PRODUCT OPERATOR

Question 1

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \, .$$

Find the value of

$$\prod_{r=3}^{16}\left[1+\frac{4}{r-2}\right].$$

$\boxed{3060}$

USING THE DEFINITION OF THE PRODUCT OPERATOR

$$\prod_{r=3}^{16}\left[1+\frac{4}{r-2}\right] = \prod_{r=3}^{16}\left[\frac{r-2+4}{r-2}\right]$$

$$= \prod_{r=3}^{16}\left[\frac{r+2}{r-2}\right]$$

AS THERE ARE NOT TOO MANY TERMS, WE MAY WRITE THEM OUT

$$= \frac{\cancel{5}}{1} \times \frac{\cancel{6}}{2} \times \frac{\cancel{7}}{3} \times \frac{\cancel{8}}{4} \times \frac{\cancel{9}}{\cancel{5}} \times \frac{\cancel{10}}{\cancel{6}} \times \frac{\cancel{11}}{\cancel{7}} \times \frac{\cancel{12}}{\cancel{8}} \times \frac{\cancel{13}}{\cancel{9}} \times \frac{\cancel{14}}{\cancel{10}} \times \frac{15}{\cancel{11}} \times \frac{16}{\cancel{12}} \times \frac{17}{\cancel{13}} \times \frac{18}{\cancel{14}}$$

$$= \frac{15 \times 16 \times 17 \times 18}{1 \times 2 \times 3 \times 4}$$

$$= \frac{(5 \times \cancel{3})(\cancel{2} \times \cancel{4} \times 2) \times 17 \times 18}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}}$$

$$= 5 \times 2 \times 17 \times 18$$

$$= 3060$$

Question 2

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method

$$\prod_{r=1}^{n}\left[\frac{r+1}{r}\right].$$

$\boxed{n+1}$

WRITE OUT A FEW TERMS & LOOK FOR A PATTERN

$$\prod_{r=1}^{n}\left(\frac{r+1}{r}\right) = \frac{2}{\cancel{1}} \times \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \ldots \times \frac{n+1}{\cancel{n}}$$

$$= \frac{n+1}{1}$$

$$= n+1$$

Question 3

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method

$$\prod_{r=2}^{n}\left[\frac{r^2-1}{r^2}\right].$$

$$\boxed{\frac{n+1}{2n}}$$

$$\prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right) = \prod_{r=2}^{n}\frac{(r+1)(r-1)}{r^2} = \left[\prod_{r=2}^{n}\left(\frac{r+1}{r}\right)\right]\left[\prod_{r=2}^{n}\left(\frac{r-1}{r}\right)\right]$$

$$= \left[\frac{3}{2} \times \frac{4}{3} \times \frac{5}{4} \times \cdots \times \frac{n+1}{n}\right]\left[\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{n-1}{n}\right]$$

$$= \frac{n+1}{2} \times \frac{1}{n} = \frac{n+1}{2n}$$

Question 4

The product operator $\prod$, is defined as

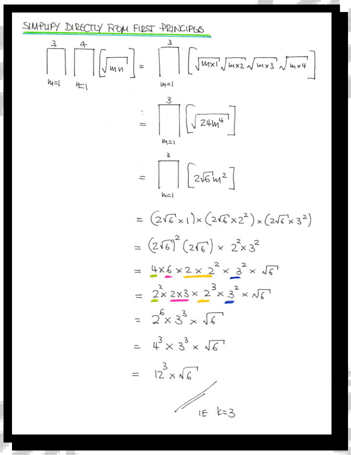
$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Use a clear method to show that

$$\prod_{m=1}^{3}\prod_{n=1}^{4}\left[\sqrt{mn}\right] = k^3\sqrt{6},$$

where k is a positive integer to be found.

$\boxed{}$, $\boxed{}$, $\boxed{k = 12}$


SIMPLIFY DIRECTLY FROM FIRST PRINCIPLES

$$\prod_{m=1}^{3}\prod_{n=1}^{4}\left[\sqrt{mn}\right] = \prod_{m=1}^{3}\left[\sqrt{m\times1}\sqrt{m\times2}\sqrt{m\times3}\sqrt{m\times4}\right]$$

$$= \prod_{m=1}^{3}\left[\sqrt{24m^4}\right]$$

$$= \prod_{m=1}^{3}\left[2\sqrt{6}m^2\right]$$

$$= (2\sqrt{6}\times1)\times(2\sqrt{6}\times2^2)\times(2\sqrt{6}\times3^2)$$

$$= (2\sqrt{6})^2(2\sqrt{6})\times2^2\times3^2$$

$$= 4\times6\times2\times2^2\times3^2\times\sqrt{6}$$

$$= 2^2\times2\times3\times2^3\times3^2\times\sqrt{6}$$

$$= 2^6\times3^3\times\sqrt{6}$$

$$= 4^3\times3^3\times\sqrt{6}$$

$$= 12^3\times\sqrt{6}$$

IE $k=3$

Question 5

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \, .$$

Given that $k \in \mathbb{N}$, use a detailed method to find the value of

$$\prod_{r=2}^{2^k-1}\left[\log_r (r+1)\right].$$

$\boxed{}$, $\boxed{k}$

GENERATE SOME TERMS TO SEE A PATTERN

$$\prod_{r=2}^{2^k-1} \log_r(r+1) = \log_2 3 \times \log_3 4 \times \log_4 5 \times \ldots \times \log_{2^k-2}(2^k-1) \times \log_{2^k-1}(2^k)$$

CHANGE THE BASE OF THE LOGS TO ANY BASE, SAY a, e ETC

$$\ldots = \frac{\cancel{\log_a 3}}{\log_a 2} \times \frac{\cancel{\log_a 4}}{\cancel{\log_a 3}} \times \frac{\cancel{\log_a 5}}{\cancel{\log_a 4}} \times \ldots \times \frac{\cancel{\log_a (2^k-1)}}{\cancel{\log_a (2^k-2)}} \times \frac{\log_a (2^k)}{\cancel{\log_a (2^k-1)}}$$

$$\ldots = \frac{\log_a (2^k)}{\log_a 2} = \frac{k\cancel{\log_a 2}}{\cancel{\log_a 2}}$$

$$\therefore \prod_{r=2}^{2^k-1} \log_r(r+1) = k$$

Question 6

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times ... \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{n=2}^{\infty}\left[1+\frac{1}{n^2-1}\right].$$

$\boxed{}$, $\boxed{2}$

MANIPULATE THE ARGUMENT OF THE OPERATOR FIRST

$$\prod_{n=2}^{\infty}\left[1+\frac{1}{n^2-1}\right] = \prod_{n=2}^{\infty}\frac{n^2-1+1}{n^2-1} = \prod_{n=2}^{\infty}\frac{n}{(n-1)(n+1)}$$

$$= \prod_{n=2}^{\infty}\left[\frac{n}{n-1}\times\frac{n}{n+1}\right]$$

INTRODUCE LIMITS TO INFINITY

$$= \lim_{k\to\infty}\left[\prod_{n=2}^{k}\left[\frac{n}{n-1}\times\frac{n}{n+1}\right]\right]$$

WRITE SOME TERMS OUT

$$= \lim_{k\to\infty}\left[\left(\frac{2}{1}\times\frac{2}{3}\right)\left(\frac{3}{2}\times\frac{3}{4}\right)\left(\frac{4}{3}\times\frac{4}{5}\right)\left(\frac{5}{4}\times\frac{5}{6}\right)\times\cdots\times\left(\frac{k}{k-1}\right)\left(\frac{k}{k+1}\right)\right]$$

$$= \lim_{k\to\infty}\left[\left(\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\cdots\times\frac{k}{k-1}\right)\left(\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\cdots\times\frac{k}{k+1}\right)\right]$$

$$= \lim_{k\to\infty}\left[\frac{k}{1}\times\frac{2}{k+1}\right]$$

$$= \lim_{k\to\infty}\left[\frac{2k}{k+1}\right]$$

$$= \lim_{k\to\infty}\left[\frac{2}{1+\frac{1}{k}}\right]$$

$$= 2$$

Question 7

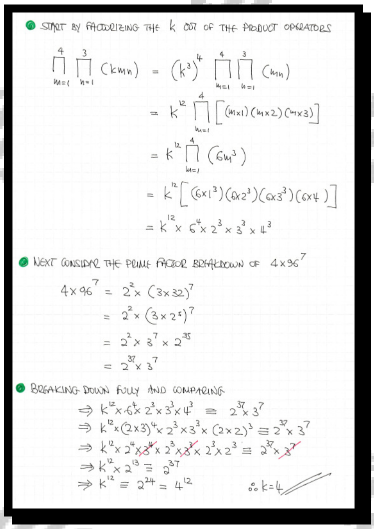
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Use a clear method to determine the value of k given that

$$\prod_{m=1}^{4}\prod_{n=1}^{3}\left[kmn\right] = 4 \times 96^7 .$$

$\boxed{}$, $\boxed{k=4}$



START BY FACTORIZING THE k OUT OF THE PRODUCT OPERATORS

$$\prod_{m=1}^{4}\prod_{n=1}^{3}(kmn) = \left(k^3\right)^4 \prod_{m=1}^{4}\prod_{n=1}^{3}(mn)$$
$$= k^{12}\prod_{m=1}^{4}\left[(m\times1)(m\times2)(m\times3)\right]$$
$$= k^{12}\prod_{m=1}^{4}\left(6m^3\right)$$
$$= k^{12}\left[(6\times1^3)(6\times2^3)(6\times3^3)(6\times4)\right]$$
$$= k^{12}\times 6^4\times 2^3\times 3^3\times 4^3$$

NEXT CONSIDER THE PRIME FACTOR BREAKDOWN OF 4×96^7

$$4\times96^7 = 2^2\times(3\times32)^7$$
$$= 2^2\times\left(3\times2^5\right)^7$$
$$= 2^2\times3^7\times2^{35}$$
$$= 2^{37}\times3^7$$

BREAKING DOWN FULLY AND COMPARING

$$\Rightarrow k^{12}\times6^4\times2^3\times3^3\times4^3 \equiv 2^{37}\times3^7$$
$$\Rightarrow k^{12}\times(2\times3)^4\times2^3\times3^3\times(2\times2)^3 \equiv 2^{37}\times3^7$$
$$\Rightarrow k^{12}\times2^4\times\cancel{3^4}\times2^3\times\cancel{3^3}\times2^3\times2^3 \equiv 2^{37}\times\cancel{3^7}$$
$$\Rightarrow k^{12}\times2^{13} \equiv 2^{37}$$
$$\Rightarrow k^{12} \equiv 2^{24} = 4^{12}$$

$\therefore k=4$

Question 8

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method

$$\prod_{r=1}^{n}\left[\frac{r}{2r+1}\right].$$

$\boxed{\text{PX-F}}$, $\boxed{\dfrac{2^n (n!)^2}{(2n+1)!}}$

GENERATING A FEW TERMS AND LOOK FOR PATTERNS

$$\prod\left(\frac{r}{2r+1}\right) = \frac{1}{3} \times \frac{2}{5} \times \frac{3}{7} \times \frac{4}{9} \times \cdots \frac{n-2}{2n-3} \times \frac{n-1}{2n-1} \times \frac{n}{2n+1}$$

$$= \frac{n!}{3 \times 5 \times 7 \times 9 \times \cdots (2n-3)(2n-1)(2n+1)}$$

CREATE MORE FACTORIALS AS FOLLOWS

$$= \frac{n! \times 2 \times 4 \times 6 \times \ldots \times (2n-4)(2n-2)(2n)}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times \cdots \times (2n-4)(2n-3)(2n-2)(2n-1)2n(2n+1)}$$

$$= \frac{n! \times 2^n \times [1 \times 2 \times 3 \times \cdots (n-2)(n-1)n]}{(2n+1)!}$$

$$= \frac{n! \times 2^n \times n!}{(2n+1)!}$$

$$= \frac{2^n (n!)^2}{(2n+1)!}$$

Question 9

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{n=2}^{\infty}\left[1-\frac{1}{2-2^n}\right].$$

$\boxed{\text{MMD}}$, $\boxed{}$, $\boxed{2}$

MANIPULATE THE PRODUCT OPERATOR AS FOLLOWS

$$\prod_{n=2}^{\infty}\left[1-\frac{1}{2-2^n}\right] = \prod_{n=2}^{\infty}\left[\frac{(2-2^n)-1}{2-2^n}\right]$$

$$= \prod_{n=2}^{\infty}\left[\frac{1-2^n}{2-2^n}\right]$$

$$= \prod_{n=2}^{\infty}\left[\frac{2^n-1}{2^n-2}\right]$$

$$= \prod_{n=2}^{\infty}\left[\frac{2^n-1}{2(2^{n-1}-1)}\right]$$

TAKING LIMITS TO INFINITY

$$= \lim_{k\to\infty}\left[\prod_{n=2}^{k}\left[\frac{2^n-1}{2(2^{n-1}-1)}\right]\right]$$

FACTORIZE $\frac{1}{2}$ OUT OF THE PRODUCT $(k-1)$ TIMES, AS n RUNS FROM 2 TO k

$$= \lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\prod_{n=2}^{k}\left[\frac{2^n-1}{2^{n-1}-1}\right]\right]$$

NEXT WRITE THE PRODUCT EXPLICITLY & LOOK FOR A PATTERN

$$= \lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\left[\frac{3}{1}\times\frac{7}{3}\times\frac{15}{7}\times\frac{31}{15}\times\cdots\times\frac{2^k-1}{2^{k-1}-1}\right]\right]$$

$$= \lim_{k\to\infty}\left[\frac{1}{2^{k-1}}\times(2^k-1)\right]$$

$$= \lim_{k\to\infty}\left[\frac{2^k-1}{2^{k-1}}\right]$$

$$= \lim_{k\to\infty}\left[\frac{\frac{2^k}{2^{k-1}}-\frac{1}{2^{k-1}}}{\frac{2^{k-1}}{2^{k-1}}}\right]$$

$$= \lim_{k\to\infty}\left[\frac{2-\frac{1}{2^{k-1}}}{1}\right]$$

$$= 2$$

ALTERNATIVE

$$\lim_{k\to\infty}\left[\frac{2^k-1}{2^{k-1}}\right]$$

$$= \lim_{k\to\infty}\left[2\times\frac{2^k-1}{2^k}\right]$$

$$= \lim_{k\to\infty}\left[2\left(1-\frac{1}{2^k}\right)\right]$$

$$= 2$$

Question 10

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Given that e is Euler's number, use a detailed method to find the exact value of

$$\prod_{r=1}^{\infty}\left[\frac{\sqrt[(2r)]{\mathrm{e}}}{\sqrt[(2r+1)]{\mathrm{e}}}\right].$$

$\boxed{}$, $\boxed{\frac{1}{2}\mathrm{e}}$

GENERATE SOME TERMS

$$\prod_{r=1}^{\infty}\frac{\sqrt[2r]{e}}{\sqrt[2r+1]{e}} = \frac{\sqrt[2]{e}}{\sqrt[3]{e}} \times \frac{\sqrt[4]{e}}{\sqrt[5]{e}} \times \frac{\sqrt[6]{e}}{\sqrt[7]{e}} \times \frac{\sqrt[8]{e}}{\sqrt[9]{e}} \times \ldots$$

$$= \frac{e^{\frac{1}{2}}}{e^{\frac{1}{3}}} \times \frac{e^{\frac{1}{4}}}{e^{\frac{1}{5}}} \times \frac{e^{\frac{1}{6}}}{e^{\frac{1}{7}}} \times \frac{e^{\frac{1}{8}}}{e^{\frac{1}{9}}} \times \ldots$$

$$= e^{\frac{1}{2}} \times e^{-\frac{1}{3}} \times e^{\frac{1}{4}} \times e^{-\frac{1}{5}} \times e^{\frac{1}{6}} \times e^{-\frac{1}{7}} \times \ldots$$

$$= e^{\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots}$$

THIS IS "ALMOST" THE ALTERNATING HARMONIC WHICH CONVERGES TO $\ln 2$

$$\Rightarrow \frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\ldots$$

$$= -\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$= 1-\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\ldots\right]$$

$$= 1-\ln 2$$

THUS WE NOW HAVE

$$\prod_{r=1}^{\infty}\frac{\sqrt[2r]{e}}{\sqrt[2r+1]{e}} = e^{1-\ln 2} = e^{1} \times e^{-\ln 2} = e \times \frac{1}{2} = \frac{1}{2}e$$

Question 11

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Simplify, showing a clear method, the following expression.

$$\prod_{r=1}^{n}\left[\frac{2r}{2r+1}\right].$$

Give the final answer as a single simplified fraction.

$\boxed{}$, $\boxed{\dfrac{4^n (n!)^2}{(2n+1)!}}$

WRITE A FEW TERMS OUT SEEKING PATTERNS

$$\prod_{r=1}^{n}\left[\frac{2r}{2r+1}\right] = \frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \cdots \times \frac{2n-4}{2n-3} \times \frac{2n-2}{2n-1} \times \frac{2n}{2n+1}$$

$$= \frac{2^n\left[1 \times 2 \times 3 \times \ldots \times (n-2)(n-1)n\right]}{3 \times 5 \times 7 \times \ldots \times (2n-3)(2n-1)(2n+1)}$$

$$= \frac{2^n \times n!}{3 \times 5 \times 7 \times \ldots \times (2n-3)(2n-1)(2n+1)}$$

IN ORDER TO "COMPLETE" A FACTORIAL EXPRESSION AT THE DENOMINATOR MULTIPLY "TOP & BOTTOM" BY THE EVEN "MISSING" TERMS

$$= \frac{2^n \times n! \times \left[2 \times 4 \times 6 \ldots (2n-4)(2n-2)(2n)\right]}{2 \times 3 \times 4 \times 5 \times 6 \times 7 \times \ldots \times (2n-4)(2n-3)(2n-2)(2n-1)(2n)(2n+1)}$$

$$= \frac{2^n \times n! \times 2^n\left[1 \times 2 \times 3 \times \ldots \times (n-2)(n-1)n\right]}{(2n+1)!}$$

$$= \frac{(2^n)^2 \times n! \times n!}{(2n+1)!}$$

$$= \frac{4^n \times (n!)^2}{(2n+1)!}$$

Question 12

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{r=2}^{\infty}\left[1-\frac{2}{r(r+1)}\right].$$

$\boxed{}$, $\boxed{\frac{1}{3}}$

START BY FACTORIZING THE ARGUMENT OF THE PRODUCT OPERATOR

$$\prod_{r=2}^{\infty}\left[1-\frac{2}{r(r+1)}\right] = \prod_{r=2}^{\infty}\left[\frac{r(r+1)-2}{r(r+1)}\right]$$

$$= \prod_{r=2}^{\infty}\left[\frac{r^2+r-2}{r(r+1)}\right]$$

$$= \prod_{r=2}^{\infty}\left[\frac{(r-1)(r+2)}{r(r+1)}\right]$$

TAKING LIMITS

$$= \lim_{k\to\infty}\left[\prod_{r=2}^{k}\frac{(r-1)(r+2)}{(r+1)r}\right]$$

WRITING A FEW TERMS OUT TO SEE IF THERE IS ANY CANCELLING PATTERN

$$\ldots = \lim_{k\to\infty}\left[\frac{1\times 4}{3\times 2}\times\frac{2\times 5}{4\times 3}\times\frac{3\times 6}{5\times 4}\times\frac{4\times 7}{6\times 5}\times\ldots\times\frac{(k-2)(k+1)}{k(k-1)}\times\frac{(k-1)(k+2)}{(k+1)k}\right]$$

$$= \lim_{k\to\infty}\left[\frac{k+2}{3k}\right]$$

$$= \lim_{k\to\infty}\left[\frac{1}{3}+\frac{2}{3k}\right]$$

$$= \frac{1}{3}$$

Question 13

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the value of

$$\prod_{r=1}^{\infty}\left[e^{\frac{(-1)^{r+1}}{r}} \right].$$

V, $\boxed{}$, $\boxed{2}$

STARTING WITH THE DEFINITION OF THE PRODUCT OPERATOR

$$\prod_{r=1}^{\infty} e^{\frac{(-1)^{r+1}}{r}} = e^{1} \times e^{-\frac{1}{2}} \times e^{\frac{1}{3}} \times e^{-\frac{1}{4}} \times e^{\frac{1}{5}} \times \ldots$$

COMPACT THE EXPRESSION INTO A SINGLE EXPONENTIAL

$$= e^{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}}$$

THE EXPONENT IS A WELL KNOWN CONVERGENT SERIES; CONVERGES TO $\ln 2$

$$= e^{\ln 2}$$

$$= 2$$

Question 14

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \,.$$

By showing a detailed method prove that

$$\prod_{k=1}^{n}\left[\frac{2k-1}{2k+2}\right] = \binom{2n+1}{n}\frac{1}{4^n\left(2n+1\right)}.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\prod_{k=1}^{n}\left(\frac{2k-1}{2k+2}\right) = \frac{1}{4}\times\frac{3}{6}\times\frac{5}{8}\times\cdots\times\frac{2n-5}{2n-2}\times\frac{2n-3}{2n}\times\frac{2n-1}{2n+2}$$

$$= \frac{1\times3\times5\times\cdots\times(2n-5)\times(2n-3)\times(2n-1)}{2^n\left[2\times3\times4\times\cdots\times(n-1)n(n+1)\right]} = \frac{1\times3\times5\times\ldots\times(2n-5)(2n-3)(2n-1)}{2^n(n+1)!}$$

$$= \frac{1\times2\times3\times4\times5\times6\times\cdots\times(2n-5)(2n-4)(2n-3)(2n-2)(2n-1)(2n)}{2^n(n+1)!\times2\times4\times6\times\ldots\times(2n-4)(2n-2)(2n)}$$

$$= \frac{(2n)!}{2^n(n+1)!\times2^n\left[1\times2\times3\times\cdots\times(n-2)(n-1)(n)\right]} = \frac{2n!}{2^n\times(n+1)!\times2^n\times n!}$$

$$= \frac{(2n)!}{2^{2n}\times n!\,(n+1)!} = \frac{2n!\,(2n+1)}{4^n\,n!\,(n+1)!\,(2n+1)} = \frac{(2n+1)!}{4^n\,n!\,(n+1)!\,(2n+1)}$$

$$= \frac{(2n+1)!}{(n+1)!\,n!}\times\frac{1}{4^n(2n+1)} = \binom{2n+1}{n}\frac{1}{4^n(2n+1)}$$

Question 15

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Evaluate, showing a clear method

$$\prod_{r=2}^{\infty}\left[\frac{r^3-1}{r^3+1}\right].$$

$\boxed{}$, $\boxed{\frac{2}{3}}$

$$\prod_{r=2}^{\infty}\left[\frac{r^3-1}{r^3+1}\right] = \lim_{k\to\infty}\prod_{r=2}^{k}\frac{r^3-1}{r^3+1}$$

- USING THE SUM/DIFFERENCE OF CUBES IDENTITIES

$$\boxed{\begin{aligned} A^3-B^3 &\equiv (A-B)(A^2+AB+B^2) \\ A^3+B^3 &\equiv (A+B)(A^2-AB+B^2) \end{aligned}}$$

$$= \lim_{k\to\infty}\left[\prod_{r=2}^{k}\left[\frac{(r-1)(r^2+r+1)}{(r+1)(r^2-r+1)}\right]\right]$$

- GENERATE A FEW TERMS & LOOK FOR PATTERNS

$$= \lim_{k\to\infty}\left[\frac{1\times7}{3\times3}\times\frac{2\times13}{4\times7}\times\frac{3\times21}{5\times13}\times\frac{4\times31}{6\times21}\times\frac{5\times43}{7\times31}\times\ldots\times\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}\right]$$

- REGROUP SOME OF THESE TERMS

$$= \lim_{k\to\infty}\left[\left[\frac{1}{\cancel{3}}\times\frac{2}{\cancel{4}}\times\frac{\cancel{3}}{\cancel{5}}\times\frac{\cancel{4}}{\cancel{6}}\times\frac{\cancel{5}}{\cancel{7}}\times\ldots\times\frac{\cancel{k-2}}{k}\times\frac{\cancel{k-1}}{k+1}\right]\left[\frac{\cancel{7}}{3}\times\frac{\cancel{13}}{\cancel{7}}\times\frac{\cancel{21}}{\cancel{13}}\times\frac{\cancel{31}}{\cancel{21}}\times\frac{\cancel{43}}{\cancel{31}}\times\ldots\times\frac{k^2+k+1}{\cancel{k^2-k+1}}\right]\right]$$

$$= \lim_{k\to\infty}\left[\frac{1\times2}{k(k+1)}\times\frac{k^2+k+1}{3}\right] = \frac{2}{3}\lim_{k\to\infty}\left[\frac{k^2+k+1}{k(k+1)}\right]$$

$$= \frac{2}{3}\lim_{k\to\infty}\left[\frac{k^2+k+1}{k^2+k}\right]$$

- DIVIDE "TOP & BOTTOM" BY k^2

$$= \frac{2}{3}\lim_{k\to\infty}\left[\frac{1+\frac{1}{k}+\frac{1}{k^2}}{1+\frac{1}{k}}\right]$$

$$= \frac{2}{3}\times1$$

$$= \frac{2}{3}$$

Question 16

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}\left[u_i\right] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Solve the equation

$$\prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)}.$$

You may assume that the left hand side of the equation converges.

$\boxed{x=-1}$, $\boxed{x=-1}$

WORK AS FOLLOWS

$$\Rightarrow \prod_{r=1}^{\infty}\left[\sqrt[2r]{2^x}\right] = 2^{-(x+2)}$$

$$\Rightarrow \sqrt{2^x}\,\sqrt[4]{2^x}\,\sqrt[8]{2^x}\,\sqrt[16]{2^x}\ldots = 2^{-x-2}$$

$$\Rightarrow (2^x)^{\frac{1}{2}}(2^x)^{\frac{1}{4}}(2^x)^{\frac{1}{8}}(2^x)^{\frac{1}{16}}\ldots = 2^{-x-2}$$

$$\Rightarrow 2^{\frac{1}{2}x}\,2^{\frac{1}{4}x}\,2^{\frac{1}{8}x}\,2^{\frac{1}{16}x}\ldots = 2^{-x-2}$$

TAKING LOGARITHMS, BASE TWO, ON BOTH SIDES

$$\Rightarrow \log_2\left[2^{\frac{1}{2}x}\,2^{\frac{1}{4}x}\,2^{\frac{1}{8}x}\,2^{\frac{1}{16}x}\ldots\right] = \log_2\left(2^{-x-2}\right)$$

$$\Rightarrow \log_2 2^{\frac{1}{2}x} + \log_2 2^{\frac{1}{4}x} + \log_2 2^{\frac{1}{8}x} + \log_2 2^{\frac{1}{16}x} + \ldots = \log_2\left(2^{-x-2}\right)$$

$$\Rightarrow \tfrac{1}{2}x\cancel{\log_2 2} + \tfrac{1}{4}x\cancel{\log_2 2} + \tfrac{1}{8}x\cancel{\log_2 2} + \tfrac{1}{16}x\cancel{\log_2 2} + \ldots = (-x-2)\cancel{\log_2 2}$$

$$\Rightarrow \tfrac{1}{2}x + \tfrac{1}{4}x + \tfrac{1}{8}x + \tfrac{1}{16}x + \ldots = -x-2$$

$$\Rightarrow x\Big(\underbrace{\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\ldots}_{\text{G.P with } S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1}\Big) = -x-2$$

$$\Rightarrow x = -x-2$$

$$\Rightarrow 2x = -2$$

$$\Rightarrow x = -1$$

Question 17

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \,.$$

By showing a detailed method prove that if n is even

$$\prod_{r=1}^{n}\left[\cos\frac{2\pi}{n} + \cos\frac{2\pi}{n}\cot\frac{(2r-1)\pi}{n}\right] = 1\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY REWRITING INTO SINES & COSINES

$$\prod_{r=1}^{n}\left[\cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\cot\frac{(2r-1)\pi}{n}\right]$$

$$= \prod_{r=1}^{n}\left[\cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) \times \frac{\cos\left[\frac{\pi}{n}(2r-1)\right]}{\sin\left[\frac{\pi}{n}(2r-1)\right]}\right]$$

ADD THE FRACTIONS INSIDE THE PRODUCT OPERATOR

$$= \prod_{r=1}^{n}\left[\frac{\cos\left(\frac{2\pi}{n}\right)\sin\left[\frac{\pi}{n}(2r-1)\right] + \sin\left(\frac{2\pi}{n}\right)\cos\left[\frac{\pi}{n}(2r-1)\right]}{\sin\left[\frac{\pi}{n}(2r-1)\right]}\right]$$

NOW THE NUMERATOR IS MERELY THE TRIGONOMETRIC EXPANSION OF "$\sin(A+B)$"

$$= \prod_{r=1}^{n}\left[\frac{\sin\left[\frac{\pi}{n}(2r-1) + \frac{2\pi}{n}\right]}{\sin\left[\frac{\pi}{n}(2r-1)\right]}\right]$$

$$= \prod_{r=1}^{n}\left[\frac{\sin\left[\frac{\pi}{n}(2r-1+2)\right]}{\sin\left[\frac{\pi}{n}(2r-1)\right]}\right]$$

$$= \prod_{r=1}^{n}\left[\frac{\sin\left[\frac{\pi}{n}(2r+1)\right]}{\sin\left[\frac{\pi}{n}(2r-1)\right]}\right]$$

NOW WRITE THE TERMS OF THE PRODUCT EXPLICITLY

$$= \frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}} \times \frac{\sin\frac{5\pi}{n}}{\sin\frac{3\pi}{n}} \times \frac{\sin\frac{7\pi}{n}}{\sin\frac{5\pi}{n}} \times \cdots \times \frac{\sin\left(\frac{\pi}{n}(2n+1)\right)}{\sin\left(\frac{\pi}{n}(2n-1)\right)}$$

$$= \frac{\sin\left[\frac{\pi}{n}(2n+1)\right]}{\sin\frac{\pi}{n}}$$

$$= \frac{\sin\left[2\pi + \frac{\pi}{n}\right]}{\sin\frac{\pi}{n}}$$

BY THE PERIODICITY PROPERTY OF SINE (OR THE COMPOUND ANGLE IDENTITY)

$\sin(A+2\pi) \equiv \sin A$

$\sin(A+B)$

$\sin\left(2\pi+\frac{\pi}{n}\right) = \cancel{\sin 2\pi\cos\frac{\pi}{n}} + \cos 2\pi \sin\frac{\pi}{n}$

$\sin\left(2\pi+\frac{\pi}{n}\right) = \sin\frac{\pi}{n}$

$$= \frac{\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}$$

$$= 1$$

Question 18

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k} [u_i] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \, .$$

A sequence of numbers, $P(1)$, $P(2)$, $P(3)$ ... $P(n)$ is defined by the equation

$$P(n) = \frac{9}{10} \prod_{r=1}^{n} \left[1 + \left[\sum_{k=1}^{r} 10^k \right]^{-1} \right].$$

Express $P(n)$ in a simplified form not involving a sigma or product operators.

$\boxed{}$, $\boxed{P(n) = 1 - 0.1^{n+1}}$

$$\boxed{P(n) = \frac{9}{10} \prod_{r=1}^{n} \left[1 + \left[\sum_{k=1}^{r} 10^k \right]^{-1} \right]}$$

GENERATE A FEW TERMS AND LOOK FOR A PATTERN

- $P(1) = \frac{9}{10} \prod_{r=1}^{1} \left[1 + \frac{1}{\sum_{k=1}^{r} 10^k} \right] = \frac{9}{10}\left[1 + \frac{1}{10}\right]$
- $P(2) = \frac{9}{10} \prod_{r=1}^{2} \left[1 + \frac{1}{\sum_{k=1}^{r} 10^k} \right] = \frac{9}{10}\left[1 + \frac{1}{10^1}\right]\left[1 + \frac{1}{10^1+10^2}\right]$
- $P(3) = \frac{9}{10} \prod_{r=1}^{3} \left[1 + \frac{1}{\sum_{k=1}^{r} 10^k} \right] = \frac{9}{10}\left[1 + \frac{1}{10^1}\right]\left[1 + \frac{1}{10^1+10^2}\right]\left[1 + \frac{1}{10^1+10^2+10^3}\right]$

SIMPLIFYING $P(1)$, $P(2)$, $P(3)$ FURTHER

- $P(1) = \frac{9}{10} \times \frac{11}{10} = \frac{99}{100} = 0.99$
- $P(2) = \frac{9}{10} \times \frac{11}{10} \times \left(1 + \frac{1}{110}\right) = \frac{9}{10} \times \frac{\cancel{11}^{1}}{10} \times \frac{111}{\cancel{110}_{10}} = \frac{999}{1000} = 0.999$
- $P(3) = \frac{9}{10} \times \frac{11}{10} \times \frac{111}{110} \times \left(1 + \frac{1}{1110}\right) = \frac{9}{10} \times \frac{\cancel{11}^{1}}{10} \times \frac{\cancel{111}^{1}}{\cancel{110}_{10}} \times \frac{1111}{\cancel{1110}_{10}} = \frac{9999}{10000} = 0.9999$

& SIMILARLY

- $P(4) = \frac{9}{10} \times \frac{\cancel{11}^{1}}{10} \times \frac{\cancel{111}^{1}}{\cancel{110}_{10}} \times \frac{\cancel{1111}^{1}}{\cancel{1110}_{10}} \times \frac{11111}{\cancel{11110}_{10}} = \frac{99999}{10000} = 0.99999$

REWRITING THESE TERMS

- $P(1) = 0.99 = 0.9 + 0.09$
- $P(2) = 0.999 = 0.9 + 0.09 + 0.009$
- $P(3) = 0.9999 = 0.9 + 0.09 + 0.009 + 0.0009$
- $P(4) = 0.99999 = 0.9 + 0.09 + 0.009 + 0.00009 + 0.00009$

THESE TERMS FORM A SEQUENCE OF THE PARTIAL SUMS OF A GEOMETRIC PROGRESSION WITH

$a = 0.9$
$r = 0.1$
(SHIFTED BY 1).

$$\Rightarrow S'_n = \frac{a(1 - r^{n+1})}{1 - r}$$

$$\Rightarrow S'_n = \frac{0.9(1 - 0.1^{n+1})}{1 - 0.1}$$

$$\Rightarrow S'_n = \frac{\cancel{0.9}(1 - 0.1^{n+1})}{\cancel{0.9}}$$

$$\Rightarrow S'_n = 1 - 0.1^{n+1}$$

$$\therefore \frac{9}{10} \prod_{r=1}^{n} \left[1 + \left(\sum_{k=1}^{r} 10^k \right)^{-1} \right] = 1 - 0.1^{n+1}$$

Question 19

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Show by a detailed method that

$$\prod_{r=1}^{\infty}\left[1+\left(\tfrac{1}{4}\right)^{2^r}\right] = \frac{16}{15}.$$

$\boxed{\text{proof}}$

$$\boxed{\prod_{r=1}^{\infty}\left[1+\left(\tfrac{1}{4}\right)^{2^r}\right] \qquad \text{CAREFUL } a^{b^c} \neq a^{bc}}$$

- LET $x=\frac{1}{4}$ & LOOK FOR A PATTERN IN $\prod_{r=1}^{k}\left[1+x^{2^r}\right]$
  - $k=1$: $(1+x^2)$
  - $k=2$: $(1+x^2)(1+x^4) = 1+x^2+x^4+x^6$
  - $k=3$: $(1+x^2)(1+x^4)(1+x^8) = (1+x^2+x^4+x^6)(1+x^8)$ $= 1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+x^{14}$
  - $k=4$: $(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$ $=(1+x^{16})(1+x^2+x^4+x^6+\cdots+x^{14})$ $=1+x^2+x^4+x^6+x^8+\cdots+x^{14}+x^{16}+x^{18}+\cdots+x^{30}$
- $\prod_{r=1}^{k}\left[1+x^{2^r}\right] = 1+x^2+x^4+x^6+x^8+\cdots\cdots+x^{2^{2(k+1)-2}}$
- $\prod_{r=1}^{\infty}\left[1+\left(\frac{1}{4}\right)^{2^r}\right] = \lim_{k\to\infty}\Big[\underbrace{1+0.25^2+0.25^4+0.25^6+0.25^8+\cdots+0.25^{2^{2(k+1)-2}}}_{\text{THIS IS THE SUM TO INFINITY OF A GEOMETRIC PROGRESSION WITH FIRST TERM 1 \& COMMON RATIO } 0.25^2}\Big]$

NOTE $S_\infty = \frac{a}{1-r}$

$$= \frac{1}{1-0.25^2} = \frac{1}{1-\frac{1}{16}}$$

MULTIPLY TOP & BOTTOM BY 16

$$= \frac{16}{16-1} = \frac{16}{15}$$

// AS REQUIRED

Question 20

The product operator $\prod$, is defined as

$$\prod_{r=1}^{k}[u_r] \;=\; u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k \,.$$

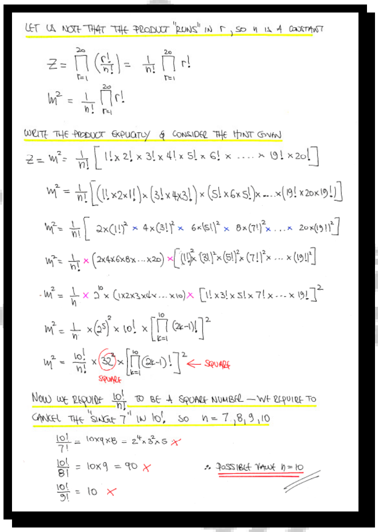
The integer Z is a **square number** and defined as

$$Z = \prod_{r=1}^{20}\left(\frac{r!}{n!}\right),\ \{n \in \mathbb{N} : 1 \le n \le 20\}.$$

By considering the terms inside the product operator in pairs, or otherwise, determine a possible value of n.

You must show a detailed method in this question.

$\boxed{}$, $\boxed{n = 10}$



LET US NOTE THAT THE PRODUCT "RUNS" IN r, SO n IS A CONSTANT

$$Z = \prod_{r=1}^{20}\left(\frac{r!}{n!}\right) = \frac{1}{n!}\prod_{r=1}^{20} r!$$

$$m^2 = \frac{1}{n!}\prod_{r=1}^{20} r!$$

WRITE THE PRODUCT EXPLICITLY & CONSIDER THE HINT GIVEN

$$Z = m^2 = \frac{1}{n!}\left[1! \times 2! \times 3! \times 4! \times 5! \times 6! \times \ldots \times 19! \times 20!\right]$$

$$m^2 = \frac{1}{n!}\left[(1! \times 2 \times 1!) \times (3! \times 4 \times 3!) \times (5! \times 6 \times 5!) \times \ldots \times (19! \times 20 \times 19!)\right]$$

$$m^2 = \frac{1}{n!}\left[2 \times (1!)^2 \times 4 \times (3!)^2 \times 6 \times (5!)^2 \times 8 \times (7!)^2 \times \ldots \times 20 \times (19!)^2\right]$$

$$m^2 = \frac{1}{n!} \times (2 \times 4 \times 6 \times 8 \times \ldots \times 20) \times \left[(1!)^2 \times (3!)^2 \times (5!)^2 \times (7!)^2 \times \ldots \times (19!)^2\right]$$

$$m^2 = \frac{1}{n} \times 2^{10} \times (1 \times 2 \times 3 \times 4 \times \ldots \times 10) \times \left[1! \times 3! \times 5! \times 7! \times \ldots \times 19!\right]^2$$

$$m^2 = \frac{1}{n} \times (2^5)^2 \times 10! \times \left[\prod_{k=1}^{10}(2k-1)!\right]^2$$

$$m^2 = \frac{10!}{n!} \times \underbrace{32^2}_{\text{SQUARE}} \times \left[\prod_{k=1}^{10}(2k-1)!\right]^2 \leftarrow \text{SQUARE}$$

NOW WE REQUIRE $\frac{10!}{n!}$ TO BE A SQUARE NUMBER — WE REQUIRE TO CANCEL THE "SINGLE 7" IN $10!$, SO $n = 7, 8, 9, 10$

$\frac{10!}{7!} = 10 \times 9 \times 8 = 2^4 \times 3^2 \times 5$ ✗

$\frac{10!}{8!} = 10 \times 9 = 90$ ✗

$\frac{10!}{9!} = 10$ ✗

$\therefore$ POSSIBLE VALUE $n = 10$

Question 21

$$I=\int_0^1\left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\left(\frac{1}{x+r}\right)\right]dx\,.$$

Show by a detailed method that

$$I=a\times b!,$$

where a and b are positive integers to be found.

The product operator $\prod$***, is defined as***

$$\prod_{i=1}^{k}[u_i] \;=\; u_1\times u_2\times u_3\times u_4\times\ldots\times u_{k-1}\times u_k\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{\int_0^1\left[\prod_{r=1}^{10}(x+r)\right]\left[\sum_{r=1}^{10}\frac{1}{x+r}\right]dx}$$

- LET $u=\prod_{r=1}^{10}(x+r)=(x+1)(x+2)(x+3)\ldots(x+10)$

$$\ln u=\ln\left[(x+1)(x+2)(x+3)\ldots(x+10)\right]$$
$$\ln u=\ln(x+1)+\ln(x+2)+\ln(x+3)+\ldots+\ln(x+10)$$
$$\frac{1}{u}\frac{du}{dx}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+10}$$
$$\frac{du}{dx}\times\frac{1}{u}=\sum_{r=1}^{10}\frac{1}{x+r}$$
$$\frac{du}{dx}=u\sum_{r=1}^{10}\frac{1}{x+r}$$
$$\frac{du}{dx}=\prod_{r=1}^{10}(x+r)\sum_{r=1}^{10}\frac{1}{x+r}$$

- $\frac{dv}{dx}=\sum_{r=1}^{\infty}\frac{1}{x+r}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+10}$

$$v=\ln(x+1)+\ln(x+2)+\ln(x+3)+\cdots+\ln(x+10)$$
$$v=\ln\left[(x+1)(x+2)(x+3)\ldots(x+10)\right]$$
$$v=\ln\left[\prod_{r=1}^{10}(x+r)\right]$$

- BY PARTS

$$\int_0^1\underbrace{\left[\prod_{r=1}^{10}(x+r)\right]}_{u}\underbrace{\left[\sum_{r=1}^{\infty}\frac{1}{x+r}\right]}_{\frac{dv}{dx}}dx=\left[\overbrace{\prod_{r=1}^{10}(x+r)}^{u}\,\overbrace{\ln\prod_{r=1}^{10}(x+r)}^{v}\right]_0^1$$
$$-\int_0^1\underbrace{\prod_{r=1}^{10}(x+r)\sum_{r=1}^{10}\frac{1}{x+r}}_{\frac{du}{dx}}\,\underbrace{\ln\left[\prod_{r=1}^{10}(x+r)\right]}_{v}\,dx$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\int_0^1\frac{du}{dx}v\,dx$$
$$=\Big[uv\Big]_{x=0}^{x=1}-\int_{x=0}^{x=1}\frac{du}{dx}\ln(u)\,dx$$
$$=\Big[uv\Big]_{x=0}^{x=1}-\int_{x=0}^{x=1}\ln u\,du$$

EARLIER
$$v=\ln\left[\prod_{r=1}^{10}(x+r)\right]$$
$$u=\prod_{r=1}^{10}(x+r)$$
$$\therefore v=\ln(u)$$

STANDARD RESULTS OR BY PARTS
$$\int y\ln y\,dy=y\ln y-y+C$$

$$=\Big[uv\Big]_{x=0}^{x=1}-\left\{\Big[u\ln u-u\Big]_{x=0}^{x=1}\right\}$$
$$=\Big[uv\Big]_{x=0}^{x=1}-\Big[u\ln u\Big]_{x=0}^{x=1}+\Big[u\Big]_{x=0}^{x=1}$$
$$=\cancel{\Big[uv\Big]_{x=0}^{x=1}}-\cancel{\Big[uv\Big]_{x=0}^{x=1}}+\left[\prod_{r=1}^{10}(x+r)\right]_0^1$$
$$=\prod_{r=1}^{10}(1+r)-\prod_{r=1}^{10}r=11!-10!$$
$$=11\times10!-10!$$
$$=10\times10!$$

Question 22

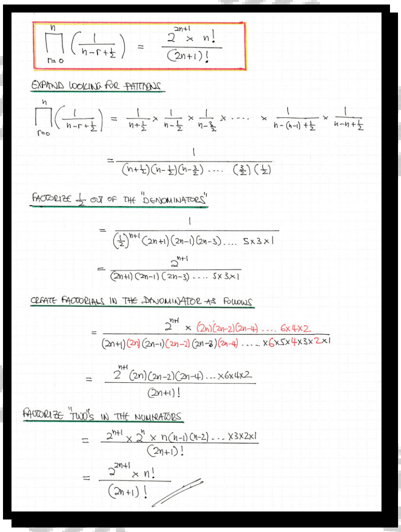
The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Show in detail that

$$\prod_{r=1}^{n}\left(\frac{1}{n-r+\frac{1}{2}}\right) = \frac{2^{2n+1} \times n!}{(2n+1)!} .$$

$\boxed{}$, $\boxed{\text{proof}}$



$$\boxed{\prod_{r=0}^{n}\left(\frac{1}{n-r+\frac{1}{2}}\right) = \frac{2^{2n+1} \times n!}{(2n+1)!}}$$

EXPAND LOOKING FOR PATTERNS

$$\prod_{r=0}^{n}\left(\frac{1}{n-r+\frac{1}{2}}\right) = \frac{1}{n+\frac{1}{2}} \times \frac{1}{n-\frac{1}{2}} \times \frac{1}{n-\frac{3}{2}} \times \cdots \times \frac{1}{n-(n-1)+\frac{1}{2}} \times \frac{1}{n-n+\frac{1}{2}}$$

$$= \frac{1}{(n+\frac{1}{2})(n-\frac{1}{2})(n-\frac{3}{2}) \ldots (\frac{3}{2})(\frac{1}{2})}$$

FACTORIZE $\frac{1}{2}$ OUT OF THE "DENOMINATORS"

$$= \frac{1}{(\frac{1}{2})^{n+1}(2n+1)(2n-1)(2n-3) \ldots 5 \times 3 \times 1}$$

$$= \frac{2^{n+1}}{(2n+1)(2n-1)(2n-3) \ldots 5 \times 3 \times 1}$$

CREATE FACTORIALS IN THE DENOMINATOR AS FOLLOWS

$$= \frac{2^{n+1} \times (2n)(2n-2)(2n-4) \ldots 6 \times 4 \times 2}{(2n+1)(2n)(2n-1)(2n-2)(2n-3)(2n-4) \ldots \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

$$= \frac{2^{n+1}(2n)(2n-2)(2n-4) \ldots \times 6 \times 4 \times 2}{(2n+1)!}$$

FACTORIZE "TWO'S" IN THE NUMERATORS

$$= \frac{2^{n+1} \times 2^{n} \times n(n-1)(n-2) \ldots \times 3 \times 2 \times 1}{(2n+1)!}$$

$$= \frac{2^{2n+1} \times n!}{(2n+1)!}$$

Question 23

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Find the sum to infinity of the following expression

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right].$$

$\boxed{\text{SP-K}}$, $\boxed{\sqrt[8]{\frac{5}{4}}-1}$

START BY WRITING A FEW TERMS EXPLICITLY & LOOK FOR A PATTERN

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \prod_{r=1}^{1}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{2}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{3}\left(\frac{8r-7}{40r}\right) + \prod_{r=1}^{4}\left(\frac{8r-7}{40r}\right) + \ldots$$

$$= \frac{1}{40} + \frac{1}{40} \times \frac{9}{80} + \frac{1}{40} \times \frac{9}{80} \times \frac{17}{120} + \frac{1}{40} \times \frac{9}{80} + \frac{17}{120} \times \frac{25}{160} + \ldots$$

$$= \frac{1}{40} + \frac{1 \times 9}{40 \times 80} + \frac{1 \times 9 \times 17}{40 \times 80 \times 120} + \frac{1 \times 9 \times 17 \times 25}{40 \times 80 \times 120 \times 160}$$

$$= \frac{1}{40 \times 1} + \frac{1 \times 9}{40^2 (1 \times 2)} + \frac{1 \times 9 \times 17}{40^3 (1 \times 2 \times 3)} + \frac{1 \times 9 \times 17 \times 25}{40^4 (1 \times 2 \times 3 \times 4)}$$

THIS RESEMBLES A BINOMIAL EXPANSION DUE TO THE FACTORIALS AT THE DENOMINATOR
THE NEXT ITEM IS TO "CREATE" NUMERATORS OF THE FORM $n(n-1)(n-2)(n-3)\ldots$

BY INSPECTION THIS WILL OCCUR AS $-\frac{1}{8}, -\frac{9}{8}, -\frac{17}{8}, -\frac{25}{8}$

THUS TRY AND ADJUST THE SERIES

$$= \frac{1}{(-8)(-5)\,1!} + \frac{1 \times 9}{(-8)^2(-5)^2\,2!} + \frac{1 \times 9 \times 17}{(-8)^3(-5)^3\,3!} + \frac{1 \times 9 \times 17 \times 25}{(-8)^4(-5)^4\,4!}$$

$$\sum_{k=1}^{\infty}\left[\prod_{r=1}^{k}\left(\frac{8r-7}{40r}\right)\right] = \underbrace{\frac{-\frac{1}{8}}{1!}\left(-\frac{1}{5}\right) + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)}{2!}\left(-\frac{1}{5}\right)^2 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)}{3!}\left(-\frac{1}{5}\right)^3 + \frac{\left(-\frac{1}{8}\right)\left(-\frac{9}{8}\right)\left(-\frac{17}{8}\right)\left(-\frac{25}{8}\right)}{4!}\left(-\frac{1}{5}\right)^4 + \ldots}_{\text{THIS IS A BINOMIAL EXPANSION WITH THE "1" MISSING AT THE FRONT}}$$

$$= \left(1 - \frac{1}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \left(\frac{4}{5}\right)^{-\frac{1}{8}} - 1$$

$$= \sqrt[8]{\frac{5}{4}} - 1$$

Question 24

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

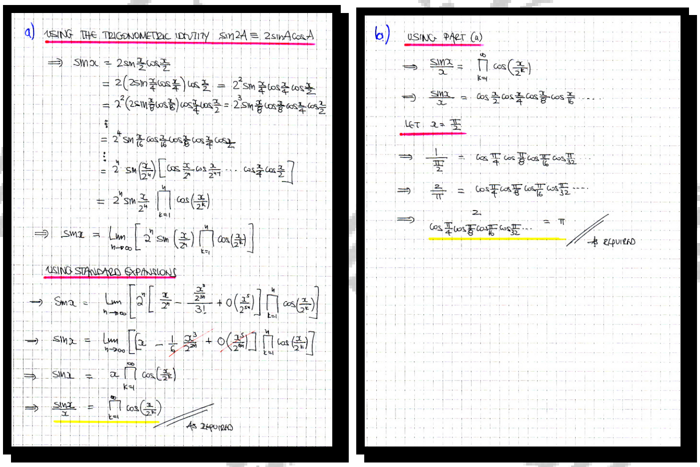
a) By considering the sine double angle identity show that

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty}\left[\cos\left(\frac{x}{2^k}\right)\right].$$

b) Deduce that

$$\frac{2}{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{32}\right)\ldots} = \pi .$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 25

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

By writing $\sin x$ as an infinite factorized polynomial, where each of the factors represents a zero of $\sin x$, derive Wallis's formula

$$\frac{\pi}{2} = \prod_{r=1}^{\infty} \frac{4r^2}{4r^2-1} .$$

You may assume without proof that

$$\lim_{x \to 0}\left[\frac{\sin x}{x}\right] = 1 .$$

$\boxed{}$, $\boxed{\text{proof}}$

START BY LOOKING AT THE ZEROS OF $\sin x$

$\Rightarrow \sin x = x(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)(x-3\pi)(x+3\pi)\ldots$

$\Rightarrow \frac{\sin x}{x} = (x^2-\pi^2)(x^2-4\pi^2)(x^2-9\pi^2)(x^2-16\pi^2)(x^2-25\pi^2)\ldots$

$\Rightarrow \frac{\sin x}{x} = (-\pi^2)\left[1-\frac{x^2}{\pi^2}\right](-4\pi^2)\left[1-\frac{x^2}{4\pi^2}\right](-9\pi^2)\left[1-\frac{x^2}{9\pi^2}\right]\ldots$

$\Rightarrow \frac{\sin x}{x} = A\left[1-\frac{x^2}{\pi^2}\right]\left[1-\frac{x^2}{4\pi^2}\right]\left[1-\frac{x^2}{9\pi^2}\right]\left[1-\frac{x^2}{16\pi^2}\right]\ldots$

$\boxed{\text{NOTE THAT } \lim_{x\to 0}\frac{\sin x}{x} = 1}$

TO EVALUATE THIS CONSTANT "A" IN THE ABOVE EQUATION, TAKE THE LIMIT AS $x \to 0$, SO $A=1$

$\Rightarrow \frac{\sin x}{x} = \left[1-\frac{x^2}{\pi^2}\right]\left[1-\frac{x^2}{4\pi^2}\right]\left[1-\frac{x^2}{9\pi^2}\right]\left[1-\frac{x^2}{16\pi^2}\right]\ldots$

LET $x = \pi/2$

$\Rightarrow \frac{1}{\pi/2} = \left[1-\frac{\pi^2/4}{\pi^2}\right]\left[1-\frac{\pi^2/4}{4\pi^2}\right]\left[1-\frac{\pi^2/4}{9\pi^2}\right]\left[1-\frac{\pi^2/4}{16\pi^2}\right]\ldots$

$\Rightarrow \frac{2}{\pi} = \left(1-\frac{1}{4}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{36}\right)\left(1-\frac{1}{64}\right)\left(1-\frac{1}{100}\right)\ldots$

$\Rightarrow \frac{2}{\pi} = \frac{3}{4} \times \frac{15}{16} \times \frac{35}{36} \times \frac{63}{64} \times \frac{99}{100} \times \ldots$

$\Rightarrow \frac{\pi}{2} = \frac{4}{3} \times \frac{16}{15} \times \frac{36}{35} \times \frac{64}{63} \times \frac{100}{99} \times \ldots$

$\Rightarrow \frac{\pi}{2} = \frac{2\times 2}{1\times 3} \times \frac{4\times 4}{3\times 5} \times \frac{6\times 6}{5\times 7} \times \frac{8\times 8}{7\times 9} \times \frac{10\times 10}{9\times 11} \times \ldots$

$\Rightarrow \frac{\pi}{2} = \prod_{r=1}^{\infty}\left[\frac{2r}{2r-1} \times \frac{2r}{2r+1}\right]$

$\Rightarrow \frac{\pi}{2} = \prod_{r=1}^{\infty}\left[\frac{4r^2}{4r^2-1}\right]$ //

OR $2\prod_{r=1}^{\infty}\frac{4r^2}{4r^2-1} = \pi$

Question 26

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

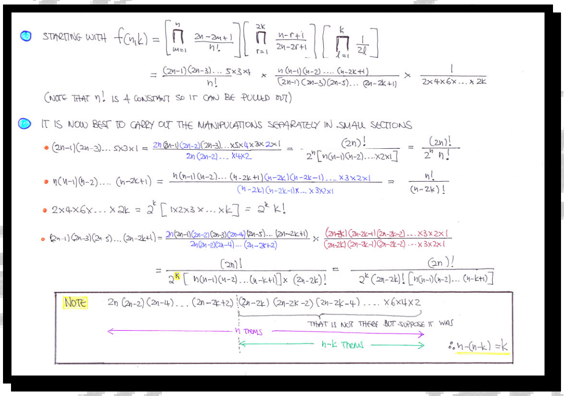
The function $f(n,k)$ is defined as

$$f(n,k)=\left[\prod_{m=1}^{n}\left[\frac{2n-2m+1}{n!}\right]\right]\left[\prod_{r=1}^{k}\left[\frac{n-r+1}{2n-2r+1}\right]\right]\left[\prod_{l=1}^{k}\left[\frac{1}{(2l)}\right]\right], \quad n \geq 2k .$$

Show by a detailed method that

$$f(n,k)=\frac{(2n-2k)!}{2^n \; k!(n-k)! \; (n-2k)!}$$

$\boxed{}$, $\boxed{\text{proof}}$



STARTING WITH $f(n,k)=\left[\prod_{m=1}^{n}\frac{2n-2m+1}{n!}\right]\left[\prod_{r=1}^{2k}\frac{n-r+1}{2n-2r+1}\right]\left[\prod_{l=1}^{k}\frac{1}{2l}\right]$

$$= \frac{(2n-1)(2n-3)\ldots 5\times 3\times 1}{n!} \times \frac{n(n-1)(n-2)\ldots(n-2k+1)}{(2n-1)(2n-3)(2n-5)\ldots(2n-2k+1)} \times \frac{1}{2\times 4\times 6\times \ldots \times 2k}$$

(NOTE THAT $n!$ IS A CONSTANT SO IT CAN BE PULLED OUT)

IT IS NOW BEST TO CARRY OUT THE MANIPULATIONS SEPARATELY IN SMALL SECTIONS

- $(2n-1)(2n-3)\ldots 5\times 3\times 1 = \frac{2n(2n-1)(2n-2)(2n-3)\ldots\times 5\times 4\times 3\times 2\times 1}{2n(2n-2)\ldots\times 4\times 2} = \frac{(2n)!}{2^n[n(n-1)(n-2)\ldots\times 2\times 1]} = \frac{(2n)!}{2^n n!}$
- $n(n-1)(n-2)\ldots(n-2k+1) = \frac{n(n-1)(n-2)\ldots(n-2k+1)(n-2k)(n-2k-1)\ldots\times 3\times 2\times 1}{(n-2k)(n-2k-1)\times\ldots\times 3\times 2\times 1} = \frac{n!}{(n-2k)!}$
- $2\times 4\times 6\times\ldots\times 2k = 2^k[1\times 2\times 3\times\ldots\times k] = 2^k k!$
- $(2n-1)(2n-3)(2n-5)\ldots(2n-2k+1) = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)\ldots(2n-2k+1)}{2n(2n-2)(2n-4)\ldots(2n-2k+2)} \times \frac{(2n-2k)(2n-2k-1)(2n-2k-2)\ldots\times 3\times 2\times 1}{(2n-2k)(2n-2k-1)(2n-2k-2)\ldots\times 3\times 2\times 1}$

$$= \frac{(2n)!}{2^k[n(n-1)(n-2)\ldots(n-k+1)]\times(2n-2k)!} = \frac{(2n)!}{2^k(2n-2k)![n(n-1)(n-2)\ldots(n-k+1)]}$$

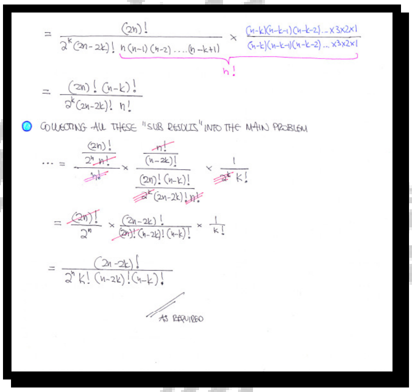
NOTE $2n(2n-2)(2n-4)\ldots(2n-2k+2)(2n-2k)(2n-2k-2)(2n-2k-4)\ldots\times 6\times 4\times 2$

THAT IS NOT THERE BUT SUPPOSE IT WAS

n TERMS

$n-k$ TERMS

$\therefore n-(n-k)=k$



$$= \frac{(2n)!}{2^k(2n-2k)!\,n(n-1)(n-2)\ldots(n-k+1)} \times \frac{(n-k)(n-k-1)(n-k-2)\ldots\times 3\times 2\times 1}{(n-k)(n-k-1)(n-k-2)\ldots\times 3\times 2\times 1}$$

$n!$

$$= \frac{(2n)!\,(n-k)!}{2^k(2n-2k)!\,n!}$$

COLLECTING ALL THESE "SUB RESULTS" INTO THE MAIN PROBLEM

$$\ldots = \frac{\frac{(2n)!}{2^n \cancel{n!}}}{\cancel{n!}} \times \frac{\frac{\cancel{n!}}{(n-2k)!}}{\frac{(2n)!(n-k)!}{\cancel{2^k}(2n-2k)!\cancel{n!}}} \times \frac{1}{\cancel{2^k}k!}$$

$$= \frac{\cancel{(2n)!}}{2^n} \times \frac{(2n-2k)!}{\cancel{(2n)!}(n-2k)!(n-k)!} \times \frac{1}{k!}$$

$$= \frac{(2n-2k)!}{2^n k!(n-2k)!(n-k)!}$$

AS REQUIRED