  HAS SIMPLE POLES AT  $z = \frac{1}{2}$  AND  $z = 2$

•  $\text{Res}\left(z = \frac{1}{2}\right) = \lim_{z \rightarrow \frac{1}{2}} \left[ (z - \frac{1}{2}) f(z) \right] = \lim_{z \rightarrow \frac{1}{2}} \left[ (z - \frac{1}{2}) \cdot \frac{z}{(2z-1)(z-2)} \right]$

$$= \lim_{z \rightarrow \frac{1}{2}} \left[ \frac{z}{2(z-2)} \right] = \frac{\frac{1}{2}}{2(-\frac{3}{2})} = -\frac{1}{6}$$

•  $\text{Res}\left(z = 2\right) = \lim_{z \rightarrow 2} \left[ (z - 2) f(z) \right] = \lim_{z \rightarrow 2} \left[ (z - 2) \cdot \frac{z}{(2z-1)(z-2)} \right]$

$$= \lim_{z \rightarrow 2} \left[ \frac{z}{2z-1} \right] = \frac{2}{3}$$

## Question 6

$$f(z) \equiv \frac{1 - e^{iz}}{z^3}, \quad z \in \mathbb{C}.$$

- a) Find the first four terms in the Laurent expansion of  $f(z)$ , and hence state the residue of the pole of  $f(z)$ .
- b) Determine the residue of the pole of  $f(z)$  by an alternative method

$$\boxed{113}, \quad \boxed{\text{res}(z=0) = \frac{1}{2}}$$

a) FIRST, GET THE RESIDUE FROM THE LAURENT EXPANSION

$$\begin{aligned} f(z) &= \frac{1 - e^{iz}}{z^3} = \frac{1}{z^3} \left[ 1 - \left( 1 + (iz) + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots \right) \right] \\ &= \frac{1}{z^3} \left[ -iz - \frac{1}{2}(iz)^2 - \frac{1}{6}(iz)^3 - \frac{1}{24}(iz)^4 + \dots \right] \\ &= \frac{1}{z^3} \left[ -iz + \frac{1}{2}z^2 + \frac{1}{6}iz^3 - \frac{1}{24}z^4 + \dots \right] \\ &= -\frac{i}{z^2} + \frac{1}{2z} + \frac{1}{6} + \frac{1}{24}z + \dots \end{aligned}$$

∴ RESIDUE OF THE DOUBLE POLE AT  $z=0$  IS  $\frac{1}{2}$

b) NEXT, USING THE FORMULA FOR THE RESIDUE OF POLE OF ORDER  $n$

$$\text{Res}\left(\frac{f}{g}, c\right) = \frac{1}{(n-1)!} \lim_{z \rightarrow c} \left[ \frac{d^{n-1}}{dz^{n-1}} \left[ (z-c)^n f(z) \right] \right]$$

$$\begin{aligned} \text{Res}\left(\frac{f}{g}, 0\right) &= \frac{1}{1!} \lim_{z \rightarrow 0} \left[ \frac{d}{dz} \left[ z^3 \times \frac{1 - e^{iz}}{z^3} \right] \right] \quad \text{NOTE THAT IT IS A DOUBLE POLE NOT TRIPLE} \\ &= \lim_{z \rightarrow 0} \left[ \frac{d}{dz} \left[ 1 - e^{iz} \right] \right] \\ &= \lim_{z \rightarrow 0} \left[ z(-ie^{iz}) - (1 - e^{iz}) \right] = \lim_{z \rightarrow 0} \left[ -iz e^{iz} - 1 + e^{iz} \right] \end{aligned}$$

THIS IS "0/0" TYPE, SO BY L'HOSPITAL RULE WE OBTAIN

$$\begin{aligned} &= \lim_{z \rightarrow 0} \left[ \frac{(-ie^{iz} + ie^{iz}) + e^{iz}}{1} \right] = \lim_{z \rightarrow 0} \left[ \frac{e^{iz}}{1} \right] \\ &= \lim_{z \rightarrow 0} \left[ \frac{e^{iz}}{1} \right] = \frac{1}{1} = 1 \end{aligned}$$

∴ RESIDUE

## Question 7

$$f(z) \equiv \frac{z^2 + 4}{z^3 + 2z^2 + 2z}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of  $f(z)$ .

$$\boxed{\text{res}(z=0)=2}, \quad \boxed{\text{res}(z=-1+i)=\frac{1}{2}(-1+3i)}, \quad \boxed{\text{res}(z=-1-i)=-\frac{1}{2}(1+3i)}$$

$f(z) = \frac{z^2 + 4}{z^3 + 2z^2 + 2z}$   
 $\therefore$  SIMPLE POLES AT  $z=0, z=-1+i, z=-1-i$   
 $\bullet \lim_{z \rightarrow 0} [z \cdot f(z)] = \lim_{z \rightarrow 0} \left[ \frac{z^2 + 4}{z^2 + 2z} \right] = \frac{4}{2} = 2$   
 $\bullet \lim_{z \rightarrow -1+i} [(z+1-i) f(z)] = \lim_{z \rightarrow -1+i} \left[ \frac{z^2 + 4}{z(z+1+i)} \right] = \frac{(-1+i)^2 + 4}{(-1+i)(-1+i+1)} = \frac{1-2i-1+4}{2i(-1+i)} = \frac{4-2i}{-2+2i} = \frac{2-i}{-1-i} = \frac{(2-i)(-1+i)}{(-1-i)(-1+i)} = \frac{-2+2i-1+i}{1-1-i+i} = \frac{-3+3i}{-i} = \frac{3(1-i)}{-i} = \frac{3}{2}(-1+3i)$   
 $\bullet \lim_{z \rightarrow -1-i} [(z+1+i) f(z)] = \lim_{z \rightarrow -1-i} \left[ \frac{z^2 + 4}{z(z+1-i)} \right] = \frac{(-1-i)^2 + 4}{(-1-i)(-1-i-1)} = \frac{1+2i-1+4}{-2-2i} = \frac{4+2i}{-2-2i} = \frac{2+i}{-1-i} = \frac{(2+i)(-1+i)}{(-1-i)(-1+i)} = \frac{-2+2i-1+i}{1-1-i+i} = \frac{-3+i}{-i} = \frac{3}{2}(-1-3i)$

## Question 8

$$f(z) \equiv \frac{\tan 3z}{z^4}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$ .

$$\boxed{9}, \quad \boxed{\text{res}(z=0)=9}$$

$f(z) = \frac{\tan 3z}{z^4}$   
 $\tan 3z = 3z + \frac{9}{3!}z^3 + O(z^5) = 3z + \frac{3}{2}z^3 + O(z^5)$   
 $\therefore f(z) = \frac{3z + \frac{3}{2}z^3 + O(z^5)}{z^4} = \frac{3}{z} + \frac{3}{2}z + O(z^3)$   
 $\therefore$  RESIDUE OF THE POLE AT THE ORIGIN IS 3

## Question 9

$$f(z) \equiv \frac{z^2 - 2z}{(z^2 + 4)(z + 1)^2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of  $f(z)$ .

$$\boxed{\operatorname{res}(z = 2i) = \frac{1}{25}(7 + i)}, \quad \boxed{\operatorname{res}(z = -2i) = \frac{1}{25}(7 - i)}, \quad \boxed{\operatorname{res}(z = -1) = -\frac{14}{25}}$$

Handwritten solution for Question 9:

$f(z) = \frac{z^2 - 2z}{(z^2 + 4)(z + 1)^2} = \frac{z(z - 2)}{(z + 2i)(z - 2i)(z + 1)^2}$

Residue at  $z = 2i$ :

$$\lim_{z \rightarrow 2i} \left[ (z - 2i) \frac{z^2 - 2z}{(z + 2i)(z - 2i)(z + 1)^2} \right] = \lim_{z \rightarrow 2i} \left[ \frac{z(z - 2)}{(z + 2i)(z + 1)^2} \right] = \frac{2i(2i - 2)}{(2i + 2i)(2i + 1)^2} = \frac{-4}{4(1 - 4)} = \frac{-1}{1 \times 5 \times 5} = \frac{-1}{25}$$

Residue at  $z = -2i$ :

$$\lim_{z \rightarrow -2i} \left[ (z + 2i) \frac{z^2 - 2z}{(z + 2i)(z - 2i)(z + 1)^2} \right] = \lim_{z \rightarrow -2i} \left[ \frac{z(z - 2)}{(z - 2i)(z + 1)^2} \right] = \frac{-2i(-2i - 2)}{(-2i - 2i)(-2i + 1)^2} = \frac{4}{4(1 - 4)} = \frac{1}{25}$$

Residue at  $z = -1$ :

$$\lim_{z \rightarrow -1} \left[ \frac{d}{dz} \left( \frac{z^2 - 2z}{(z + 2i)(z - 2i)} \right) \right] = \lim_{z \rightarrow -1} \left[ \frac{z^2 - 2z}{(z + 2i)(z - 2i)} \right] = \frac{(-1)^2 - 2(-1)}{(-1 + 2i)(-1 - 2i)} = \frac{1 + 2}{1 - 4} = \frac{3}{-3} = -1$$

## Question 10

$$f(z) \equiv \frac{1}{e^z - 1}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$ , at the origin.

$$\boxed{\operatorname{res}(z = 0) = 1}$$

Handwritten solution for Question 10:

$f(z) = \frac{1}{e^z - 1} = \frac{1}{1 - (-e^z)} = \frac{1}{1 - (-1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots)} = \frac{1}{1 - (-1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots)}$

$= \frac{1}{2 - z - \frac{z^2}{2!} - \frac{z^3}{3!} - \dots} = \frac{1}{2} \left( 1 + \frac{1}{2}z + O(z^2) \right)^{-1}$

$= \frac{1}{2} \left[ 1 - \frac{1}{2}z + O(z^2) \right] = \frac{1}{2} - \frac{1}{4}z + O(z^2)$

$\therefore \text{Residue } 1$

## Question 11

$$f(z) \equiv \frac{z}{(3z^2 - 10iz - 3)^2}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of  $f(z)$ .

$$\boxed{\operatorname{res}\left(z = 3i\right) = \frac{5}{256}}, \quad \boxed{\operatorname{res}\left(z = \frac{1}{3}i\right) = -\frac{5}{256}}$$

$f(z) = \frac{z}{(3z^2 - 10iz - 3)^2}$

$3z^2 - 10iz - 3 = 3\left[z^2 - \frac{10}{3}iz - 1\right] = 3\left[\left(z - \frac{5}{3}i\right)^2 - \frac{16}{9}\right]$

$= 3\left[\left(z - \frac{5}{3}i\right)^2 - \left(\frac{4}{3}\right)^2\right] = 3\left[\left(z - \frac{5}{3}i\right) - \frac{4}{3}\right]\left[\left(z - \frac{5}{3}i\right) + \frac{4}{3}\right]$

$= 3(z - 3i)(z - \frac{1}{3}i)$

$= (z - 3i)(3z - 1)$

$f(z)$  has double poles at  $z = 3i$  &  $\frac{1}{3}i$

$\bullet \lim_{z \rightarrow 3i} \left[ \frac{d}{dz} \left( \frac{z}{(3z-1)^2 (z-\frac{1}{3}i)^2} \right) \right] = \lim_{z \rightarrow 3i} \left[ \frac{d}{dz} \left( \frac{z}{(3z-1)^2} \right) \right]$

$= \lim_{z \rightarrow 3i} \left[ \frac{(3z-1)^{-2} \cdot 1 - z \cdot 2(3z-1)^{-3} \cdot 3}{(3z-1)^4} \right] = \lim_{z \rightarrow 3i} \left[ \frac{(3z-1) - 6z}{(3z-1)^3} \right]$

$= \lim_{z \rightarrow 3i} \left[ \frac{-1 - 3z}{(3z-1)^3} \right] = \frac{-1 - 9i}{(9i-1)^3} = \frac{-10i}{(8i)^3} = \frac{-10i}{-512i} = \frac{5}{256}$

$\bullet \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{d}{dz} \left( \frac{z}{(z-3i)^2 (3z-1)^2} \right) \right] = \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{d}{dz} \left( \frac{z}{9(z-3i)^2} \right) \right]$

$= \frac{1}{9} \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{(z-3i)^{-2} \cdot 1 - z \cdot 2(z-3i)^{-3} \cdot 1}{(z-3i)^4} \right] = \frac{1}{9} \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{(z-3i) - 2z}{(z-3i)^3} \right]$

$= \frac{1}{9} \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{-z - 3i}{(z-3i)^3} \right] = -\frac{1}{9} \lim_{z \rightarrow \frac{1}{3}i} \left[ \frac{\frac{1}{3} - 3i}{(\frac{1}{3} - 3i)^3} \right]$

$= -\frac{1}{9} \cdot \frac{\frac{1}{3} - 3i}{\frac{1}{27} - \frac{27i}{27}} = -\frac{10}{512i} = -\frac{5}{256i}$

## Question 12

$$f(z) = \frac{\cot z \coth z}{z^3}, \quad z \in \mathbb{C}.$$

Find the residue of the pole of  $f(z)$  at  $z = 0$ .

$$\boxed{\phantom{000}}, \quad \boxed{\text{res}(z=0) = -\frac{7}{45}}$$

1) BEST TO FIND THE RESIDUE BY EXPANSION FOR  $f(z) = \frac{\cot z \coth z}{z^3}$

$$f(z) = \frac{1}{z^3} \times \frac{\cot z}{\sinh z} \times \frac{\cosh z}{\sinh z}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

$$= \frac{1}{z^3} \times \frac{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}{1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)} \times \frac{1 + \frac{z^2}{2} + \frac{z^4}{24} + O(z^6)}{1 + \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)}$$

2) BETTER IN ORDER TO COMPLETE THE EXPANSION

$$= \frac{1}{z^3} \left[ 1 - \frac{1}{6}z^2 + O(z^4) \right] \left[ 1 - \frac{1}{6}z^2 + O(z^4) \right]^{-1}$$

$$= \frac{1}{z^3} \left[ 1 - \frac{1}{6}z^2 + O(z^4) \right] \left[ 1 + \frac{1}{6}z^2 + O(z^4) \right]$$

$$= \frac{1}{z^3} \left[ 1 + \frac{1}{36}z^4 - \frac{1}{6}z^4 - \frac{1}{36}z^4 + O(z^6) \right]$$

$$= \frac{1}{z^3} \left[ 1 - \frac{1}{18}z^4 + O(z^6) \right]$$

$$= \frac{1}{z^3} - \frac{1}{18}z + O(z^3)$$

∴ Residue is  $-\frac{1}{18}$

### Question 13

$$f(z) \equiv \frac{z^6 + 1}{2z^5 - 5z^4 + 2z^3}, \quad z \in \mathbb{C}.$$

Find the residue of each of the three poles of  $f(z)$ .

$$\boxed{res\left(z=\frac{1}{2}\right)=-\frac{65}{24}}, \quad \boxed{res(z=2)=\frac{65}{24}}, \quad \boxed{res(z=0)=\frac{21}{8}}$$

[illegible]

## Question 14

$$f(z) \equiv \frac{4}{z^2(1-2i)+6zi-(1+2i)}, \quad z \in \mathbb{C}.$$

Find the residue of each of the two poles of  $f(z)$ .

$$\boxed{\operatorname{res}(z=2-i)=i}, \quad \boxed{\operatorname{res}\left(z=\frac{1}{5}(2-i)\right)=-i}$$

$f(z) = \frac{4}{z^2(1-2i) + 6zi - (1+2i)}$   
 BY THE QUADRATIC FORMULA  
 $z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)} = \frac{-6i \pm \sqrt{36 + 4(1-2i)(-1-2i)}}{2(1-2i)}$   
 $= \frac{-6i \pm 4i}{2(1-2i)} = \frac{-2i \pm 4i}{2(1-2i)} = \frac{-2i}{2(1-2i)} = \frac{-i}{1-2i} = \frac{-i(1+2i)}{(1-2i)(1+2i)} = \frac{-i-2}{1+4} = \frac{-2-i}{5} = -\frac{2}{5} - \frac{i}{5}$   
 $\sqrt{5}$  HAS SAME ROOTS AT  $z=2-i$  &  $\frac{1}{5}(2-i)$   
 $\bullet \lim_{z \rightarrow 2-i} \left[ (z-2+i) \frac{4}{z^2(1-2i) + 6zi - (1+2i)} \right] = \frac{0}{0} \dots$  BY L'HOSPITAL...  
 $= \lim_{z \rightarrow 2-i} \left[ \frac{4}{2z(1-2i) + 6i} \right] = \frac{4}{2(2-i)(1-2i) + 6i} = \frac{4}{2(2-i)(1-2i) + 6i}$   
 $= \frac{4}{2(2-i)(1-2i) + 6i} = \frac{4}{2(2-i)(1-2i) + 6i} = \frac{4}{2(2-i)(1-2i) + 6i}$   
 $\lim_{z \rightarrow \frac{1}{5}(2-i)} \left[ \left( z - \frac{1}{5}(2-i) \right) \frac{4}{z^2(1-2i) + 6zi - (1+2i)} \right] = \frac{0}{0} \dots$  BY L'HOSPITAL...  
 $= \lim_{z \rightarrow \frac{1}{5}(2-i)} \left[ \frac{4}{2z(1-2i) + 6i} \right] = \frac{4}{2 \cdot \frac{1}{5}(2-i)(1-2i) + 6i} = \frac{4}{\frac{2}{5}(2-i)(1-2i) + 6i} = \frac{4}{\frac{2}{5}(2-i)(1-2i) + 6i}$   
 $= \frac{4}{\frac{2}{5}(2-i)(1-2i) + 6i} = \frac{4}{\frac{2}{5}(2-i)(1-2i) + 6i} = \frac{4}{\frac{2}{5}(2-i)(1-2i) + 6i}$



## Question 15

$$f(z) \equiv \frac{ze^{kz}}{z^4 + 1}, \quad z \in \mathbb{C}, \quad k \in \mathbb{R}, \quad k > 0.$$

Show that the sum of the residues of the four poles of  $f(z)$ , is

$$\sin\left(\frac{k}{\sqrt{2}}\right) \sinh\left(\frac{k}{\sqrt{2}}\right).$$

, proof

IT IS BEST TO WORK WITH EXPONENTIALS IN THIS QUESTION

$f(z) = \frac{ze^{kz}}{z^4+1}$  HAS FOUR POLES AT:

$z^4 = -1 \Rightarrow e^{i(\pi+2n\pi)} = e^{i\pi(2n+1)}$   $n=0,1,2,3$

$z = e^{i\frac{\pi}{4}(2n+1)} = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$   $(\cos -2, -1, 0, 1)$

CALCULATE THREE RESIDUES USING A GENERAL METHOD - LET A POLE BE AT  $z=z_0$

$\text{Res}\left(\frac{f(z)}{z}, z_0\right) = \lim_{z \rightarrow z_0} \left[ \frac{(z-z_0)ze^{kz}}{z^4+1} \right]$

THIS MUST PRODUCE ZERO OVER ZERO AS  $z=z_0$  MUST BE A FACTOR OF THE DENOMINATOR SO WE PROCEED BY L'HOSPITAL RULE

$\text{Res}\left(\frac{f(z)}{z}, z_0\right) = \lim_{z \rightarrow z_0} \left[ \frac{\frac{d}{dz}[(z-z_0)ze^{kz}]}{\frac{d}{dz}(z^4+1)} \right] = \lim_{z \rightarrow z_0} \left[ \frac{ze^{kz} + (z-z_0)ke^{kz}}{4z^3} \right]$

$\text{Res}\left(\frac{f(z)}{z}, z_0\right) = \frac{ze^{kz}}{4z^3} = \frac{e^{kz}}{4z^2}$

NOW OBTAIN THE RESIDUE AT EACH OF THE FOUR POLES

$z_0 = e^{i\frac{\pi}{4}}$	over	$\frac{e^{kz}}{4z^2} = \frac{e^{k(\frac{\pi}{4})}}{4e^{i\frac{\pi}{2}}} = \frac{e^{k(\frac{\pi}{4})}}{4e^{i\frac{\pi}{2}}}$
$z_0 = e^{i\frac{3\pi}{4}}$	over	$\frac{e^{kz}}{4z^2} = \frac{e^{k(\frac{3\pi}{4})}}{4e^{i\frac{3\pi}{2}}} = \frac{e^{k(\frac{3\pi}{4})}}{4e^{i\frac{3\pi}{2}}}$
$z_0 = e^{i\frac{5\pi}{4}}$	over	$\frac{e^{kz}}{4z^2} = \frac{e^{k(\frac{5\pi}{4})}}{4e^{i\frac{5\pi}{2}}} = \frac{e^{k(\frac{5\pi}{4})}}{4e^{i\frac{5\pi}{2}}}$
$z_0 = e^{i\frac{7\pi}{4}}$	over	$\frac{e^{kz}}{4z^2} = \frac{e^{k(\frac{7\pi}{4})}}{4e^{i\frac{7\pi}{2}}} = \frac{e^{k(\frac{7\pi}{4})}}{4e^{i\frac{7\pi}{2}}}$

ADD UP THE 4 RESIDUES - LET  $a = \frac{k}{\sqrt{2}}$  FOR SIMPLICITY

SUM OF 4 RESIDUES =  $\frac{e^{k(\frac{\pi}{4})}}{4i} + \frac{e^{k(\frac{3\pi}{4})}}{-4i} + \frac{e^{k(\frac{5\pi}{4})}}{-4i} + \frac{e^{k(\frac{7\pi}{4})}}{4i}$

$= \frac{1}{4i} \left[ e^{ka} - e^{-ka} - e^{-ka} + e^{ka} \right]$

$= \frac{1}{4i} \left[ e^{ka} - e^{-ka} - e^{-ka} + e^{ka} \right]$

$= \frac{1}{4i} \left[ e^{ka} - e^{-ka} - e^{-ka} + e^{ka} \right]$

$= \frac{1}{4i} \left[ (e^{ka} - e^{-ka}) - (e^{-ka} - e^{ka}) \right]$

$= \frac{1}{4i} \left[ (e^{ka} - e^{-ka}) - (e^{-ka} - e^{ka}) \right]$

$= \frac{1}{4i} \times 2 \sinh(ka) \times 2 \sinh(ka)$

$= \frac{1}{4i} \times 2i \sinh(ka) \times 2 \sinh(ka)$

$= \sinh(ka) \sinh(ka)$

$= \sinh\left(\frac{k}{\sqrt{2}}\right) \sinh\left(\frac{k}{\sqrt{2}}\right)$

AS REQUESTED

Created by T. Madas

# REAL INTEGRALS

Created by T. Madas

Created by T. Madas

# UNIT CIRCLE CONTOUR

Created by T. Madas

## Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta.$$

$$\boxed{V}, \boxed{\phantom{00}}, \boxed{-\frac{2\pi}{3}}$$

STEP 1: USING THE COUPOLE (S) OF  $z = e^{i\theta}$  ( $0 \leq \theta < 2\pi$ )

- $dz = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) = \frac{1}{2} (z + \frac{1}{z})$
- $4\cos\theta - 5 = 4 \times \frac{1}{2} (z + \frac{1}{z}) - 5$   
 $= 2z + \frac{2}{z} - 5$   
 $= \frac{1}{z} (2z^2 - 5z + 2)$   
 $= \frac{1}{z} (2z - 1)(z - 2)$

TRANSFORMING THE INTEGRAL

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta = \int_0^{2\pi} \frac{1}{\frac{1}{z} (2z - 1)(z - 2)} \left( \frac{1}{2} dz \right)$$

$$= \int_0^{2\pi} \frac{-1}{2(z - 1)(z - 2)} dz$$

VIA THE RESIDUE THEOREM (S)

$z = 1$  &  $z = 2$  or which ONLY THE ONE  
 AT  $z = 1$  IS INSIDE  $\Gamma$  - FIND RESIDUE

$$\lim_{z \rightarrow 1} \left[ (z - 1) \frac{-1}{2(z - 1)(z - 2)} \right] = \lim_{z \rightarrow 1} \left[ \frac{-1}{2(z - 2)} \right]$$

$$= \lim_{z \rightarrow 1} \left[ \frac{-1}{2(1 - 2)} \right]$$

$$= \lim_{z \rightarrow 1} \left[ \frac{-1}{2(-1)} \right]$$

$$= \frac{1}{2}$$

BY THE RESIDUE THEOREM WE HAVE:

$$\int_{\Gamma} f(z) dz = 2\pi i \times (\text{SUM OF RESIDUES})$$

$$\int_{\Gamma} \frac{-1}{2(z - 1)(z - 2)} dz = 2\pi i \times \frac{1}{2}$$

$$\int_0^{2\pi} \frac{1}{4\cos\theta - 5} d\theta = -\frac{2\pi}{3}$$

## Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \frac{1}{2 + \cos \theta} d\theta.$$

$$\boxed{\frac{2\pi}{\sqrt{3}}}$$

$\int_0^{2\pi} \frac{1}{2 + \cos \theta} d\theta = \int_0^{2\pi} \frac{1}{2 + \frac{1}{2}(e^{i\theta} + e^{-i\theta})} d\theta$

THIS IS A CONSIDERABLE SIMPLIFICATION IF YOU TAKE THE CONT. ORBIT (20) BECAUSE

IF  $z$  LIES ON THIS ORBIT  
 $z = e^{i\theta}$   
 $dz = i e^{i\theta} d\theta$   
 $d\theta = \frac{dz}{iz}$

TRANSFORM THE INTEGRAL  
 $= \int_{\gamma} \frac{1}{2 + \frac{1}{2}(z + \frac{1}{z})} \frac{dz}{iz} = \int_{\gamma} \frac{-2i}{z^2 + 4z + 1} dz = \int_{\gamma} \frac{-2i}{z^2 + 4z + 1} dz$

LOOK FOR THE POLES OF  $f(z)$   
 $f(z) = \frac{-2i}{z^2 + 4z + 1} = \frac{-2i}{(z + 2 + \sqrt{3})(z + 2 - \sqrt{3})}$

$f(z)$  HAS SIMPLE POLES AT  $z = -2 + \sqrt{3}$ , BUT ONLY THE ONE AT  $z = -2 + \sqrt{3}$  IS INSIDE  $\gamma$  — CALCULATE THE RESIDUE

$\text{Res}(f; -2 + \sqrt{3}) = \lim_{z \rightarrow -2 + \sqrt{3}} [(z + 2 - \sqrt{3}) f(z)]$   
 $= \lim_{z \rightarrow -2 + \sqrt{3}} \left[ (z + 2 - \sqrt{3}) \times \frac{-2i}{(z + 2 + \sqrt{3})(z + 2 - \sqrt{3})} \right]$

BY THE RESIDUE THEOREM  
 $\int_{\gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \gamma)$   
 $\int_{\gamma} \frac{-2i}{z^2 + 4z + 1} dz = 2\pi i \times \left( \frac{-2i}{2\sqrt{3}} \right)$   
 $\int_0^{2\pi} \frac{1}{2 + \cos \theta} d\theta = \frac{2\pi}{\sqrt{3}}$

## Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta.$$

$$\boxed{\frac{5\pi}{512}}$$

• USING THE CONTOUR  $|z|=1$  or  $z=e^{i\theta}$ ,  $0 \leq \theta < 2\pi$

Thus  $z=e^{i\theta}$   
 $dz = i e^{i\theta} d\theta$

• Hence the integral becomes

$$\int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta$$

$$= \int_0^{2\pi} (\cos \theta \sin \theta)^6 \, d\theta = \int_0^{2\pi} \left(\frac{1}{2} \sin 2\theta\right)^6 \, d\theta$$


$$= \frac{1}{64} \int_0^{2\pi} \sin^6 2\theta \, d\theta = \frac{1}{64} \int_0^{2\pi} \left[\frac{1}{2i} (e^{i2\theta} - e^{-i2\theta})\right]^6 \, d\theta$$

$$= \frac{1}{64} \left(\frac{1}{2i}\right)^6 \int_0^{2\pi} \left[(e^{i2\theta})^6 - (e^{-i2\theta})^6\right]^6 \, d\theta$$

$$= \frac{1}{64} \times \frac{1}{64} \int_0^{2\pi} \left(z^2 - \frac{1}{z^2}\right)^6 \left(\frac{dz}{i2z}\right) \quad \leftarrow \boxed{dz = \frac{dz}{i e^{i\theta}}}$$

$$= \frac{1}{64^2} \int_0^{2\pi} \frac{1}{2} \left(z^2 - \frac{1}{z^2}\right)^6 \, dz$$

• EXPAND BINOMIALLY

$$= \frac{1}{2 \times 64^2} \int_0^{2\pi} \left[ z^{12} - 6z^8 + 15z^4 - 20 + \frac{15}{z^4} - \frac{6}{z^8} + \frac{1}{z^{12}} \right] dz$$


$$= \frac{1}{2 \times 64^2} \int_0^{2\pi} \left[ z^{12} - 6z^8 + 15z^4 - 20 + \frac{15}{z^4} - \frac{6}{z^8} + \frac{1}{z^{12}} \right] dz$$

• As the integrand is in SCALAR FORM the only CONTRIBUTION is FROM THE  $\frac{1}{z}$  TERM i.e. THE RESIDUE IS  $-20$

$$\Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$$

$$\Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = 2\pi i \times \frac{i}{24} \times (-20)$$

$$\Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = \frac{5\pi \times 2^3}{2^4} = \frac{5\pi}{2}$$

$$\Rightarrow \int_0^{2\pi} \cos^6 \theta \sin^6 \theta \, d\theta = \frac{5\pi}{512}$$

## Question 4

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{5 + 4 \sin \theta} d\theta.$$

$$\boxed{\phantom{000}}, \boxed{\frac{2\pi}{3}}$$

START BY PARAMETERISING THE INTEGRAL ONTO A UNIT CIRCLE, GEOMETRIC AT THE ORIGIN IS  $z = e^{i\theta}$ , THE CURVE GIVEN BELOW, LABELED AS  $\Gamma$

$$\int_0^{2\pi} \frac{1}{5 + 4 \sin \theta} d\theta$$

$$= \oint_{\Gamma} \frac{1}{5 + 4 \sin \frac{1}{2} \left( z - \frac{1}{z} \right)} \frac{dz}{iz}$$

$$= \oint_{\Gamma} \frac{dz}{5 + 2 \left( z - \frac{1}{z} \right) (iz)} = \oint_{\Gamma} \frac{dz}{5 + 2iz - 2} = \oint_{\Gamma} \frac{dz}{3 + 2iz - 2} = \oint_{\Gamma} \frac{dz}{1 + 2iz}$$

FACTORISE THE DENOMINATOR OR IN THE QUADRATIC FORMULA (GAVE THE SAME)

$$1 + 2iz = (z - \frac{1}{2})^2 = -z^2 + z = -9$$

$$z = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$f(z) = \frac{1}{2(z - \frac{1}{2})^2}$$
 HAS SIMPLE POLES AT  $-2i$  &  $2i$  BUT ONLY THE ONE AT  $z = \frac{1}{2}$  IS INSIDE  $\Gamma$  — NOTE THAT  $f(z) = \frac{1}{2(z - \frac{1}{2})^2}$

BY THE RESIDUE THEOREM

$$\int_0^{2\pi} \frac{1}{5 + 4 \sin \theta} d\theta = \oint_{\Gamma} \frac{1}{1 + 2iz} dz = 2\pi i \times \sum \text{RESIDUES INSIDE } \Gamma$$

COMPUTE THE RESIDUE AT THE POLE

$$\lim_{z \rightarrow \frac{1}{2}} \left[ \left( z - \frac{1}{2} \right) f(z) \right] = \lim_{z \rightarrow \frac{1}{2}} \left[ \frac{1}{2(z - \frac{1}{2})} \right]$$

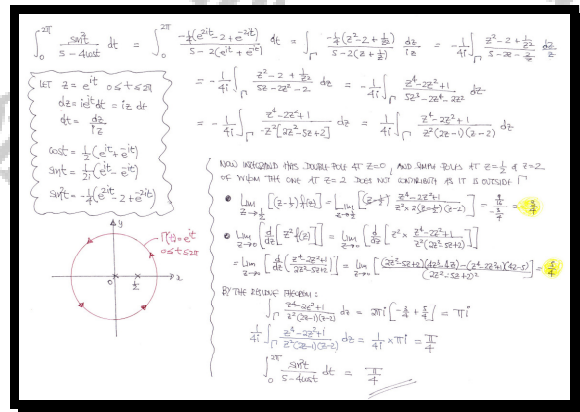
$$= \lim_{z \rightarrow \frac{1}{2}} \left[ \frac{1}{2(z - \frac{1}{2})} \right] = \frac{1}{2(-\frac{1}{2})} = -1$$

$$\therefore \int_0^{2\pi} \frac{1}{5 + 4 \sin \theta} d\theta = 2\pi i \times \frac{1}{2} = \frac{2\pi i}{2} = \pi i$$

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{\sin^2 t}{5-4\cos t} dt.$$

$$\frac{\pi}{4}$$





## Question 6

By integrating a suitable complex function over an appropriate contour find the exact value of

$$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta.$$

$$\frac{5\pi}{32}$$

Handwritten solution for Question 6:

Consider  $z = e^{i\theta}$ , it is on the standard unit circle, for  $0 \leq \theta < 2\pi$   
 $dz = i e^{i\theta} d\theta$   
 $d\theta = \frac{dz}{iz}$

Consider  $\frac{1}{(5-3\sin\theta)^2} = \frac{1}{(5-\frac{3}{2}(z-\frac{1}{z}))^2}$   
 $= \frac{1}{(\frac{10z-3(z^2-1)}{2z})^2} = \frac{4z^2}{(10z-3z^2+3)^2}$   
 $= \frac{4z^2}{(-3z^2+10z+3)^2}$

Thus  $\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \int_{\Gamma} \frac{4z^2}{(-3z^2+10z+3)^2} \frac{dz}{iz} = \int_{\Gamma} \frac{4z}{(-3z^2+10z+3)^2} dz$

Now  $-3z^2+10z+3 = 3(z-\frac{5}{3}i-1)(z+\frac{1}{3}i-1)$   
 $= 3(z-\frac{5}{3}i-1)(z+\frac{1}{3}i-1)$   
 $= 3(z-\frac{5}{3}i-1)(z+\frac{1}{3}i-1)$

The integrand has double poles at  $z = \frac{5}{3}i-1$  and  $z = -\frac{1}{3}i-1$

Calculate residues  
 $\lim_{z \rightarrow \frac{5}{3}i-1} \left[ \frac{d}{dz} \left( \frac{4z}{(-3z^2+10z+3)^2} \right) \right] = \frac{4}{3} \lim_{z \rightarrow \frac{5}{3}i-1} \left[ \frac{1}{(-3z^2+10z+3)} \right]$   
 $= \frac{4}{3} \lim_{z \rightarrow \frac{5}{3}i-1} \left[ \frac{(z-\frac{1}{3}i-1)}{(-3z^2+10z+3)} \right] = \frac{4}{3} \lim_{z \rightarrow \frac{5}{3}i-1} \left[ \frac{(z-\frac{1}{3}i-1)}{(-3(z-\frac{5}{3}i-1)(z+\frac{1}{3}i-1))} \right]$   
 $= \frac{4}{3} \lim_{z \rightarrow \frac{5}{3}i-1} \left[ \frac{1}{(-3(z+\frac{1}{3}i-1))} \right] = \frac{4}{3} \times \frac{1}{-3(-\frac{1}{3}i)} = \frac{4}{3} \times \frac{1}{i} = \frac{4}{3}(-i)$

By the residue theorem  
 $\int_{\Gamma} \frac{4z}{(-3z^2+10z+3)^2} dz = 2\pi i \times \sum(\text{residues inside } \Gamma) = 2\pi i \times \left( \frac{4}{3}(-i) \right) = \frac{8\pi}{3}$

$\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2} d\theta = \frac{8\pi}{3}$

## Question 7

$$I = \int_0^{2\pi} \frac{1}{3 - 2\cos x + \sin x} dx.$$

By integrating a suitable complex function over an appropriate contour find the exact value of  $I$ .

 $\pi$ 

**Left Page:**

Let  $z = e^{ix}$ , then  $dz = ie^{ix} dx$  and  $dx = \frac{dz}{iz}$ . The integral becomes:

$$I = \int_{|z|=1} \frac{1}{3 - 2(z + z^{-1}) + \frac{z - z^{-1}}{i}} \cdot \frac{dz}{iz}$$

Simplify the denominator:

$$3 - 2(z + z^{-1}) + \frac{z - z^{-1}}{i} = \frac{3i - 2i(z + z^{-1}) + (z - z^{-1})}{i}$$

$$= \frac{3i - 2iz - 2i/z + z - 1/z}{i} = \frac{3i^2 - 2iz^2 - 2i + z^2 - 1}{iz}$$

$$= \frac{-3 - 2iz^2 - 2i + z^2 - 1}{iz} = \frac{z^2 - 2iz - 4}{iz}$$

Thus the integral is:

$$I = \int_{|z|=1} \frac{1}{z^2 - 2iz - 4} dz$$

Find poles by solving  $z^2 - 2iz - 4 = 0$ :

$$z = \frac{2i \pm \sqrt{(-2i)^2 + 16}}{2} = \frac{2i \pm \sqrt{-4 + 16}}{2} = \frac{2i \pm \sqrt{12}}{2} = i \pm \sqrt{3}$$

The pole inside the unit circle is  $z = i - \sqrt{3}$ .

Residue at  $z = i - \sqrt{3}$ :

$$\text{Res} = \lim_{z \rightarrow i - \sqrt{3}} (z - (i - \sqrt{3})) \frac{1}{z^2 - 2iz - 4} = \lim_{z \rightarrow i - \sqrt{3}} \frac{1}{2z - 2i} = \frac{1}{2(i - \sqrt{3}) - 2i} = \frac{1}{-2\sqrt{3} - i}$$

By the residue theorem:

$$I = 2\pi i \times \text{Res} = 2\pi i \times \frac{1}{-2\sqrt{3} - i} = \pi$$

**Right Page:**

By the residue theorem:

$$I = 2\pi i \times \sum \text{Residues inside } \Gamma$$

Residue at  $z = i - \sqrt{3}$ :

$$\text{Res} = \lim_{z \rightarrow i - \sqrt{3}} (z - (i - \sqrt{3})) \frac{1}{z^2 - 2iz - 4} = \lim_{z \rightarrow i - \sqrt{3}} \frac{1}{2z - 2i} = \frac{1}{2(i - \sqrt{3}) - 2i} = \frac{1}{-2\sqrt{3} - i}$$

Thus:

$$I = 2\pi i \times \frac{1}{-2\sqrt{3} - i} = \pi$$

By integrating a suitable complex function over an appropriate contour find the exact value of  $I$ .

$$\frac{\pi}{12}$$



Created by T. Madas

# SEMI CIRCLE CONTOUR

Created by T. Madas

**Jordan's Lemma**

Suppose that  $f(z) \rightarrow 0$  uniformly, as  $|z| \rightarrow \infty$ , for  $0 \leq \arg z \leq \pi$ .

If  $\alpha > 0$ , then  $\int_{\gamma_R} f(z) e^{i\alpha z} dz \rightarrow 0$  as  $R \rightarrow \infty$ , where  $\gamma_R(\theta) = R e^{i\theta}$ , for  $0 \leq \theta \leq \pi$ .

**Proof**

Given  $\varepsilon > 0$  we may always pick  $R_0$ , so that if  $R > R_0$ ,  $|f(z)| < \varepsilon$ ,  $\forall z \in \gamma_R$ .

Thus

$$\begin{aligned} \left| \int_{\gamma_R} e^{i\alpha z} f(z) dz \right| &= \left| \int_0^\pi e^{i\alpha R(\cos\theta + i\sin\theta)} f(R e^{i\theta}) i e^{i\theta} d\theta \right| = \\ &= \left| \int_0^\pi e^{i\alpha R \cos\theta} e^{-\alpha R \sin\theta} f(R e^{i\theta}) i e^{i\theta} d\theta \right| \leq \int_0^\pi \left| e^{i\alpha R \cos\theta} e^{-\alpha R \sin\theta} f(R e^{i\theta}) i e^{i\theta} \right| d\theta = \\ &= \int_0^\pi \left| e^{i\alpha R \cos\theta} \right| \left| e^{-\alpha R \sin\theta} \right| \left| f(R e^{i\theta}) \right| \left| i e^{i\theta} \right| d\theta = \int_0^\pi e^{-\alpha R \sin\theta} \left| f(R e^{i\theta}) \right| d\theta \leq \\ &= \varepsilon R \int_0^\pi e^{-\alpha R \sin\theta} d\theta = 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R \sin\theta} d\theta \quad \left[ \text{since } \sin\theta \text{ is even about } \frac{\pi}{2} \right] \end{aligned}$$

Now by **Jordan's Inequality**

$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1, \text{ if } 0 < \theta \leq \frac{\pi}{2}$$

$$\sin\theta \geq \frac{2\theta}{\pi} \Rightarrow e^{-\sin\theta} \leq e^{-\frac{2\theta}{\pi}}, \text{ if } 0 < \theta \leq \frac{\pi}{2}$$

Hence

$$\begin{aligned} 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\alpha R \sin\theta} d\theta &\leq 2\varepsilon R \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi}\alpha R \theta} d\theta = 2\varepsilon R \left[ -\frac{\pi}{2\alpha R} e^{-\frac{2}{\pi}\alpha R \theta} \right]_0^{\frac{\pi}{2}} = \\ &= \frac{\varepsilon\pi}{\alpha} \left[ e^{-\frac{2}{\pi}\alpha R \theta} \right]_0^{\frac{\pi}{2}} = \frac{\varepsilon\pi}{\alpha} \left[ 1 - e^{-\alpha R} \right] \rightarrow 0 \text{ since as } R \rightarrow \infty, \varepsilon \rightarrow 0 \quad \square \end{aligned}$$

## Question 1

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx.$$

$$\boxed{V}, \boxed{\frac{\pi}{e}}$$

● CONSIDER  $\oint_{\Gamma} f(z) dz$ , where  $f(z) = \frac{e^{iz}}{1+z^2}$  AND  $\Gamma$  IS THE "STANDARD" SEMICIRCULAR CONTOUR, SHOWN BELOW

●  $f(z)$  HAS SIMPLE POLES AT  $\pm i$ , OF WHICH ONLY THE ONE AT  $i$  IS INSIDE  $\Gamma$

● CALCULATE THE RESIDUE OF THIS POLE

$$\begin{aligned} \lim_{z \rightarrow i} [(z-i)f(z)] &= \lim_{z \rightarrow i} \left[ (z-i) \frac{e^{iz}}{z^2+1} \right] \\ &= \lim_{z \rightarrow i} \left[ (z-i) \frac{e^{iz}}{(z-i)(z+i)} \right] \\ &= \frac{e^{-1}}{2i} \end{aligned}$$

● BY THE RESIDUE THEOREM

$$\begin{aligned} \Rightarrow \oint_{\Gamma} f(z) dz &= 2\pi i \times (\text{RESIDUES INSIDE } \Gamma) \\ \Rightarrow \oint_{\Gamma} \frac{e^{iz}}{1+z^2} dz &= 2\pi i \times \frac{e^{-1}}{2i} \end{aligned}$$

$\Rightarrow \int_{-R}^R \frac{e^{iz}}{1+z^2} dz + \int_{\text{ARC}} \frac{e^{iz}}{1+z^2} dz = \frac{\pi}{e}$

↑ ALONG THE STRAIGHT LINE      ↑ ALONG THE ARC

● NOW  $g(z) = \frac{e^{iz}}{1+z^2}$  SATISFIES JORDAN'S LEMMA, SO AS  $R \rightarrow \infty$  THE INTEGRAL AROUND  $\gamma$  VANISHES.

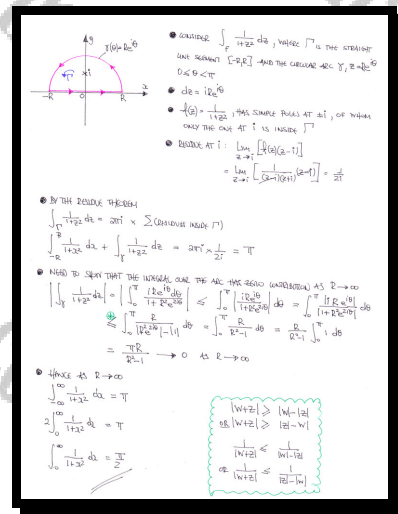
$$\begin{aligned} \Rightarrow \int_{-\infty}^{\infty} \frac{e^{iz}}{1+z^2} dz &= \frac{\pi}{e} \\ \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} + i \frac{\sin x}{1+x^2} dx &= \frac{\pi}{e} \\ \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx &= \frac{\pi}{e} \end{aligned}$$

## Question 2

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^2} dx.$$

$$\frac{\pi}{2}$$



## Question 3

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{(x^2 + 4)^2} dx.$$

$$\boxed{\phantom{000}}, \quad \frac{\pi}{32}$$

**Handwritten Solution:**

**Left Page:**

Consider  $\int_{\Gamma} \frac{1}{(z^2+4)^2} dz$  over the semi-circular contour  $\Gamma$  (shown below).

The integrand has double poles at  $z = \pm 2i$ ; only the one at  $+2i$  is inside  $\Gamma$  — calculate the residue of this pole.

By Residue Theorem:

$$\int_{\Gamma} \frac{1}{(z^2+4)^2} dz = 2\pi i \times \left( \text{Residue inside } \Gamma \right)$$

**Right Page:**

Next consider the contribution along  $\gamma$  as  $R \rightarrow \infty$ .

Hence as  $R \rightarrow \infty$ :

$$\int_0^{\infty} \frac{1}{(x^2+4)^2} dx = \frac{\pi}{32}$$



## Question 4

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{1}{1+x^4} dx.$$

$$\frac{\pi\sqrt{2}}{4}$$

**CONSIDER**  $\int_{\Gamma} \frac{1}{1+z^4} dz$  OVER THE CONTOUR  $\Gamma$  SHOWN BELOW

AT  $z = e^{i\pi/4}$   
 $\lim_{z \rightarrow e^{i\pi/4}} \left[ \frac{1}{1+z^4} (z - e^{i\pi/4}) \right] = 0$   
 AS  $(z - e^{i\pi/4})$  IS  
 ALSO A FACTOR OF  
 OF  $z^4 + 1$

BY L'HOSPITAL'S RULE  
 $= \lim_{z \rightarrow e^{i\pi/4}} \left[ \frac{1}{4z^3} \right] = \frac{1}{4e^{i3\pi/4}} = \frac{1}{4} e^{-i3\pi/4}$   
 $= \frac{1}{4} \left[ \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right] = \frac{1}{4} \left[ -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right] = -\frac{\sqrt{2}}{8} (1+i)$

● THIS BY THE RESIDUE THEOREM  
 $\Rightarrow \int_{\Gamma} \frac{1}{1+z^4} dz = 2\pi i \times \sum \text{RESIDUES INSIDE } \Gamma$   
 $\Rightarrow \int_{\Gamma} \frac{1}{1+z^4} dz = \frac{1}{1+24} + \frac{1}{1+24} = 2\pi i \left[ \frac{\sqrt{2}}{8} (1-i) - \frac{\sqrt{2}}{8} (1+i) \right]$

PARAMETRIZE  $\Gamma$ :  $z = e^{i\theta}$   
 $dz = i e^{i\theta} d\theta$   
 $\Rightarrow \int_{\Gamma} \frac{1}{1+z^4} dz = \int_0^{2\pi} \frac{1}{1+e^{i4\theta}} i e^{i\theta} d\theta = \frac{\sqrt{2}}{2} \pi \left[ (-1-i) - (-1+i) \right]$   
 $\Rightarrow \int_{\Gamma} \frac{1}{1+z^4} dz = \frac{\sqrt{2}}{2} \pi$

**LOCUS FOR POLES**  
 $1+z^4=0$   
 $z^4 = -1 = e^{i(\pi+2k\pi)}$   
 $z = e^{i(\pi/4 + k\pi/2)}$   
 OF WHICH THE PAIRS  
 AT  $z = e^{i\pi/4}$  &  $z = e^{i3\pi/4}$   
 ARE INSIDE  $\Gamma$

**CALCULATE RESIDUES AT**  $\lim_{z \rightarrow e^{i\pi/4}} \left[ \frac{1}{4z^3} (z - e^{i\pi/4}) \right]$   
 $\lim_{z \rightarrow e^{i\pi/4}} \left[ \frac{1}{4z^3} (z - e^{i\pi/4}) \right] = 0$   
 AS  $(z - e^{i\pi/4})$  IS  
 ALSO A FACTOR OF  
 OF  $z^4 + 1$

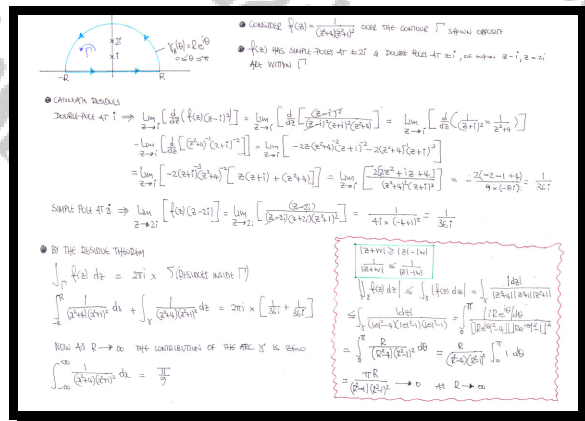
BY L'HOSPITAL'S RULE  
 $= \lim_{z \rightarrow e^{i\pi/4}} \left[ \frac{1}{4z^3} \right] = \frac{1}{4e^{i3\pi/4}} = \frac{1}{4} e^{-i3\pi/4} = \frac{1}{4} \left[ \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} \right]$   
 $= \frac{1}{4} \left[ -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right] = -\frac{\sqrt{2}}{8} (1+i)$

**CONSIDER THE CONTRIBUTION OF THE INTEGRAL OVER  $\gamma'$**   
 $\int_{\gamma'} \frac{1}{1+z^4} dz \leq \int_0^{\infty} \frac{1}{1+x^4} dx$   
 $\Rightarrow \int_{\gamma'} \frac{1}{1+z^4} dz \leq \int_0^{\infty} \frac{1}{1+x^4} dx$   
 $\Rightarrow \int_{\gamma'} \frac{1}{1+z^4} dz \rightarrow 0$  AS  $R \rightarrow \infty$

● AS  $R \rightarrow \infty$   
 $\int_{\gamma'} \frac{1}{1+z^4} dz = \frac{\sqrt{2}}{2} \pi$   
 $\Rightarrow \int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\sqrt{2}}{2} \pi$

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+1)^2} dx.$$

$$\frac{\pi}{9}$$


## Question 6

By integrating a suitable complex function over an appropriate contour find

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 4x + 5)^2} dx.$$

$$\frac{\pi}{2}$$

**Consider**  $\int_C \frac{1}{(z^2 + 4z + 5)^2} dz$  using the standard contour  $C$  shown below.

The contour  $C$  is a semi-circle in the upper half-plane with radius  $R$ , centered at  $-2$  on the real axis. The contour is traversed counter-clockwise. The poles of the integrand are at  $z = -2 \pm i$ . Only the pole at  $z = -2 + i$  is inside the contour.

**By the Residue Theorem**

$$\int_C \frac{1}{(z^2 + 4z + 5)^2} dz = 2\pi i \times \text{Residue at } z = -2 + i$$

**Calculate the residue:**

$$\lim_{z \rightarrow -2+i} \frac{d}{dz} \left[ (z - (-2+i)) \frac{1}{(z^2 + 4z + 5)^2} \right]$$

$$= \lim_{z \rightarrow -2+i} \frac{d}{dz} \left[ \frac{1}{(z^2 + 4z + 5)^2} \right] = \lim_{z \rightarrow -2+i} \left[ -\frac{2}{(z^2 + 4z + 5)^3} \right]$$

$$= -\frac{2}{(2)^3} = -\frac{1}{4i}$$

**Also consider the contribution of the arc  $\gamma$  as  $R \rightarrow \infty$ :**

$$\int_{\gamma} \frac{1}{(z^2 + 4z + 5)^2} dz \leq \int_0^\pi \frac{1}{(R^2 e^{i\theta} - 4R e^{i\theta} + 5)^2} R d\theta$$

Now  $|R^2 e^{i\theta} - 4R e^{i\theta} + 5| \geq |R^2 - 4R| = R(R - 4)$

$$\frac{1}{|R^2 e^{i\theta} - 4R e^{i\theta} + 5|^2} \leq \frac{1}{(R(R - 4))^2}$$

$$\leq \frac{1}{R^2 (R - 4)^2}$$

Thus as  $R \rightarrow \infty$

$$\int_{\gamma} \frac{1}{(z^2 + 4z + 5)^2} dz \rightarrow 0$$

**Thus as  $R \rightarrow \infty$**

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 4x + 5)^2} dx = \frac{\pi}{2}$$

## Question 7

Given that  $k > 0$  find the exact value of

$$\int_{-\infty}^{\infty} \frac{x \cos kx}{x^2 + 2x + 5} dx \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x \sin kx}{x^2 + 2x + 5} dx.$$

$$\boxed{\phantom{000000}}, \quad \boxed{\frac{1}{2} \pi e^{-2k} (2 \sin k - \cos k)}, \quad \boxed{\frac{1}{2} \pi e^{-2k} (\sin k + 2 \cos k)}$$

CONSIDER  $f(z) = \frac{ze^{ikz}}{z^2 + 2z + 5}$  OVER A SEMICIRCULAR CONTOUR  $\Gamma$ , SHOWN BELOW

$z^2 + 2z + 5 = (z+1)^2 + 4 = (z+1)^2 - (2i)^2 = (z+1-2i)(z+1+2i)$

$f(z)$  HAS SIMPLE POLES AT  $-1+2i$ , BUT ONLY THE ONE AT  $-1+2i$  IS INSIDE  $\Gamma$ , SO WE NEED ITS RESIDUE

$\lim_{z \rightarrow -1+2i} \left[ (z+1-2i) \frac{ze^{ikz}}{(z+1-2i)(z+1+2i)} \right] = \frac{(-1+2i)e^{ik(-1+2i)}}{(-1+2i)(-1+2i)}$

$= \frac{1}{4} (-1+2i)^{-2} e^{-k} e^{2ik} = \frac{e^{-k}}{4} (-1+2i)^{-2} (\cos k - i \sin k)$

BY THE RESIDUE THEOREM

$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$

$\int_{-\infty}^{\infty} \frac{ze^{ikz}}{z^2 + 2z + 5} dz + \int_{\text{arc}} \frac{ze^{ikz}}{z^2 + 2z + 5} dz = 2\pi i \times \frac{e^{-k}}{4} (-1+2i)^{-2} (\cos k - i \sin k)$

$\int_{-\infty}^{\infty} \frac{ze^{ikz}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (-1+2i)^{-2} (\cos k - i \sin k)$

HOW AS  $R \rightarrow \infty$ , THE CONTRIBUTION OF THE INTEGRAL OVER  $\gamma$  IS ZERO BY JORDAN'S LEMMA WITHHOLD STRIPS

If  $|k| \rightarrow 0$  AS  $|z| \rightarrow \infty$ ,  $0 < \arg z < \pi$ , THEN  $\int_{\gamma} f(z) dz \rightarrow 0$ , AS  $R \rightarrow \infty$ , SO UNLESS  $k > 0$   $f(z) = e^{ikz}$

$\left[ \text{HOLD } f(z) = \frac{z}{z^2 + 2z + 5} \rightarrow 0 \text{ AS } |z| \rightarrow \infty \right]$

FINALLY WE HAVE AS  $R \rightarrow \infty$

$\int_{-\infty}^{\infty} \frac{ze^{ikz}}{z^2 + 2z + 5} dz = \frac{\pi}{2} e^{-k} (-1+2i)^{-2} (\cos k - i \sin k)$

$\int_{-\infty}^{\infty} \frac{z \cos kx}{x^2 + 2x + 5} dx = \frac{\pi}{2} e^{-k} [\cos k + 2 \sin k]$

SEPARATING REAL & IMAGINARY

$\int_{-\infty}^{\infty} \frac{z \cos kx}{x^2 + 2x + 5} dx = \frac{\pi}{2} e^{-k} (2 \cos k - \cos k)$

$\int_{-\infty}^{\infty} \frac{z \sin kx}{x^2 + 2x + 5} dx = \frac{\pi}{2} e^{-k} (\sin k + 2 \cos k)$

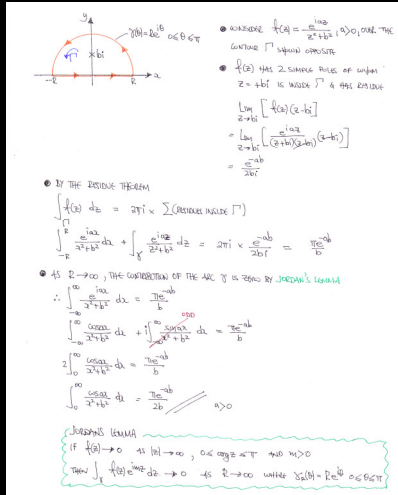
## Question 8

By integrating a suitable complex function over an appropriate contour find

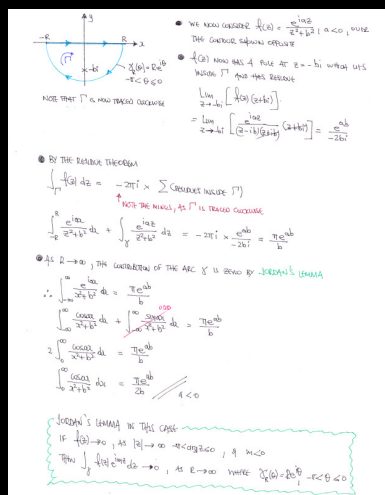
a) ...  $\int_0^{\infty} \frac{\cos ax}{x^2 + b^2} dx, a > 0.$

b) ...  $\int_0^{\infty} \frac{\cos ax}{x^2 + b^2} dx, a < 0.$

$$\frac{\pi e^{-ab}}{2b}, a > 0, \quad \frac{\pi e^{ab}}{2b}, a < 0$$



Handwritten solution for part (a) of Question 8. The solution uses the residue theorem with a contour in the upper half-plane. The function  $f(z) = \frac{e^{iaz}}{z^2 + b^2}$  is integrated over the contour. The integral over the semi-circle goes to zero as  $R \rightarrow \infty$ . The integral over the real axis is the desired integral. The residue at  $z = ib$  is calculated, and the final result is  $\frac{\pi e^{-ab}}{2b}$ .



Handwritten solution for part (b) of Question 8. The solution uses the residue theorem with a contour in the lower half-plane. The function  $f(z) = \frac{e^{iaz}}{z^2 + b^2}$  is integrated over the contour. The integral over the semi-circle goes to zero as  $R \rightarrow \infty$ . The integral over the real axis is the desired integral. The residue at  $z = -ib$  is calculated, and the final result is  $\frac{\pi e^{ab}}{2b}$ .

## Question 9

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} dx, \quad a > 0.$$

$$\boxed{\frac{1}{2} \pi e^{-ab}}$$

Consider  $f(z) = \frac{ze^{iaz}}{z^2 + b^2}$ ,  $a > 0$ , over  $\Gamma$  (shown) contour.  
 $f(z)$  has simple poles at  $\pm bi$ , of which  $z = bi$  is inside  $\Gamma$ , with residue:  
 $\lim_{z \rightarrow bi} \left[ \frac{ze^{iaz}}{(z-bi)(z+bi)} (z-bi) \right]$   
 $= \lim_{z \rightarrow bi} \left[ \frac{ze^{iaz}}{z+bi} \right] = \frac{be^{-ab}}{2bi} = \frac{1}{2} e^{-ab}$

By the residue theorem  
 $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$   
 $\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2 + b^2} dx + \int_{\gamma} \frac{ze^{iaz}}{z^2 + b^2} dz = 2\pi i \times \frac{1}{2} e^{-ab}$

As  $R \rightarrow \infty$ , the contribution of the integral over  $\gamma$  is zero, by Jordan's lemma  
 $\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2 + b^2} dx = i\pi e^{-ab}$   
 $\int_0^{\infty} \frac{2x \sin ax}{x^2 + b^2} dx = i\pi e^{-ab}$   
 $2i \int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} dx = i\pi e^{-ab}$   
 $\int_0^{\infty} \frac{x \sin ax}{x^2 + b^2} dx = \frac{\pi}{2} e^{-ab} \quad a > 0$

Jordan's lemma  
 If  $f(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ ,  $0 < \arg z \leq \pi$ ,  $a > 0$   
 Then  $\int_{\gamma} f(z) e^{iaz} dz \rightarrow 0$  as  $R \rightarrow \infty$ , where  $\gamma_R(t) = Re^{it}$ ,  $0 \leq t \leq \pi$

## Question 10

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{(1-x^2)\cos \alpha x}{(1+x^2)^2} dx, \alpha > 0.$$

$$\boxed{\frac{1}{2}\pi\alpha e^{-\alpha}}$$

• CONSIDER  $f(z) = \frac{(1-z^2)e^{i\alpha z}}{(1+z^2)^2}$ ,  $\alpha > 0$  over the contour  $\Gamma$

•  $f(z)$  HAS DOUBLE POLES AT  $\pm i$ , of which ONLY THE ONE AT  $+i$  IS INSIDE  $\Gamma$

CALCULATE ITS RESIDUE

$$\begin{aligned} \lim_{z \rightarrow i} \frac{d}{dz} \left[ (z-i)^2 \frac{(1-z^2)e^{i\alpha z}}{(z-i)^2(z+i)^2} \right] &= \lim_{z \rightarrow i} \frac{d}{dz} \left[ \frac{(1-z^2)e^{i\alpha z}}{(z+i)^2} \right] \\ &= \lim_{z \rightarrow i} \left[ \frac{(z+i)^2 \frac{d}{dz} [(1-z^2)e^{i\alpha z}] - (1-z^2)e^{i\alpha z} \cdot 2(z+i)}{(z+i)^4} \right] \\ &= \lim_{z \rightarrow i} \left[ \frac{(z+i)^2 [i\alpha(1-z^2)e^{i\alpha z} - 2z(1-z^2)e^{i\alpha z}]}{(z+i)^4} \right] \\ &= \lim_{z \rightarrow i} \left[ \frac{(z+i)^2 [2i\alpha(1-z^2) - 2z(1-z^2)]}{(z+i)^4} \right] \\ &= \frac{2i\alpha(1-i^2) - 2i(1-i^2)}{(2i)^2} = \frac{e^{-\alpha} [4 - \ln 4]}{-2i} = \frac{\pi e^{-\alpha}}{2i} \end{aligned}$$

• BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{Residues inside } \Gamma)$$

$$\int_{-R}^R \frac{(1-x^2)e^{i\alpha x}}{(1+x^2)^2} dx + \int_{\gamma} \frac{(1-z^2)e^{i\alpha z}}{(1+z^2)^2} dz = 2\pi i \times \frac{\pi e^{-\alpha}}{2i}$$

• AS  $R \rightarrow \infty$ , THE CONTRIBUTION OF THE INTERVAL OVER  $\gamma$  IS ZERO, AS IT SATISFIES JORDAN'S LEMMA

• THIS GIVES

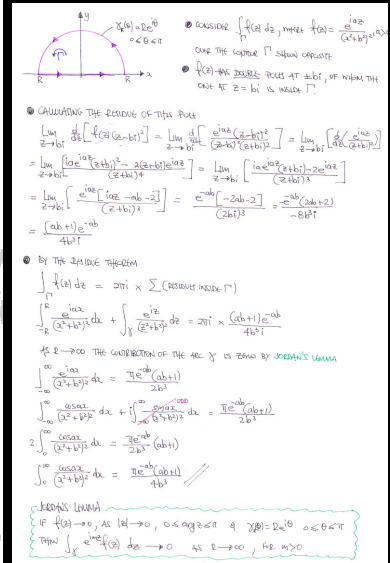
$$\begin{aligned} \int_{-\infty}^{\infty} \frac{(1-x^2)e^{i\alpha x}}{(1+x^2)^2} dx &= \pi\alpha e^{-\alpha} \\ \int_{-\infty}^{\infty} \frac{(1-x^2)\cos \alpha x}{(1+x^2)^2} dx + i \int_{-\infty}^{\infty} \frac{(1-x^2)\sin \alpha x}{(1+x^2)^2} dx &= \pi\alpha e^{-\alpha} \\ \int_0^{\infty} \frac{(1-x^2)\cos \alpha x}{(1+x^2)^2} dx &= \frac{\pi\alpha e^{-\alpha}}{2} \end{aligned}$$

## Question 11

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} dx, \quad a > 0.$$

$$\frac{\pi e^{-ab}(ab+1)}{4b^3}$$



• Consider  $\oint_{\Gamma} f(z) dz$ , where  $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$   
 over the contour  $\Gamma$  shown above.  
 •  $f(z)$  has double poles at  $\pm ib$ , of which the one at  $z = ib$  is inside  $\Gamma$ .

• Calculating the residue of this pole  

$$\lim_{z \rightarrow ib} \frac{d}{dz} \left[ (z - ib) f(z) \right] = \lim_{z \rightarrow ib} \frac{d}{dz} \left[ \frac{e^{iaz}}{(z+ib)^2} \right] = \lim_{z \rightarrow ib} \left[ \frac{ia e^{iaz}}{(z+ib)^2} - \frac{2e^{iaz}}{(z+ib)^3} \right]$$

$$= \lim_{z \rightarrow ib} \left[ \frac{ia e^{ia(ib)}}{(ib+ib)^2} - \frac{2e^{ia(ib)}}{(ib+ib)^3} \right] = \lim_{z \rightarrow ib} \left[ \frac{ia e^{-ab}}{(2ib)^2} - \frac{2e^{-ab}}{(2ib)^3} \right]$$

$$= \lim_{z \rightarrow ib} \left[ \frac{ia e^{-ab}}{(2ib)^2} - \frac{2e^{-ab}}{(2ib)^3} \right] = \frac{e^{-ab}}{(2ib)^2} \left[ -2ab - 2 \right] = \frac{e^{-ab}}{(2ib)^2} (-2ab - 2)$$

$$= \frac{e^{-ab}}{(2ib)^2} (-2ab - 2) = \frac{e^{-ab}}{4b^2} (-ab - 1)$$

• By the residue theorem  

$$\oint_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$$

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2} dx + \int_{\gamma} \frac{e^{iaz}}{(z^2+b^2)^2} dz = 2\pi i \times \frac{e^{-ab}}{4b^2} (-ab - 1)$$

As  $\epsilon \rightarrow 0$  the contribution of the arc  $\gamma$  is zero by Jordan's lemma

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{(x^2+b^2)^2} dx = \frac{\pi e^{-ab}(ab+1)}{2b^3}$$

$$\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx + i \int_0^{\infty} \frac{\sin ax}{(x^2+b^2)^2} dx = \frac{\pi e^{-ab}(ab+1)}{2b^3}$$

$$2 \int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi e^{-ab}(ab+1)}{2b^3}$$

$$\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx = \frac{\pi e^{-ab}(ab+1)}{4b^3} //$$

Jordan's lemma  
 If  $f(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ ,  $0 < \arg z < \pi$  and  $|f(z)| \leq \frac{M}{|z|^n}$ ,  $n > 0$   
 then  $\int_{\gamma} f(z) dz \rightarrow 0$  as  $\epsilon \rightarrow \infty$ , for  $n > 0$



## Question 12

By integrating a suitable complex function over an appropriate contour find

$$\int_0^{\infty} \frac{\cos x}{1+x^6} dx.$$

$$\frac{\pi}{6e} \left[ 1 + \sqrt{e} \left[ \cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right] \right]$$

**Left Page:**

- Consider  $f(z) = \frac{e^{iz}}{1+z^6}$  and the contour  $\Gamma$  shown opposite.
- $f(z)$  has 6 simple poles  $z^6 = -1 = e^{i(1+2n)\pi} \Rightarrow e^{i\pi(2n+1)}$   
 $z = e^{i\pi(2n+1)/6}$
- of these poles  $z = e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6}$  are inside  $\Gamma$
- TO CALCULATE RESIDUES USING  $\lim_{z \rightarrow z_0} [f(z)(z-z_0)]$
- CONSIDER A SIMILAR CASE  
 $\lim_{z \rightarrow z_0} \left[ \frac{z^2}{1+z^2} (z-z_0) \right] = 0$  SINCE  $(z-z_0)$  IS A FACTOR OF  $1+z^2$
- USING L'HOPITAL'S RULE  
 $= \lim_{z \rightarrow z_0} \left[ \frac{2z}{2z} (z-z_0) \right] = \frac{2z_0}{2z_0} = 1$  (SINCE  $z-z_0$  IS A FACTOR OF  $1+z^2$ )
- POLE AT  $z = e^{i\pi/6}$  RESIDUE  $\frac{e^{i\pi/6}}{6e^{i\pi/2}} = \frac{e^{i\pi/6}}{6i}$
- POLE AT  $z = e^{i\pi/2}$  RESIDUE  $\frac{e^{i\pi/2}}{6e^{i\pi/2}} = \frac{1}{6}$
- POLE AT  $z = e^{i5\pi/6}$  RESIDUE  $\frac{e^{i5\pi/6}}{6e^{i5\pi/2}} = \frac{e^{i5\pi/6}}{6(-i)}$
- NOW BY THE RESIDUE THEOREM  $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$

**Right Page:**

POLE AT  $z = e^{i\pi/6}$  RESIDUE  $\frac{e^{i\pi/6}}{6e^{i\pi/2}} = \frac{e^{i\pi/6}}{6i}$

POLE AT  $z = e^{i\pi/2}$  RESIDUE  $\frac{e^{i\pi/2}}{6e^{i\pi/2}} = \frac{1}{6}$

POLE AT  $z = e^{i5\pi/6}$  RESIDUE  $\frac{e^{i5\pi/6}}{6e^{i5\pi/2}} = \frac{e^{i5\pi/6}}{6(-i)}$

NOW BY THE RESIDUE THEOREM  $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$

JOORDAN'S LEMMA  
 If  $f(z) \rightarrow 0$  as  $|z| \rightarrow \infty$  then  $\int_{\Gamma_R} f(z) dz \rightarrow 0$  as  $R \rightarrow \infty$

Then  $\int_0^{\infty} \frac{\cos x}{1+x^6} dx = \frac{\pi}{6e} \left[ 1 + \sqrt{e} \left[ \cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right] \right]$

## Question 13

By integrating a suitable complex function over an appropriate contour find an exact simplified value for

$$\int_{-\infty}^{\infty} \frac{1}{ax^2 + bx + c} dx,$$

where  $a$ ,  $b$  and  $c$  are real constants such that  $a > 0$  and  $b^2 - 4ac < 0$ .

$$\boxed{V}, \boxed{\phantom{00}}, \boxed{\frac{2\pi}{\sqrt{4ac - b^2}}}$$

CONSIDER THE INTEGRAL OF  $f(z) = \frac{1}{az^2 + bz + c}$  AROUND THE CONTOUR  $\Gamma$  SHOWN BELOW

DO THE QUADRATIC FORMULA ON THE DENOMINATOR TO FIND THE POLES

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a} \quad \text{WITH } \Delta = b^2 - 4ac < 0$$

$$z = \frac{-b \pm \sqrt{-\Delta}}{2a}$$

ONE OF THE POLES IS ON THE TOP HALF OF THE PLANE AND THE OTHER AT THE BOTTOM HALF - PICK R SUFFICIENTLY LARGE, SO THE POLE AT  $z = z_0$  AT THE TOP IS INSIDE  $\Gamma$

Res  $\left[ f; z_0 \right] = \lim_{z \rightarrow z_0} (z - z_0) f(z) = \lim_{z \rightarrow z_0} \left[ \frac{z - z_0}{az^2 + bz + c} \right]$

BY USING THE RULE FOR THE LIMIT OF A FRACTION WITH ZERO OVER ZERO

$$= \lim_{z \rightarrow z_0} \left[ \frac{1}{2az + b} \right] = \frac{1}{2az_0 + b} = \frac{1}{2a \left( \frac{-b + \sqrt{-\Delta}}{2a} \right) + b}$$

$$= \frac{1}{-b + \sqrt{-\Delta}} = \frac{1}{\sqrt{-\Delta}}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} \frac{1}{az^2 + bz + c} dz = 2\pi i \times \sum (\text{Sum of Residues of } f \text{ inside } \Gamma)$$

$$\int_{\Gamma} \frac{1}{az^2 + bz + c} dz = 2\pi i \times \frac{1}{\sqrt{-\Delta}} = \frac{2\pi i}{\sqrt{4ac - b^2}}$$

PARAMETRISE AROUND  $\Gamma$  AS  $z = Re^{i\theta}$ ,  $\theta$  FROM 0 TO  $2\pi$   $dz = iRe^{i\theta} d\theta$

$$\left| \int_{\Gamma} f(z) dz \right| = \left| \int_0^{2\pi} \frac{1}{aR^2e^{2i\theta} + bRe^{i\theta} + c} (iRe^{i\theta} d\theta) \right| = \left| \int_0^{2\pi} \frac{iRe^{i\theta}}{aR^2e^{2i\theta} + bRe^{i\theta} + c} d\theta \right|$$

$$\leq \int_0^{2\pi} \left| \frac{Re^{i\theta}}{aR^2e^{2i\theta} + bRe^{i\theta} + c} \right| d\theta = \int_0^{2\pi} \frac{R}{|aR^2e^{2i\theta} + bRe^{i\theta} + c|} d\theta$$

... SIMPLIFIED INEQUALITY...

$$\leq \int_0^{2\pi} \frac{R}{|aR^2 - bR + c|} d\theta = \frac{2\pi R}{|aR^2 - bR + c|}$$

$$= \frac{2\pi R}{aR^2 - bR + c} = \frac{2\pi}{aR - b + \frac{c}{R}}$$

FINALLY AS  $R \rightarrow \infty$  THE FRACTION  $\rightarrow 0$

$$\int_{-\infty}^{\infty} \frac{1}{az^2 + bz + c} dz = \frac{2\pi i}{\sqrt{4ac - b^2}} \quad b^2 - 4ac < 0$$

## Question 14

$$I = \int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2 + 1} dx.$$

By integrating  $\frac{\ln(z+i)}{z^2+1}$  over a semicircular contour find the exact value of  $I$ .

$$I = \pi \ln 2$$

Consider  $\int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz$  over the standard semicircular contour  $\Gamma$ , shown below.

$z = Re^{i\theta}$   
 $0 \leq \theta \leq \pi$

$\oint_{\Gamma} f(z) dz = 2\pi i \sum \text{Residues inside } \Gamma$

$\oint_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = 2\pi i \left[ \frac{\ln(i+i)}{2i} \right] = \pi \ln 2$

By the residue theorem

$\int_{\Gamma} \frac{\ln(z+i)}{z^2+1} dz = 2\pi i \left[ \frac{\ln(i+i)}{2i} \right]$

Firstly consider the contribution of  $f(z)$  over the arc  $\gamma_R$  as  $R \rightarrow \infty$

Thus as  $R \rightarrow \infty$

$\int_{\gamma_R} f(z) dz \sim \pi \ln 2 + i \frac{\pi^2}{2}$

We need to subtract the integral over the real axis

$\int_{-\infty}^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \int_{-\infty}^0 \frac{\ln(x^2+1)}{x^2+1} dx + \int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2 + i \frac{\pi^2}{2}$

$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2 + i \frac{\pi^2}{2}$

$\int_{-\infty}^0 \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2 + i \frac{\pi^2}{2}$

$\int_{-\infty}^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2 + i \frac{\pi^2}{2}$

$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2$

$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1} dx = \pi \ln 2$

Created by T. Madas

# **SEMI CIRCLE CONTOUR WITH HOLE**

Created by T. Madas

### Question 1

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

V, V R, proof

CONSIDER THE INTEGRAL OF  $f(z) = \frac{e^{iz}}{z}$  OVER THE CONTOUR  $\Gamma$  SHOWN BELOW

ALONG THE  $z$  AXIS

<p>ABOVE <math>\Gamma_1</math></p> <p><math>z = -R</math></p> <p><math>dz = -dx</math></p>	<p>ABOVE <math>\Gamma_2</math></p> <p><math>z = \epsilon e^{i\theta}</math></p> <p><math>dz = i\epsilon e^{i\theta} d\theta</math></p> <p><math>\theta</math> RANG FROM <math>0</math> TO <math>\pi</math></p>	<p>ABOVE <math>\Gamma_3</math></p> <p><math>z = R e^{i\theta}</math></p> <p><math>dz = iR e^{i\theta} d\theta</math></p> <p><math>\theta</math> RANG FROM <math>0</math> TO <math>\pi</math></p>
--	--	--

NOT ACTUALLY NEEDED HERE

$f(z)$  HAS A SIMPLE POLE AT  $z=0$ , WHICH DOES NOT LIE INSIDE  $\Gamma$ , SO BY THE RESIDUE/CASORATI THEOREM WE HAVE

$$\oint_{\Gamma} f(z) dz = 0$$

$$\Rightarrow \int_{-R}^{-\epsilon} \frac{e^{iz}}{z} dz + \int_{\epsilon}^R \frac{e^{iz}}{z} dz + \int_{\Gamma_2} \frac{e^{iz}}{z} dz + \int_{\Gamma_4} \frac{e^{iz}}{z} dz = 0$$

NO NEED TO PAIRWISE THESE

$$\Rightarrow \int_{-R}^{-\epsilon} \frac{e^{iz}}{z} dz + \int_{\epsilon}^R \frac{e^{iz}}{z} dz + \int_0^{\pi} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta + \int_0^{\pi} \frac{e^{iR e^{i\theta}}}{R e^{i\theta}} iR e^{i\theta} d\theta = 0$$

AS  $R \rightarrow \infty$ , THE CONTRIBUTION OF THE "BIG SEMICIRCLE" VANISHES, BY JORDAN'S LEMMA

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + i \int_0^{\pi} e^{i\epsilon e^{i\theta}} d\theta + \int_0^{\pi} \frac{e^{iR e^{i\theta}}}{R e^{i\theta}} iR e^{i\theta} d\theta = 0$$

USE AS  $\epsilon \rightarrow 0$ ,  $e^{i\epsilon e^{i\theta}} \rightarrow 1$ , SO WE HAVE

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + i \int_0^{\pi} 1 d\theta + \int_0^{\pi} \frac{e^{iR e^{i\theta}}}{R e^{i\theta}} iR e^{i\theta} d\theta = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx + i(\pi) = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = -i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = -i\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

AND SINCE  $\frac{\sin x}{x}$  IS ODD

$$\Rightarrow 2 \int_0^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} //$$

## Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-a}).$$

**v**,  , **proof**

CONSIDER  $f(z) = \frac{e^{iz}}{z(z^2 + a^2)}$  INTEGRATED OVER THE CONTOUR SHOWN BELOW

$f(z)$  HAS SIMPLE POLES AT  $z=0$ ,  $z=ia$  &  $z=-ia$ , OF WHICH ONLY THE ONE AT  $z=ia$  IS INSIDE  $\Gamma$  — COMPUTE THE RESIDUE

$$\text{Res}[f(z)] = \lim_{z \rightarrow ia} [(z-ia)f(z)] = \lim_{z \rightarrow ia} \left[ (z-ia) \frac{e^{iz}}{z(z+ia)(z-ia)} \right]$$

$$= \lim_{z \rightarrow ia} \frac{e^{iz}}{z(z+ia)} = \frac{e^{-a}}{ia(2ia)} = -\frac{e^{-a}}{2a^2}$$

APPLY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES OF } f \text{ INSIDE } \Gamma)$$

$$\left\{ \int_{\epsilon}^R + \int_{\text{arc}} + \int_{-R}^{-\epsilon} \right\} f(z) dz = 2\pi i \times \frac{e^{-a}}{-2a^2}$$

NOW AS  $R \rightarrow \infty$  & LOOKING AT THE INTEGRAL OF  $f(z)$  OVER  $\gamma_1$

$$\int_{\gamma_1} \frac{e^{iz}}{z(z^2 + a^2)} dz \rightarrow 0 \text{ AS } R \rightarrow \infty \text{ BY JOHNSON'S LEMMA}

JOHNSON'S LEMMA  
IF  $g(z) \rightarrow 0$  UNIFORMLY AS  $|z| \rightarrow \infty$ , FOR  $0 < \theta < 2\pi$ , THEN ONE OBTAINS

$$\int_{\gamma_2} e^{iz} g(z) dz \rightarrow 0 \text{ AS } R \rightarrow \infty, \text{ WHERE } \gamma_2 \text{ IS } z = Re^{i\theta}, 0 < \theta < 2\pi$$$$

NEXT CONSIDER  $f(z)$  INTEGRATED OVER  $\gamma_2$  — FURTHERMORE FIRST

$$z = \epsilon e^{i\theta}, dz = i\epsilon e^{i\theta} d\theta, \theta \text{ RAYS FROM } 0 \text{ TO } 2\pi \text{ (CLOCKWISE)}$$

$$\int_{\gamma_2} f(z) dz = \int_0^{2\pi} \frac{e^{i(\epsilon e^{i\theta})}}{\epsilon e^{i\theta} (\epsilon^2 e^{2i\theta} + a^2)} (i\epsilon e^{i\theta} d\theta)$$

$$= i \int_0^{2\pi} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon^2 e^{2i\theta} + a^2} d\theta$$

NOW AS  $\epsilon \rightarrow 0$  THIS INTEGRAL TENDS TO

$$= i \int_0^{2\pi} \frac{1}{a^2} d\theta = -\frac{\pi i}{a^2}$$

FINALLY AS  $R \rightarrow \infty$  &  $\epsilon \rightarrow 0$  USE HADAMARD — NOTE  $R \rightarrow \infty$  &  $\epsilon \rightarrow 0$

$$\Rightarrow \int_{\gamma_1} \frac{e^{iz}}{z(z^2 + a^2)} dz + \int_{\gamma_2} \frac{e^{iz}}{z(z^2 + a^2)} dz - \frac{\pi i}{a^2} = \frac{2\pi i e^{-a}}{-2a^2}$$

$$\Rightarrow \int_{\gamma_1} \frac{e^{iz}}{z(z^2 + a^2)} dz = \left( \frac{\pi e^{-a}}{a^2} + \frac{\pi i}{a^2} \right) i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x(x^2 + a^2)} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{a^2} (1 - e^{-a}) i$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{a^2} (1 - e^{-a})$$

$$\Rightarrow 2 \int_0^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{a^2} (1 - e^{-a})$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2} (1 - e^{-a})$$

By integrating a suitable complex function over an appropriate contour show that

M, proof

[illegible]

## Question 4

$$\int_0^{\infty} \frac{\ln x}{1+x^4} dx.$$

- a) Find the value of the above improper integral, by integrating

$$f(z) = \frac{\log z}{1+z^4}, \quad z \in \mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

- b) State the value of

$$\int_0^{\infty} \frac{1}{1+x^4} dx.$$

$$\frac{\pi^2 \sqrt{2}}{16}, \quad \frac{\pi \sqrt{2}}{4}$$

**Handwritten Solution (Left Page):**

Consider  $f(z) = \frac{\log z}{1+z^4}$  over a keyhole contour. The branch cut is along the positive real axis. The contour consists of a large circle  $R \rightarrow \infty$ , a small circle  $r \rightarrow 0$ , and two horizontal segments just above and below the cut. The integral over the large circle vanishes as  $R \rightarrow \infty$ . The integral over the small circle vanishes as  $r \rightarrow 0$ . The integral over the upper segment is  $\int_0^{\infty} \frac{\log x}{1+x^4} dx$ . The integral over the lower segment is  $\int_{\infty}^0 \frac{\log x + 2\pi i}{1+x^4} dx = -\int_0^{\infty} \frac{\log x + 2\pi i}{1+x^4} dx$ . The total integral is  $\int_0^{\infty} \frac{\log x}{1+x^4} dx - \int_0^{\infty} \frac{\log x + 2\pi i}{1+x^4} dx = -2\pi i \int_0^{\infty} \frac{1}{1+x^4} dx$ . The integral over the contour is also equal to  $2\pi i$  times the sum of residues inside the contour. The residues are at  $z = e^{i\pi/4}$  and  $z = e^{3i\pi/4}$ . The sum of residues is  $\frac{\pi \sqrt{2}}{4}$ . Therefore,  $-2\pi i \int_0^{\infty} \frac{1}{1+x^4} dx = 2\pi i \cdot \frac{\pi \sqrt{2}}{4}$ , which gives  $\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi^2 \sqrt{2}}{16}$ .

**Handwritten Solution (Right Page):**

Consider  $f(z) = \frac{\log z}{1+z^4}$  over a semicircular contour in the third or fourth quadrant. The branch cut is along the positive real axis. The contour consists of a large semicircle  $R \rightarrow \infty$  and a small semicircle  $r \rightarrow 0$ . The integral over the large semicircle vanishes as  $R \rightarrow \infty$ . The integral over the small semicircle vanishes as  $r \rightarrow 0$ . The integral over the upper segment is  $\int_0^{\infty} \frac{\log x}{1+x^4} dx$ . The integral over the lower segment is  $\int_{\infty}^0 \frac{\log x + 2\pi i}{1+x^4} dx = -\int_0^{\infty} \frac{\log x + 2\pi i}{1+x^4} dx$ . The total integral is  $\int_0^{\infty} \frac{\log x}{1+x^4} dx - \int_0^{\infty} \frac{\log x + 2\pi i}{1+x^4} dx = -2\pi i \int_0^{\infty} \frac{1}{1+x^4} dx$ . The integral over the contour is also equal to  $2\pi i$  times the sum of residues inside the contour. The residues are at  $z = e^{i\pi/4}$  and  $z = e^{3i\pi/4}$ . The sum of residues is  $\frac{\pi \sqrt{2}}{4}$ . Therefore,  $-2\pi i \int_0^{\infty} \frac{1}{1+x^4} dx = 2\pi i \cdot \frac{\pi \sqrt{2}}{4}$ , which gives  $\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi^2 \sqrt{2}}{16}$ .



## Question 5

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} dx.$$

Find the value of the above improper integral, by integrating

$$f(z) = \frac{(\log z)^2}{1+z^2}, \quad z \in \mathbb{C},$$

over a semicircular contour with a branch cut starting at the origin and oriented in some arbitrary direction in the third or fourth quadrant.

$$\frac{\pi^3}{8}$$

CONSIDER  $f(z) = \frac{(\log z)^2}{z^2+1}$  ON A SEMICIRCULAR CONTOUR, WHERE THE ORIGIN IS A BRANCH POINT & THE BRANCH CUT IS TAKEN ARBITRARILY IN THE 3<sup>RD</sup> OR 4<sup>TH</sup> QUADRANT.

NOTE THAT IF THE BRANCH OF THE FOLIUM ALONG THE  $z$ -AXIS THE NEGATIVE PART AS THE BRANCH CUT (ARBITRARILY NOT)

(a) THIS SINGULAR POINT AT  $z = \pm i$  OF WHICH ONLY THE ONE AT  $z = i$  IS INSIDE  $\Gamma$  - CALCULATE ITS RESIDUE

$$\lim_{z \rightarrow i} \left[ (z-i) f(z) \right] = \lim_{z \rightarrow i} \left[ (z-i) \frac{(\log z)^2}{(z-i)(z+i)} \right] = \frac{(\log i)^2}{2i}$$

$$= \frac{(\log e^{i\pi/2})^2}{2i} = \frac{(i\pi/2)^2}{2i} = -\frac{\pi^2}{8}$$

BY THE RESIDUE THEOREM FOR POSITIVE

$$\Rightarrow \oint_{\Gamma} f(z) dz = 2\pi i \times \left( \text{RESIDUE INSIDE } \Gamma \right) = 2\pi i \times \left( -\frac{\pi^2}{8} \right) = -\frac{\pi^3}{4}$$

NOW EXAMINE THE CONTRIBUTION OF  $\gamma_1$  AS  $R \rightarrow \infty$

$$\left| \int_{\gamma_1} f(z) dz \right| \leq \int_{\gamma_1} |f(z)| |dz| = \int_0^{\pi} \left| \frac{(\log z)^2}{z^2+1} \right| |dz|$$

ON  $\gamma_1$   $z = Re^{i\theta}$   
 $dz = iRe^{i\theta} d\theta$   
 $0 \leq \theta \leq \pi$

$$= \int_0^{\pi} \frac{|\log z|^2}{|z^2+1|} |dz| = \int_0^{\pi} \frac{(\log R)^2}{R^2 e^{2i\theta} + 1} R d\theta$$

NOW APPLY THE FOLLOWING INEQUALITIES

ON NUMERATOR $ z \pm w  \leq  z  +  w $	ON DENOMINATOR $ z \pm w  \geq   z  -  w  $
--	--

$$\leq \int_0^{\pi} \frac{R \left[ (\log R)^2 + (\log e^{i\theta})^2 \right]}{R^2 e^{2i\theta} - 1} R d\theta = \int_0^{\pi} \frac{R \left[ (\log R)^2 + \theta^2 \right]}{R^2 e^{2i\theta} - 1} R d\theta$$

$$= \frac{R}{R^2-1} \int_0^{\pi} \left[ \log R + \theta \right]^2 d\theta = \frac{R}{R^2-1} \left[ \log R + \theta \right]^3 \Big|_0^{\pi}$$

$$= \frac{R}{R^2-1} \left[ \left( \log R + \pi \right)^3 - \left( \log R \right)^3 \right] \rightarrow 0 \quad \text{AS } R \rightarrow \infty$$

APPLY A SIMILAR LIMITING PROCESS FOR THE CONTRIBUTION OF  $\gamma_2$  AS  $R \rightarrow \infty$

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \int_{\gamma_2} |f(z)| |dz| = \int_0^{\pi} \left| \frac{(\log z)^2}{z^2+1} \right| |dz|$$

ON  $\gamma_2$   $z = \epsilon e^{i\theta}$   
 $dz = i\epsilon e^{i\theta} d\theta$   
 $0 \leq \theta \leq \pi$

$$= \int_0^{\pi} \frac{|\log z|^2}{|z^2+1|} |dz| = \int_0^{\pi} \frac{(\log \epsilon)^2}{\epsilon^2 e^{2i\theta} + 1} \epsilon d\theta$$

USING THE SAME INEQUALITIES FROM PREVIOUS LIMITING PROCESS OF  $\gamma_1$

$$\leq \int_0^{\pi} \frac{\epsilon \left[ (\log \epsilon)^2 + (\log e^{i\theta})^2 \right]}{\epsilon^2 e^{2i\theta} - 1} \epsilon d\theta = \int_0^{\pi} \frac{\epsilon \left[ (\log \epsilon)^2 + \theta^2 \right]}{\epsilon^2 e^{2i\theta} - 1} \epsilon d\theta$$

$$= \frac{\epsilon}{\epsilon^2-1} \int_0^{\pi} \left[ \log \epsilon + \theta \right]^2 d\theta = \frac{\epsilon}{\epsilon^2-1} \left[ \log \epsilon + \theta \right]^3 \Big|_0^{\pi}$$

$$= \frac{\epsilon}{\epsilon^2-1} \left[ \left( \log \epsilon + \pi \right)^3 - \left( \log \epsilon \right)^3 \right] \rightarrow 0 \quad \text{AS } \epsilon \rightarrow 0$$

(NOTE THAT  $\epsilon \rightarrow 0$  FASTER THAN  $\log \epsilon \rightarrow -\infty$  OR  $|\log \epsilon| \rightarrow +\infty$ )

SUMMARISES THE RESULTS AS  $R \rightarrow \infty$  &  $\epsilon \rightarrow 0$

$$\int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz = -\frac{\pi^3}{4}$$

ON THE POSITIVE  $z$ -AXIS  $z = x = x e^{i0}$   
 $\log z = \log(x e^{i0})$   
 $\log z = \log x + i0$   
 $\log z = \log x$   
 $\log z = \log x$

ON THE NEGATIVE  $z$ -AXIS  $z = -x = x e^{i\pi}$   
 $\log z = \log(x e^{i\pi})$   
 $\log z = \log x + i\pi$   
 $\log z = \log x + i\pi$

COMBINE BOTH A LIMITING PROCESS

$$= 2 \int_0^{\infty} \frac{(\log x)^2}{x^2+1} dx - \pi^2 \left( \frac{\pi}{2} \right) + 2\pi i \int_0^{\infty} \frac{\log x}{x^2+1} dx = -\frac{\pi^3}{4} + 0$$

$$= 2 \int_0^{\infty} \frac{(\log x)^2}{x^2+1} dx - \frac{\pi^3}{4} = -\frac{\pi^3}{4}$$

$$\Rightarrow \int_0^{\infty} \frac{(\log x)^2}{x^2+1} dx = \frac{\pi^3}{8}$$



Created by T. Madas

# KEYHOLE CONTOUR

(Branch Cuts)

Created by T. Madas

## Question 1

$$f(z) = \frac{\log z}{1+z^2}, \quad z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

proof

Handwritten mathematical notes showing the proof of the integral of  $\frac{1}{1+x^2}$  from 0 to  $\infty$  using complex analysis. The notes include a diagram of a contour in the complex plane, a branch cut for the logarithm, and detailed calculations for the integral over the contour and the limit as the radius goes to infinity.

**Left Page:**

- Now  $f(z) = \frac{\log z}{1+z^2}$  has a branch point at  $z=0$  because of the  $\log z$ , so we must make a branch cut from  $z=0$  to  $\infty$  and it is sensible to do this on the positive  $x$ -axis. Consider the branch cut and only way to find the value of the branch integral is to use the residue theorem.
- By the residue theorem:  $\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$
- Now the contribution of  $\gamma_2(z) = Re^{i\theta}$  as  $R \rightarrow \infty$ : Let  $z = Re^{i\theta}$ ,  $0 < \theta < 2\pi$ ,  $dz = iRe^{i\theta} d\theta$ . Then  $\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{\log(Re^{i\theta})}{1+R^2e^{2i\theta}} iRe^{i\theta} d\theta \right| \leq \int_0^{2\pi} \frac{|\log R + i\theta| |iRe^{i\theta}|}{1+R^2} d\theta \leq \frac{2\pi R}{1+R^2} \rightarrow 0$  as  $R \rightarrow \infty$ .
- Now on the contribution of  $\gamma_1(z) = x$  inside  $\Gamma$ :  $\lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz = \lim_{R \rightarrow \infty} \int_0^R \frac{\log x}{1+x^2} dx = \int_0^{\infty} \frac{\log x}{1+x^2} dx$ . But  $\log x = \ln x + i0$ , so  $\int_0^{\infty} \frac{\log x}{1+x^2} dx = \int_0^{\infty} \frac{\ln x}{1+x^2} dx$ . This is the real part of the integral we want.

**Right Page:**

- Now the contribution of  $\gamma_2(z) = Re^{i\theta}$  as  $R \rightarrow \infty$ : Let  $z = Re^{i\theta}$ ,  $0 < \theta < 2\pi$ ,  $dz = iRe^{i\theta} d\theta$ . Then  $\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{\log(Re^{i\theta})}{1+R^2e^{2i\theta}} iRe^{i\theta} d\theta \right| \leq \int_0^{2\pi} \frac{|\log R + i\theta| |iRe^{i\theta}|}{1+R^2} d\theta \leq \frac{2\pi R}{1+R^2} \rightarrow 0$  as  $R \rightarrow \infty$ .
- Now the contribution of  $\gamma_1(z) = x$  inside  $\Gamma$ :  $\lim_{R \rightarrow \infty} \int_{\gamma_1} f(z) dz = \lim_{R \rightarrow \infty} \int_0^R \frac{\log x}{1+x^2} dx = \int_0^{\infty} \frac{\log x}{1+x^2} dx$ . But  $\log x = \ln x + i0$ , so  $\int_0^{\infty} \frac{\log x}{1+x^2} dx = \int_0^{\infty} \frac{\ln x}{1+x^2} dx$ . This is the real part of the integral we want.

## Question 2

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \pi \operatorname{cosec}(p\pi), \quad 0 < p < 1.$$

**V**, proof

The handwritten proof is divided into three main sections:

- Left Section:**
  - Defines  $f(z) = \frac{z^{p-1}}{1+z}$  and notes that  $0 < p < 1$  implies a branch cut along the positive real axis.
  - Considers a keyhole contour  $\Gamma$  in the complex plane, consisting of a large circle  $C_R$ , a small circle  $C_r$ , and two horizontal segments  $\gamma_1$  and  $\gamma_2$ .
  - Shows that  $f(z)$  has a simple pole at  $z = -1$ .
  - Uses the residue theorem to relate the integral over the contour to  $2\pi i \operatorname{Res}(f, -1)$ .
  - Parameterizes the circles and segments, showing that the integrals over  $C_R$  and  $C_r$  vanish as  $R \rightarrow \infty$  and  $r \rightarrow 0$ .
- Middle Section:**
  - Parameterizes the horizontal segments  $\gamma_1$  and  $\gamma_2$  using  $z = re^{i\theta}$ .
  - Shows that the integral over  $\gamma_2$  is  $e^{i2\pi p}$  times the integral over  $\gamma_1$ .
  - Combines the results to get  $(1 - e^{i2\pi p}) \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = 2\pi i \operatorname{Res}(f, -1)$ .
- Right Section:**
  - Calculates the residue at  $z = -1$  using the limit definition:  $\operatorname{Res}(f, -1) = \lim_{z \rightarrow -1} (z+1)f(z) = (-1)^{p-1} = e^{i\pi(p-1)}$ .
  - Substitutes the residue into the equation from the middle section.
  - Solves for the real integral, yielding  $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \pi \operatorname{cosec}(p\pi)$ .

## Question 3

By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{p\pi}{2}\right), \quad 0 < p < 2.$$

proof

The handwritten proof is divided into two main sections. The left section shows the derivation of the integral formula for  $\int_0^{\infty} \frac{x^{p-1}}{1+x^2} dx$  using the residue theorem. It starts with the function  $f(z) = \frac{z^{p-1}}{1+z^2}$  and a keyhole contour  $\Gamma$  in the complex plane. The contour consists of a large circle  $C_R$ , a small circle  $C_\epsilon$ , and two horizontal segments  $L_1$  and  $L_2$  along the real axis. The integral over the contour is zero, and the integrals over the circles  $C_R$  and  $C_\epsilon$  vanish as  $R \rightarrow \infty$  and  $\epsilon \rightarrow 0$ . The integrals over  $L_1$  and  $L_2$  are related by a factor of  $e^{2\pi i p}$  due to the branch cut. This leads to the formula  $\int_0^{\infty} \frac{x^{p-1}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{p\pi}{2}\right)$ .

The right section shows the evaluation of the integral for a specific value of  $p$ ,  $p = \frac{1}{2}$ . It uses the same contour and function  $f(z) = \frac{z^{-1/2}}{1+z^2}$ . The integral over the contour is zero, and the integrals over the circles  $C_R$  and  $C_\epsilon$  vanish. The integrals over  $L_1$  and  $L_2$  are related by a factor of  $e^{-\pi i}$  due to the branch cut. This leads to the formula  $\int_0^{\infty} \frac{x^{-1/2}}{1+x^2} dx = \frac{\pi}{2} \operatorname{cosec}\left(\frac{\pi}{4}\right) = \frac{\pi}{2} \sqrt{2}$ .



## Question 4

$$f(z) = \frac{\log z}{(z+1)(z+2)}, \quad z \in \mathbb{C}.$$

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\int_0^{\infty} \frac{1}{(x+1)(x+2)} dx = \ln 2.$$

proof

**Handwritten Solution:**

Since  $f(z) = \frac{\log z}{(z+1)(z+2)}$  has a branch cut along the positive real axis, we must take a branch cut from  $0$  to  $\infty$  along the positive real axis. We choose the branch where  $\arg z = 0$  for  $z > 0$ .

Consider the contour  $\Gamma$  in the complex plane, consisting of a large circle  $C_R$  of radius  $R$ , a small circle  $C_\epsilon$  of radius  $\epsilon$ , and two line segments along the real axis:  $\gamma_1$  from  $\epsilon$  to  $R$  and  $\gamma_2$  from  $R$  to  $\epsilon$ . The contour is traversed counter-clockwise.

By the residue theorem,  $\oint_{\Gamma} f(z) dz = 2\pi i \sum \text{Residues}$ .

The residues are at  $z = -1$  and  $z = -2$ :

$$\text{Res}_{z=-1} f(z) = \lim_{z \rightarrow -1} (z+1) f(z) = \frac{\log(-1)}{(-1+2)} = \frac{i\pi}{1} = i\pi$$

$$\text{Res}_{z=-2} f(z) = \lim_{z \rightarrow -2} (z+2) f(z) = \frac{\log(-2)}{(-2+1)} = \frac{\log 2 + i\pi}{-1} = -\log 2 - i\pi$$

Thus,  $\oint_{\Gamma} f(z) dz = 2\pi i (i\pi - \log 2 - i\pi) = -2\pi i \log 2$ .

Now, we evaluate the integral over the contour segments:

On  $\gamma_1$  (from  $\epsilon$  to  $R$ ),  $\arg z = 0$ , so  $\log z = \ln x$ . The integral is  $\int_{\epsilon}^R \frac{\ln x}{(x+1)(x+2)} dx$ .

On  $\gamma_2$  (from  $R$  to  $\epsilon$ ),  $\arg z = 2\pi$ , so  $\log z = \ln x + 2\pi i$ . The integral is  $\int_R^{\epsilon} \frac{\ln x + 2\pi i}{(x+1)(x+2)} dx = -\int_{\epsilon}^R \frac{\ln x + 2\pi i}{(x+1)(x+2)} dx$ .

On the large circle  $C_R$ ,  $|f(z)| \sim \frac{\ln R}{R^2}$ , so the integral  $\rightarrow 0$  as  $R \rightarrow \infty$ .

On the small circle  $C_\epsilon$ ,  $|f(z)| \sim \frac{\ln \epsilon}{\epsilon^2}$ , so the integral  $\rightarrow 0$  as  $\epsilon \rightarrow 0$ .

Therefore,  $\int_0^{\infty} \frac{\ln x}{(x+1)(x+2)} dx = -\int_0^{\infty} \frac{\ln x + 2\pi i}{(x+1)(x+2)} dx$ .

Adding these two equations, we get  $2 \int_0^{\infty} \frac{\ln x}{(x+1)(x+2)} dx = -2\pi i \log 2$ .

Thus,  $\int_0^{\infty} \frac{\ln x}{(x+1)(x+2)} dx = -\pi i \log 2$ .

Since the original integral is real, we take the real part:  $\int_0^{\infty} \frac{1}{(x+1)(x+2)} dx = \ln 2$ .

## Question 5

$$f(z) = \frac{\log z}{(z+a)(z+b)}, \quad z \in \mathbb{C},$$

where  $a \in \mathbb{R}^+$ ,  $b \in \mathbb{R}^+$  with  $b > a$ .

By integrating  $f(z)$  over a suitable contour  $\Gamma$ , show that

$$\int_0^\infty \frac{1}{(x+a)(x+b)} dx = \frac{1}{b-a} \ln\left(\frac{b}{a}\right).$$

Q.E.D., proof

CONSIDER  $f(z) = \frac{\log z}{(z+a)(z+b)}$  AND A SUITABLE CONTOUR  $\Gamma$

AS  $f(z)$  CONTAINS  $\log z$ , WHICH HAS A BRANCH POINT AT  $z=0$ , WE MUST TAKE A BRANCH OF  $\log z$  FROM  $z=0$  TO INFINITY, IN ANY SUITABLE DIRECTION.

IN THIS CASE WE MUST TAKE THE BRANCH CUT ALONG THE POSITIVE  $x$ -AXIS AND HAVE:

- IT IS CONVENIENT TO TAKE  $\arg z = 0$  ON THE POSITIVE  $x$ -AXIS.
- THE LIMIT  $\arg z \rightarrow 2\pi$  AS  $z \rightarrow 0$  FROM ABOVE.
- THE LIMIT  $\arg z \rightarrow 0$  AS  $z \rightarrow 0$  FROM BELOW.

USE THE KEYHOLE CONTOUR SHOWN BELOW

THE ARGUMENT  $\arg z$  DOES NOT CHANGE AS  $z$  MOVES FROM  $0$  TO  $2\pi$ , INSTEAD OF THE CIRCLE  $|z|=R$ , BECAUSE OF THE BRANCH CUT.

$\log z = \ln|z| + i\arg z$  as  $\theta \rightarrow 0$  or  $2\pi$

FEEL THE SAME RESULT AT  $z=a$  &  $z=b$ , WHICH ARE NOT INSIDE  $\Gamma$

CALCULATE THE RESIDUES AT EACH POLE

- $\lim_{z \rightarrow -a} (z+a) f(z) = \frac{\log(-a)}{(b-a)}$
- $\lim_{z \rightarrow -b} (z+b) f(z) = \frac{\log(-b)}{(a-b)}$

USE THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \sum (\text{RESIDUES OF } f(z) \text{ INSIDE } \Gamma)$$

$$\left\{ \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z}{(z+a)(z+b)} dz \right\} = 2\pi i \left[ \frac{\log(-a)}{b-a} - \frac{\log(-b)}{a-b} \right]$$

$$= \frac{2\pi i}{b-a} (\log(-a) - \log(-b))$$

$$= -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right) \quad (b > a)$$

NEED TO SHOW NOW THAT THE CONTRIBUTION ALONG  $\gamma_\epsilon$  IS ZERO, AS  $R \rightarrow \infty$

$\int_{\gamma_\epsilon} f(z) dz \rightarrow 0$  as  $\epsilon \rightarrow 0$

NEED SHOW THE FOLLOWING INEQUALITIES

- $|z+w| \leq |z| + |w| \Rightarrow \log|z+w| \leq \log|z| + \log|w|$
- $|z+w| \geq ||z| - |w|| \Rightarrow \log|z+w| \geq \log|z| - \log|w|$  ON THE BOUNDARY

RETURNING TO THE INTEGRAL OVER  $\gamma_\epsilon$

$$\int_{\gamma_\epsilon} f(z) dz = \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z}{(z+a)(z+b)} dz$$

$$= \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz - \int_0^\infty \frac{\log z + 2\pi i}{(z+a)(z+b)} dz$$

$$= -2\pi i \int_0^\infty \frac{1}{(z+a)(z+b)} dz$$

NEED TO SHOW THAT  $\int_0^\infty \frac{1}{(z+a)(z+b)} dz \rightarrow 0$  AS  $R \rightarrow \infty$

NEED TO SHOW THAT  $\int_0^\infty \frac{1}{(z+a)(z+b)} dz \rightarrow 0$  AS  $R \rightarrow \infty$

NEED TO SHOW THAT  $\int_0^\infty \frac{1}{(z+a)(z+b)} dz \rightarrow 0$  AS  $R \rightarrow \infty$

$\int_{\gamma_\epsilon} f(z) dz \rightarrow 0$  AS  $\epsilon \rightarrow 0$

CONSIDER THE  $\gamma_\epsilon$  AND  $\gamma_R$  AS  $\epsilon \rightarrow 0$  AND  $R \rightarrow \infty$

THE INTEGRAL OVER  $\gamma_\epsilon$  IS

$$\int_{\gamma_\epsilon} f(z) dz = \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z}{(z+a)(z+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right)$$

THE INTEGRAL OVER  $\gamma_R$  IS

$$\int_{\gamma_R} f(z) dz = \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z + 2\pi i}{(z+a)(z+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right)$$

THE INTEGRAL OVER  $\gamma_\epsilon$  IS

$$\int_{\gamma_\epsilon} f(z) dz = \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z}{(z+a)(z+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right)$$

THE INTEGRAL OVER  $\gamma_R$  IS

$$\int_{\gamma_R} f(z) dz = \int_0^\infty \frac{\log z}{(z+a)(z+b)} dz + \int_{\infty}^0 \frac{\log z + 2\pi i}{(z+a)(z+b)} dz = -\frac{2\pi i}{b-a} \ln\left(\frac{b}{a}\right)$$



By integrating a suitable complex function over an appropriate contour show that

$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^3} dx = \frac{\pi}{3}.$$

**Created by T. Madas**

### Question 7

$$\int_0^{\infty} \frac{(\ln x)^2}{1+x^2} dx.$$

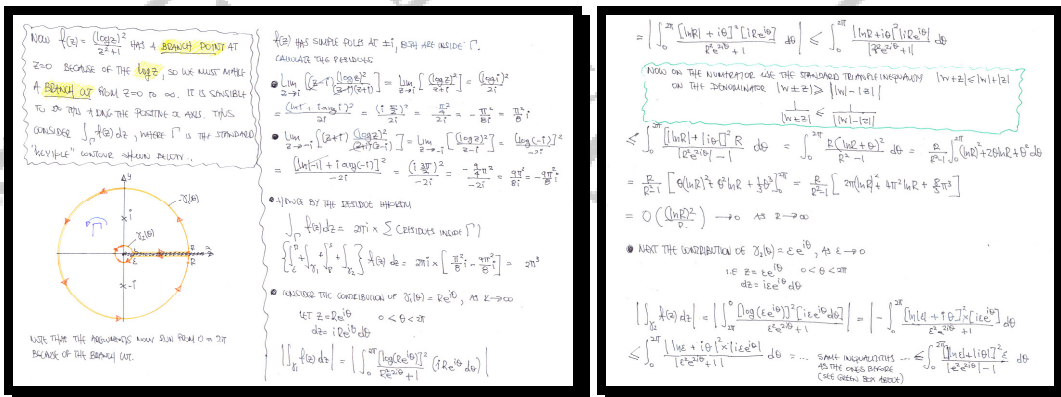
An attempt is made to find the value of the above improper integral, by integrating

$$f(z) = \frac{(\ln z)^2}{1+z^2}, \quad z \in \mathbb{C},$$

over the standard “keyhole” contour with a branch cut taken on the positive  $x$  axis.

- a) Show that such attempt fails.
- b) Calculate the value of the two integrals that can be found during this attempt.

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}, \quad \int_0^{\infty} \frac{\ln x}{1+x^2} dx = 0$$



[illegible]

### Question 8

Use a substitution followed by integration of a suitable complex function over an appropriate contour, to show that

$$\int_0^{\frac{1}{2}\pi} (\tan x)^\alpha dx = \frac{1}{2}\pi \sec\left(\frac{1}{2}\pi\alpha\right), \quad -1 < \alpha < 1.$$

proof

[illegible]

Created by T. Madas

# SPECIAL CONTOURS

Created by T. Madas

## Question 1

Consider the contour  $\Gamma$  located in the first quadrant, defined as the boundary of a quarter circular sector of radius  $R$ , with centre at the origin  $O$ .

By integrating a suitable complex function over  $\Gamma$  show that

$$\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi\sqrt{2}}{4}.$$

$$\boxed{\phantom{000}}, \quad \boxed{\frac{\pi\sqrt{2}}{4}}$$

LET US NOTE THAT THE SEMICIRCLED CONTOUR FOR THIS IS A "TOP HALF" SEMICIRCLE", HOWEVER WE SHALL USE A QUARTER CIRCLE HERE AS INSTRUCTED. — SOMETIMES ADVANTAGE IS TAKEN RESIDUES TO COMPUTE

CONSIDER  $\int_{\Gamma} \frac{1}{1+z^4} dz$  OVER THE CONTOUR SHOWN BELOW

POLES:  $z^4 = 1 \Rightarrow z = 1, i, -1, -i$   
 $z^4 = e^{i(2\pi k)}$   
 $z = e^{i(2\pi k/4)}$   
 $z = e^{i(\pi/2 k)}$   
 ONLY POLE INSIDE  $\Gamma$  IS AT  $z = e^{i\pi/4}$

PARAMETRISE EACH SECTION

ALONG  $\alpha$ :  $z = x$ ,  $dz = dx$ ,  $2 \text{ REAL } 0 \leq x \leq R$   
 ALONG  $\beta$ :  $z = Re^{i\theta}$ ,  $dz = iRe^{i\theta} d\theta$ ,  $2 \text{ REAL } 0 \leq \theta \leq \pi/2$   
 ALONG THE ARC  $\gamma$ :  $z = Re^{i\theta}$ ,  $dz = iRe^{i\theta} d\theta$ ,  $2 \text{ REAL } 0 \leq \theta \leq \pi/2$

AS  $R \rightarrow \infty$ , THE INTEGRAL OVER THE ARC VANISHES (SADDLE POINT)

OTHER REAL OR IMAGINARY YIELDS

FIND THE CORRESPONDING CLAIM THAT THE INTEGRAL OVER  $\gamma$  VANISHES

ALONG THE "SERIAL INEQUALITY" (JUST THAT  $z = q + iw$ )

LET  $z = u + iv$ ,  $|z| = \sqrt{u^2 + v^2}$   
 $|z+1| \leq \frac{1}{|z-1|}$   
 $\frac{1}{1+|z|^4} \leq \frac{1}{|z|^4} - 1 = \frac{1}{|z|^4} - 1$   
 HENCE THE INTEGRAL VANISHES

... =  $\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi\sqrt{2}}{4}$

AND THE CLAIM IS JUSTIFIED

## Question 2

By integrating a suitable complex function over a contour defined as the outline of a circular sector subtending an angle of  $\frac{1}{3}\pi$  at the origin, find an exact value for

$$\int_0^{\infty} \frac{1}{1+x^6} dx.$$

*No credit will be given for integration over alternative contours.*

$$\mathbf{V}, \quad \square, \quad \frac{\pi}{3}$$

CONSIDER  $f(z) = \frac{1}{z+1}$  OVER THE ARC  $\Gamma$ , DEFINED BELOW

$f(z)$  HAS SIMPLE POLES WHEN  $z = -1$ ,  $\neq \pm \sqrt[n]{1}$ ,  $\in \mathbb{R}$ ,  $\neq \pm \sqrt[n]{1}$

ONLY  $z = \pm \sqrt[n]{1}$  IS INSIDE  $\Gamma$

$\lim_{z \rightarrow \pm \sqrt[n]{1}} [(z - \sqrt[n]{1}) \times \frac{1}{z+1}] = \lim_{z \rightarrow \pm \sqrt[n]{1}} \frac{z - \sqrt[n]{1}}{z+1}$

BY L'HOSPITAL'S RULE

$$= \lim_{z \rightarrow \pm \sqrt[n]{1}} \left[ \frac{1}{2\sqrt[n]{1}} \right] = \frac{1}{2\sqrt[n]{1}} = \frac{1}{2} e^{\pm \frac{2\pi i}{n}}$$

BY THE RESIDUE THEOREM

$$\int_{\Gamma} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \Gamma)$$

$$\left\{ \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} \right\} f(z) dz = 2\pi i \times \frac{1}{2} e^{\frac{2\pi i}{n}}$$

NOTE: LOOKING AT THE CONTRIBUTION OF  $f(z)$  AROUND  $\gamma$

$$\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_{\epsilon}^R \frac{1}{(z e^{2\pi i/n} + 1)} (i 2\pi e^{2\pi i/n} dz) \right|$$

AROUND  $\gamma_2$   
 $z = z e^{2\pi i/n}$   
 $dz = i 2\pi e^{2\pi i/n} dz$   
 $\gamma$  RUNS C TO  $\gamma_2$

HOW USING THE INEQUALITY

$$\leq \int_{\epsilon}^R \left| \frac{i 2\pi e^{2\pi i/n}}{(z e^{2\pi i/n} + 1)} \right| dz = \int_{\epsilon}^R \frac{i 2\pi e^{2\pi i/n}}{|z e^{2\pi i/n} + 1|} dz = \int_{\epsilon}^R \frac{i 2\pi |z| e^{2\pi i/n}}{|z e^{2\pi i/n} + 1|} dz$$

$$= \int_{\epsilon}^R \frac{2\pi}{|z e^{2\pi i/n} + 1|} dz \leftarrow \text{NOT USING THE INEQUALITY}$$

$|z e^{2\pi i/n} + 1| > |z| - |1| > 0$   
 $\frac{1}{|z e^{2\pi i/n} + 1|} \leq \frac{1}{|z| - |1|}$

$$\leq \int_{\epsilon}^R \frac{2\pi}{|z e^{2\pi i/n} + 1|} dz$$

$$= \int_{\epsilon}^R \frac{2\pi}{z+1} dz = \frac{2\pi}{2} \left( \frac{1}{\epsilon} \right) - \frac{2\pi}{2} (1) = O\left(\frac{1}{\epsilon}\right)$$

$\rightarrow 0$   
As  $R \rightarrow \infty$

THIS AS  $R \rightarrow \infty$  WE HAVE

$$\int_{\gamma_1} \frac{1}{z+1} dz + \int_{\gamma_2} \frac{1}{z+1} dz = \frac{1}{2} 2\pi i \times e^{\frac{2\pi i}{n}}$$

$z = z e^{2\pi i/n}$   
 $dz = dz$   
 $z$  RUNS FROM 0 TO  $\infty$

$z = z e^{\frac{2\pi i}{n}}$   
 $dz = dz e^{\frac{2\pi i}{n}}$   
 $z$  RUNS FROM  $\infty$  TO 0

THEOREM 10.1

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \int_{\epsilon}^R \frac{1}{z^2+1} dz = \frac{1}{2} \pi i \times e^{\frac{2\pi i}{n}}$$

$$\left( 1 - e^{\frac{2\pi i}{n}} \right) \int_{\gamma} \frac{1}{z^2+1} dz = \frac{1}{2} \pi i \times e^{\frac{2\pi i}{n}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}}}{1 - e^{\frac{2\pi i}{n}}}$$

THEOREM 10.2

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

THEOREM 10.3

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$

$$\int_{\gamma} \frac{1}{z^2+1} dz = \frac{\pi i}{2} \times \frac{e^{\frac{2\pi i}{n}} - 1}{1 - e^{\frac{2\pi i}{n}}}$$



proof

[illegible]



By integrating a suitable complex function over an appropriate contour show that

**proof**

$f(z) = \frac{\log z}{z^2 + 1}$  for a **BRANCH POINT** at  $z=0$   
 BECAUSE OF THE **LOG**, INVERSE THE CONVENTIONAL  
 BRANCH CUT WITH A **BRANCH CUT** ALONG  
 THE POSITIVE  $z$ -AXIS. FIRST, RECALL THE LOGS  
 UNUSUAL QD, SO WE MUST CHOOSE  $\int_{\gamma_1} f(z) dz$

● BY THE RESIDUE THEOREM  
 $\int_{\gamma_1} f(z) dz = 2\pi i \times \sum (\text{RESIDUES INSIDE } \gamma_1)$   
 $\left[ \int_0^R + \int_{\text{arc}} + \int_{-R}^0 \right] f(z) dz = 2\pi i \times \frac{\ln i + i\pi}{2i}$

● NEXT THE CONNECTION OF  $\gamma_2 = \gamma_1 + 2\pi i$  AS  $R \rightarrow \infty$   
 LET  $z = Re^{i\theta}$   $0 \leq \theta \leq 2\pi$   
 $dz = iRe^{i\theta} d\theta$   
 $\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{\ln(Re^{i\theta})}{R^2 e^{2i\theta}} (iRe^{i\theta} d\theta) \right|$   
 $= \left| \int_0^{2\pi} \frac{[\ln R + i\theta]}{R^2 e^{2i\theta}} (iRe^{i\theta} d\theta) \right|$   
 $\leq \int_0^{2\pi} \frac{[\ln R + |\theta|]}{R^2 e^{2i\theta}} d\theta$   
 $\leq \int_0^{2\pi} \frac{[\ln R + |\theta|]}{[R^2 + 2i\theta]} d\theta$

NOW WE CONSIDER THE TRIANGLE INEQUALITY  
 $|\ln R + i\theta| \leq |\ln R| + |\theta|$   
 $\Rightarrow \frac{2\pi \ln R}{R^2 + 2i\theta} \leq \frac{|\ln R + i\theta|}{R^2 + 2i\theta} \leq \frac{|\ln R| + |\theta|}{|R^2 + 2i\theta|}$   
 $\Rightarrow \frac{2\pi \ln R}{R^2 + 2i\theta} \leq \frac{|\ln R| + |\theta|}{|R^2 + 2i\theta|}$   
 $\Rightarrow \frac{2\pi \ln R}{R^2 + 2i\theta} \leq \frac{|\ln R| + |\theta|}{R^2 + 2i\theta}$   
 $\Rightarrow \frac{2\pi \ln R}{R^2 + 2i\theta} \leq \frac{|\ln R| + |\theta|}{R^2 + 2i\theta}$

$= \frac{\pi \ln R}{R^2 + 2i\theta} + \frac{\pi}{R^2 + 2i\theta} \cdot \frac{1}{2} = O\left(\frac{\ln R}{R^2}\right) + O\left(\frac{1}{R^2}\right)$   
 $\rightarrow 0$  AS  $R \rightarrow \infty$

● NEXT THE CONNECTION OF  $\gamma_2 = \gamma_1 + 2\pi i$  AS  $R \rightarrow \infty$   
 $\left| \int_{\gamma_2} f(z) dz \right| = \left| \int_0^{2\pi} \frac{\log(Re^{i\theta})}{R^2 e^{2i\theta}} (iRe^{i\theta} d\theta) \right|$   
 $= \left| \int_0^{2\pi} \frac{[\ln R + i\theta]}{R^2 e^{2i\theta}} (iRe^{i\theta} d\theta) \right|$   
 $\leq \int_0^{2\pi} \frac{[\ln R + |\theta|]}{R^2 e^{2i\theta}} d\theta$   
 $\leq \int_0^{2\pi} \frac{[\ln R + |\theta|]}{R^2 + 2i\theta} d\theta$

SMALL INEQUALITIES COME FROM AS THE ONE IN  $\gamma_1$  (SEE PREVIOUS  
 TWO THE INEQUALITIES IN THE GREEN BOXES)

$\leq \frac{\pi}{R^2} \left( \frac{|\ln R + i\theta|}{|R^2 + 2i\theta|} \right) \leq \frac{\pi}{R^2} \left( \frac{|\ln R + i\theta|}{R^2 - 2i\theta} \right)$   
 $= \frac{\pi \ln R}{R^2 - 2i\theta} + \frac{\pi}{R^2 - 2i\theta} \cdot \frac{1}{2} \rightarrow 0$  AS  $R \rightarrow \infty$

SINCE THE DISC INEQUALITIES  $\rightarrow 0$   
 FIRST DISC INEQUALITY  $\rightarrow 0$  SINCE  $C \rightarrow 0$  FASTER THAN  
 $|\ln R| \rightarrow \infty$   
 SECOND DISC INEQUALITY  $\rightarrow 0$  SINCE IT JUST REMAINS  $C$

## Question 6

By integrating a suitable complex function over an appropriate contour show that

$$\int_{-\infty}^{\infty} \operatorname{sech} x \, dx = \pi.$$

proof

The handwritten proof is divided into two main sections, each enclosed in a box.

**Left Section:**

- Starts with the integral:  $\int_{-\infty}^{\infty} \operatorname{sech} x \, dx = \int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx = \int_{-\infty}^{\infty} \frac{2}{e^x + e^{-x}} dx$
- Notes to find poles:  $e^z + e^{-z} = 0 \Rightarrow e^{2z} = -1 \Rightarrow 2z = i\pi(2n+1) \Rightarrow z = \frac{i\pi(2n+1)}{2}$ . The pole in the upper half-plane is  $z = \frac{i\pi}{2}$ .
- Considers the contour integral  $\oint_{\Gamma} \frac{f(z)}{g(z)} dz$  where  $\Gamma$  is the contour shown in the diagram. The diagram shows a rectangular contour in the complex plane with vertices at  $-R$ ,  $R$ ,  $R+i\pi$ , and  $-R+i\pi$ . The contour is traversed counter-clockwise.
- Uses the residue theorem:  $\oint_{\Gamma} \frac{f(z)}{g(z)} dz = 2\pi i \times \sum (\text{residues inside } \Gamma)$ . The residue at  $z = \frac{i\pi}{2}$  is calculated as  $\lim_{z \rightarrow \frac{i\pi}{2}} \left( \frac{f(z)}{g'(z)} \right) = \frac{2}{e^{\frac{i\pi}{2}} + e^{-\frac{i\pi}{2}}} = \frac{2}{i + (-i)} = \frac{2}{0}$ . (Note: The handwritten calculation shows a different approach, leading to  $\frac{1}{2}$ ).
- Concludes that  $\int_{-\infty}^{\infty} \operatorname{sech} x \, dx = \pi$ .

**Right Section:**

- Starts with the integral:  $\int_{-\infty}^{\infty} \frac{2}{e^x + e^{-x}} dx = \int_{-\infty}^{\infty} \frac{2}{e^x + e^{-x}} dx$
- Uses the substitution  $u = e^x$  to transform the integral into  $\int_{-\infty}^{\infty} \frac{2}{u + \frac{1}{u}} \frac{1}{u} du = \int_{-\infty}^{\infty} \frac{2}{u^2 + 1} du$ .
- Calculates the integral using the arctangent function:  $\int_{-\infty}^{\infty} \frac{2}{u^2 + 1} du = 2 \left[ \arctan u \right]_{-\infty}^{\infty} = 2 \left( \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 2\pi$ .
- Concludes that  $\int_{-\infty}^{\infty} \operatorname{sech} x \, dx = \pi$ .

## Question 7

It is required to evaluate the integral

$$\int_0^{\infty} e^{-x^2} \cos x \, dx.$$

- a) Show that the above integral can be written as

$$\frac{1}{2} e^{-\frac{1}{4}} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx$$

- b) By integrating the complex function  $f(z) = e^{-z^2}$ , over a rectangular contour with vertices at  $(-R, 0)$ ,  $(R, 0)$ ,  $(R, \frac{1}{2}i)$  and  $(-R, \frac{1}{2}i)$ , show that

$$\int_0^{\infty} e^{-x^2} \cos x \, dx = \frac{1}{2} e^{-\frac{1}{4}} \sqrt{\pi}.$$

You may assume without proof that

$$\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

proof

d)  $\int_0^{\infty} e^{-x^2} \cos x \, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \cos x \, dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} \frac{e^{ix} + e^{-ix}}{2} dx = \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} (e^{ix} + e^{-ix}) dx$

$= \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} e^{ix} dx + \frac{1}{4} \int_{-\infty}^{\infty} e^{-x^2} e^{-ix} dx = \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}i)^2} dx + \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx$

$= \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}i)^2} dx + \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx$

b) NOW CONSIDER  $\int_{\gamma} e^{-z^2} dz$  WHERE  $\gamma$  IS THE CONTOUR BELOW

As  $f(z) = e^{-z^2}$  HAS NO POLES INSIDE  $\gamma$  BY CAUCHY'S THEOREM

$\Rightarrow \int_{\gamma} e^{-z^2} dz = 0$

$\Rightarrow \int_{-R}^R e^{-x^2} dx + \int_R^{R+\frac{1}{2}i} e^{-z^2} dz + \int_{R+\frac{1}{2}i}^{-R+\frac{1}{2}i} e^{-z^2} dz + \int_{-R+\frac{1}{2}i}^{-R} e^{-z^2} dz = 0$

$\left\{ \begin{array}{l} z = R + iy \\ dz = i dy \\ y \text{ from } 0 \text{ to } \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{l} z = R + iy \\ dz = i dy \\ y \text{ from } \frac{1}{2} \text{ to } 0 \end{array} \right\} \left\{ \begin{array}{l} z = -R + iy \\ dz = i dy \\ y \text{ from } 0 \text{ to } \frac{1}{2} \end{array} \right\} \left\{ \begin{array}{l} z = -R + iy \\ dz = i dy \\ y \text{ from } \frac{1}{2} \text{ to } 0 \end{array} \right\}$

As  $R \rightarrow \infty$ , THE 2ND & 4TH INTEGRALS VANISH AS THEY ARE  $O(e^{-R^2})$

THIS IS  $e \rightarrow \infty$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx + \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = 0$

$\Rightarrow \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = - \int_{-\infty}^{\infty} e^{-x^2} dx$

$\Rightarrow \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$

$\Rightarrow \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = \sqrt{\pi}$

$\Rightarrow \frac{1}{4} \int_{-\infty}^{\infty} e^{-(x+\frac{1}{2}i)^2} dx = \frac{1}{4} \sqrt{\pi}$

$\Rightarrow \int_0^{\infty} e^{-x^2} \cos x \, dx = \frac{1}{4} \sqrt{\pi}$

## Question 8

$$f(z) \equiv \frac{1}{z}, \quad z \in \mathbb{C}, \quad z \neq 0.$$

By considering the integral of  $f(z)$  over two different suitably parameterized closed paths, show that

$$\int_0^{2\pi} \frac{1}{9\cos^2 \theta + 4\sin^2 \theta} d\theta = \frac{\pi}{3}.$$

,  proof

PICK A PATH WHICH CIRCUMVENTS THE SINGULARITY AT THE ORIGIN, SAY A UNIT CIRCLE AT THE ORIGIN

$$\oint_C \frac{1}{z} dz = 2\pi i \quad \leftarrow \text{STANDARD RESULT}$$

OR PARAMETERISE AS  $z = e^{i\theta}$ ,  $dz = i e^{i\theta} d\theta$ ,  $\theta$  FROM 0 TO  $2\pi$

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} (i e^{i\theta} d\theta) = \int_0^{2\pi} i d\theta = 2\pi i$$

NEXT SELECT TWO ORIENTED FIRST WITH AN ORIENTED PATH ENCLOSING 0, SAY  $\Gamma$

$$\oint_{\Gamma} \frac{1}{z} dz = \oint_{\Gamma} \frac{1}{x+iy} (dx+idy) = \oint_{\Gamma} \frac{x-iy}{x^2+y^2} (dx+idy) = 2\pi i$$

SPLIT INTO REAL & IMAGINARY & WORK WITH IMAGINARY

$$\oint_{\Gamma} \frac{1}{x^2+y^2} [x dx + i x dy - i y dx + y dy] = 2\pi i$$

$$\oint_{\Gamma} \frac{x dx + y dy}{x^2+y^2} + i \oint_{\Gamma} \frac{-y dx + x dy}{x^2+y^2} = 2\pi i$$

THE FIRST TERM IS 0

$$\oint_{\Gamma} \frac{-y dx + x dy}{x^2+y^2} = 2\pi$$

NOW LET  $\Gamma$  BE A PARAMETERISED SQUARE - SEE BELOW

- $x = 2\cos\theta$
- $dx = -2\sin\theta d\theta$
- $y = 2\sin\theta$
- $dy = 2\cos\theta d\theta$
- $\theta$  RANGES FROM 0 TO  $2\pi$

$\int_0^{2\pi} \frac{-2\sin\theta(-2\sin\theta d\theta) + 2\cos\theta(2\cos\theta d\theta)}{9\cos^2\theta + 4\sin^2\theta} = 2\pi$   
 $\int_0^{2\pi} \frac{4\sin^2\theta + 4\cos^2\theta}{9\cos^2\theta + 4\sin^2\theta} d\theta = 2\pi$   
 $\int_0^{2\pi} \frac{4(\cos^2\theta + \sin^2\theta)}{9\cos^2\theta + 4\sin^2\theta} d\theta = 2\pi$   
 $\int_0^{2\pi} \frac{4}{9\cos^2\theta + 4\sin^2\theta} d\theta = 2\pi$   
 $\therefore \int_0^{2\pi} \frac{1}{9\cos^2\theta + 4\sin^2\theta} d\theta = \frac{\pi}{3}$

## Question 9

The complex number  $z = c + a \cos \theta + i b \sin \theta$ ,  $0 \leq \theta < 2\pi$ , traces a closed contour  $C$ , where  $a$ ,  $b$  and  $c$  are positive real numbers with  $a > c$ .

By considering

$$\oint_C \frac{1}{z} dz,$$

show that

$$\int_0^{2\pi} \frac{a + c \cos \theta}{(c + a \cos \theta)^2 + (b \sin \theta)^2} d\theta = \frac{2\pi}{b}.$$

proof

$\oint_C \frac{a + c \cos \theta}{(a \cos \theta + c)^2 + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{b}$ ,  $a, b, c \in \mathbb{R}$ ,  $a > c$ ,  $a, b, c > 0$

• LET  $z = c + a \cos \theta + i b \sin \theta$ ,  $\theta \in [0, 2\pi]$   
 $a > c$

$dz = -a \sin \theta d\theta + i b \cos \theta d\theta$   
 $dz = (-a \sin \theta + i b \cos \theta) d\theta$

• NOW CONSIDER THE INTEGRAL

$\Rightarrow \oint_C \frac{1}{z} dz = 2\pi i$  (SINCE  $z$  IS INSIDE  $C$  AS  $a > c$ )

$\Rightarrow \int_0^{2\pi} \frac{1}{[c + a \cos \theta] + i [b \sin \theta]} (-a \sin \theta + i b \cos \theta) d\theta = 2\pi i$

$\Rightarrow \int_0^{2\pi} \frac{[-a \sin \theta + i b \cos \theta] [c + a \cos \theta - i b \sin \theta]}{[(c + a \cos \theta) + i b \sin \theta] [(c + a \cos \theta) - i b \sin \theta]} d\theta = 2\pi i$

$\Rightarrow \int_0^{2\pi} \frac{(-a \sin \theta + i b \cos \theta) [c + a \cos \theta - i b \sin \theta]}{(c + a \cos \theta)^2 + b^2 \sin^2 \theta} d\theta = 2\pi i$

$\Rightarrow \int_0^{2\pi} \frac{[-a(c + a \cos \theta) \sin \theta + b^2 \sin \theta \cos \theta] + i [bc \cos \theta + ab \cos^2 \theta + ab \sin^2 \theta]}{(c + a \cos \theta)^2 + b^2 \sin^2 \theta} d\theta = 2\pi i$

$= 2\pi i$

• NOW SEPARATING THE INTEGRAL INTO REAL & IMAGINARY

$\int_0^{2\pi} \frac{(b^2 - a^2) \sin \theta \cos \theta - a c \sin \theta}{(a \cos \theta + c)^2 + b^2 \sin^2 \theta} d\theta + i \int_0^{2\pi} \frac{ab + bc \cos \theta}{(a \cos \theta + c)^2 + b^2 \sin^2 \theta} d\theta = 2\pi i$

• THE REAL INTEGRAL YIELDS ZERO

LOOKING AT THE IMAGINARY INTEGRAL

$i b \int_0^{2\pi} \frac{a + c \cos \theta}{(a \cos \theta + c)^2 + b^2 \sin^2 \theta} d\theta = 2\pi i$

$\int_0^{2\pi} \frac{a + c \cos \theta}{(a \cos \theta + c)^2 + b^2 \sin^2 \theta} d\theta = \frac{2\pi}{b}$