THE WAVE EQUATION

WAVE EQUATION

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z=z(x,t)$$

Propagating Waves

Question 1

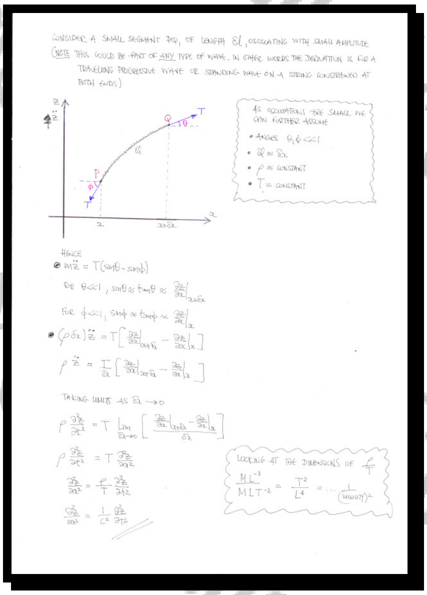
It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0 .$$

a) Derive the above partial differential equation from first principles, for standing waves or propagating waves, where c is a positive constant.

b) Show further that if z represents the vertical displacement of propagating wave then c represents the propagating speed.

$\boxed{\text{proof}}$



CONSIDER A SMALL SEGMENT PQ, OF LENGTH δl, OSCILLATING WITH SMALL AMPLITUDE

(NOTE THIS COULD BE PART OF ANY TYPE OF WAVE. IN OTHER WORDS THE DERIVATION IS FOR A TRAVELLING PROGRESSIVE WAVE OR STANDING WAVE ON A STRING CONSTRAINED AT BOTH ENDS)

AS OSCILLATIONS ARE SMALL WE CAN FURTHER ASSUME
- ANGLES $\theta, \phi \ll 1$
- $\delta l \approx \delta x$
- $\rho \approx$ CONSTANT
- $T =$ CONSTANT

HENCE

- $m\ddot{z} = T(\sin\theta - \sin\phi)$

FOR $\theta \ll 1$, $\sin\theta \approx \tan\theta \approx \left.\frac{\partial z}{\partial x}\right|_{x+\delta x}$

FOR $\phi \ll 1$, $\sin\phi \approx \tan\phi \approx \left.\frac{\partial z}{\partial x}\right|_{x}$

- $(\rho\delta x)\ddot{z} = T\left[\left.\frac{\partial z}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial z}{\partial x}\right|_{x}\right]$

$\rho\ddot{z} = \frac{T}{\delta x}\left[\left.\frac{\partial z}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial z}{\partial x}\right|_{x}\right]$

TAKING LIMITS AS $\delta x \to 0$

$\rho\frac{\partial^2 z}{\partial t^2} = T\lim_{\delta x \to 0}\left[\frac{\left.\frac{\partial z}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial z}{\partial x}\right|_{x}}{\delta x}\right]$

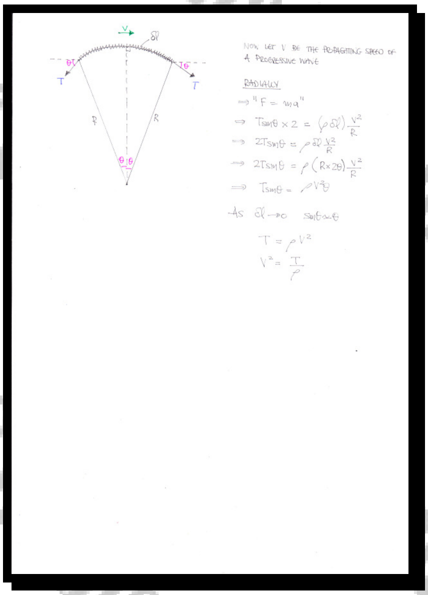
$\rho\frac{\partial^2 z}{\partial t^2} = T\frac{\partial^2 z}{\partial x^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 z}{\partial t^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ //

LOOKING AT THE DIMENSIONS OF $\frac{\rho}{T}$

$\frac{ML^{-3}}{MLT^{-2}} = \frac{T^2}{L^4} = \dots \frac{1}{(\text{VELOCITY})^2}$



NOW LET V BE THE PROPAGATING SPEED OF A PROGRESSIVE WAVE

RADIALLY

$\Rightarrow$ "$F = ma$"

$\Rightarrow T\sin\theta \times 2 = (\rho\delta l)\frac{V^2}{R}$

$\Rightarrow 2T\sin\theta = \rho\,\delta l\,\frac{V^2}{R}$

$\Rightarrow 2T\sin\theta = \rho(R\times 2\theta)\frac{V^2}{R}$

$\Rightarrow T\sin\theta = \rho V^2\theta$

AS $\delta l \to 0$ $\sin\theta \approx \theta$

$T = \rho V^2$

$V^2 = \frac{T}{\rho}$

Question 2

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

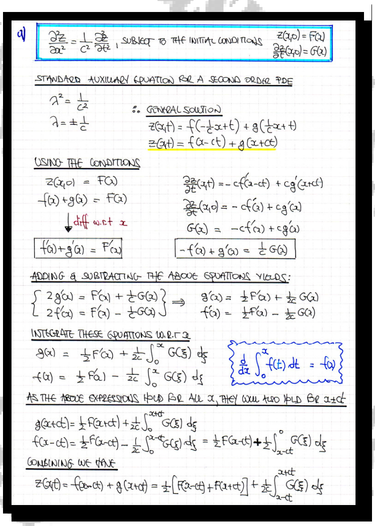
It is further given further that

$$F(x) = \begin{cases} 1-x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0.$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$.

c) Sketch the wave profiles for $t = 0$ and $t = \frac{2}{c}$.

, solution below



a) $\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \text{ SUBJECT TO THE INITIAL CONDITIONS } \begin{matrix} z(x,0) = F(x) \\ \frac{\partial z}{\partial t}(x,0) = G(x) \end{matrix}}$

STANDARD AUXILIARY EQUATION FOR A SECOND ORDER PDE

$\lambda^2 = \frac{1}{c^2}$

$\lambda = \pm\frac{1}{c}$

∴ GENERAL SOLUTION

$z(x,t) = f(-\frac{1}{c}x + t) + g(\frac{1}{c}x + t)$

$z(x,t) = f(x-ct) + g(x+ct)$

USING THE CONDITIONS

$z(x,0) = F(x)$

$f(x) + g(x) = F(x)$

$\Big\downarrow$ diff w.r.t x

$\boxed{f'(x) + g'(x) = F'(x)}$

$\frac{\partial z}{\partial t}(x,t) = -cf'(x-ct) + cg'(x+ct)$

$\frac{\partial z}{\partial t}(x,0) = -cf'(x) + cg'(x)$

$G(x) = -cf'(x) + cg'(x)$

$\boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$

ADDING & SUBTRACTING THE ABOVE EQUATIONS YIELDS:

$\left\{\begin{matrix} 2g'(x) = F'(x) + \frac{1}{c}G(x) \\ 2f'(x) = F'(x) - \frac{1}{c}G(x) \end{matrix}\right\} \Rightarrow \begin{matrix} g'(x) = \frac{1}{2}F'(x) + \frac{1}{2c}G(x) \\ f'(x) = \frac{1}{2}F'(x) - \frac{1}{2c}G(x) \end{matrix}$

INTEGRATE THESE EQUATIONS W.R.T x

$g(x) = \frac{1}{2}F'(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

$\left\{ \frac{d}{dx}\int_0^x f(t)\,dt = f(x) \right\}$

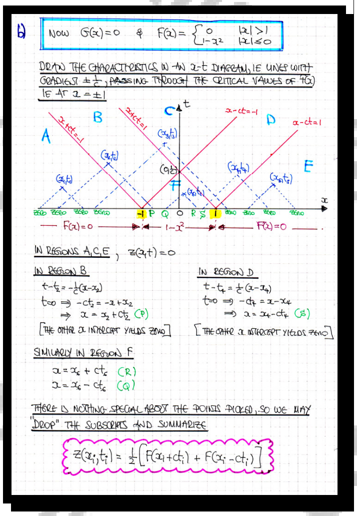
AS THE ABOVE EXPRESSIONS HOLD FOR ALL x, THEY WILL ALSO HOLD FOR $x \pm ct$

$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$

$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2}\int_{x-ct}^{0} G(\xi)\,d\xi$

COMBINING WE HAVE

$z(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$



b) $\boxed{\text{NOW } G(x) = 0 \text{ \& } F(x) = \begin{cases} 0 & |x| > 1 \\ 1-x^2 & |x| \leq 0 \end{cases}}$

DRAW THE CHARACTERISTICS IN AN x-t DIAGRAM, IE LINES WITH GRADIENT $\pm\frac{1}{c}$, PASSING THROUGH THE CRITICAL VALUES OF $F(x)$ IE AT $x = \pm 1$

IN REGIONS A, C, E, $z(x,t) = 0$

IN REGION B

$t - t_2 = -\frac{1}{c}(x - x_2)$

$t=0 \Rightarrow -ct_2 = -x + x_2$

$\Rightarrow x = x_2 + ct_2$ (P)

[THE OTHER x INTERCEPT YIELDS ZERO]

IN REGION D

$t - t_4 = \frac{1}{c}(x - x_4)$

$t=0 \Rightarrow -ct_4 = x - x_4$

$\Rightarrow x = x_4 - ct_4$ (S)

[THE OTHER x INTERCEPT YIELDS ZERO]

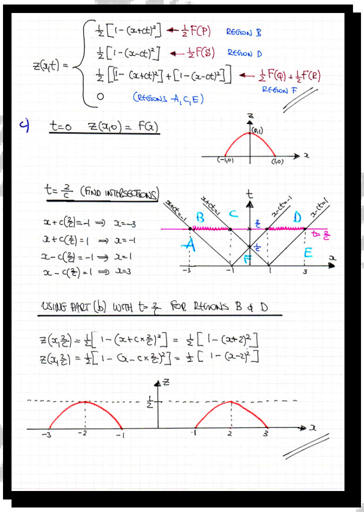
SIMILARLY IN REGION F

$x = x_6 + ct_6$ (R)

$x = x_6 - ct_6$ (Q)

THERE IS NOTHING SPECIAL ABOUT THE POINTS PICKED, SO WE MAY "DROP" THE SUBSCRIPTS AND SUMMARIZE

$\left\{ z(x_i, t_i) = \frac{1}{2}\left[F(x_i + ct_i) + F(x_i - ct_i)\right] \right\}$



$z(x,t) = \begin{cases} \frac{1}{2}\left[1-(x+ct)^2\right] \leftarrow \frac{1}{2}F(P) & \text{REGION B} \\ \frac{1}{2}\left[1-(x-ct)^2\right] \leftarrow \frac{1}{2}F(S) & \text{REGION D} \\ \frac{1}{2}\left[\left[1-(x+ct)^2\right] + \left[1-(x-ct)^2\right]\right] \leftarrow \frac{1}{2}F(Q) + \frac{1}{2}F(R) & \text{REGION F} \\ 0 & \text{(REGIONS A, C, E)} \end{cases}$

c) $t=0 \quad z(x,0) = F(x)$

$t = \frac{2}{c}$ (FIND INTERSECTIONS)

$x + c\left(\frac{2}{c}\right) = -1 \Rightarrow x = -3$

$x + c\left(\frac{2}{c}\right) = 1 \Rightarrow x = -1$

$x - c\left(\frac{2}{c}\right) = -1 \Rightarrow x = 1$

$x - c\left(\frac{2}{c}\right) = 1 \Rightarrow x = 3$

USING PART (b) WITH $t = \frac{2}{c}$ FOR REGIONS B & D

$z(x,\frac{2}{c}) = \frac{1}{2}\left[1-(x+c\times\frac{2}{c})^2\right] = \frac{1}{2}\left[1-(x+2)^2\right]$

$z(x,\frac{2}{c}) = \frac{1}{2}\left[1-(x-c\times\frac{2}{c})^2\right] = \frac{1}{2}\left[1-(x-2)^2\right]$

Question 3

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi,$$

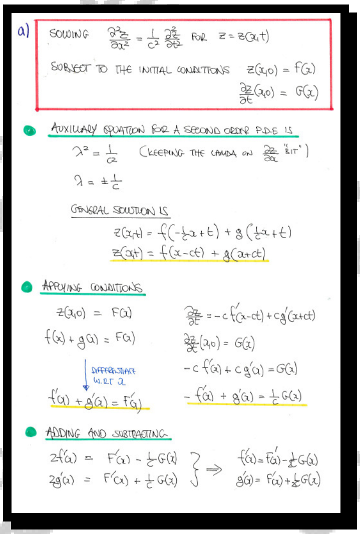
for $-\infty < x < \infty$, $t \geq 0$.

b) Given further that

$$F(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0,$$

sketch the wave profiles for $t = \frac{n}{c}$, $n = 0, 1, 2, 3, 4$.

MM , solution below



a) SOLVING $\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ FOR $z = z(x,t)$

SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$

$\frac{\partial z}{\partial t}(x,0) = G(x)$

- AUXILIARY EQUATION FOR A SECOND ORDER P.D.E IS

$\lambda^2 = \frac{1}{c^2}$ (KEEPING THE LAMDA ON $\frac{\partial z}{\partial x}$ "BIT")

$\lambda = \pm\frac{1}{c}$

GENERAL SOLUTION IS

$z(x,t) = f\left(-\frac{1}{c}x + t\right) + g\left(\frac{1}{c}x + t\right)$

$z(x,t) = f(x-ct) + g(x+ct)$

- APPLYING CONDITIONS

$z(x,0) = F(x)$

$f(x) + g(x) = F(x)$

DIFFERENTIATE W.R.T x

$f'(x) + g'(x) = F'(x)$

$\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

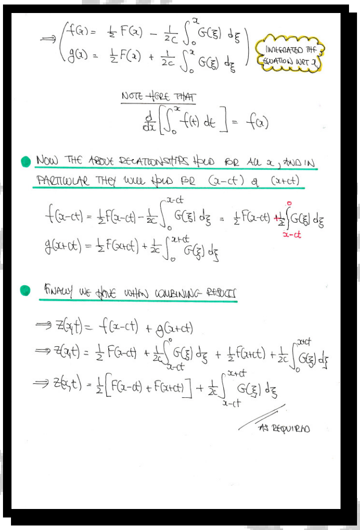
$\frac{\partial z}{\partial t}(x,0) = G(x)$

$-cf'(x) + cg'(x) = G(x)$

$-f'(x) + g'(x) = \frac{1}{c}G(x)$

- ADDING AND SUBTRACTING

$$\left.\begin{aligned} 2f'(x) &= F'(x) - \tfrac{1}{c}G(x) \\ 2g'(x) &= F'(x) + \tfrac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x) \\ g'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x) \end{aligned}$$



$$\Rightarrow \begin{pmatrix} f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi \\ g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi \end{pmatrix}$$

INTEGRATED THE EQUATION WRT x

NOTE HERE THAT

$$\frac{d}{dx}\left[\int_0^x f(t)\,dt\right] = f(x)$$

- NOW THE ABOVE RELATIONSHIPS HOLD FOR ALL x, AND IN PARTICULAR THEY WILL HOLD FOR $(x-ct)$ & $(x+ct)$

$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$

$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$

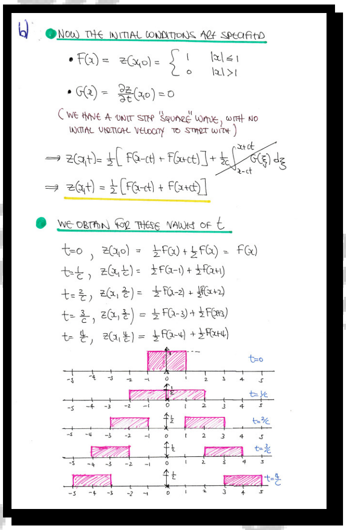
- FINALLY WE HAVE WHEN COMBINING RESULTS

$\Rightarrow z(x,t) = f(x-ct) + g(x+ct)$

$\Rightarrow z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$

$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$

AS REQUIRED



b) - NOW THE INITIAL CONDITIONS ARE SPECIFIED

- $F(x) = z(x,0) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$
- $G(x) = \frac{\partial z}{\partial t}(x,0) = 0$

(WE HAVE A UNIT STEP "SQUARE" WAVE, WITH NO INITIAL VERTICAL VELOCITY TO START WITH)

$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \cancel{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi}$

$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$

- WE OBTAIN FOR THESE VALUES OF t

$t=0$, $z(x,0) = \frac{1}{2}F(x) + \frac{1}{2}F(x) = F(x)$

$t=\frac{1}{c}$, $z(x,\frac{1}{c}) = \frac{1}{2}F(x-1) + \frac{1}{2}F(x+1)$

$t=\frac{2}{c}$, $z(x,\frac{2}{c}) = \frac{1}{2}F(x-2) + \frac{1}{2}F(x+2)$

$t=\frac{3}{c}$, $z(x,\frac{3}{c}) = \frac{1}{2}F(x-3) + \frac{1}{2}F(x+3)$

$t=\frac{4}{c}$, $z(x,\frac{4}{c}) = \frac{1}{2}F(x-4) + \frac{1}{2}F(x+4)$

Question 4

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \; c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

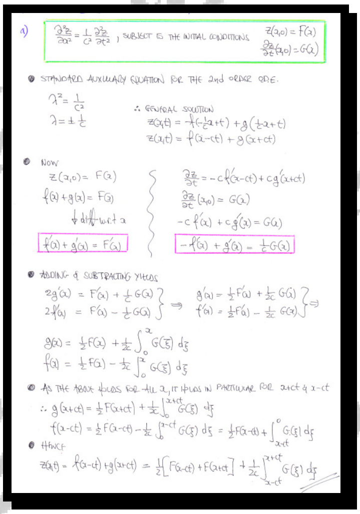
It is further given further that

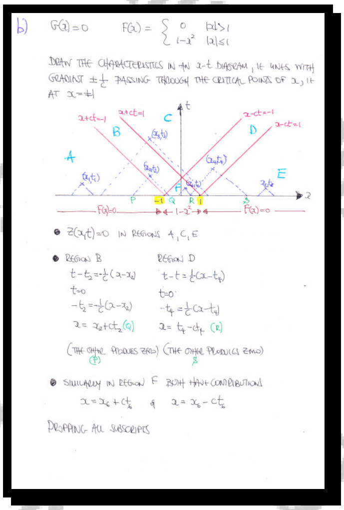
$$F(x) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0.$$

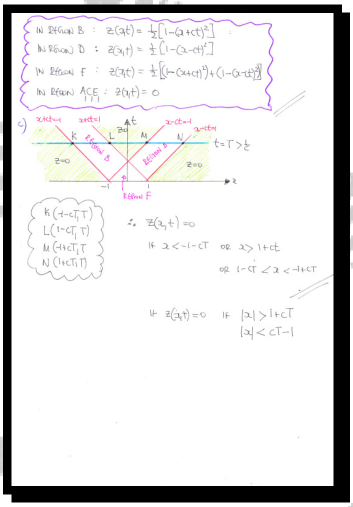
b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$.

c) Given that $t = T > \frac{1}{c}$, determine expressions for $z(x,t)$.

solution below







Question 5

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

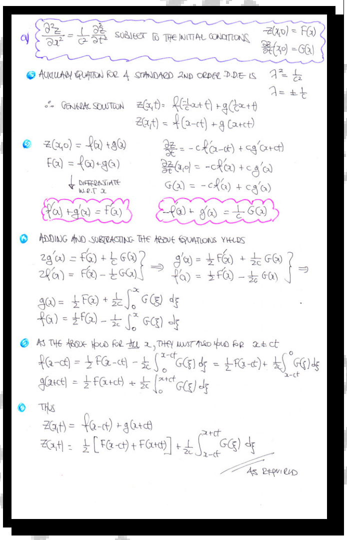
It is further given further that

$$F(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0.$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$.

c) Sketch the wave profiles for $t = 0$ and $t = \frac{1}{2c}$.

solution below



a) $\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$, $\frac{\partial z}{\partial t}(x,0) = G(x)$

- AUXILIARY EQUATION FOR A STANDARD 2ND ORDER P.D.E IS $\lambda^2 = \frac{1}{c^2}$, $\lambda = \pm\frac{1}{c}$

∴ GENERAL SOLUTION $z(x,t) = f(-\frac{1}{c}x+t) + g(\frac{1}{c}x+t)$

$z(x,t) = f(x-ct) + g(x+ct)$

- $z(x,0) = f(x) + g(x)$ $\qquad$ $\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

$F(x) = f(x) + g(x)$ $\qquad$ $\frac{\partial z}{\partial t}(x,0) = -cf'(x) + cg'(x)$

↓ DIFFERENTIATE W.R.T x $\qquad$ $G(x) = -cf'(x) + cg'(x)$

$f'(x) + g'(x) = F'(x)$ $\qquad$ $-f'(x) + g'(x) = \frac{1}{c}G(x)$

- ADDING AND SUBTRACTING THE ABOVE EQUATIONS YIELDS

$$\left.\begin{aligned} 2g'(x) &= F'(x) + \tfrac{1}{c}G(x) \\ 2f'(x) &= F'(x) - \tfrac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} g'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x) \\ f'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x) \end{aligned}\right\} \Rightarrow$$

$g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

- AS THE ABOVE HOLD FOR ALL x, THEY MUST ALSO HOLD FOR $x \pm ct$

$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^0 G(\xi)\,d\xi$

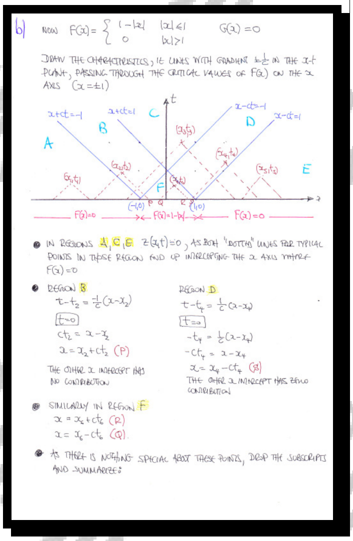
$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$

- THUS

$z(x,t) = f(x-ct) + g(x+ct)$

$z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$

AS REQUIRED



b) NOW $F(x) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ $\qquad$ $G(x) = 0$

DRAW THE CHARACTERISTICS, IE LINES WITH GRADIENT $\pm\frac{1}{c}$ IN THE x-t PLANE, PASSING THROUGH THE CRITICAL VALUES OF $F(x)$ ON THE x AXIS $(x = \pm 1)$

- IN REGIONS A, C, E $z(x,t) = 0$, AS BOTH "DOTTED" LINES FOR TYPICAL POINTS IN THOSE REGION END UP INTERCEPTING THE x AXIS WHERE $F(x) = 0$

- REGION B $\qquad$ REGION D

$t - t_2 = -\frac{1}{c}(x - x_2)$ $\qquad$ $t - t_4 = \frac{1}{c}(x - x_4)$

$\boxed{t=0}$ $\qquad$ $\boxed{t=0}$

$ct_2 = x - x_2$ $\qquad$ $-t_4 = \frac{1}{c}(x - x_4)$

$x = x_2 + ct_2$ (P) $\qquad$ $-ct_4 = x - x_4$

THE OTHER x INTERCEPT HAS NO CONTRIBUTION $\qquad$ $x = x_4 - ct_4$ (S)

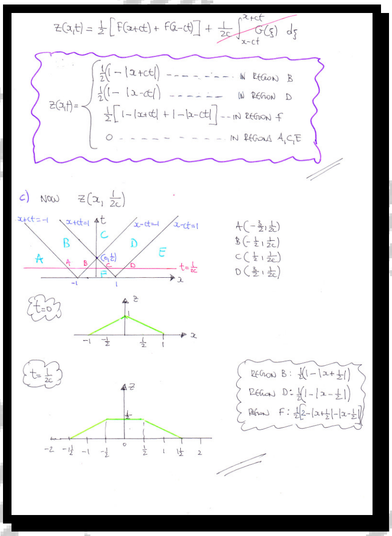
THE OTHER x INTERCEPT HAS ZERO CONTRIBUTION

- SIMILARLY IN REGION F

$x = x_6 + ct_6$ (R)

$x = x_6 - ct_6$ (Q)

- AS THERE IS NOTHING SPECIAL ABOUT THESE POINTS, DROP THE SUBSCRIPTS AND SUMMARIZE:



$z(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \cancel{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi}$

$$z(x,t) = \begin{cases} \frac{1}{2}(1 - |x+ct|) & \text{---------- IN REGION B} \\ \frac{1}{2}(1 - |x-ct|) & \text{---------- IN REGION D} \\ \frac{1}{2}\left[1 - |x+ct| + 1 - |x-ct|\right] & \text{-- IN REGION F} \\ 0 & \text{---------- IN REGIONS A, C, E} \end{cases}$$

c) NOW $z(x, \frac{1}{2c})$

A$(-\frac{3}{2}, \frac{1}{2c})$

B$(-\frac{1}{2}, \frac{1}{2c})$

C$(\frac{1}{2}, \frac{1}{2c})$

D$(\frac{3}{2}, \frac{1}{2c})$

$t=0$

$t=\frac{1}{2c}$

REGION B: $\frac{1}{2}(1 - |x+\frac{1}{2}|)$

REGION D: $\frac{1}{2}(1 - |x-\frac{1}{2}|)$

REGION F: $\frac{1}{2}[2 - |x+\frac{1}{2}| - |x-\frac{1}{2}|]$

Question 6

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \text{ and } \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

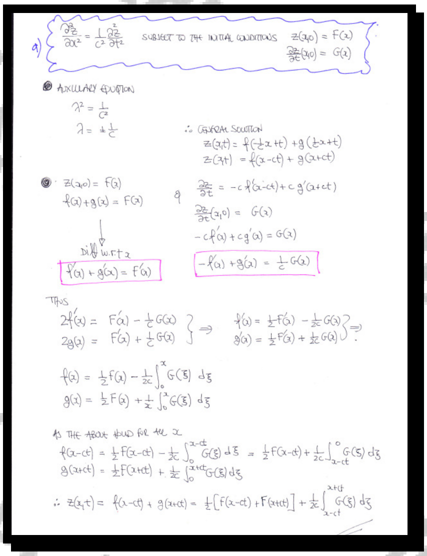
for $-\infty < x < \infty$, $t \geq 0$.

b) Given further that

$$F(x) = 0 \quad \text{and} \quad G(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases},$$

sketch the wave profiles for $t = \frac{n}{c}$, $n = 0, 1, 2$.

solution below



a) $\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$

$\frac{\partial z}{\partial t}(x,0) = G(x)$

● AUXILLARY EQUATION

$\lambda^2 = \frac{1}{c^2}$

$\lambda = \pm\frac{1}{c}$

∴ GENERAL SOLUTION

$z(x,t) = f(-\frac{1}{c}x + t) + g(\frac{1}{c}x + t)$

$z(x,t) = f(x-ct) + g(x+ct)$

● $z(x,0) = F(x)$

$f(x) + g(x) = F(x)$

Diff w.r.t x

$\boxed{f'(x) + g'(x) = F'(x)}$

& $\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

$\frac{\partial z}{\partial t}(x,0) = G(x)$

$-cf'(x) + cg'(x) = G(x)$

$\boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$

THUS

$\left.\begin{array}{l} 2f'(x) = F'(x) - \frac{1}{c}G(x) \\ 2g(x) = F'(x) + \frac{1}{c}G(x) \end{array}\right\} \Rightarrow \left.\begin{array}{l} f'(x) = \frac{1}{2}F'(x) - \frac{1}{2c}G(x) \\ g'(x) = \frac{1}{2}F'(x) + \frac{1}{2c}G(x) \end{array}\right\} \Rightarrow$

$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\, d\xi$

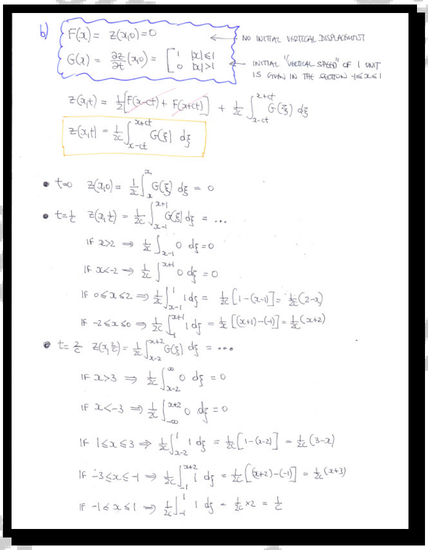
$g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\, d\xi$

AS THE ABOVE HOLD FOR ALL x

$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\, d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^0 G(\xi)\, d\xi$

$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\, d\xi$

$\therefore z(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi$



b) $F(x) = z(x,0) = 0$ ← NO INITIAL VERTICAL DISPLACEMENT

$G(x) = \frac{\partial z}{\partial t}(x,0) = \left[\begin{array}{ll} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{array}\right.$ ← INITIAL "VERTICAL SPEED" OF 1 UNIT IS GIVEN IN THE SECTION $-1 \leq x \leq 1$

$z(x,t) = \frac{1}{2}\left[\cancel{F(x-ct)} + \cancel{F(x+ct)}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi$

$\boxed{z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi}$

- $t=0$ $z(x,0) = \frac{1}{2c}\int_x^x G(\xi)\, d\xi = 0$
- $t=\frac{1}{c}$ $z(x,\frac{1}{c}) = \frac{1}{2c}\int_{x-1}^{x+1} G(\xi)\, d\xi = \dots$

IF $x>2 \Rightarrow \frac{1}{2c}\int_{x-1} 0\, d\xi = 0$

IF $x<-2 \Rightarrow \frac{1}{2c}\int^{x+1} 0\, d\xi = 0$

IF $0 \leq x \leq 2 \Rightarrow \frac{1}{2c}\int_{x-1}^{1} 1\, d\xi = \frac{1}{2c}\left[1-(x-1)\right] = \frac{1}{2c}(2-x)$

IF $-2 \leq x \leq 0 \Rightarrow \frac{1}{2c}\int_{-1}^{x+1} 1\, d\xi = \frac{1}{2c}\left[(x+1)-(-1)\right] = \frac{1}{2c}(x+2)$

- $t=\frac{2}{c}$ $z(x,\frac{2}{c}) = \frac{1}{2c}\int_{x-2}^{x+2} G(\xi)\, d\xi = \dots$

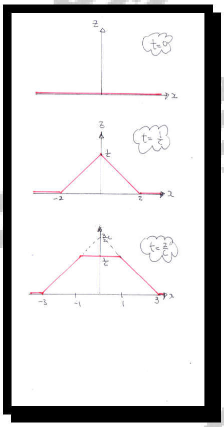
IF $x>3 \Rightarrow \frac{1}{2c}\int_{x-2}^{\infty} 0\, d\xi = 0$

IF $x<-3 \Rightarrow \frac{1}{2c}\int_{-\infty}^{x+2} 0\, d\xi = 0$

IF $1 \leq x \leq 3 \Rightarrow \frac{1}{2c}\int_{x-2}^{1} 1\, d\xi = \frac{1}{2c}\left[1-(x-2)\right] = \frac{1}{2c}(3-x)$

IF $-3 \leq x \leq -1 \Rightarrow \frac{1}{2c}\int_{-1}^{x+2} 1\, d\xi = \frac{1}{2c}\left[(x+2)-(-1)\right] = \frac{1}{2c}(x+3)$

IF $-1 \leq x \leq 1 \Rightarrow \frac{1}{2c}\int_{-1}^{1} 1\, d\xi = \frac{1}{2c} \times 2 = \frac{1}{c}$



Question 7

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = 0 \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

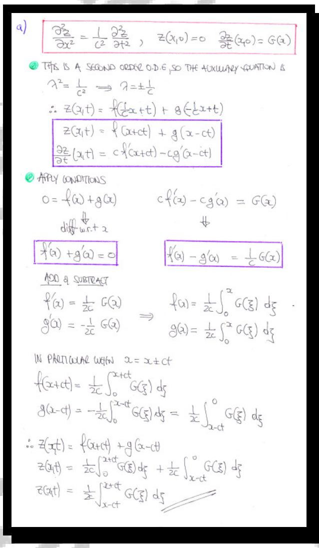
for $-\infty < x < \infty$, $t \geq 0$.

It is further given further that

$$G(x) = \begin{cases} \cos\left(\frac{1}{2}\pi x\right) & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$.

solution below



a) $\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z(x,0) = 0 \quad \frac{\partial z}{\partial t}(x,0) = G(x)}$

- THIS IS A SECOND ORDER O.D.E, SO THE AUXILIARY EQUATION IS

$\lambda^2 = \frac{1}{c^2} \Longrightarrow \lambda = \pm\frac{1}{c}$

$\therefore z(x,t) = f(\frac{1}{c}x+t) + g(-\frac{1}{c}x+t)$

$$\boxed{\begin{aligned} z(x,t) &= f(x+ct) + g(x-ct) \\ \frac{\partial z}{\partial t}(x,t) &= cf'(x+ct) - cg'(x-ct) \end{aligned}}$$

- APPLY CONDITIONS

$0 = f(x) + g(x) \qquad cf'(x) - cg'(x) = G(x)$

$\Downarrow$ diff w.r.t x $\qquad \Downarrow$

$\boxed{f'(x) + g'(x) = 0} \qquad \boxed{f'(x) - g'(x) = \frac{1}{c}G(x)}$

ADD & SUBTRACT

$f'(x) = \frac{1}{2c}G(x)$

$g'(x) = -\frac{1}{2c}G(x)$

$\Longrightarrow$

$f(x) = \frac{1}{2c}\int_0^x G(\xi)\, d\xi$

$g(x) = \frac{1}{2c}\int_0^x G(\xi)\, d\xi$

IN PARTICULAR WHEN $x = x \pm ct$

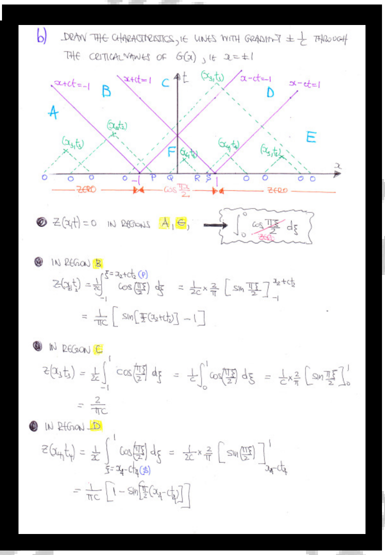
$f(x+ct) = \frac{1}{2c}\int_0^{x+ct} G(\xi)\, d\xi$

$g(x-ct) = -\frac{1}{2c}\int_0^{x-ct} G(\xi)\, d\xi = \frac{1}{2c}\int_{x-ct}^0 G(\xi)\, d\xi$

$\therefore z(x,t) = f(x+ct) + g(x-ct)$

$z(x,t) = \frac{1}{2c}\int_0^{x+ct} G(\xi)\, d\xi + \frac{1}{2c}\int_{x-ct}^0 G(\xi)\, d\xi$

$z(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi$



b) DRAW THE CHARACTERISTICS, IE LINES WITH GRADIENT $\pm\frac{1}{c}$ THROUGH THE CRITICAL VALUES OF $G(x)$, IE $x = \pm 1$

- $z(x,t) = 0$ IN REGIONS A, E, $\longrightarrow \int_0^0 \cos\frac{\pi\xi}{2}\, d\xi$ ZERO

- IN REGION B

$z(x_2,t_2) = \frac{1}{2c}\int_{-1}^{\xi = x_2+ct_2 \,(P)} \cos\left(\frac{\pi\xi}{2}\right) d\xi = \frac{1}{2c} \times \frac{2}{\pi}\left[\sin\frac{\pi\xi}{2}\right]_{-1}^{x_2+ct_2}$

$= \frac{1}{\pi c}\left[\sin\left[\frac{\pi}{2}(x_2+ct_2)\right] - 1\right]$

- IN REGION C

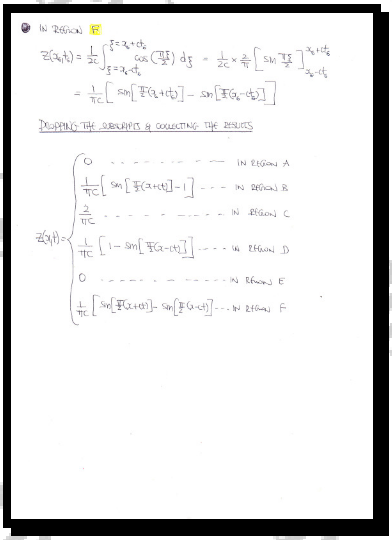
$z(x_3t_3) = \frac{1}{2c}\int_{-1}^{1} \cos\left(\frac{\pi\xi}{2}\right) d\xi = \frac{1}{c}\int_0^1 \cos\left(\frac{\pi\xi}{2}\right) d\xi = \frac{1}{c} \times \frac{2}{\pi}\left[\sin\frac{\pi\xi}{2}\right]_0^1$

$= \frac{2}{\pi c}$

- IN REGION D

$z(x_4,t_4) = \frac{1}{2c}\int_{\xi = x_4-ct_4 \,(S)}^{1} \cos\left(\frac{\pi\xi}{2}\right) d\xi = \frac{1}{2c} \times \frac{2}{\pi}\left[\sin\left(\frac{\pi\xi}{2}\right)\right]_{x_4-ct_4}^{1}$

$= \frac{1}{\pi c}\left[1 - \sin\left[\frac{\pi}{2}(x_4-ct_4)\right]\right]$



- IN REGION F

$z(x_6,t_6) = \frac{1}{2c}\int_{\xi = x_6-ct_6}^{\xi = x_6+ct_6} \cos\left(\frac{\pi\xi}{2}\right) d\xi = \frac{1}{2c} \times \frac{2}{\pi}\left[\sin\frac{\pi\xi}{2}\right]_{x_6-ct_6}^{x_6+ct_6}$

$= \frac{1}{\pi c}\left[\sin\left[\frac{\pi}{2}(x_6+ct_6)\right] - \sin\left[\frac{\pi}{2}(x_6-ct_6)\right]\right]$

DROPPING THE SUBSCRIPTS & COLLECTING THE RESULTS

$$z(x,t) = \begin{cases} 0 & \text{IN REGION A} \\ \frac{1}{\pi c}\left[\sin\left[\frac{\pi}{2}(x+ct)\right] - 1\right] & \text{IN REGION B} \\ \frac{2}{\pi c} & \text{IN REGION C} \\ \frac{1}{\pi c}\left[1 - \sin\left[\frac{\pi}{2}(x-ct)\right]\right] & \text{IN REGION D} \\ 0 & \text{IN REGION E} \\ \frac{1}{\pi c}\left[\sin\left[\frac{\pi}{2}(x+ct)\right] - \sin\left[\frac{\pi}{2}(x-ct)\right]\right] & \text{IN REGION F} \end{cases}$$

Question 8

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

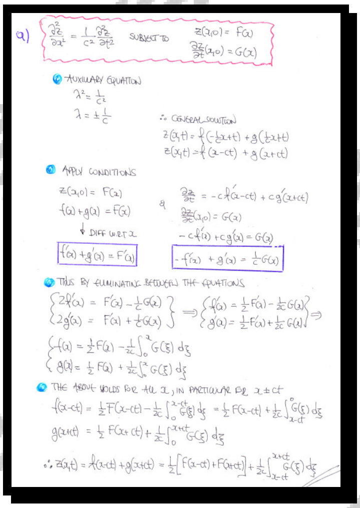
for $-\infty < x < \infty$, $t \geq 0$.

It is further given further that

$$F(x) = \begin{cases} \cos x & |x| < \frac{1}{2}\pi \\ 0 & |x| \geq \frac{1}{2}\pi \end{cases} \quad \text{and} \quad G(x) = 0.$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$, and hence show that there is a region of x-t plane where $z(x,t)$ represents a stationary wave.

solution below



a) $$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{SUBJECT TO} \quad z(x,0) = F(x), \quad \frac{\partial z}{\partial t}(x,0) = G(x)}$$

- AUXILIARY EQUATION

$$\lambda^2 = \frac{1}{c^2}$$
$$\lambda = \pm\frac{1}{c}$$

∴ GENERAL SOLUTION

$$z(x,t) = f\left(-\tfrac{1}{c}x + t\right) + g\left(\tfrac{1}{c}x + t\right)$$
$$z(x,t) = f(x-ct) + g(x+ct)$$

- APPLY CONDITIONS

$$z(x,0) = F(x)$$
$$f(x) + g(x) = F(x)$$
$$\downarrow \text{DIFF W.R.T } x$$
$$\boxed{f'(x) + g'(x) = F'(x)}$$

&

$$\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$$
$$\frac{\partial z}{\partial t}(x,0) = G(x)$$
$$-cf'(x) + cg'(x) = G(x)$$
$$\boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$$

- THUS BY ELIMINATING BETWEEN THE EQUATIONS

$$\left\{\begin{array}{l} 2f'(x) = F'(x) - \frac{1}{c}G(x) \\ 2g'(x) = F'(x) + \frac{1}{c}G(x) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} f'(x) = \frac{1}{2}F'(x) - \frac{1}{2c}G(x) \\ g'(x) = \frac{1}{2}F'(x) + \frac{1}{2c}G(x) \end{array}\right\} \Rightarrow$$

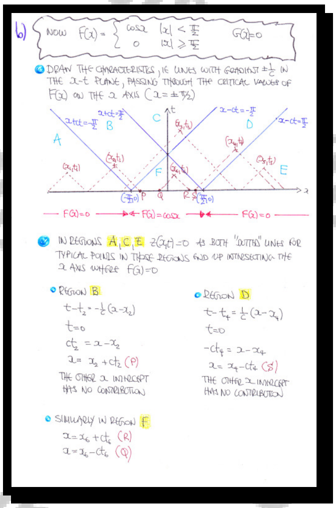
$$\left\{\begin{array}{l} f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi \\ g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi \end{array}\right.$$

- THE ABOVE HOLDS FOR ALL x, IN PARTICULAR FOR $x \pm ct$

$$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$$

$$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

$$\therefore z(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$



b) NOW $$F(x) = \begin{cases} \cos x & |x| < \frac{\pi}{2} \\ 0 & |x| \geqslant \frac{\pi}{2} \end{cases} \qquad G(x) = 0$$

- DRAW THE CHARACTERISTICS, IE LINES WITH GRADIENT $\pm\frac{1}{c}$ IN THE x-t PLANE, PASSING THROUGH THE CRITICAL VALUES OF $F(x)$ ON THE x AXIS $(x = \pm\pi/2)$

- IN REGIONS A, C, E $z(x,t) = 0$ AS BOTH "DOTTED" LINES FOR TYPICAL POINTS IN THOSE REGIONS END UP INTERSECTING THE x AXIS WHERE $F(x) = 0$

- REGION B

$$t - t_2 = -\frac{1}{c}(x - x_2)$$
$$t = 0$$
$$ct_2 = x - x_2$$
$$x = x_2 + ct_2 \quad (P)$$

THE OTHER x INTERCEPT HAS NO CONTRIBUTION

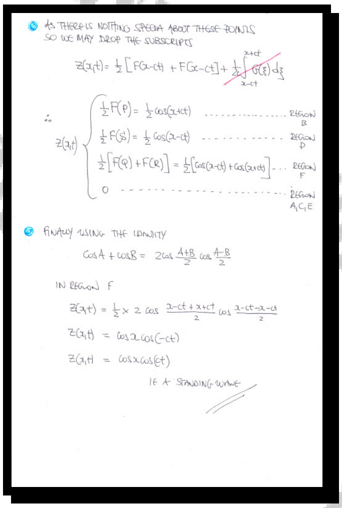
- REGION D

$$t - t_4 = \frac{1}{c}(x - x_4)$$
$$t = 0$$
$$-ct_4 = x - x_4$$
$$x = x_4 - ct_4 \quad (S)$$

THE OTHER x INTERCEPT HAS NO CONTRIBUTION

- SIMILARLY IN REGION F

$$x = x_6 + ct_6 \quad (R)$$
$$x = x_6 - ct_6 \quad (Q)$$



- AS THERE IS NOTHING SPECIAL ABOUT THESE POINTS SO WE MAY DROP THE SUBSCRIPTS

$$z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x-ct)\right] + \cancel{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi}$$

$$\therefore \; z(x,t) \begin{cases} \frac{1}{2}F(P) = \frac{1}{2}\cos(x+ct) & \ldots\ldots\ldots \text{REGION B} \\ \frac{1}{2}F(S) = \frac{1}{2}\cos(x-ct) & \ldots\ldots\ldots \text{REGION D} \\ \frac{1}{2}\left[F(Q) + F(R)\right] = \frac{1}{2}\left[\cos(x-ct) + \cos(x+ct)\right] & \ldots \text{REGION F} \\ 0 & \ldots\ldots\ldots \text{REGION A, C, E} \end{cases}$$

- FINALLY USING THE IDENTITY

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

IN REGION F

$$z(x,t) = \frac{1}{2} \times 2\cos\frac{x-ct+x+ct}{2}\cos\frac{x-ct-x-ct}{2}$$
$$z(x,t) = \cos x\cos(-ct)$$
$$z(x,t) = \cos x\cos(ct)$$

IE A STANDING WAVE

Question 9

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2},$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-t) + \frac{1}{2}F(x+t) + \frac{1}{2}\int_{x-t}^{x+t} G(\xi)\, d\xi,$$

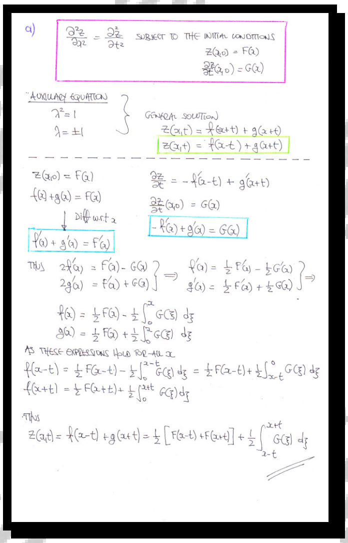
for $-\infty < x < \infty$, $t \geq 0$.

b) Given further that

$$F(x) = 0 \quad \text{and} \quad G(x) = \begin{cases} 1 & x < 0 \\ x+1 & x \geq 0 \end{cases},$$

use the method of the characteristics in the x-t plane to solve the equation of part **(b)** and hence sketch the wave profile when $t = 1$.

solution below



a) $\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}}$ SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$, $\frac{\partial z}{\partial t}(x,0) = G(x)$

AUXILIARY EQUATION

$\lambda^2 = 1$

$\lambda = \pm 1$

GENERAL SOLUTION

$z(x,t) = f(x+t) + g(x+t)$

$\boxed{z(x,t) = f(x-t) + g(x+t)}$

$z(x,0) = F(x)$

$f(x) + g(x) = F(x)$

$\downarrow$ Diff w.r.t x

$\boxed{f'(x) + g'(x) = F'(x)}$

$\frac{\partial z}{\partial t} = -f'(x-t) + g'(x+t)$

$\frac{\partial z}{\partial t}(x,0) = G(x)$

$\boxed{-f'(x) + g'(x) = G(x)}$

THUS $\left.\begin{array}{l} 2f'(x) = F'(x) - G(x) \\ 2g'(x) = F'(x) + G(x) \end{array}\right\} \Rightarrow \left.\begin{array}{l} f'(x) = \frac{1}{2}F'(x) - \frac{1}{2}G'(x) \\ g'(x) = \frac{1}{2}F'(x) + \frac{1}{2}G(x) \end{array}\right\} \Rightarrow$

$f(x) = \frac{1}{2}F(x) - \frac{1}{2}\int_0^x G(\xi)\, d\xi$

$g(x) = \frac{1}{2}F(x) + \frac{1}{2}\int_0^x G(\xi)\, d\xi$

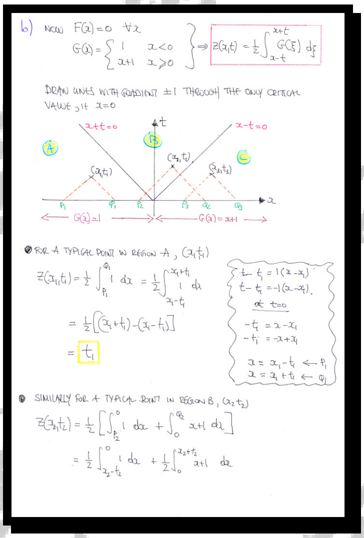
AS THESE EXPRESSIONS HOLD FOR ALL x

$f(x-t) = \frac{1}{2}F(x-t) - \frac{1}{2}\int_0^{x-t} G(\xi)\, d\xi = \frac{1}{2}F(x-t) + \frac{1}{2}\int_{x-t}^0 G(\xi)\, d\xi$

$f(x+t) = \frac{1}{2}F(x+t) + \frac{1}{2}\int_0^{x+t} G(\xi)\, d\xi$

THUS

$z(x,t) = f(x-t) + g(x+t) = \frac{1}{2}\left[F(x-t) + F(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} G(\xi)\, d\xi$



b) NOW $\left.\begin{array}{l} F(x) = 0 \quad \forall x \\ G(x) = \begin{cases} 1 & x < 0 \\ x+1 & x \geq 0 \end{cases} \end{array}\right\} \Rightarrow \boxed{z(x,t) = \frac{1}{2}\int_{x-t}^{x+t} G(\xi)\, d\xi}$

DRAW LINES WITH GRADIENT ± 1 THROUGH THE ONLY CRITICAL VALUE, IE $x=0$

- FOR A TYPICAL POINT IN REGION A, (x_1,t_1)

$z(x_1,t_1) = \frac{1}{2}\int_{P_1}^{Q_1} 1\, dx = \frac{1}{2}\int_{x_1-t_1}^{x_1+t_1} 1\, dx$

$= \frac{1}{2}\left[(x_1+t_1) - (x_1-t_1)\right]$

$= \boxed{t_1}$

$t - t_1 = 1(x - x_1)$

$t - t_1 = -1(x - x_1)$.

at $t=0$

$-t_1 = x - x_1$

$-t_1 = -x + x_1$

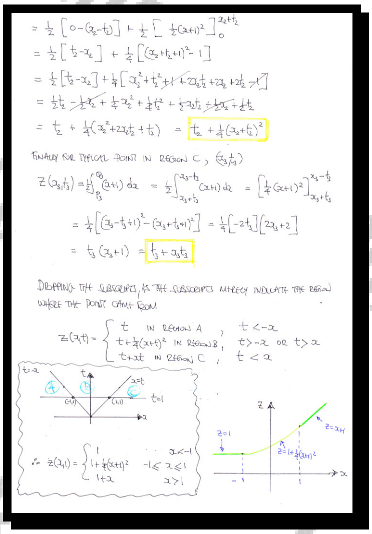
$x = x_1 - t_1 \leftarrow P_1$

$x = x_1 + t_1 \leftarrow Q_1$

- SIMILARLY FOR A TYPICAL POINT IN REGION B, (x_2, t_2)

$z(x_2,t_2) = \frac{1}{2}\left[\int_{P_2}^{0} 1\, dx + \int_0^{Q_2} x+1\, dx\right]$

$= \frac{1}{2}\int_{x_2-t_2}^{0} 1\, dx + \frac{1}{2}\int_0^{x_2+t_2} x+1\, dx$



$= \frac{1}{2}\left[0 - (x_2 - t_2)\right] + \frac{1}{2}\left[\frac{1}{2}(x+1)^2\right]_0^{x_2+t_2}$

$= \frac{1}{2}\left[t_2 - x_2\right] + \frac{1}{4}\left[(x_2+t_2+1)^2 - 1\right]$

$= \frac{1}{2}\left[t_2 - x_2\right] + \frac{1}{4}\left[x_2^2 + t_2^2 \cancel{+1} + 2x_2t_2 + 2x_2 + 2t_2 \cancel{-1}\right]$

$= \frac{1}{2}t_2 \cancel{-\frac{1}{2}x_2} + \frac{1}{4}x_2^2 + \frac{1}{4}t_2^2 + \frac{1}{2}x_2t_2 \cancel{+\frac{1}{2}x_2} + \frac{1}{2}t_2$

$= t_2 + \frac{1}{4}(x_2^2 + 2x_2t_2 + t_2) = \boxed{t_2 + \frac{1}{4}(x_2+t_2)^2}$

FINALLY FOR TYPICAL POINT IN REGION C, (x_3,t_3)

$z(x_3,t_3) = \frac{1}{2}\int_{P_3}^{Q_3}(x+1)\, dx = \frac{1}{2}\int_{x_3+t_3}^{x_3-t_3}(x+1)\, dx = \left[\frac{1}{4}(x+1)^2\right]_{x_3+t_3}^{x_3-t_3}$

$= \frac{1}{4}\left[(x_3-t_3+1)^2 - (x_3+t_3+1)^2\right] = \frac{1}{4}\left[-2t_3\right]\left[2x_3+2\right]$

$= t_3(x_3+1) = \boxed{t_3 + x_3t_3}$

DROPPING THE SUBSCRIPTS, AS THE SUBSCRIPTS MERELY INDICATE THE REGION WHERE THE POINT CAME FROM

$z(x,t) = \begin{cases} t & \text{IN REGION A}, & t < -x \\ t + \frac{1}{4}(x+t)^2 & \text{IN REGION B}, & t > -x \text{ OR } t > x \\ t + xt & \text{IN REGION C}, & t < x \end{cases}$

$\therefore\ z(x,1) = \begin{cases} 1 & x < -1 \\ 1 + \frac{1}{4}(x+1)^2 & -1 \leq x \leq 1 \\ 1 + x & x > 1 \end{cases}$

Question 10

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \ c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

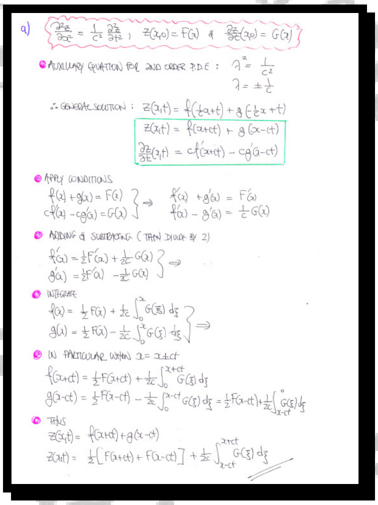
It is further given further that

$$G(x) = \begin{cases} \cos x & |x| \leq \dfrac{\pi}{2} \\ 0 & |x| > \dfrac{\pi}{2} \end{cases}$$

b) Indicate in the different regions of the x-t plane expressions for $z(x,t)$.

c) Show that if $t < \dfrac{\pi}{2c}$ there exists a range of values of x, over which $z(x,t)$ represents a stationary wave.

solution overleaf



a) $\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z(x,0) = F(x) \ \& \ \frac{\partial z}{\partial t}(x,0) = G(x)}$

- AUXILLARY EQUATION FOR 2ND ORDER P.D.E : $\lambda^2 = \frac{1}{c^2}$

$$\lambda = \pm\frac{1}{c}$$

$\therefore$ GENERAL SOLUTION : $z(x,t) = f(\tfrac{1}{c}x+t) + g(-\tfrac{1}{c}x+t)$

$$\boxed{\begin{aligned} z(x,t) &= f(x+ct) + g(x-ct) \\ \frac{\partial z}{\partial t}(x,t) &= cf'(x+ct) - cg'(x-ct) \end{aligned}}$$

- APPLY CONDITIONS

$$\left.\begin{aligned} f(x)+g(x) &= F(x) \\ cf'(x)-cg'(x) &= G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f'(x)+g'(x) &= F'(x) \\ f'(x)-g'(x) &= \tfrac{1}{c}G(x) \end{aligned}$$

- ADDING & SUBTRACTING (THEN DIVIDE BY 2)

$$\left.\begin{aligned} f'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x) \\ g'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x) \end{aligned}\right\} \Rightarrow$$

- INTEGRATE

$$\left.\begin{aligned} f(x) &= \tfrac{1}{2}F(x) + \tfrac{1}{2c}\int_0^x G(\xi)\,d\xi \\ g(x) &= \tfrac{1}{2}F(x) - \tfrac{1}{2c}\int_0^x G(\xi)\,d\xi \end{aligned}\right\} \Rightarrow$$

- IN PARTICULAR WHEN $x = x \pm ct$

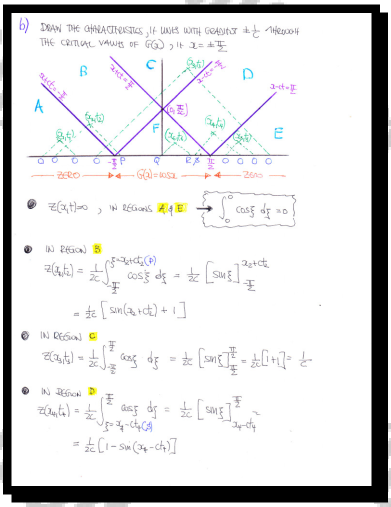
$$f(x+ct) = \tfrac{1}{2}F(x+ct) + \tfrac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

$$g(x-ct) = \tfrac{1}{2}F(x-ct) - \tfrac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \tfrac{1}{2}F(x-ct) + \tfrac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$$

- THUS

$$z(x,t) = f(x+ct) + g(x-ct)$$

$$z(x,t) = \tfrac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$



b) DRAW THE CHARACTERISTICS, IE LINES WITH GRADIENT $\pm\frac{1}{c}$ THROUGH THE CRITICAL VALUES OF $G(x)$, IE $x = \pm\frac{\pi}{2}$

- $z(x,t)=0$, IN REGIONS A & E $\rightarrow \int_0^0 \cos\xi \, d\xi = 0$

- IN REGION B

$$z(x_2,t_2) = \frac{1}{2c}\int_{-\frac{\pi}{2}}^{\xi=x_2+ct_2\,(P)} \cos\xi \, d\xi = \frac{1}{2c}\Big[\sin\xi\Big]_{-\frac{\pi}{2}}^{x_2+ct_2}$$

$$= \frac{1}{2c}\left[\sin(x_2+ct_2) + 1\right]$$

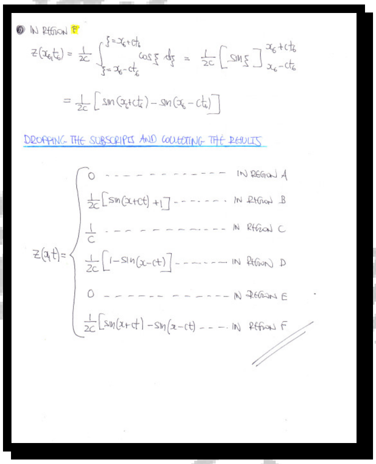
- IN REGION C

$$z(x_3,t_3) = \frac{1}{2c}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\xi \, d\xi = \frac{1}{2c}\Big[\sin\xi\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2c}[1+1] = \frac{1}{c}$$

- IN REGION D

$$z(x_4,t_4) = \frac{1}{2c}\int_{\xi=x_4-ct_4\,(S)}^{\frac{\pi}{2}} \cos\xi \, d\xi = \frac{1}{2c}\Big[\sin\xi\Big]_{x_4-ct_4}^{\frac{\pi}{2}} =$$

$$= \frac{1}{2c}\left[1 - \sin(x_4-ct_4)\right]$$



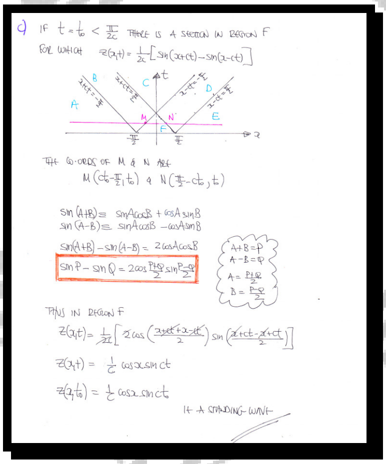
- IN REGION F

$$z(x_6,t_6) = \frac{1}{2c}\int_{\xi=x_6-ct_6}^{\xi=x_6+ct_6} \cos\xi \, d\xi = \frac{1}{2c}\Big[\sin\xi\Big]_{x_6-ct_6}^{x_6+ct_6}$$

$$= \frac{1}{2c}\left[\sin(x_6+ct_6) - \sin(x_6-ct_6)\right]$$

DROPPING THE SUBSCRIPTS AND COLLECTING THE RESULTS

$$z(x,t) = \begin{cases} 0 & \text{IN REGION A} \\ \frac{1}{2c}\left[\sin(x+ct)+1\right] & \text{IN REGION B} \\ \frac{1}{c} & \text{IN REGION C} \\ \frac{1}{2c}\left[1-\sin(x-ct)\right] & \text{IN REGION D} \\ 0 & \text{IN REGION E} \\ \frac{1}{2c}\left[\sin(x+ct)-\sin(x-ct)\right. & \text{IN REGION F} \end{cases}$$



c) IF $t = t_0 < \frac{\pi}{2c}$ THERE IS A SECTION IN REGION F FOR WHICH $z(x,t) = \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right]$

THE CO-ORDS OF M & N ARE

$$M\left(ct_0-\tfrac{\pi}{2}, t_0\right) \ \& \ N\left(\tfrac{\pi}{2}-ct_0, t_0\right)$$

$$\sin(A+B) \equiv \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) \equiv \sin A\cos B - \cos A\sin B$$

$$\sin(A+B) - \sin(A-B) = 2\cos A\cos B$$

$$\boxed{\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}}$$

$A+B=P$
$A-B=Q$
$A=\frac{P+Q}{2}$
$B=\frac{P-Q}{2}$

THUS IN REGION F

$$z(x,t) = \frac{1}{\cancel{2}c}\left[\cancel{2}\cos\left(\frac{x\cancel{+ct}+x\cancel{-ct}}{2}\right)\sin\left(\frac{\cancel{x}+ct-\cancel{x}+ct}{2}\right)\right]$$

$$z(x,t) = \frac{1}{c}\cos x\sin ct$$

$$z(x,t_0) = \frac{1}{c}\cos x\sin ct_0$$

IT A STANDING WAVE

Question 11

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \; c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

It is further given further that

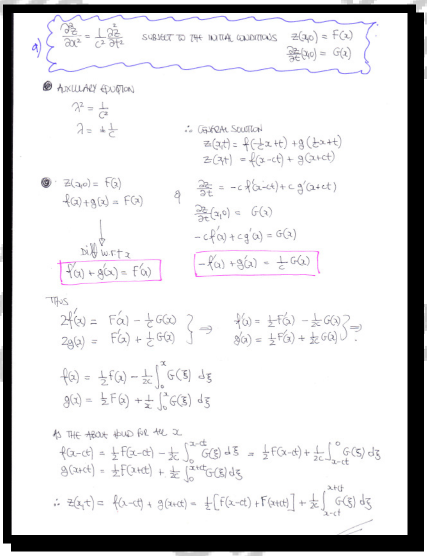
$$F(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad G(x) = 0.$$

b) Use the result of part **(a)** with the method of characteristics to determine expressions for

$$z(x,t), \quad \text{for } t < \frac{1}{c}, \quad t = \frac{1}{c} \quad \text{and } t > \frac{1}{c}$$

c) Sketch the wave profiles for $t = \frac{n}{2c}$, $n = 0, 1, 2, 3$.

solution overleaf



a) $\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$ SUBJECT TO THE INITIAL CONDITIONS $z(x,0) = F(x)$, $\frac{\partial z}{\partial t}(x,0) = G(x)$

- AUXILIARY EQUATION

$$\lambda^2 = \frac{1}{c^2}$$
$$\lambda = \pm\frac{1}{c}$$

∴ GENERAL SOLUTION

$$z(x,t) = f\left(-\tfrac{1}{c}x+t\right) + g\left(\tfrac{1}{c}x+t\right)$$
$$z(x,t) = f(x-ct) + g(x+ct)$$

- $z(x,0) = F(x)$

$$f(x)+g(x) = F(x)$$

$\downarrow$ Diff w.r.t x

$$\boxed{f'(x) + g'(x) = F'(x)}$$

& $\frac{\partial z}{\partial t} = -cf'(x-ct)+cg'(x+ct)$

$$\frac{\partial z}{\partial t}(x,0) = G(x)$$
$$-cf'(x)+cg'(x) = G(x)$$
$$\boxed{-f'(x)+g'(x) = \frac{1}{c}G(x)}$$

THUS

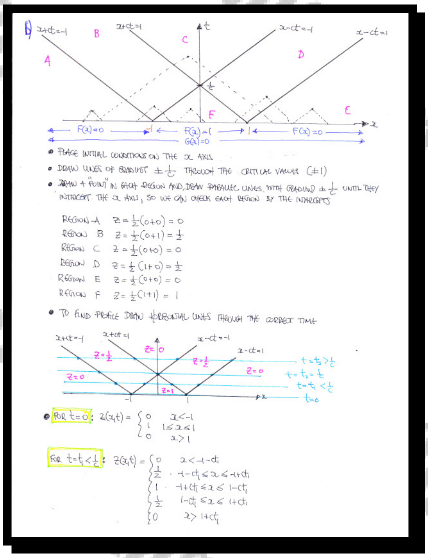
$$\left.\begin{aligned}2f'(x) &= F'(x) - \tfrac{1}{c}G(x)\\ 2g(x) &= F'(x) + \tfrac{1}{c}G(x)\end{aligned}\right\} \Rightarrow \left.\begin{aligned}f'(x) &= \tfrac{1}{2}F'(x) - \tfrac{1}{2c}G(x)\\ g'(x) &= \tfrac{1}{2}F'(x) + \tfrac{1}{2c}G(x)\end{aligned}\right\} \Rightarrow$$

$$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$$
$$g(x) = \frac{1}{2}F(x) + \frac{1}{x}\int_0^x G(\xi)\,d\xi$$

AS THE ABOVE HOLD FOR ALL x

$$f(x-ct) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi = \frac{1}{2}F(x-ct) + \frac{1}{2c}\int_{x-ct}^0 G(\xi)\,d\xi$$
$$g(x+ct) = \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$$

$$\therefore z(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$



b)

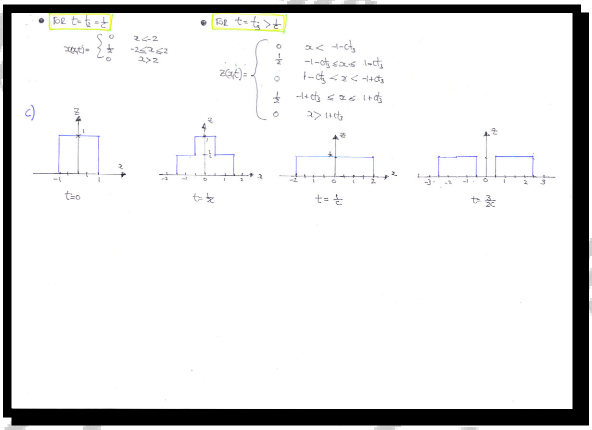
- PLACE INITIAL CONDITIONS ON THE x AXIS
- DRAW LINES OF GRADIENT $\pm\frac{1}{c}$ THROUGH THE CRITICAL VALUES (± 1)
- DRAW A "POINT" IN EACH REGION AND DRAW PARALLEL LINES WITH GRADIENT $\pm\frac{1}{c}$ UNTIL THEY INTERCEPT THE x AXIS, SO WE CAN CHECK EACH REGION BY THE INTERCEPTS

REGION A $z = \frac{1}{2}(0+0) = 0$
REGION B $z = \frac{1}{2}(0+1) = \frac{1}{2}$
REGION C $z = \frac{1}{2}(0+0) = 0$
REGION D $z = \frac{1}{2}(1+0) = \frac{1}{2}$
REGION E $z = \frac{1}{2}(0+0) = 0$
REGION F $z = \frac{1}{2}(1+1) = 1$

- TO FIND PROFILE DRAW HORIZONTAL LINES THROUGH THE CORRECT TIME

- $\boxed{\text{FOR } t=0}$: $z(x,t) = \begin{cases} 0 & x<-1 \\ 1 & 1\leq x\leq 1 \\ 0 & x>1 \end{cases}$

$\boxed{\text{FOR } t=t_1<\frac{1}{c}}$: $z(x,t) = \begin{cases} 0 & x<-1-ct_1 \\ \frac{1}{2} & -1-ct_1\leq x\leq -1+ct_1 \\ 1 & -1+ct_1\leq x\leq 1-ct_1 \\ \frac{1}{2} & 1-ct_1\leq x\leq 1+ct_1 \\ 0 & x>1+ct_1 \end{cases}$



- $\boxed{\text{FOR } t=t_2=\frac{1}{c}}$

$$x(x,t) = \begin{cases} 0 & x<-2 \\ \frac{1}{2} & -2\leq x\leq 2 \\ 0 & x>2 \end{cases}$$

- $\boxed{\text{FOR } t=t_3>\frac{1}{c}}$

$$z(x,t) = \begin{cases} 0 & x< -1-ct_3 \\ \frac{1}{2} & -1-ct_3\leq x\leq 1-ct_3 \\ 0 & 1-ct_3 < x < -1+ct_3 \\ \frac{1}{2} & -1+ct_3 \leq x\leq 1+ct_3 \\ 0 & x> 1+ct_3 \end{cases}$$

c)

Question 12

It is given that $u = u(x,t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

subject to the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

a) Derive D'Alembert's solution

$$u(x,t) = \frac{1}{2} f(x-t) + \frac{1}{2} f(x+t) + \frac{1}{2}\int_{x-t}^{x+t} G(\zeta)\, d\zeta,$$

for $-\infty < x < \infty$, $t \geq 0$.

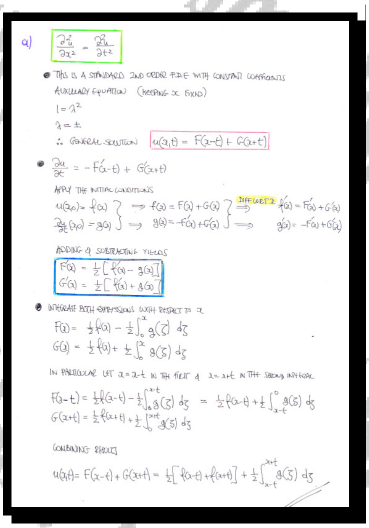
It is further given further that

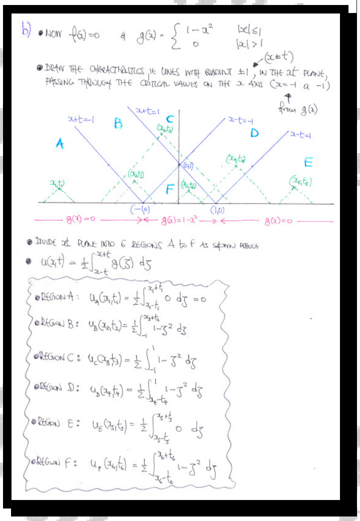
$$f(x) = 0 \quad \text{and} \quad g(x) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

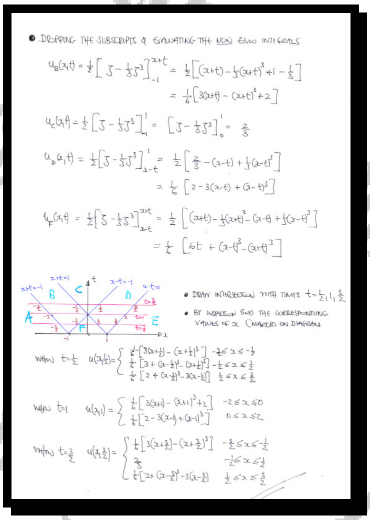
b) Use the result of part **(a)** with the method of characteristics to determine expressions for

$$u(x,t), \quad \text{for } t = \tfrac{1}{2},\ 1,\ \tfrac{3}{2}.$$

solution below







Question 13

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \; c > 0,$$

subject to the initial conditions

$$z(x,0) = F(x) \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = G(x).$$

a) Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\, d\xi,$$

for $-\infty < x < \infty$, $t \geq 0$.

It is further given further that

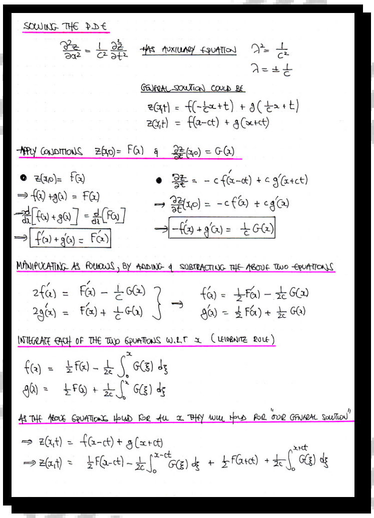
$$F(x) = 0 \quad \text{and} \quad G(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}.$$

b) Use the result of part **(a)** with the method of characteristics to determine expressions for

$$z(x,t), \quad \text{for } t = \frac{1}{2c}, \quad t = \frac{1}{c} \quad \text{and } t = \frac{3}{2c}$$

c) Sketch the wave profiles for $t = \frac{n}{2c}$, $n = 0, 1, 2, 3$.

$\boxed{}$, $\boxed{\text{solution overleaf}}$



SOLVING THE P.D.E

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$ HAS AUXILIARY EQUATION $\lambda^2 = \frac{1}{c^2}$

$\lambda = \pm \frac{1}{c}$

GENERAL SOLUTION COULD BE

$z(x,t) = f(-\frac{1}{c}x + t) + g(\frac{1}{c}x + t)$

$z(x,t) = f(x-ct) + g(x+ct)$

APPLY CONDITIONS $z(x,0) = F(x)$ & $\frac{\partial z}{\partial t}(x,0) = G(x)$

- $z(x,0) = F(x)$

$\Rightarrow f(x) + g(x) = F(x)$

$\Rightarrow \frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[F(x)\right]$

$\Rightarrow \boxed{f'(x) + g'(x) = F'(x)}$

- $\frac{\partial z}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

$\Rightarrow \frac{\partial z}{\partial t}(x,0) = -cf'(x) + cg'(x)$

$\Rightarrow \boxed{-f'(x) + g'(x) = \frac{1}{c}G(x)}$

MANIPULATING AS FOLLOWS, BY ADDING & SUBTRACTING THE ABOVE TWO EQUATIONS

$$\left.\begin{aligned} 2f'(x) &= F'(x) - \frac{1}{c}G(x) \\ 2g'(x) &= F'(x) + \frac{1}{c}G(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f'(x) &= \frac{1}{2}F'(x) - \frac{1}{2c}G(x) \\ g'(x) &= \frac{1}{2}F'(x) + \frac{1}{2c}G(x) \end{aligned}$$

INTEGRATE EACH OF THE TWO EQUATIONS W.R.T x (LEIBNITZ RULE)

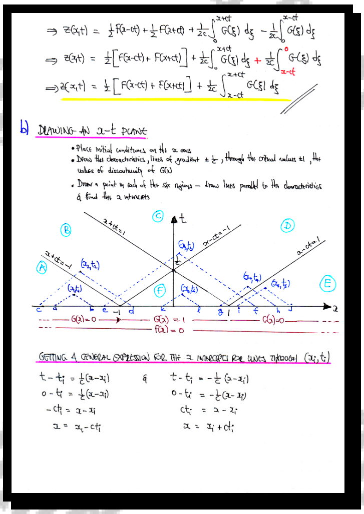
$f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

$g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_0^x G(\xi)\,d\xi$

AS THE ABOVE EQUATIONS HOLD FOR ALL x THEY WILL HOLD FOR "OUR GENERAL SOLUTION"

$\Rightarrow z(x,t) = f(x-ct) + g(x+ct)$

$\Rightarrow z(x,t) = \frac{1}{2}F(x-ct) - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi$



$\Rightarrow z(x,t) = \frac{1}{2}F(x-ct) + \frac{1}{2}F(x+ct) + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi - \frac{1}{2c}\int_0^{x-ct} G(\xi)\,d\xi$

$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_0^{x+ct} G(\xi)\,d\xi + \frac{1}{2c}\int_{x-ct}^{0} G(\xi)\,d\xi$

$\Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$

b) DRAWING AN x–t PLANE

- Place initial conditions on the x axis
- Draw the characteristics, lines of gradient $\pm\frac{1}{c}$, through the critical values ±1, the values of discontinuity of $G(x)$
- Draw a point in each of the six regions – draw lines parallel to the characteristics & find the x intercepts

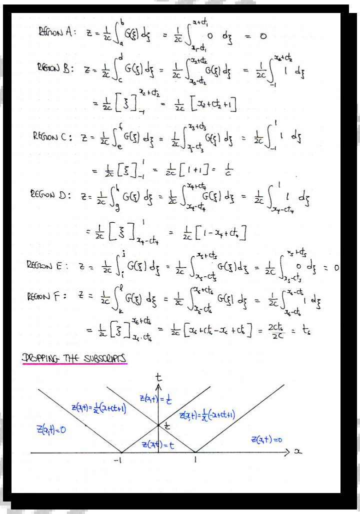
GETTING A GENERAL EXPRESSION FOR THE x INTERCEPTS FOR LINES THROUGH (x_i, t_i)

$t - t_i = \frac{1}{c}(x - x_i)$ & $t - t_i = -\frac{1}{c}(x - x_i)$

$0 - t_i = \frac{1}{c}(x - x_i)$ $\quad$ $0 - t_i = -\frac{1}{c}(x - x_i)$

$-ct_i = x - x_i$ $\quad$ $ct_i = x - x_i$

$x = x_i - ct_i$ $\quad$ $x = x_i + ct_i$



REGION A: $z = \frac{1}{2c}\int_a^b G(\xi)\,d\xi = \frac{1}{2c}\int_{x_1-ct_1}^{x+ct_1} 0\,d\xi = 0$

REGION B: $z = \frac{1}{2c}\int_c^d G(\xi)\,d\xi = \frac{1}{2c}\int_{x_2-ct_2}^{x_2+ct_2} G(\xi)\,d\xi = \frac{1}{2c}\int_{-1}^{x_2+ct_2} 1\,d\xi$

$= \frac{1}{2c}\left[\xi\right]_{-1}^{x_2+ct_2} = \frac{1}{2c}\left[x_2 + ct_2 + 1\right]$

REGION C: $z = \frac{1}{2c}\int_e^f G(\xi)\,d\xi = \frac{1}{2c}\int_{x_3-ct_3}^{x_3+ct_3} G(\xi)\,d\xi = \frac{1}{2c}\int_{-1}^{1} 1\,d\xi$

$= \frac{1}{2c}\left[\xi\right]_{-1}^{1} = \frac{1}{2c}\left[1+1\right] = \frac{1}{c}$

REGION D: $z = \frac{1}{2c}\int_g^h G(\xi)\,d\xi = \frac{1}{2c}\int_{x_4-ct_4}^{x_4+ct_4} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_4-ct_4}^{1} 1\,d\xi$

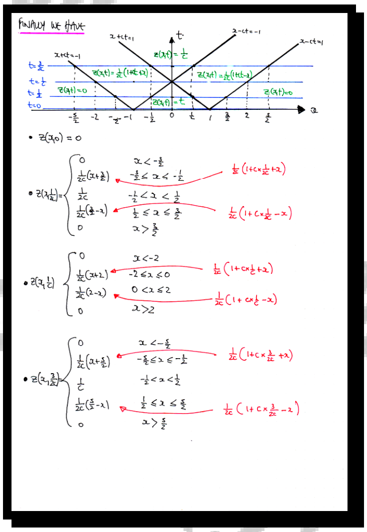
$= \frac{1}{2c}\left[\xi\right]_{x_4-ct_4}^{1} = \frac{1}{2c}\left[1 - x_4 + ct_4\right]$

REGION E: $z = \frac{1}{2c}\int_i^j G(\xi)\,d\xi = \frac{1}{2c}\int_{x_5-ct_5}^{x_5+ct_5} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_5-ct_5}^{x_5+ct_5} 0\,d\xi = 0$

REGION F: $z = \frac{1}{2c}\int_k^l G(\xi)\,d\xi = \frac{1}{2c}\int_{x_6-ct_6}^{x_6+ct_6} G(\xi)\,d\xi = \frac{1}{2c}\int_{x_6-ct_6}^{x_6-ct_6} 1\,d\xi$

$= \frac{1}{2c}\left[\xi\right]_{x_6-ct_6}^{x_6+ct_6} = \frac{1}{2c}\left[x_6 + ct_6 - x_6 + ct_6\right] = \frac{2ct_6}{2c} = t_6$

DROPPING THE SUBSCRIPTS



FINALLY WE HAVE

- $z(x,0) = 0$

- $z(x,\frac{1}{2c}) = \begin{cases} 0 & x < -\frac{3}{2} \\ \frac{1}{2c}(x+\frac{3}{2}) & -\frac{3}{2} \leqslant x \leqslant -\frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2c}(\frac{3}{2}-x) & \frac{1}{2} \leqslant x \leqslant \frac{3}{2} \\ 0 & x > \frac{3}{2} \end{cases}$

$\frac{1}{2c}(1 + c \times \frac{1}{2c} + x)$

$\frac{1}{2c}(1 + c \times \frac{1}{2c} - x)$

- $z(x,\frac{1}{c}) \begin{cases} 0 & x < -2 \\ \frac{1}{2c}(x+2) & -2 \leqslant x \leqslant 0 \\ \frac{1}{2c}(2-x) & 0 < x \leqslant 2 \\ 0 & x > 2 \end{cases}$

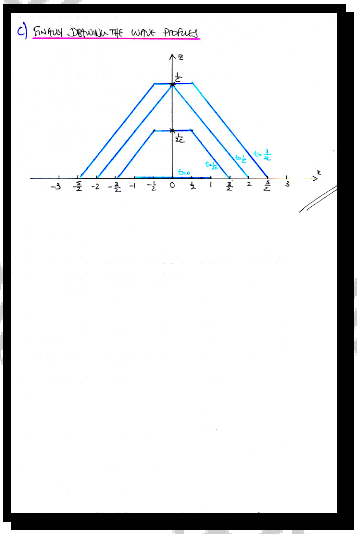
$\frac{1}{2c}(1 + c \times \frac{1}{c} + x)$

$\frac{1}{2c}(1 + c \times \frac{1}{c} - x)$

- $z(x,\frac{3}{2c}) = \begin{cases} 0 & x < -\frac{5}{2} \\ \frac{1}{2c}(x+\frac{5}{2}) & -\frac{5}{2} \leqslant x \leqslant -\frac{1}{2} \\ \frac{1}{c} & -\frac{1}{2} < x < \frac{1}{2} \\ \frac{1}{2c}(\frac{5}{2}-x) & \frac{1}{2} \leqslant x \leqslant \frac{5}{2} \\ 0 & x > \frac{5}{2} \end{cases}$

$\frac{1}{2c}(1 + c \times \frac{3}{2c} + x)$

$\frac{1}{2c}(1 + c \times \frac{3}{2c} - x)$



c) FINALLY DRAWING THE WAVE PROFILES

Question 14

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{4}\frac{\partial^2 z}{\partial t^2},$$

subject to the initial conditions

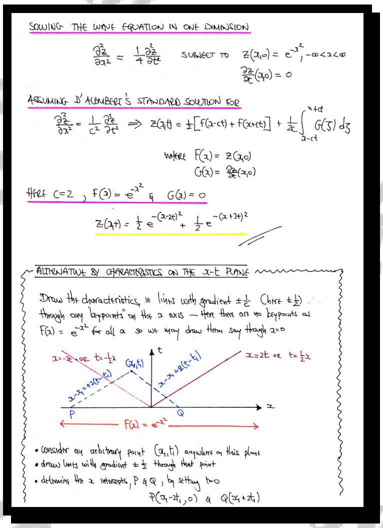
$$z(x,0) = \mathrm{e}^{-x^2}, \ -\infty < x < \infty \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = 0 .$$

a) Determine the solution of this wave equation.

b) Sketch the wave profiles for $t = i$, $i = 0, 1, 2, 3$.

You may use without proof the standard D'Alembert's solution for the wave equation.

$\boxed{}$, $\boxed{z(x,t) = \frac{1}{2}\mathrm{e}^{-(x-2t)^2} + \frac{1}{2}\mathrm{e}^{-(x+2t)^2}}$



SOLVING THE WAVE EQUATION IN ONE DIMENSION

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{4}\frac{\partial^2 z}{\partial t^2} \quad \text{SUBJECT TO} \quad z(x,0) = e^{-x^2}, -\infty<x<\infty$$
$$\frac{\partial z}{\partial t}(x,0) = 0$$

ASSUMING D'ALEMBERT'S STANDARD SOLUTION FOR

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \Rightarrow z(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

WHERE $F(x) = z(x,0)$
$G(x) = \frac{\partial z}{\partial t}(x,0)$

HERE $c=2$, $F(x) = e^{-x^2}$ & $G(x) = 0$

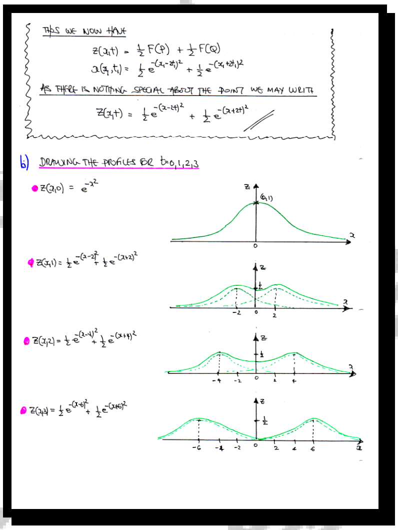
$$z(x,t) = \frac{1}{2}e^{-(x-2t)^2} + \frac{1}{2}e^{-(x+2t)^2}$$

ALTERNATIVE BY CHARACTERISTICS ON THE x-t PLANE

Draw the characteristics, ie lines with gradient $\pm\frac{1}{c}$ (here $\pm\frac{1}{2}$) through any "keypoints" on the x axis — here there are no keypoints as $F(x) = e^{-x^2}$ for all x so we may draw them say through $x=0$

- consider an arbitrary point (x_1,t_1) anywhere on this plane
- draw lines with gradient $\pm\frac{1}{2}$ through that point
- determine the x intercepts, P & Q, by setting $t=0$

$P(x_1-2t_1,0)$ & $Q(x_1+2t_1)$



THUS WE NOW HAVE

$$z(x,t) = \frac{1}{2}F(P) + \frac{1}{2}F(Q)$$
$$z(x_1,t_1) = \frac{1}{2}e^{-(x_1-2t_1)^2} + \frac{1}{2}e^{-(x_1+2t_1)^2}$$

AS THERE IS NOTHING SPECIAL ABOUT THE POINT WE MAY WRITE

$$z(x,t) = \frac{1}{2}e^{-(x-2t)^2} + \frac{1}{2}e^{-(x+2t)^2}$$

b) DRAWING THE PROFILES FOR $t=0,1,2,3$

- $z(x,0) = e^{-x^2}$
- $z(x,1) = \frac{1}{2}e^{-(x-2)^2} + \frac{1}{2}e^{-(x+2)^2}$
- $z(x,2) = \frac{1}{2}e^{-(x-4)^2} + \frac{1}{2}e^{-(x+4)^2}$
- $z(x,3) = \frac{1}{2}e^{-(x-6)^2} + \frac{1}{2}e^{-(x+6)^2}$

Question 15

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad c > 0,$$

subject to the initial conditions.

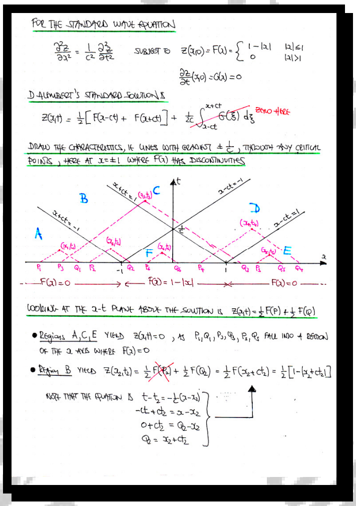
$$z(x,0) = \begin{cases} 1-|x| & |x| \le 1 \\ 0 & |x| > 1 \end{cases} \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = 0.$$

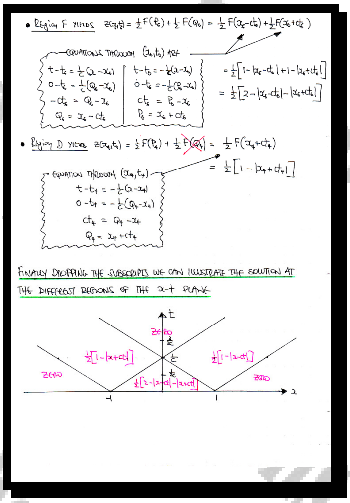
a) Display the values of $z(x,t)$ in the regions of an (x,t) plane diagram.

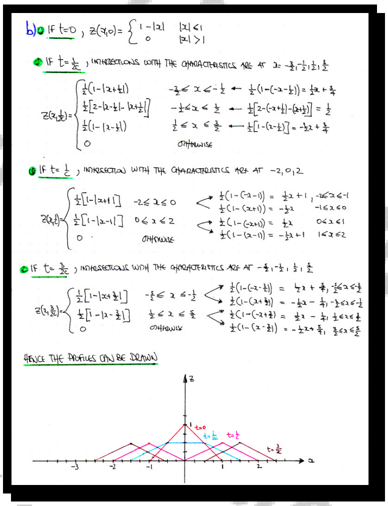
b) Sketch the wave profiles for $t = 0$, $t = \frac{1}{2c}$, $t = \frac{1}{c}$ and $t = \frac{3}{2c}$.

You may use without proof the standard D'Alembert's solution for the wave equation.

☐ , solution below







Question 16

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{4}\frac{\partial^2 z}{\partial t^2},$$

subject to the initial conditions.

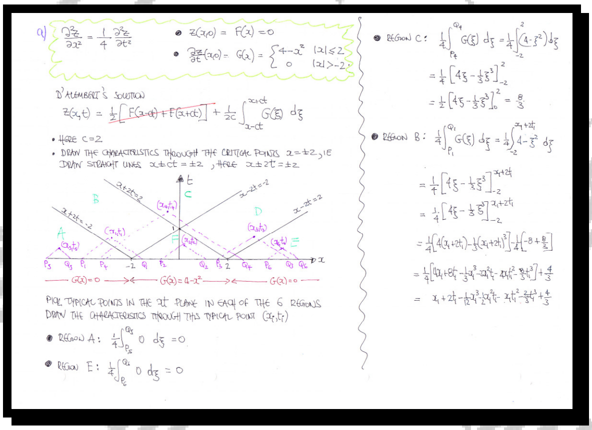
$$z(x,0) = 0 \quad \text{and} \quad \frac{\partial z}{\partial t}(x,0) = \begin{cases} 4 - x^2 & |x| \le 2 \\ 0 & |x| > 2 \end{cases}.$$

a) Display the values of $z(x,t)$ in the regions of an (x,t) plane diagram.

b) Determine expressions for $z\left(x,\frac{1}{2}\right)$ and $z\left(x,\frac{3}{2}\right)$.

You may use without proof the standard D'Alembert's solution for the wave equation.

solution below



a) $\frac{\partial^2 z}{\partial x^2} = \frac{1}{4}\frac{\partial^2 z}{\partial t^2}$

- $z(x,0) = F(x) = 0$
- $\frac{\partial z}{\partial t}(x,0) = G(x) = \begin{cases} 4-x^2 & |x| \le 2 \\ 0 & |x| > -2 \end{cases}$

D'ALEMBERT'S SOLUTION

$z(x,t) = \cancel{\frac{1}{2}\left[F(x-ct) + F(x+ct)\right]} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$

- HERE $c=2$
- DRAW THE CHARACTERISTICS THROUGH THE CRITICAL POINTS $x = \pm 2$, IE DRAW STRAIGHT LINES $x \pm ct = \pm 2$, HERE $x \pm 2t = \pm 2$

PICK TYPICAL POINTS IN THE xt PLANE IN EACH OF THE 6 REGIONS DRAW THE CHARACTERISTICS THROUGH THIS TYPICAL POINT (x_i,t_i)

- REGION A: $\frac{1}{4}\int_{P_5}^{Q_5} 0\,d\xi = 0$
- REGION E: $\frac{1}{4}\int_{P_6}^{Q_6} 0\,d\xi = 0$
- REGION C: $\frac{1}{4}\int_{P_4}^{Q_4} G(\xi)\,d\xi = \frac{1}{4}\int_{-2}^{2}(4-\xi^2)\,d\xi$

$= \frac{1}{4}\left[4\xi - \frac{1}{3}\xi^3\right]_{-2}^{2}$

$= \frac{1}{2}\left[4\xi - \frac{1}{3}\xi^3\right]_{0}^{2} = \frac{8}{3}$

- REGION B: $\frac{1}{4}\int_{P_1}^{Q_1} G(\xi)\,d\xi = \frac{1}{4}\int_{-2}^{x_1+2t_1} 4-\xi^2\,d\xi$

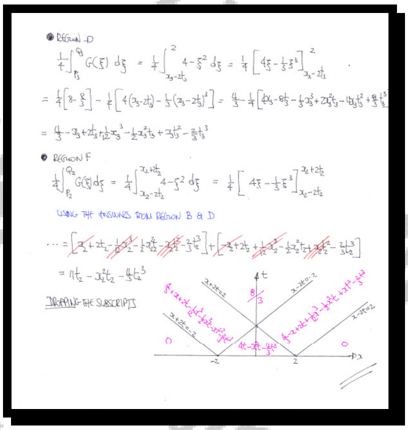
$= \frac{1}{4}\left[4\xi - \frac{1}{3}\xi^3\right]_{-2}^{x_1+2t_1}$

$= \frac{1}{4}\left[4\xi - \frac{1}{3}\xi^3\right]_{-2}^{x_1+2t_1}$

$= \frac{1}{4}\left[4(x_1+2t_1) - \frac{1}{3}(x_1+2t_1)^3\right] - \frac{1}{4}\left[-8+\frac{8}{3}\right]$

$= \frac{1}{4}\left[4x_1 + 8t_1 - \frac{1}{3}x_1^3 - 2x_1^2t_1 - 4x_1t_1^2 - \frac{8}{3}t_1^3\right] + \frac{4}{3}$

$= x_1 + 2t_1 - \frac{1}{12}x_1^3 - \frac{1}{2}x_1^2t_1 - x_1t_1^2 - \frac{2}{3}t_1^3 + \frac{4}{3}$



- REGION D

$\frac{1}{4}\int_{P_3}^{Q_3} G(\xi)\,d\xi = \frac{1}{4}\int_{x_3-2t_3}^{2} 4-\xi^2\,d\xi = \frac{1}{4}\left[4\xi - \frac{1}{3}\xi^3\right]_{x_3-2t_3}^{2}$

$= \frac{1}{4}\left[8-\frac{8}{3}\right] - \frac{1}{4}\left[4(x_3-2t_3) - \frac{1}{3}(x_3-2t_3)^3\right] = \frac{4}{3} - \frac{1}{4}\left[4x_3 - 8t_3 - \frac{1}{3}x_3^3 + 2x_3^2t_3 - 4x_3t_3^2 + \frac{8}{3}t_3^3\right]$

$= \frac{4}{3} - x_3 + 2t_3 + \frac{1}{12}x_3^3 - \frac{1}{2}x_3^2t_3 + x_3t_3^2 - \frac{2}{3}t_3^3$

- REGION F

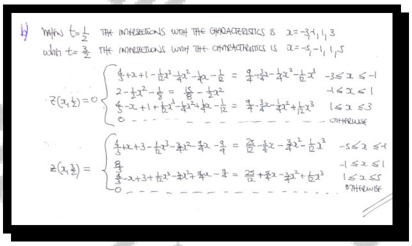
$\frac{1}{4}\int_{P_2}^{Q_2} G(\xi)\,d\xi = \frac{1}{4}\int_{x_2-2t_2}^{x_2+2t_2} 4-\xi^2\,d\xi = \frac{1}{4}\left[4\xi - \frac{1}{3}\xi^3\right]_{x_2-2t_2}^{x_2+2t_2}$

USING THE ANSWERS FROM REGION B & D

$\dots = \left[\cancel{x_2} + 2t_2 - \cancel{\frac{1}{12}x_2^3} - \frac{1}{2}x_2^2t_2 - \cancel{x_2t_2^2} - \frac{2}{3}t_2^3\right] + \left[\cancel{-x_2} + 2t_2 + \cancel{\frac{1}{12}x_2^3} - \frac{1}{2}x_2^2t_2 + \cancel{x_2t_2^2} - \frac{2}{3}t_2^3\right]$

$= 4t_2 - x_2^2t_2 - \frac{4}{3}t_2^3$

DROPPING THE SUBSCRIPTS

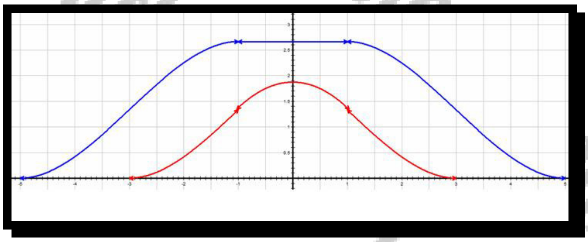


b) WHEN $t=\frac{1}{2}$ THE INTERSECTIONS WITH THE CHARACTERISTICS IS $x = -3, -1, 1, 3$

WHEN $t = \frac{3}{2}$ THE INTERSECTIONS WITH THE CHARACTERISTICS IS $x = -5, -1, 1, 5$

$$z(x,\tfrac{1}{2}) = 0\begin{cases} \frac{4}{3}+x+1-\frac{1}{12}x^3-\frac{1}{4}x^2-\frac{1}{4}x-\frac{1}{12} = \frac{9}{4}+\frac{3}{4}x-\frac{1}{4}x^2-\frac{1}{12}x^3 & -3 \le x \le -1 \\ 2-\frac{1}{2}x^2-\frac{1}{8} = \frac{15}{8}-\frac{1}{2}x^2 & -1 \le x \le 1 \\ \frac{4}{3}-x+1+\frac{1}{12}x^3-\frac{1}{4}x^2+\frac{1}{4}x-\frac{1}{12} = \frac{9}{4}-\frac{3}{4}x-\frac{1}{4}x^2+\frac{1}{12}x^3 & 1 \le x \le 3 \\ 0 & \text{OTHERWISE} \end{cases}$$

$$z(x,\tfrac{3}{2}) = \begin{cases} \frac{4}{3}+x+3-\frac{1}{12}x^3-\frac{3}{4}x^2-\frac{9}{4}x-\frac{9}{4} = \frac{25}{12}-\frac{5}{4}x-\frac{3}{4}x^2-\frac{1}{12}x^3 & -5 \le x \le -1 \\ \frac{8}{3} & -1 \le x \le 1 \\ \frac{4}{3}-x+3+\frac{1}{12}x^3-\frac{3}{4}x^2+\frac{9}{4}x-\frac{9}{4} = \frac{25}{12}+\frac{5}{4}x-\frac{3}{4}x^2+\frac{1}{12}x^3 & 1 \le x \le 5 \\ 0 & \text{OTHERWISE} \end{cases}$$



WAVE EQUATION

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z = z\left(x,t\right)$$

Standing Waves

Question 1

It is given that $z = z(x,t)$ satisfies the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \ c > 0,$$

subject to the conditions

$$z(x,0) = F(x) \ , \ \frac{\partial z}{\partial t}(x,0) = G(x) \ \text{and} \ \ z(0,t) = z(L,t) = 0 .$$

Derive the solution

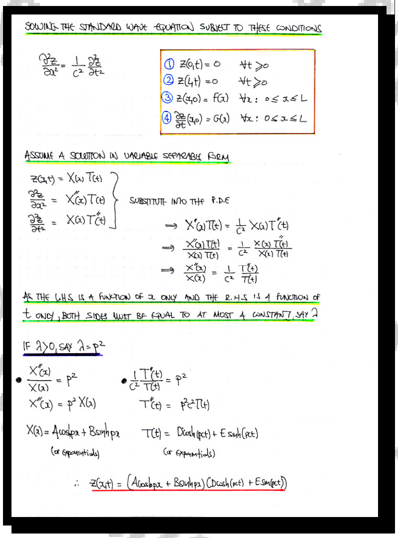
$$z(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[P_n \cos\left(\frac{n\pi ct}{L}\right) + Q_n \sin\left(\frac{n\pi ct}{L}\right)\right],$$

where

$$P_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right) \ dx \quad \text{and} \quad Q_n = \frac{2}{n\pi c}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right) \ dx$$

$\boxed{\text{[illegible]}}$, $\boxed{\text{proof}}$

[solution overleaf]



SOLVING THE STANDARD WAVE EQUATION SUBJECT TO THESE CONDITIONS

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

① $z(0,t) = 0 \quad \forall t \geqslant 0$
② $z(L,t) = 0 \quad \forall t \geqslant 0$
③ $z(x,0) = F(x) \quad \forall x : 0 \leqslant x \leqslant L$
④ $\frac{\partial z}{\partial t}(x,0) = G(x) \quad \forall x : 0 \leqslant x \leqslant L$

ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\left.\begin{aligned} z(x,t) &= X(x)T(t) \\ \frac{\partial^2 z}{\partial x^2} &= X''(x)T(t) \\ \frac{\partial^2 z}{\partial t^2} &= X(x)T''(t) \end{aligned}\right\} \text{SUBSTITUTE INTO THE P.D.E}$$

$$\Rightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

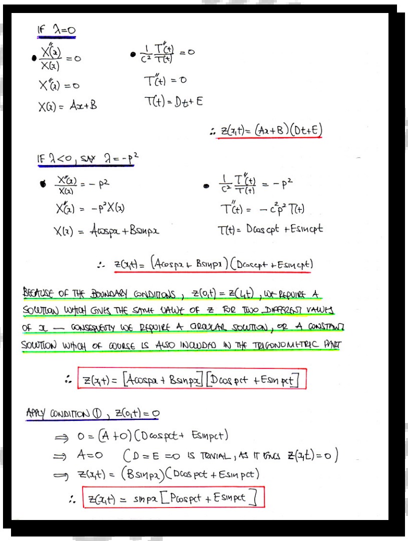
$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES MUST BE EQUAL TO AT MOST A CONSTANT, SAY λ

IF $\lambda > 0$, SAY $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$
  $X''(x) = p^2 X(x)$
  $X(x) = A\cosh px + B\sinh px$
  (or exponentials)
- $\frac{1}{c^2}\frac{T''(t)}{T(t)} = p^2$
  $T''(t) = p^2c^2T(t)$
  $T(t) = D\cosh(pct) + E\sinh(pct)$
  (or exponentials)

$$\therefore \ z(x,t) = (A\cosh px + B\sinh px)(D\cosh(pct) + E\sin(pct))$$



IF $\lambda = 0$

- $\frac{X''(x)}{X(x)} = 0$
  $X''(x) = 0$
  $X(x) = Ax + B$
- $\frac{1}{c^2}\frac{T''(t)}{T(t)} = 0$
  $T''(t) = 0$
  $T(t) = Dt + E$

$$\therefore \ z(x,t) = (Ax+B)(Dt+E)$$

IF $\lambda < 0$, SAY $\lambda = -p^2$

- $\frac{X''(x)}{X(x)} = -p^2$
  $X''(x) = -p^2X(x)$
  $X(x) = A\cos px + B\sin px$
- $\frac{1}{c^2}\frac{T''(t)}{T(t)} = -p^2$
  $T''(t) = -c^2p^2T(t)$
  $T(t) = D\cos cpt + E\sin cpt$

$$\therefore \ z(x,t) = (A\cos px + B\sin px)(D\cos cpt + E\sin cpt)$$

BECAUSE OF THE BOUNDARY CONDITIONS, $z(0,t) = z(L,t)$, WE REQUIRE A SOLUTION WHICH GIVES THE SAME VALUE OF z FOR TWO DIFFERENT VALUES OF x — CONSEQUENTLY WE REQUIRE A CIRCULAR SOLUTION, OR A CONSTANT SOLUTION WHICH OF COURSE IS ALSO INCLUDED IN THE TRIGONOMETRIC PART

$$\therefore \ \boxed{z(x,t) = [A\cos px + B\sin px][D\cos pct + E\sin pct]}$$

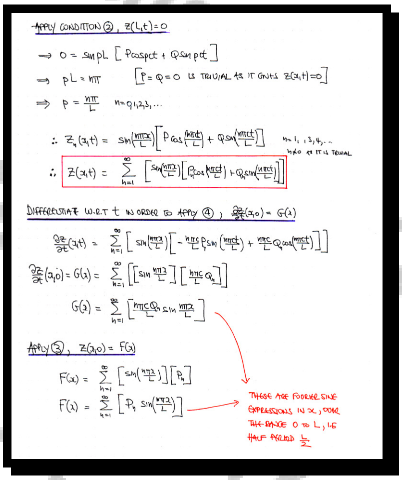
APPLY CONDITION ①, $z(0,t) = 0$

$$\Rightarrow 0 = (A+0)(D\cos pct + E\sin pct)$$

$$\Rightarrow A = 0 \quad (D = E = 0 \text{ IS TRIVIAL, AS IT GIVES } z(x,t) = 0)$$

$$\Rightarrow z(x,t) = (B\sin px)(D\cos pct + E\sin pct)$$

$$\therefore \ \boxed{z(x,t) = \sin px\,[P\cos pct + E\sin pct]}$$



APPLY CONDITION ②, $z(L,t) = 0$

$$\Rightarrow 0 = \sin pL\,[P\cos pct + Q\sin pct]$$

$$\Rightarrow pL = n\pi \qquad [P = Q = 0 \text{ IS TRIVIAL AS IT GIVES } z(x,t) = 0]$$

$$\Rightarrow p = \frac{n\pi}{L} \qquad n = 0,1,2,3,\ldots$$

$$\therefore \ z_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[P\cos\left(\frac{n\pi ct}{L}\right) + Q\sin\left(\frac{n\pi ct}{L}\right)\right] \qquad n = 1,\ ,3,4,\ldots$$

$n \neq 0$ AS IT IS TRIVIAL

$$\therefore \ \boxed{z(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\left[P_n\cos\left(\frac{n\pi ct}{L}\right) + Q_n\sin\left(\frac{n\pi ct}{L}\right)\right]\right]}$$

DIFFERENTIATE W.R.T t IN ORDER TO APPLY ④, $\frac{\partial z}{\partial t}(x,0) = G(x)$

$$\frac{\partial z}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\left[-\frac{n\pi c}{L}P_n\sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L}Q_n\cos\left(\frac{n\pi ct}{L}\right)\right]\right]$$

$$\frac{\partial z}{\partial t}(x,0) = G(x) = \sum_{n=1}^{\infty}\left[\left[\sin\frac{n\pi x}{L}\right]\left[\frac{n\pi c}{L}Q_n\right]\right]$$

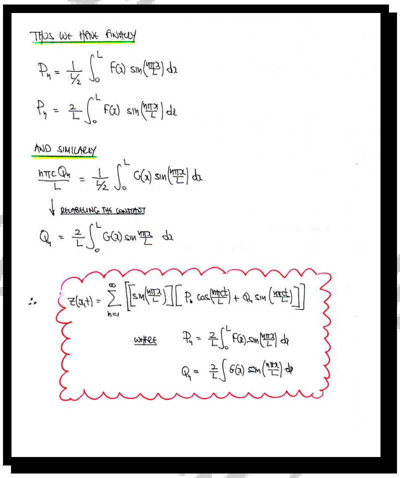
$$G(x) = \sum_{n=1}^{\infty}\left[\frac{n\pi cQ_n}{L}\sin\frac{n\pi x}{L}\right]$$

APPLY ③, $z(x,0) = F(x)$

$$F(x) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi x}{L}\right)\right][P_n]$$

$$F(x) = \sum_{n=1}^{\infty}\left[P_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

THESE ARE FOURIER SINE EXPRESSIONS IN x, OVER THE RANGE 0 TO L, I.E HALF PERIOD $\frac{L}{2}$



THUS WE HAVE FINALLY

$$P_n = \frac{1}{L/2}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$P_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

AND SIMILARLY

$$\frac{n\pi cQ_n}{L} = \frac{1}{L/2}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

↓ RELABELLING THE CONSTANT

$$Q_n = \frac{2}{L}\int_0^L G(x)\sin\frac{n\pi x}{L}\,dx$$

$$\therefore \ z(x,t) = \sum_{n=1}^{\infty}\left[\left[\sin\left(\frac{n\pi x}{L}\right)\right]\left[P_n\cos\left(\frac{n\pi ct}{L}\right) + Q_n\sin\left(\frac{n\pi ct}{L}\right)\right]\right]$$

WHERE $P_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$Q_n = \frac{2}{L}\int G(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Question 2

A taut string of length 20 units is fixed at its endpoints at $x = 0$ and at $x = 20$, and rests in a horizontal position along the x axis. The midpoint of the string is pulled by a distance of 1 unit and released from rest.

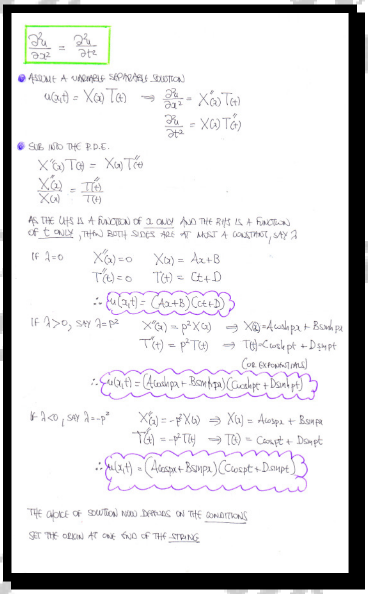
If the vertical displacement of the string u satisfies the standard wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

show that

$$u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left[\frac{n\pi x}{20}\right]\cos\left[\frac{n\pi t}{20}\right]\right],$$

$\boxed{\text{proof}}$



$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}}$$

- ASSUME A VARIABLE SEPARABLE SOLUTION

$$u(x,t) = X(x)T(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$
$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

- SUB INTO THE P.D.E.

$$X''(x)T(t) = X(x)T''(t)$$
$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$$

AS THE LHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda = 0$ $\quad X''(x) = 0 \quad X(x) = Ax + B$
$T''(t) = 0 \quad T(t) = Ct + D$

$$\therefore u(x,t) = (Ax+B)(Ct+D)$$

IF $\lambda > 0$, SAY $\lambda = p^2$ $\quad X''(x) = p^2X(x) \Rightarrow X(x) = A\cosh px + B\sinh px$
$T''(t) = p^2T(t) \Rightarrow T(t) = C\cosh pt + D\sinh pt$
(OR EXPONENTIALS)

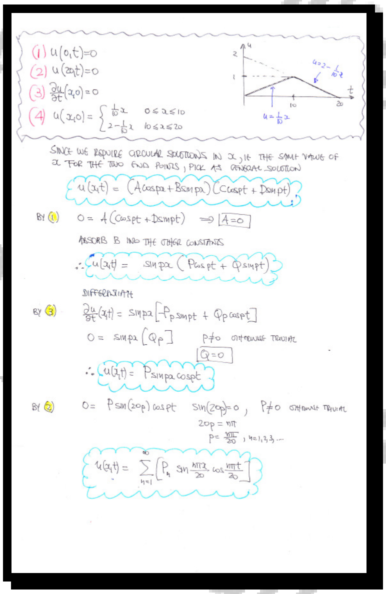
$$\therefore u(x,t) = (A\cosh px + B\sinh px)(C\cosh pt + D\sinh pt)$$

IF $\lambda < 0$, SAY $\lambda = -p^2$ $\quad X''(x) = -p^2X(x) \Rightarrow X(x) = A\cos px + B\sin px$
$T''(t) = -p^2T(t) \Rightarrow T(t) = C\cos pt + D\sin pt$

$$\therefore u(x,t) = (A\cos px + B\sin px)(C\cos pt + D\sin pt)$$

THE CHOICE OF SOLUTION NOW DEPENDS ON THE CONDITIONS

SET THE ORIGIN AT ONE END OF THE STRING



(1) $u(0,t) = 0$
(2) $u(20,t) = 0$
(3) $\frac{\partial u}{\partial t}(x,0) = 0$
(4) $u(x,0) = \begin{cases} \frac{1}{10}x & 0 \leqslant x \leqslant 10 \\ 2 - \frac{1}{10}x & 10 \leqslant x \leqslant 20 \end{cases}$

SINCE WE REQUIRE CIRCULAR SOLUTIONS IN x, IE THE SAME VALUE OF x FOR THE TWO END POINTS, PICK AS GENERAL SOLUTION

$$u(x,t) = (A\cos px + B\sin px)(C\cos pt + D\sin pt)$$

BY (1) $\quad 0 = A(C\cos pt + D\sin pt) \Rightarrow \boxed{A = 0}$

ABSORB B INTO THE OTHER CONSTANTS

$$\therefore u(x,t) = \sin px\,(P\cos pt + Q\sin pt)$$

DIFFERENTIATE

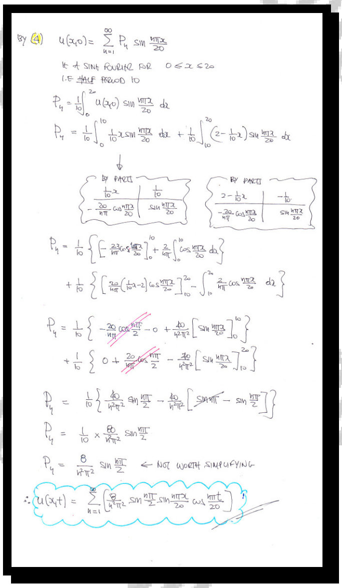
BY (3) $\quad \frac{\partial u}{\partial t}(x,t) = \sin px\left[-Pp\sin pt + Qp\cos pt\right]$

$0 = \sin px\left[Qp\right] \quad p \neq 0$ OTHERWISE TRIVIAL
$\boxed{Q = 0}$

$$\therefore u(x,t) = P\sin px\cos pt$$

BY (2) $\quad 0 = P\sin(20p)\cos pt \quad \sin(20p) = 0, \quad P \neq 0$ OTHERWISE TRIVIAL
$20p = n\pi$
$p = \frac{n\pi}{20}, \; n = 1, 2, 3, \ldots$

$$u(x,t) = \sum_{n=1}^{\infty}\left[P_n \sin\frac{n\pi x}{20}\cos\frac{n\pi t}{20}\right]$$



BY (4) $\quad u(x,0) = \sum_{n=1}^{\infty} P_n \sin\frac{n\pi x}{20}$

IE A SINE FOURIER FOR $0 \leqslant x \leqslant 20$
I.E HALF PERIOD 10

$$P_n = \frac{1}{10}\int_0^{20} u(x,0)\sin\frac{n\pi x}{20}\,dx$$

$$P_n = \frac{1}{10}\int_0^{10}\frac{1}{10}x\sin\frac{n\pi x}{20}\,dx + \frac{1}{10}\int_{10}^{20}\left(2 - \frac{1}{10}x\right)\sin\frac{n\pi x}{20}\,dx$$

BY PARTS

$$\begin{array}{c|c} \frac{1}{10}x & \frac{1}{10} \\ \hline -\frac{20}{n\pi}\cos\frac{n\pi x}{20} & \sin\frac{n\pi x}{20} \end{array}$$

BY PARTS

$$\begin{array}{c|c} 2 - \frac{1}{10}x & -\frac{1}{10} \\ \hline -\frac{20}{n\pi}\cos\frac{n\pi x}{20} & \sin\frac{n\pi x}{20} \end{array}$$

$$P_n = \frac{1}{10}\left\{\left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{20}\right]_0^{10} + \frac{2}{n\pi}\int_0^{10}\cos\frac{n\pi x}{20}\,dx\right\}$$
$$+ \frac{1}{10}\left\{\left[\frac{20}{n\pi}\left(\frac{1}{10}x - 2\right)\cos\frac{n\pi x}{20}\right]_{10}^{20} - \int_{10}^{20}\frac{2}{n\pi}\cos\frac{n\pi x}{20}\,dx\right\}$$

$$P_n = \frac{1}{10}\left\{\cancel{-\frac{20}{n\pi}\cos\frac{n\pi}{2}} - 0 + \frac{40}{n^2\pi^2}\left[\sin\frac{n\pi x}{20}\right]_0^{10}\right\}$$
$$+ \frac{1}{10}\left\{0 + \cancel{\frac{20}{n\pi}\cos\frac{n\pi}{2}} - \frac{40}{n^2\pi^2}\left[\sin\frac{n\pi x}{20}\right]_{10}^{20}\right\}$$

$$P_n = \frac{1}{10}\left\{\frac{40}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{40}{n^2\pi^2}\left[\cancel{\sin n\pi} - \sin\frac{n\pi}{2}\right]\right\}$$

$$P_n = \frac{1}{10} \times \frac{80}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$P_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \leftarrow \text{NOT WORTH SIMPLIFYING}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty}\left[\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{20}\cos\frac{n\pi t}{20}\right]$$

Question 3

Solve the wave equation

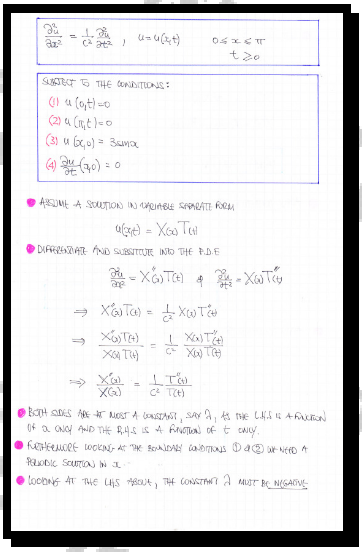
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad c > 0,$$

for $u = u(x,t)$, $0 \le x \le \pi$, $t \ge 0$,

subject to the following boundary and initial conditions.

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad u(x,0) = 3\sin x.$$

$$\boxed{u(x,t) = 3\sin x \cos ct}$$



$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad u = u(x,t) \qquad \begin{array}{l} 0 \le x \le \pi \\ t \ge 0 \end{array}}$$

SUBJECT TO THE CONDITIONS:

(1) $u(0,t) = 0$

(2) $u(\pi,t) = 0$

(3) $u(x,0) = 3\sin x$

(4) $\frac{\partial u}{\partial t}(x,0) = 0$

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$u(x,t) = X(x)T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

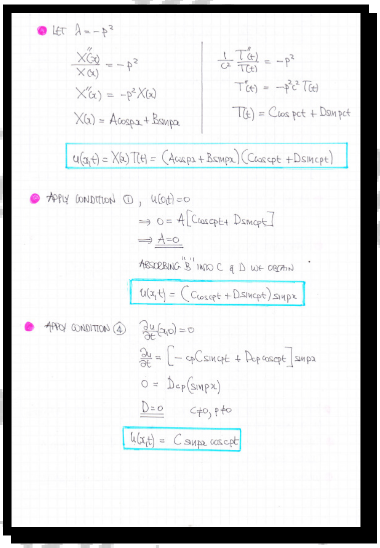
$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\Rightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

- BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY.
- FURTHERMORE LOOKING AT THE BOUNDARY CONDITIONS ① & ② WE NEED A PERIODIC SOLUTION IN x
- LOOKING AT THE LHS ABOUT, THE CONSTANT λ MUST BE NEGATIVE



- LET $\lambda = -p^2$

$$\frac{X''(x)}{X(x)} = -p^2 \qquad \Big| \qquad \frac{1}{c^2}\frac{T''(t)}{T(t)} = -p^2$$

$$X''(x) = -p^2X(x) \qquad \Big| \qquad T''(t) = -p^2c^2T(t)$$

$$X(x) = A\cos px + B\sin px \qquad \Big| \qquad T(t) = C\cos pct + D\sin pct$$

$$\boxed{u(x,t) = X(x)T(t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)}$$

- APPLY CONDITION ①, $u(0,t) = 0$

$$\Rightarrow 0 = A[C\cos cpt + D\sin cpt]$$

$$\Rightarrow \underline{\underline{A = 0}}$$

ABSORBING "B" INTO C & D WE OBTAIN

$$\boxed{u(x,t) = (C\cos cpt + D\sin cpt)\sin px}$$

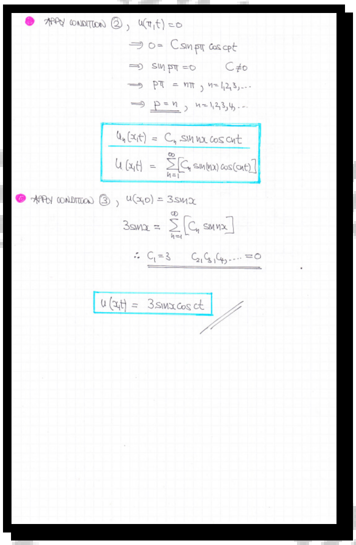
- APPLY CONDITION ④ $\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{\partial u}{\partial t} = [-cpC\sin cpt + Dcp\cos cpt]\sin px$$

$$0 = Dcp(\sin px)$$

$$\underline{\underline{D = 0}} \qquad c \ne 0, \, p \ne 0$$

$$\boxed{u(x,t) = C\sin px\cos cpt}$$



- APPLY CONDITION ②, $u(\pi,t) = 0$

$$\Rightarrow 0 = C\sin p\pi\cos cpt$$

$$\Rightarrow \sin p\pi = 0 \qquad C \ne 0$$

$$\Rightarrow p\pi = n\pi, \; n = 1,2,3,\ldots$$

$$\Rightarrow \underline{\underline{p = n}}, \; n = 1,2,3,4,\ldots$$

$$\boxed{\begin{array}{l} u_n(x,t) = C_n\sin nx\cos cnt \\ \hline u(x,t) = \sum_{n=1}^{\infty}[C_n\sin(nx)\cos(cnt)] \end{array}}$$

- APPLY CONDITION ③, $u(x,0) = 3\sin x$

$$3\sin x = \sum_{n=1}^{\infty}[C_n\sin nx]$$

$$\therefore \underline{\underline{C_1 = 3 \qquad C_2, C_3, C_4, \ldots = 0}}$$

$$\boxed{u(x,t) = 3\sin x\cos ct} \; //$$

Question 4

Solve the wave equation

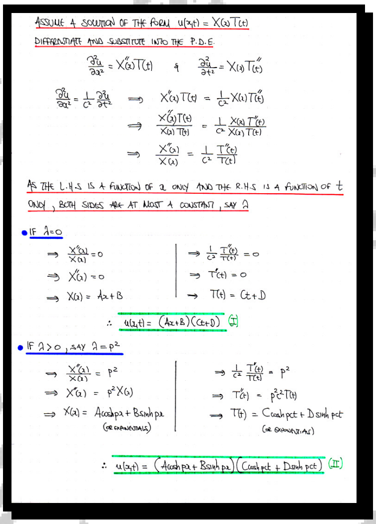
$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},\ c>0,$$

for $u=u(x,t)$, $0\le x\le 1$, $t\ge 0$,

subject to the following boundary and initial conditions.

$$u(0,t)=0,\quad u(1,t)=0,\quad \frac{\partial u}{\partial t}(x,0)=0,\quad u(x,0)=\sin(5\pi x)+2\sin(7\pi x).$$

$\boxed{}$, $\boxed{u(x,t)=\sin(5\pi x)\cos(5\pi ct)+2\sin(7\pi x)\cos(7\pi ct)}$



ASSUME A SOLUTION OF THE FORM $u(x,t)=X(x)T(t)$

DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$\frac{\partial^2 u}{\partial x^2}=X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2}=X(x)T''(t)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \implies X''(x)T(t)=\frac{1}{c^2}X(x)T''(t)$$

$$\implies \frac{X''(x)T(t)}{X(x)T(t)}=\frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\implies \frac{X''(x)}{X(x)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

- IF $\lambda=0$

$\implies \frac{X''(x)}{X(x)}=0$ | $\implies \frac{1}{c^2}\frac{T''(t)}{T(t)}=0$

$\implies X''(x)=0$ | $\implies T'(t)=0$

$\implies X(x)=Ax+B$ | $\implies T(t)=Ct+D$

$$\therefore\ u(x,t)=(Ax+B)(Ct+D) \quad \text{(I)}$$

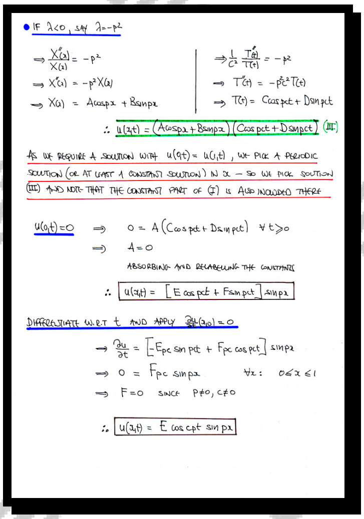
- IF $\lambda>0$, SAY $\lambda=p^2$

$\implies \frac{X''(x)}{X(x)}=p^2$ | $\implies \frac{1}{c^2}\frac{T''(t)}{T(t)}=p^2$

$\implies X''(x)=p^2X(x)$ | $\implies T''(t)=p^2c^2T(t)$

$\implies X(x)=A\cosh px+B\sinh px$ (OR EXPONENTIALS) | $\implies T(t)=C\cosh pct+D\sinh pct$ (OR EXPONENTIALS)

$$\therefore\ u(x,t)=(A\cosh px+B\sinh px)(C\cosh pct+D\sinh pct) \quad \text{(II)}$$



- IF $\lambda<0$, SAY $\lambda=-p^2$

$\implies \frac{X''(x)}{X(x)}=-p^2$ | $\implies \frac{1}{c^2}\frac{T'(t)}{T(t)}=-p^2$

$\implies X''(x)=-p^2X(x)$ | $\implies T''(t)=-p^2c^2T(t)$

$\implies X(x)=A\cos px+B\sin px$ | $\implies T(t)=C\cos pct+D\sin pct$

$$\therefore\ u(x,t)=(A\cos px+B\sin px)(C\cos pct+D\sin pct) \quad \text{(III)}$$

AS WE REQUIRE A SOLUTION WITH $u(0,t)=u(1,t)$, WE PICK A PERIODIC SOLUTION (OR AT LEAST A CONSTANT SOLUTION) IN x – SO WE PICK SOLUTION (III) AND NOTE THAT THE CONSTANT PART OF (I) IS ALSO INCLUDED THERE

$u(0,t)=0 \implies 0=A(C\cos pct+D\sin pct)\ \forall t\geqslant 0$

$\implies A=0$

ABSORBING AND RELABELLING THE CONSTANTS

$$\therefore\ \boxed{u(x,t)=\left[E\cos pct+F\sin pct\right]\sin px}$$

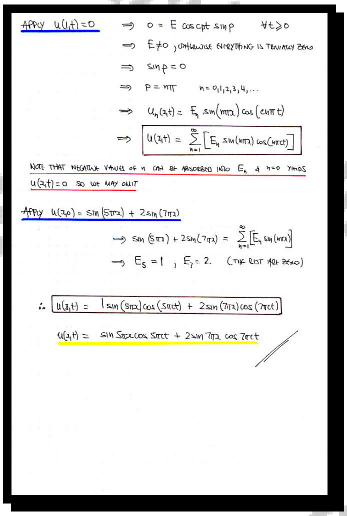
DIFFERENTIATE W.R.T t AND APPLY $\frac{\partial u}{\partial t}(x,0)=0$

$$\implies \frac{\partial u}{\partial t}=\left[-Epc\sin pct+Fpc\cos pct\right]\sin px$$

$$\implies 0=Fpc\sin px \qquad \forall x:\ 0\leqslant x\leqslant 1$$

$$\implies F=0 \quad \text{SINCE } p\neq 0,\ c\neq 0$$

$$\therefore\ \boxed{u(x,t)=E\cos cpt\sin px}$$



APPLY $u(1,t)=0 \implies 0=E\cos cpt\sin p \quad \forall t\geqslant 0$

$\implies E\neq 0$, OTHERWISE EVERYTHING IS TRIVIALLY ZERO

$\implies \sin p=0$

$\implies p=n\pi \qquad n=0,1,2,3,4,\dots$

$\implies u_n(x,t)=E_n\sin(n\pi x)\cos(cn\pi t)$

$$\implies \boxed{u(x,t)=\sum_{n=1}^{\infty}\left[E_n\sin(n\pi x)\cos(n\pi ct)\right]}$$

NOTE THAT NEGATIVE VALUES OF n CAN BE ABSORBED INTO E_n & $n=0$ YIELDS $u(x,t)=0$ SO WE MAY OMIT

APPLY $u(x,0)=\sin(5\pi x)+2\sin(7\pi x)$

$$\implies \sin(5\pi x)+2\sin(7\pi x)=\sum_{n=1}^{\infty}\left[E_n\sin(n\pi x)\right]$$

$$\implies E_5=1,\ E_7=2 \quad \text{(THE REST ARE ZERO)}$$

$$\therefore\ \boxed{u(x,t)=1\sin(5\pi x)\cos(5\pi ct)+2\sin(7\pi x)\cos(7\pi ct)}$$

$$u(x,t)=\sin 5\pi x\cos 5\pi ct+2\sin 7\pi x\cos 7\pi ct$$

Question 5

Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

for $u = u(x,t)$, $0 \le x \le 1$, $t \ge 0$,

subject to the following boundary and initial conditions.

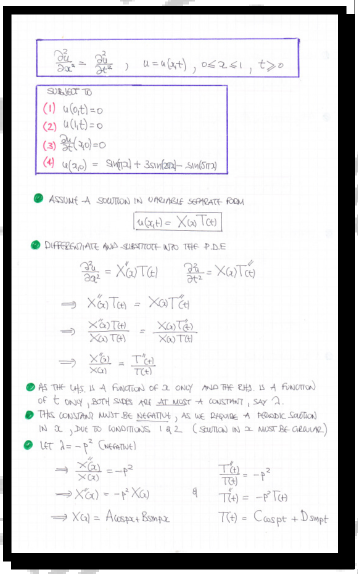
1. $u(0,t) = 0$.

2. $u(1,t) = 0$.

3. $\frac{\partial u}{\partial t}(x,0) = 0$.

4. $u(x,0) = \sin(\pi x) + 3\sin(2\pi x) - \sin(5\pi x)$.

$$\boxed{u(x,t) = \sin(5\pi x)\cos(5\pi ct) + 2\sin(7\pi x)\cos(7\pi ct)}$$



$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad u = u(x,t), \ 0 \le x \le 1, \ t \geqslant 0}$$

SUBJECT TO

(1) $u(0,t) = 0$

(2) $u(1,t) = 0$

(3) $\frac{\partial u}{\partial t}(x,0) = 0$

(4) $u(x,0) = \sin(\pi x) + 3\sin(2\pi x) - \sin(5\pi x)$

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$\boxed{u(x,t) = X(x)T(t)}$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \qquad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\Longrightarrow X''(x)T(t) = X(x)T''(t)$$

$$\Longrightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{X(x)T''(t)}{X(x)T(t)}$$

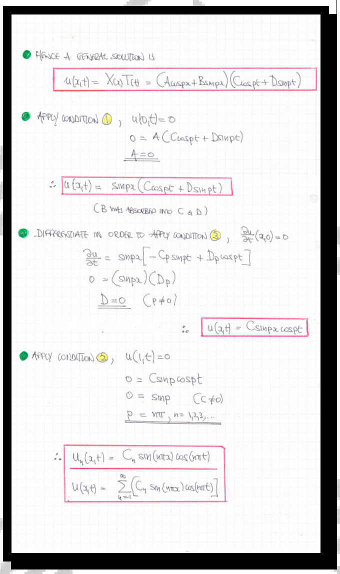
$$\Longrightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$$

- AS THE L.H.S. IS A FUNCTION OF x ONLY AND THE R.H.S. IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ.
- THIS CONSTANT MUST BE NEGATIVE, AS WE REQUIRE A PERIODIC SOLUTION IN x, DUE TO CONDITIONS 1 & 2 (SOLUTION IN x MUST BE CIRCULAR)
- LET $\lambda = -p^2$ (NEGATIVE)

$$\Longrightarrow \frac{X''(x)}{X(x)} = -p^2 \qquad \frac{T''(t)}{T(t)} = -p^2$$

$$\Longrightarrow X''(x) = -p^2 X(x) \quad \& \quad T''(t) = -p^2 T(t)$$

$$\Longrightarrow X(x) = A\cos px + B\sin px \qquad T(t) = C\cos pt + D\sin pt$$



- HENCE A GENERAL SOLUTION IS

$$\boxed{u(x,t) = X(x)T(t) = (A\cos px + B\sin px)(C\cos pt + D\sin pt)}$$

- APPLY CONDITION ①, $u(0,t) = 0$

$$0 = A(C\cos pt + D\sin pt)$$

$$\underline{\underline{A = 0}}$$

$$\therefore \boxed{u(x,t) = \sin px\,(C\cos pt + D\sin pt)}$$

(B WAS ABSORBED INTO C & D)

- DIFFERENTIATE IN ORDER TO APPLY CONDITION ③, $\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{\partial u}{\partial t} = \sin px\left[-Cp\sin pt + Dp\cos pt\right]$$

$$0 = (\sin px)(Dp)$$

$$\underline{\underline{D = 0}} \quad (p \neq 0)$$

$$\therefore \boxed{u(x,t) = C\sin px\cos pt}$$

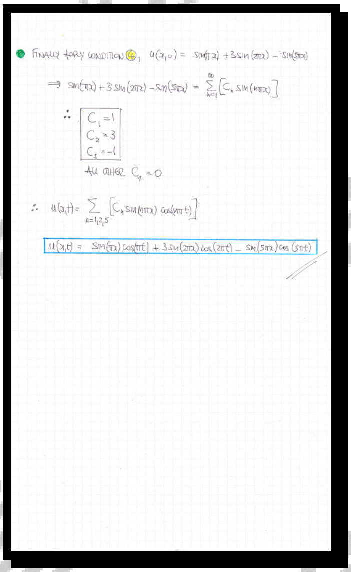
- APPLY CONDITION ②, $u(1,t) = 0$

$$0 = C\sin p\cos pt$$

$$0 = \sin p \quad (C \neq 0)$$

$$\underline{\underline{p = n\pi, \ n = 1,2,3,\ldots}}$$

$$\therefore \boxed{\begin{array}{l} u_n(x,t) = C_n\sin(n\pi x)\cos(n\pi t) \\ \hline u(x,t) = \sum_{n=1}^{\infty}\left[C_n\sin(n\pi x)\cos(n\pi t)\right] \end{array}}$$



- FINALLY APPLY CONDITION ④, $u(x,0) = \sin(\pi x) + 3\sin(2\pi x) - \sin(5\pi x)$

$$\Longrightarrow \sin(\pi x) + 3\sin(2\pi x) - \sin(5\pi x) = \sum_{n=1}^{\infty}\left[C_n\sin(n\pi x)\right]$$

$$\therefore \boxed{\begin{array}{l} C_1 = 1 \\ C_2 = 3 \\ C_5 = -1 \end{array}}$$

ALL OTHER $C_n = 0$

$$\therefore u(x,t) = \sum_{n=1,2,5}\left[C_n\sin(n\pi x)\cos(n\pi t)\right]$$

$$\boxed{u(x,t) = \sin(\pi x)\cos(\pi t) + 3\sin(2\pi x)\cos(2\pi t) - \sin(5\pi x)\cos(5\pi t)}$$

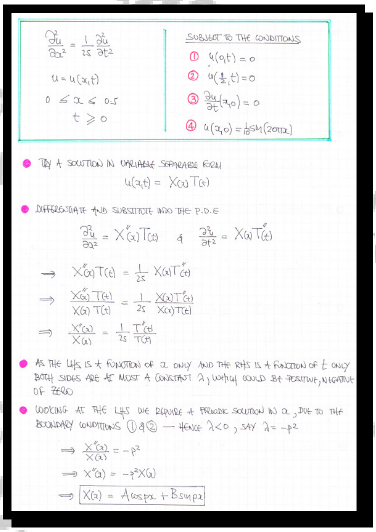
Question 6

The vertical displacements, $u = u(x,t)$, of the oscillations of a taut flexible elastic string of length 0.5 m, fixed at its endpoints is governed by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{25}\frac{\partial^2 u}{\partial t^2}, \qquad 0 \le x \le 0.5, \qquad t \ge 0.$$

Given further that the string is initially stationary, and $u(x,0) = \frac{1}{10}\sin(20\pi x)$, find a simplified expression for $u(x,t)$

$\boxed{}$, $\boxed{u(x,t) = \frac{1}{10}\sin(20\pi x)\cos(100\pi t)}$



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{25}\frac{\partial^2 u}{\partial t^2}$$

$u = u(x,t)$

$0 \leqslant x \leqslant 0.5$

$t \geqslant 0$

SUBJECT TO THE CONDITIONS

① $u(0,t) = 0$

② $u(\frac{1}{2},t) = 0$

③ $\frac{\partial u}{\partial t}(x,0) = 0$

④ $u(x,0) = \frac{1}{10}\sin(20\pi x)$

- TRY A SOLUTION IN VARIABLE SEPARABLE FORM

$$u(x,t) = X(x)T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\Rightarrow X''(x)T(t) = \frac{1}{25}X(x)T''(t)$$

$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{25}\frac{X(x)T''(t)}{X(x)T(t)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{25}\frac{T''(t)}{T(t)}$$

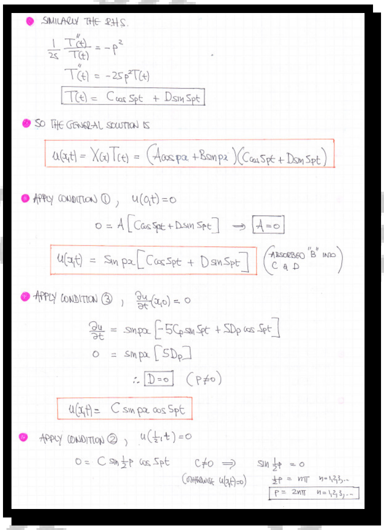
- AS THE LHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY BOTH SIDES ARE AT MOST A CONSTANT λ, WHICH COULD BE POSITIVE, NEGATIVE OF ZERO

- LOOKING AT THE LHS WE REQUIRE A PERIODIC SOLUTION IN x, DUE TO THE BOUNDARY CONDITIONS ① & ② — HENCE $\lambda < 0$, SAY $\lambda = -p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2$$

$$\Rightarrow X''(x) = -p^2 X(x)$$

$$\Rightarrow \boxed{X(x) = A\cos px + B\sin px}$$



- SIMILARLY THE RHS.

$$\frac{1}{25}\frac{T''(t)}{T(t)} = -p^2$$

$$T''(t) = -25p^2 T(t)$$

$$\boxed{T(t) = C\cos 5pt + D\sin 5pt}$$

- SO THE GENERAL SOLUTION IS

$$\boxed{u(x,t) = X(x)T(t) = (A\cos px + B\sin px)(C\cos 5pt + D\sin 5pt)}$$

- APPLY CONDITION ①, $u(0,t) = 0$

$$0 = A[C\cos 5pt + D\sin 5pt] \rightarrow \boxed{A = 0}$$

$$\boxed{u(x,t) = \sin px[C\cos 5pt + D\sin 5pt]} \quad \left(\text{ABSORBED "B" INTO } C \;\&\; D\right)$$

- APPLY CONDITION ③, $\frac{\partial u}{\partial t}(x,0) = 0$

$$\frac{\partial u}{\partial t} = \sin px[-5Cp\sin 5pt + 5Dp\cos 5pt]$$

$$0 = \sin px[5Dp]$$

$$\therefore \boxed{D = 0} \quad (p \neq 0)$$

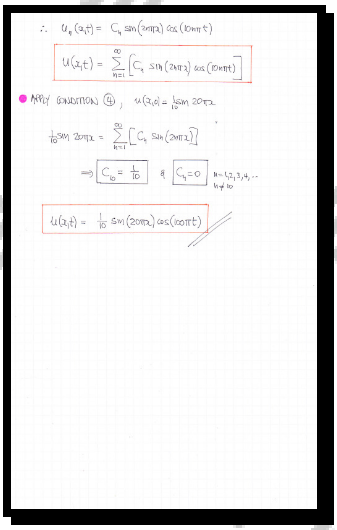
$$\boxed{u(x,t) = C\sin px\cos 5pt}$$

- APPLY CONDITION ②, $u(\frac{1}{2},t) = 0$

$$0 = C\sin\tfrac{1}{2}p\cos 5pt \qquad C \neq 0 \Rightarrow \qquad \sin\tfrac{1}{2}p = 0$$

$$(\text{OTHERWISE } u(x,t) = 0) \qquad \tfrac{1}{2}p = n\pi \quad n = 1,2,3,\dots$$

$$\boxed{p = 2n\pi \quad n = 1,2,3,\dots}$$



$$\therefore \; u_n(x,t) = C_n\sin(2n\pi x)\cos(10n\pi t)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[C_n\sin(2n\pi x)\cos(10n\pi t)\right]}$$

- APPLY CONDITION ④, $u(x,0) = \frac{1}{10}\sin 20\pi x$

$$\frac{1}{10}\sin 20\pi x = \sum_{n=1}^{\infty}\left[C_n\sin(2n\pi x)\right]$$

$$\Rightarrow \boxed{C_{10} = \frac{1}{10}} \quad \& \quad \boxed{C_n = 0} \quad \begin{matrix} n = 1,2,3,4,\dots \\ n \neq 10 \end{matrix}$$

$$\boxed{u(x,t) = \frac{1}{10}\sin(20\pi x)\cos(100\pi t)}$$

Question 7

A taut string of length L is fixed at its endpoints at $x=0$ and at $x=L$, and rests in a horizontal position along the x axis. The midpoint of the string is pulled by a small distance h and released from rest.

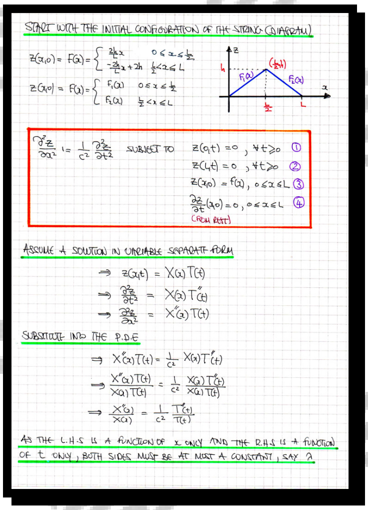
If the vertical displacement of the string z satisfies the standard wave equation

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},\ c>0,$$

show that

$$z(x,t)=\frac{8h}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{L}\right]\cos\left[\frac{(2n-1)\pi ct}{L}\right]\right].$$

MM4-B, proof



START WITH THE INITIAL CONFIGURATION OF THE STRING (DIAGRAM)

$$z(x,0)=F(x)=\begin{cases}\frac{2h}{L}x & 0\leqslant x\leqslant \frac{L}{2}\\ -\frac{2h}{L}x+2h & \frac{L}{2}<x\leqslant L\end{cases}$$

$$z(x,0)=F(x)=\begin{cases}F_1(x) & 0\leqslant x\leqslant \frac{L}{2}\\ F_2(x) & \frac{L}{2}<x\leqslant L\end{cases}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}\quad \text{SUBJECT TO}\quad \begin{array}{ll} z(0,t)=0,\ \forall t\geqslant 0 & ①\\ z(L,t)=0,\ \forall t\geqslant 0 & ②\\ z(x,0)=f(x),\ 0\leqslant x\leqslant L & ③\\ \frac{\partial z}{\partial t}(x,0)=0,\ 0\leqslant x\leqslant L & ④\\ \text{(FROM REST)} & \end{array}}$$

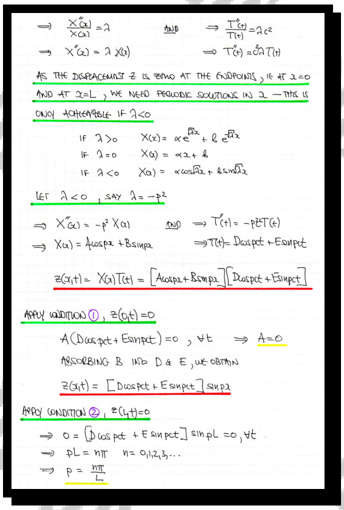
ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$\Rightarrow z(x,t)=X(x)T(t)$$
$$\Rightarrow \frac{\partial^2 z}{\partial t^2}=X(x)T''(t)$$
$$\Rightarrow \frac{\partial^2 z}{\partial x^2}=X''(x)T(t)$$

SUBSTITUTE INTO THE P.D.E

$$\Rightarrow X''(x)T(t)=\frac{1}{c^2}X(x)T''(t)$$
$$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)}=\frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$
$$\Rightarrow \frac{X''(x)}{X(x)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES MUST BE AT MOST A CONSTANT, SAY λ



$$\Rightarrow \frac{X''(x)}{X(x)}=\lambda \quad \text{AND} \quad \Rightarrow \frac{T''(t)}{T(t)}=\lambda c^2$$
$$\Rightarrow X''(x)=\lambda X(x) \qquad \Rightarrow T''(t)=c^2\lambda T(t)$$

AS THE DISPLACEMENT z IS ZERO AT THE ENDPOINTS, IE AT $x=0$ AND AT $x=L$, WE NEED PERIODIC SOLUTIONS IN x — THIS IS ONLY ACHIEVABLE IF $\lambda<0$

IF $\lambda>0$ $\quad X(x)=\alpha e^{\sqrt{\lambda}x}+\beta e^{-\sqrt{\lambda}x}$

IF $\lambda=0$ $\quad X(x)=\alpha x+\beta$

IF $\lambda<0$ $\quad X(x)=\alpha\cos\sqrt{\lambda}x+\beta\sin\sqrt{\lambda}x$

LET $\lambda<0$, SAY $\lambda=-p^2$

$$\Rightarrow X''(x)=-p^2X(x) \quad \text{AND} \quad \Rightarrow T''(t)=-p^2c^2T(t)$$
$$\Rightarrow X(x)=A\cos px+B\sin px \qquad \Rightarrow T(t)=D\cos pct+E\sin pct$$

$$z(x,t)=X(x)T(t)=\left[A\cos px+B\sin px\right]\left[D\cos pct+E\sin pct\right]$$

APPLY CONDITION ①, $z(0,t)=0$

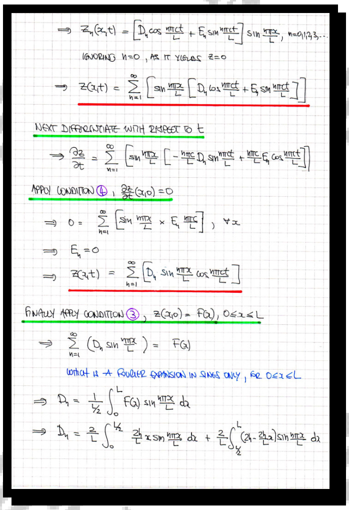
$$A(D\cos pct+E\sin pct)=0,\ \forall t \quad \Rightarrow A=0$$

ABSORBING B INTO D & E, WE OBTAIN

$$z(x,t)=\left[D\cos pct+E\sin pct\right]\sin px$$

APPLY CONDITION ②, $z(L,t)=0$

$$\Rightarrow 0=\left[D\cos pct+E\sin pct\right]\sin pL=0,\ \forall t$$
$$\Rightarrow pL=n\pi \quad n=0,1,2,3,\dots$$
$$\Rightarrow p=\frac{n\pi}{L}$$



$$\Rightarrow z_n(x,t)=\left[D_n\cos\frac{n\pi ct}{L}+E_n\sin\frac{n\pi ct}{L}\right]\sin\frac{n\pi x}{L},\ n=0,1,2,3,\dots$$

IGNORING $n=0$, AS IT YIELDS $z=0$

$$\Rightarrow z(x,t)=\sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\left[D_n\cos\frac{n\pi ct}{L}+E_n\sin\frac{n\pi ct}{L}\right]\right]$$

NEXT DIFFERENTIATE WITH RESPECT TO t

$$\Rightarrow \frac{\partial z}{\partial t}=\sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\left[-\frac{n\pi c}{L}D_n\sin\frac{n\pi ct}{L}+\frac{n\pi c}{L}E_n\cos\frac{n\pi ct}{L}\right]\right]$$

APPLY CONDITION ④, $\frac{\partial z}{\partial t}(x,0)=0$

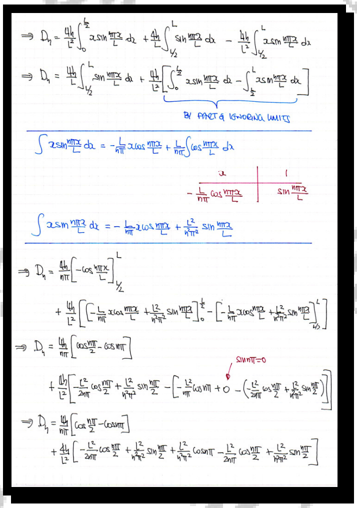
$$\Rightarrow 0=\sum_{n=1}^{\infty}\left[\sin\frac{n\pi x}{L}\times E_n\frac{n\pi c}{L}\right],\ \forall x$$
$$\Rightarrow E_n=0$$
$$\Rightarrow z(x,t)=\sum_{n=1}^{\infty}\left[D_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}\right]$$

FINALLY APPLY CONDITION ③, $z(x,0)=F(x)$, $0\leqslant x\leqslant L$

$$\Rightarrow \sum_{n=1}^{\infty}\left(D_n\sin\frac{n\pi x}{L}\right)=F(x)$$

WHICH IS A FOURIER EXPANSION IN SINES ONLY, FOR $0\leqslant x\leqslant L$

$$\Rightarrow D_n=\frac{1}{L/2}\int_0^L F(x)\sin\frac{n\pi x}{L}\,dx$$
$$\Rightarrow D_n=\frac{2}{L}\int_0^{L/2}\frac{2h}{L}x\sin\frac{n\pi x}{L}\,dx+\frac{2}{L}\int_{L/2}^{L}\left(2h-\frac{2h}{L}x\right)\sin\frac{n\pi x}{L}\,dx$$



$$\Rightarrow D_n=\frac{4h}{L^2}\int_0^{\frac{L}{2}}x\sin\frac{n\pi x}{L}\,dx+\frac{4h}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx-\frac{4h}{L^2}\int_{L/2}^{L}x\sin\frac{n\pi x}{L}\,dx$$

$$\Rightarrow D_n=\frac{4h}{L}\int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx+\frac{4h}{L^2}\underbrace{\left[\int_0^{\frac{L}{2}}x\sin\frac{n\pi x}{L}\,dx-\int_{\frac{L}{2}}^{L}x\sin\frac{n\pi x}{L}\,dx\right]}_{\text{BY PARTS \& IGNORING LIMITS}}$$

$$\int x\sin\frac{n\pi x}{L}\,dx=-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}+\frac{L}{n\pi}\int\cos\frac{n\pi x}{L}\,dx$$

$$\begin{array}{c|c} x & 1\\ \hline -\frac{L}{n\pi}\cos\frac{n\pi x}{L} & \sin\frac{n\pi x}{L}\end{array}$$

$$\int x\sin\frac{n\pi x}{L}\,dx=-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}$$

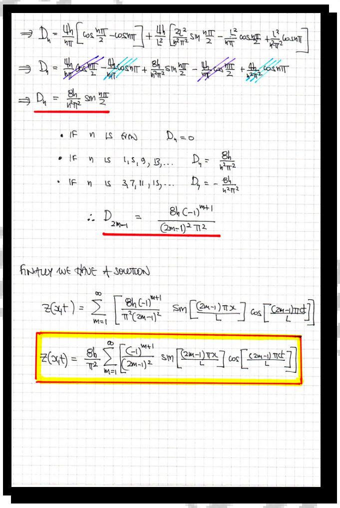
$$\Rightarrow D_n=\frac{4h}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_{L/2}^{L}$$
$$+\frac{4h}{L^2}\left[\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^{\frac{L}{2}}-\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_{L/2}^{L}\right]$$

$$\Rightarrow D_n=\frac{4h}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]$$

$\sin n\pi=0$

$$+\frac{4h}{L^2}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}-\left[-\frac{L^2}{n\pi}\cos n\pi+0-\left(-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right]$$

$$\Rightarrow D_n=\frac{4h}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]$$
$$+\frac{4h}{L^2}\left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\cos n\pi-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$



$$\Rightarrow D_n=\frac{4h}{n\pi}\left[\cos\frac{n\pi}{2}-\cos n\pi\right]+\frac{4h}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{L^2}{n\pi}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\cos n\pi\right]$$

$$\Rightarrow D_n=\cancel{\frac{4h}{n\pi}\cos\frac{n\pi}{2}}-\cancel{\frac{4h}{n\pi}\cos n\pi}+\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}-\cancel{\frac{4h}{n\pi}\cos\frac{n\pi}{2}}+\cancel{\frac{4h}{n^2\pi^2}\cos n\pi}$$

$$\Rightarrow D_n=\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

- IF n IS EVEN $\quad D_n=0$
- IF n IS $1,5,9,13,\dots$ $\quad D_n=\frac{8h}{n^2\pi^2}$
- IF n IS $3,7,11,15,\dots$ $\quad D_n=-\frac{8h}{n^2\pi^2}$

$$\therefore D_{2m-1}=\frac{8h(-1)^{m+1}}{(2m-1)^2\pi^2}$$

FINALLY WE HAVE A SOLUTION

$$z(x,t)=\sum_{m=1}^{\infty}\left[\frac{8h(-1)^{m+1}}{\pi^2(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{L}\right]\cos\left[\frac{(2m-1)\pi ct}{L}\right]\right]$$

$$\boxed{z(x,t)=\frac{8h}{\pi^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^{m+1}}{(2m-1)^2}\sin\left[\frac{(2m-1)\pi x}{L}\right]\cos\left[\frac{(2m-1)\pi ct}{L}\right]\right]}$$

Question 8

A taut string as its fixed endpoints attached to the x axis at $x=0$ and at $x=1$.

The vertical displacement of the string $u(x,t)$ satisfies a standard wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad |x| \le 1, \qquad t \ge 0,$$

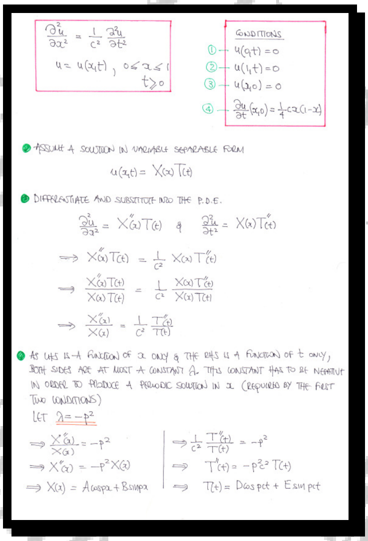
where c is a positive constant.

At time $t=0$, while the string is undisturbed, it is given a transverse velocity of magnitude $\frac{1}{4}cx(1-x)$ along its length.

Show that

$$u(x,t) = \frac{2}{\pi^4}\sum_{k=0}^{\infty}\left[\frac{\sin\left[(2k+1)\pi x\right]\sin\left[(2k+1)\pi ct\right]}{(2k+1)^4}\right].$$

proof



$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$u = u(x,t)$, $0 \le x \le 1$
$t \ge 0$

CONDITIONS

① — $u(0,t)=0$
② — $u(1,t)=0$
③ — $u(x,0)=0$
④ — $\frac{\partial u}{\partial t}(x,0) = \frac{1}{4}cx(1-x)$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$u(x,t) = X(x)T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t) \quad \& \quad \frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$$

$$\Longrightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

$$\Longrightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T''(t)}{X(x)T(t)}$$

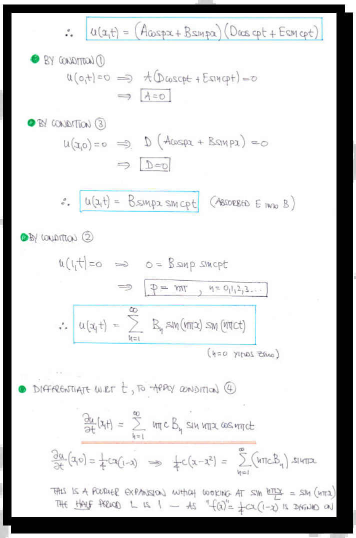
$$\Longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

- AS LHS IS A FUNCTION OF x ONLY & THE RHS IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT λ. THIS CONSTANT HAS TO BE NEGATIVE IN ORDER TO PRODUCE A PERIODIC SOLUTION IN x (REQUIRED BY THE FIRST TWO CONDITIONS)

LET $\lambda = -p^2$

$\Longrightarrow \frac{X''(x)}{X(x)} = -p^2$
$\Longrightarrow X''(x) = -p^2X(x)$
$\Longrightarrow X(x) = A\cos px + B\sin px$

$\Longrightarrow \frac{1}{c^2}\frac{T''(t)}{T(t)} = -p^2$
$\Longrightarrow T''(t) = -p^2c^2\,T(t)$
$\Longrightarrow T(t) = D\cos pct + E\sin pct$



$$\therefore \boxed{u(x,t) = (A\cos px + B\sin px)(D\cos cpt + E\sin cpt)}$$

- BY CONDITION ①

$$u(0,t)=0 \Longrightarrow A(D\cos cpt + E\sin cpt) = 0$$
$$\Longrightarrow \boxed{A=0}$$

- BY CONDITION ③

$$u(x,0)=0 \Longrightarrow D(A\cos px + B\sin px) = 0$$
$$\Longrightarrow \boxed{D=0}$$

$$\therefore \boxed{u(x,t) = B\sin px\sin cpt} \quad \text{(ABSORBED E INTO B)}$$

- BY CONDITION ②

$$u(1,t)=0 \Longrightarrow 0 = B\sin p\sin cpt$$
$$\Longrightarrow \boxed{p = n\pi\ ,\ n=0,1,2,3\ldots}$$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} B_n\sin(n\pi x)\sin(n\pi ct)}$$

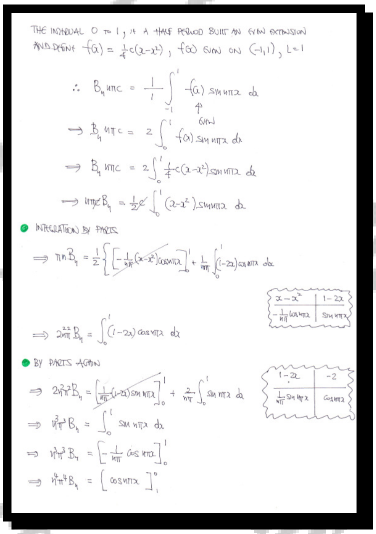
(n=0 YIELDS ZERO)

- DIFFERENTIATE W.R.T t, TO APPLY CONDITION ④

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} n\pi cB_n\sin n\pi x\cos n\pi ct$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{4}cx(1-x) \Longrightarrow \frac{1}{4}c(x-x^2) = \sum_{n=1}^{\infty}(n\pi cB_n)\sin n\pi x$$

THIS IS A FOURIER EXPANSION WHICH LOOKING AT $\sin\frac{n\pi x}{L} = \sin(n\pi x)$ THE HALF PERIOD L IS 1 — AS "$f(x)$" $= \frac{1}{4}cx(1-x)$ IS DEFINED ON



THE INTERVAL 0 TO 1, IS A HALF PERIOD BUILT AN EVEN EXTENSION AND DEFINE $f(x) = \frac{1}{4}c(x-x^2)$, $f(x)$ EVEN ON $(-1,1)$, $L=1$

$$\therefore B_n n\pi c = \frac{1}{1}\int_{-1}^{1}\underset{\substack{\uparrow\\ \text{EVEN}}}{f(x)}\sin n\pi x\,dx$$

$$\Longrightarrow B_n n\pi c = 2\int_0^1 f(x)\sin n\pi x\,dx$$

$$\Longrightarrow B_n n\pi c = 2\int_0^1 \frac{1}{4}c(x-x^2)\sin n\pi x\,dx$$

$$\Longrightarrow n\pi\cancel{c}B_n = \frac{1}{2}\cancel{c}\int_0^1 (x-x^2)\sin n\pi x\,dx$$

- INTEGRATION BY PARTS

$$\Longrightarrow \pi n B_n = \frac{1}{2}\left\{\cancel{\left[-\frac{1}{n\pi}(x-x^2)\cos n\pi x\right]_0^1} + \frac{1}{n\pi}\int_0^1(1-2x)\cos n\pi x\,dx\right.$$

$$\begin{array}{c|c} x-x^2 & 1-2x \\ \hline -\frac{1}{n\pi}\cos n\pi x & \sin n\pi x \end{array}$$

$$\Longrightarrow 2n^2\pi^2 B_n = \int_0^1(1-2x)\cos n\pi x\,dx$$

- BY PARTS AGAIN

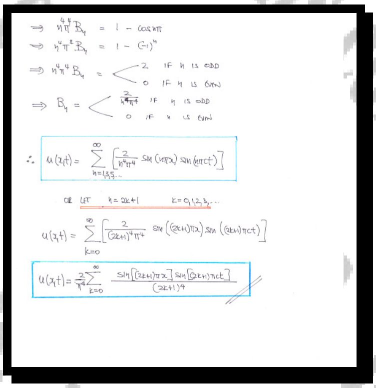
$$\Longrightarrow 2n^2\pi^2B_n = \cancel{\left[\frac{1}{n\pi}(1-2x)\sin n\pi x\right]_0^1} + \frac{2}{n\pi}\int_0^1\sin n\pi x\,dx$$

$$\begin{array}{c|c} 1-2x & -2 \\ \hline \frac{1}{n\pi}\sin n\pi x & \cos n\pi x \end{array}$$

$$\Longrightarrow n^3\pi^3B_n = \int_0^1 \sin n\pi x\,dx$$

$$\Longrightarrow n^3\pi^3B_n = \left[-\frac{1}{n\pi}\cos n\pi x\right]_0^1$$

$$\Longrightarrow n^4\pi^4B_n = \left[\cos n\pi x\right]_1^0$$



$$\Longrightarrow n^4\pi^4B_n = 1 - \cos n\pi$$

$$\Longrightarrow n^4\pi^4B_n = 1 - (-1)^n$$

$$\Longrightarrow n^4\pi^4B_n = \begin{cases} 2 & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$$

$$\Longrightarrow B_n = \begin{cases} \frac{2}{n^4\pi^4} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$$

$$\therefore \boxed{u(x,t) = \sum_{n=1,3,5\ldots}^{\infty}\left[\frac{2}{n^4\pi^4}\sin(n\pi x)\sin(n\pi ct)\right]}$$

OR LET $n = 2k+1$ $\quad k=0,1,2,3,\ldots$

$$u(x,t) = \sum_{k=0}^{\infty}\left[\frac{2}{(2k+1)^4\pi^4}\sin\left((2k+1)\pi x\right)\sin\left((2k+1)\pi ct\right)\right]$$

$$\boxed{u(x,t) = \frac{2}{\pi^4}\sum_{k=0}^{\infty}\frac{\sin\left[(2k+1)\pi x\right]\sin\left[(2k+1)\pi ct\right]}{(2k+1)^4}}$$

Question 9

A taut string of length 2 units is fixed at its endpoints at $x = \pm 1$ and rests in a horizontal position along the x axis.

At time $t = 0$, while the string is undisturbed, it is given a small transverse velocity $1 - x^2$ along its length. It is assumed that the displacement of the string

$$u(x,t), \ |x| \le 1, \ t \ge 0$$

satisfies a standard wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{4}\frac{\partial^2 u}{\partial t^2},$$

Show that

$$u(x,t) = \frac{32}{\pi^4}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{(2n-1)^4}\cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]\right],$$

and hence determine of the normal modes of the vibration of the string

$\boxed{\qquad}$, $\boxed{f_n = \frac{1}{2}(2n-1)}$

[solution overleaf]

ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$u(x,t) = X(x)T(t) \Rightarrow \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t)$$
$$\frac{\partial^2 u}{\partial t^2} = X(x)\,T''(t)$$

SUBSTITUTE INTO THE P.D.E.

$$\Rightarrow X''(x)\,T(t) = \frac{1}{4}X(x)\,T''(t)$$
$$\Rightarrow \frac{X''(x)\,T(t)}{X(x)T(t)} = \frac{1}{4}\,\frac{X(x)\,T''(t)}{X(x)\,T(t)}$$
$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{4T(t)}$$

AS THE L.H.S. IS A FUNCTION OF x ONLY AND THE RHS IS FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda > 0$, $\lambda = p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = p^2 \qquad\Big\|\qquad \Rightarrow \frac{T''(t)}{4T(t)} = p^2$$
$$\Rightarrow X''(x) = p^2 X(x) \qquad\Big\|\qquad \Rightarrow T''(t) = 4p^2 T(t)$$
$$\Rightarrow X(x) = A\cosh px + B\sinh px \qquad\Big\|\qquad \Rightarrow T(t) = D\cosh 2pt + E\sinh 2pt$$

(OR EXPONENTIALS) (OR EXPONENTIALS)

IF $\lambda = 0$

$$\Rightarrow \frac{X''(x)}{X(x)} = 0 \qquad\Big\|\qquad \Rightarrow \frac{T''(t)}{4T(t)} = 0$$
$$\Rightarrow X''(x) = 0 \qquad\Big\|\qquad \Rightarrow T''(t) = 0$$
$$\Rightarrow X(x) = Ax + B \qquad\Big\|\qquad \Rightarrow T(t) = At + B$$

IF $\lambda < 0$, $\lambda = -p^2$

$$\Rightarrow \frac{X''(x)}{X(x)} = -p^2 \qquad\Big\|\qquad \Rightarrow \frac{T''(t)}{4T(t)} = -p^2$$
$$\Rightarrow X''(x) = -p^2 X(x) \qquad\Big\|\qquad \Rightarrow T''(t) = -4p^2 T(t)$$
$$\Rightarrow X(x) = A\cos px + B\sin px \qquad\Big\|\qquad \Rightarrow T(t) = D\cos 2pt + E\sin 2pt$$

AS WE REQUIRE A SOLUTION WHICH GIVES THE SAME VALUE OF $u(x,t)$ FOR THE TWO DISTINCT VALUES OF x AT THE ENDPOINTS $(x=\pm 1)$ WE CAN ONLY PICK THE "TRIGONOMETRIC SOLUTION" & DISCARD THE OTHER TWO

$$\therefore \boxed{u(x,t) = \left[A\cos px + B\sin px\right]\left[D\cos 2pt + E\sin 2pt\right]}$$

APPLY CONDITION $u(x,0) = 0$ (UNDISTORBED INITIALLY)

$$\Rightarrow 0 = \left[A\cos px + B\sin px\right] \times D$$
$$\Rightarrow D = 0 \qquad (\text{otherwise trivial solution } A = B = 0)$$

ABSORBING "E" INTO "A" AND "B"

$$\therefore \boxed{u(x,t) = \left[A\cos px + B\sin px\right]\sin 2pt}$$

DIFFERENTIATE W.R.T t AND APPLY $\frac{\partial u}{\partial t}(x,0) = 1 - x^2$ (INITIAL TRANSVERSE VELOCITY)

$$\Rightarrow \frac{\partial u}{\partial t}(x,t) = 2p\left[A\cos px + B\sin px\right]\cos 2pt$$
$$\Rightarrow 1 - x^2 = 2p\left[A\cos px + B\sin px\right] \times 1$$
$$\Rightarrow B = 0 \quad (\text{AS THE LHS. IS AN EVEN FUNCTION IN } x)$$
$$\therefore \boxed{u(x,t) = A\cos px\,\sin 2pt}$$

APPLY BOUNDARY CONDITIONS $u(-1,t) = u(1,t) = 0$

$$\left.\begin{array}{l} 0 = A\cos(-p)\sin 2pt \\ 0 = A\cos p\,\sin 2pt \end{array}\right\} \Rightarrow A\cos p\sin 2pt = 0 \quad \forall t \geqslant 0$$
$$\Rightarrow \cos p = 0 \quad (A \neq 0)$$
$$\Rightarrow p = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$
$$\Rightarrow p = \frac{2n-1}{2}\pi \qquad n = 1,2,3,\ldots$$

$$\Rightarrow u_n(x,t) = A_n\cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]$$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{n} A_n\cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]}$$

ONE MORE CONSTANT TO EVALUATE, SO REAPPLY $\frac{\partial u}{\partial t}(x,0) = 1 - x^2$, $-1 \leqslant x \leqslant 1$

$$\Rightarrow \frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty}\left[\left[A_n(2n-1)\pi\right]\cos\left[\frac{(2n-1)\pi x}{2}\right]\cos\left[(2n-1)\pi t\right]\right]$$
$$\Rightarrow 1 - x^2 = \sum_{n=1}^{\infty}\left[\left[A_n(2n-1)\pi\right]\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]$$

THIS IS A FOURIER SERIES IN x, IN $-1 \leqslant x \leqslant 1$

$$A_n(2n-1)\pi = \frac{1}{1}\int_{-1}^{1}(1-x^2)\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$
$$A_n(2n-1)\pi = 2\int_{0}^{1}(1-x^2)\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$
$$A_n(2n-1)\pi = \int_{0}^{2} 2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx - \int_{0}^{2} 2x^2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$

CARRYING OUT THE INTEGRATIONS

- $\int_{0}^{1} 2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx = \left[2 \times \frac{2}{(2n-1)\pi} \times \sin\left[\frac{(2n-1)\pi x}{2}\right]\right]_{0}^{1}$

$$= \frac{4}{\pi(2n-1)}\sin\left[\frac{(2n-1)\pi}{2}\right] = \frac{4}{\pi(2n-1)}(-1)^{n+1}$$

- $\int_{0}^{1} -2x^2\cos\left[\frac{(2n-1)\pi x}{2}\right]dx = \ldots$ INTEGRATION BY PARTS

$-2x^2$	$-4x$
$\frac{2}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{2}\right]$	$\cos\left[\frac{(2n-1)\pi x}{2}\right]$

$$= \left[\frac{-4x^2}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi x}{2}\right]\right]_{0}^{1} + \frac{8}{(2n-1)\pi}\int_{0}^{1} x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$$
$$= \frac{-4}{(2n-1)\pi}\sin\left[\frac{(2n-1)\pi}{2}\right] - 0 + \frac{8}{(2n-1)\pi}\int_{0}^{1} x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx$$
$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{8}{(2n-1)\pi}\underbrace{\int_{0}^{1} x\sin\left[\frac{(2n-1)\pi x}{2}\right]dx}_{\text{BY PARTS AGAIN}}$$

x	1
$-\frac{2}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2}\right]$	$\sin\left[\frac{(2n-1)\pi x}{2}\right]$

$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{8}{(2n-1)\pi}\left\{\cancel{\left[-\frac{2x}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]_{0}^{1}} + \frac{2}{(2n-1)\pi}\int_{0}^{1}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right\}$$
$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{16}{(2n-1)^2\pi^2}\int_{0}^{1}\cos\left[\frac{(2n-1)\pi x}{2}\right]dx$$
$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{16}{(2n-1)^2\pi^2} \times \frac{2}{(2n-1)\pi}\left[\sin\left[\frac{(2n-1)\pi x}{2}\right]\right]_{0}^{1}$$
$$= \frac{-4(-1)^{n+1}}{(2n-1)\pi} + \frac{32}{(2n-1)^3\pi^3} \times (-1)^{n+1}$$

COLLECTING THE INTEGRATION RESULTS

$$A_n(2n-1)\pi = \cancel{\frac{4}{\pi(2n-1)}(-1)^{n+1}} - \cancel{\frac{4(-1)^{n+1}}{(2n-1)\pi}} + \frac{32(-1)^{n+1}}{(2n-1)^3\pi^3}$$

$$\boxed{A_n = \frac{32(-1)^{n+1}}{(2n-1)^4\pi^4}}$$

HENCE THE SOLUTION IS GIVEN BY

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{32(-1)^{n+1}}{(2n-1)^4\pi^4}\cos\left[\frac{(2n-1)\pi x}{2}\right]\sin\left[(2n-1)\pi t\right]\right]}$$

& THE FREQUENCIES OF NORMAL MODES OF VIBRATION ARE

$$f_n = \frac{\omega_n}{2\pi} \leftarrow \text{coefficient of } t$$
$$f_n = \frac{(2n-1)\cancel{\pi}}{2\cancel{\pi}}$$
$$f_n = n - \frac{1}{2} \qquad n = 1,2,3,4,\ldots$$

Question 10

A taut string is fixed at its endpoints at $x = 0$ and $x = L$. The string is vibrating in a resistive medium and its transverse displacement $u(x,t)$ from a horizontal position satisfies the modified wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left[\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t}\right], \quad 0 \le x \le L, \quad t \ge 0,$$

where λ and c are a positive constants.

Show that

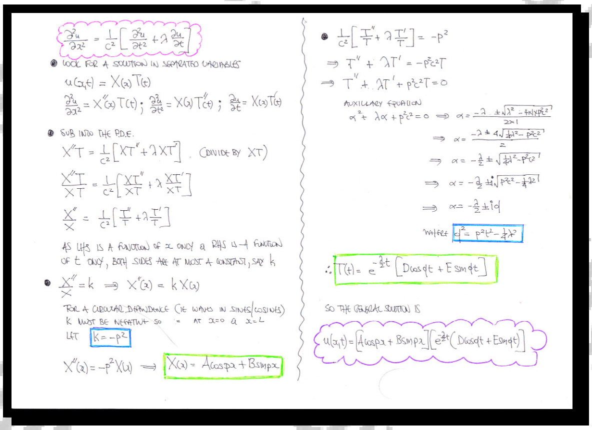
$$u(x,t) = \sum_{n=1}^{\infty}\left[P_n \, e^{-\frac{1}{2}\lambda t} \sin\left(\frac{n\pi x}{L}\right) \cos\left[q_n t - \varphi_n\right]\right],$$

where

$$q_n = \frac{(n\pi c)^2}{L} - \frac{\lambda}{4},$$

and P_n and φ_n are suitably defined constants.

proof



$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\left[\frac{\partial^2 u}{\partial t^2} + \lambda \frac{\partial u}{\partial t}\right]}$$

- LOOK FOR A SOLUTION IN SEPARATED VARIABLES

$u(x,t) = X(x)T(t)$

$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$; $\frac{\partial^2 u}{\partial t^2} = X(x)T''(t)$; $\frac{\partial u}{\partial t} = X(x)T'(t)$

- SUB INTO THE P.D.E.

$X''T = \frac{1}{c^2}\left[XT'' + \lambda XT'\right]$. (DIVIDE BY XT)

$\frac{X''T}{XT} = \frac{1}{c^2}\left[\frac{XT''}{XT} + \lambda\frac{XT'}{XT}\right]$

$\frac{X''}{X} = \frac{1}{c^2}\left[\frac{T''}{T} + \lambda\frac{T'}{T}\right]$

AS LHS IS A FUNCTION OF x ONLY & RHS IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY k

- $\frac{X''}{X} = k \implies X''(x) = kX(x)$

FOR A CIRCULAR DEPENDENCE (IE WAVES IN SINES/COSINES) k MUST BE NEGATIVE SO $=$ AT $x=0$ & $x=L$

LET $\boxed{k = -p^2}$

$X''(x) = -p^2 X(x) \implies \boxed{X(x) = A\cos px + B\sin px}$

- $\frac{1}{c^2}\left[\frac{T''}{T} + \lambda\frac{T'}{T}\right] = -p^2$

$\implies T'' + \lambda T' = -p^2c^2T$

$\implies T'' + \lambda T' + p^2c^2T = 0$

AUXILIARY EQUATION

$\alpha^2 + \lambda\alpha + p^2c^2 = 0 \implies \alpha = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\times1\times p^2c^2}}{2\times1}$

$\implies \alpha = \frac{-\lambda \pm 4\sqrt{\frac{1}{4}\lambda^2 - p^2c^2}}{2}$

$\implies \alpha = -\frac{\lambda}{2} \pm \sqrt{\frac{1}{4}\lambda^2 - p^2c^2}$

$\implies \alpha = -\frac{\lambda}{2} \pm i\sqrt{p^2c^2 - \frac{1}{4}\lambda^2}$

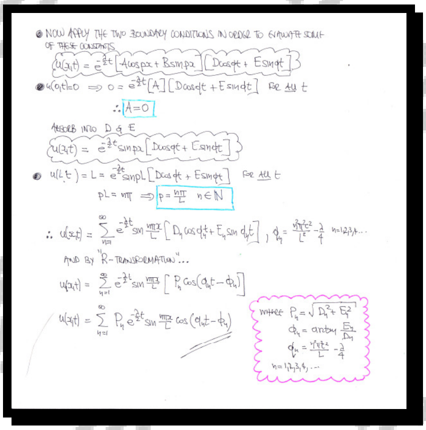
$\implies \alpha = -\frac{\lambda}{2} \pm iq$

WHERE $\boxed{q^2 = p^2t^2 - \frac{1}{4}\lambda^2}$

$\therefore \boxed{T(t) = e^{-\frac{\lambda}{2}t}\left[D\cos qt + E\sin qt\right]}$

SO THE GENERAL SOLUTION IS

$$\boxed{u(x,t) = \left[A\cos px + B\sin px\right]\left[e^{-\frac{\lambda}{2}t}\left(D\cos qt + E\sin qt\right)\right]}$$



- NOW APPLY THE TWO BOUNDARY CONDITIONS IN ORDER TO EVALUATE SOME OF THESE CONSTANTS

$$\boxed{u(x,t) = e^{-\frac{\lambda}{2}t}\left[A\cos px + B\sin px\right]\left[D\cos qt + E\sin qt\right]}$$

- $u(0,t) = 0 \implies 0 = e^{-\frac{\lambda}{2}t}[A]\left[D\cos qt + E\sin qt\right]$ FOR ALL t

$\therefore \boxed{A = 0}$

ABSORB INTO D & E

$$\boxed{u(x,t) = e^{-\frac{\lambda}{2}t}\sin px\left[D\cos qt + E\sin qt\right]}$$

- $u(L,t) = L = e^{-\frac{\lambda}{2}t}\sin pL\left[D\cos qt + E\sin qt\right]$ FOR ALL t

$pL = n\pi \implies \boxed{p = \frac{n\pi}{L} \quad n \in \mathbb{N}}$

$\therefore u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\lambda}{2}t}\sin\frac{n\pi x}{L}\left[D_n\cos q_n t + E_n\sin q_n t\right]$, $q_n = \frac{n^2\pi^2c^2}{L^2} - \frac{\lambda}{4}$ $n=1,2,3,4\ldots$

AND BY "R-TRANSFORMATION"...

$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\lambda}{2}t}\sin\frac{n\pi x}{L}\left[P_n\cos(q_n t - \phi_n)\right]$

$u(x,t) = \sum_{n=1}^{\infty} P_n e^{-\frac{\lambda}{2}t}\sin\frac{n\pi x}{L}\cos(q_n t - \phi_n)$

WHERE $P_n = \sqrt{D_n^2 + E_n^2}$

$\phi_n = \arctan\frac{E_n}{D_n}$

$q_n = \frac{n^2\pi^2c^2}{L} - \frac{\lambda}{4}$

$n = 1,2,3,4,\ldots$

Question 11

A taut uniform string lies undisturbed along the x axis.

One of its ends is fixed at $x = 0$ while the other end at $x = L$ is attached to a light ring. The ring is free to slide along a **smooth** wire at right angles to the x axis.

The vertical displacement of the string $z(x,t)$ satisfies the standard wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad 0 \le x \le L, \quad t \ge 0,$$

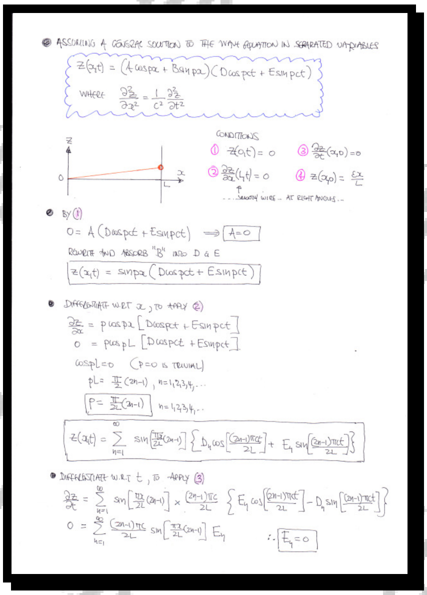
The string is released from rest and its initial displacement is given by

$$z(x,0) = \frac{\varepsilon x}{L}, \qquad 0 \le x \le L, \quad 0 < \varepsilon << 1.$$

Determine an expression for $z(x,t)$, and hence state the periods of the normal modes of vibrations of the string.

[*You may assume without proof the standard solution of the wave equation in variable separate form*]

$$\boxed{z(x,t) = \sum_{n=1}^{\infty}\left\{\frac{8\varepsilon(-1)^{n+1}}{\pi^2(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left[\frac{(2n-1)\pi ct}{2L}\right]\right\}}, \quad \boxed{T_n = \frac{4L}{(2n-1)c}}$$



- ASSUMING A GENERAL SOLUTION TO THE WAVE EQUATION IN SEPARATED VARIABLES

$$z(x,t) = (A\cos px + B\sin px)(D\cos pct + E\sin pct)$$

WHERE $\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$

CONDITIONS

① $z(0,t) = 0$

② $\frac{\partial z}{\partial x}(L,t) = 0$

③ $\frac{\partial z}{\partial t}(x,0) = 0$

④ $z(x,0) = \frac{\varepsilon x}{L}$

...SMOOTH WIRE... AT RIGHT ANGLES...

- BY ①

$0 = A(D\cos pct + E\sin pct) \Longrightarrow \boxed{A = 0}$

REWRITE AND ABSORB "B" INTO D & E

$$\boxed{z(x,t) = \sin px\,(D\cos pct + E\sin pct)}$$

- DIFFERENTIATE W.R.T x, TO APPLY ②

$\frac{\partial z}{\partial x} = p\cos px\,[D\cos pct + E\sin pct]$

$0 = p\cos pL\,[D\cos pct + E\sin pct]$

$\cos pL = 0 \quad (p = 0 \text{ IS TRIVIAL})$

$pL = \frac{\pi}{2}(2n-1), \; n = 1,2,3,4,\ldots$

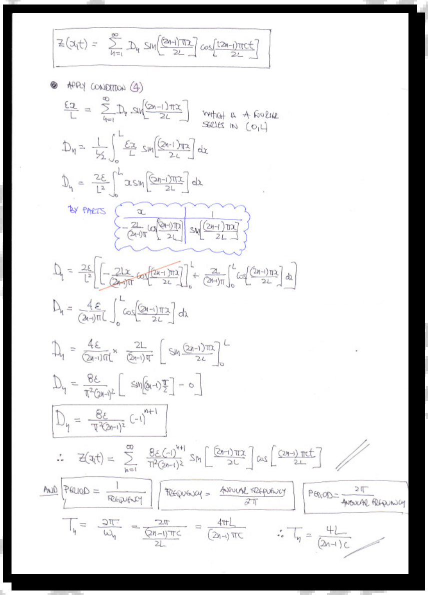
$\boxed{p = \frac{\pi}{2L}(2n-1)} \; n = 1,2,3,4,\ldots$

$$\boxed{z(x,t) = \sum_{n=1}^{\infty}\sin\left[\frac{\pi x}{2L}(2n-1)\right]\left\{D_n\cos\left[\frac{(2n-1)\pi ct}{2L}\right] + E_n\sin\left[\frac{(2n-1)\pi ct}{2L}\right]\right\}}$$

- DIFFERENTIATE W.R.T t, TO APPLY ③

$\frac{\partial z}{\partial t} = \sum_{n=1}^{\infty}\sin\left[\frac{\pi x}{2L}(2n-1)\right] \times \frac{(2n-1)\pi c}{2L}\left\{E_n\cos\left[\frac{(2n-1)\pi ct}{2L}\right] - D_n\sin\left[\frac{(2n-1)\pi ct}{2L}\right]\right\}$

$0 = \sum_{n=1}^{\infty}\frac{(2n-1)\pi c}{2L}\sin\left[\frac{\pi x}{2L}(2n-1)\right]E_n \qquad \therefore \boxed{E_n = 0}$



$$\boxed{z(x,t) = \sum_{n=1}^{\infty}D_n\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left[\frac{(2n-1)\pi ct}{2L}\right]}$$

- APPLY CONDITION ④

$\frac{\varepsilon x}{L} = \sum_{n=1}^{\infty}D_n\sin\left[\frac{(2n-1)\pi x}{2L}\right]$ WHICH IS A FOURIER SERIES IN $(0,L)$

$D_n = \frac{1}{L/2}\int_0^L \frac{\varepsilon x}{L}\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx$

$D_n = \frac{2\varepsilon}{L^2}\int_0^L x\sin\left[\frac{(2n-1)\pi x}{2L}\right]dx$

BY PARTS

x	1
$-\frac{2L}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2L}\right]$	$\sin\left[\frac{(2n-1)\pi x}{2L}\right]$

$D_n = \frac{2\varepsilon}{L^2}\left[\cancel{\left[-\frac{2Lx}{(2n-1)\pi}\cos\left[\frac{(2n-1)\pi x}{2L}\right]\right]_0^L} + \frac{2L}{(2n-1)\pi}\int_0^L\cos\left[\frac{(2n-1)\pi x}{2L}\right]dx\right]$

$D_n = \frac{4\varepsilon}{(2n-1)\pi L}\int_0^L\cos\left[\frac{(2n-1)\pi x}{2L}\right]dx$

$D_n = \frac{4\varepsilon}{(2n-1)\pi L} \times \frac{2L}{(2n-1)\pi}\left[\sin\frac{(2n-1)\pi x}{2L}\right]_0^L$

$D_n = \frac{8\varepsilon}{\pi^2(2n-1)^2}\left[\sin\left[(2n-1)\frac{\pi}{2}\right] - 0\right]$

$$\boxed{D_n = \frac{8\varepsilon}{\pi^2(2n-1)^2}(-1)^{n+1}}$$

$\therefore \; z(x,t) = \sum_{n=1}^{\infty}\frac{8\varepsilon(-1)^{n+1}}{\pi^2(2n-1)^2}\sin\left[\frac{(2n-1)\pi x}{2L}\right]\cos\left[\frac{(2n-1)\pi ct}{2L}\right]$ //

AND $\boxed{\text{PERIOD} = \frac{1}{\text{FREQUENCY}}}$ $\boxed{\text{FREQUENCY} = \frac{\text{ANGULAR FREQUENCY}}{2\pi}}$ $\boxed{\text{PERIOD} = \frac{2\pi}{\text{ANGULAR FREQUENCY}}}$

$T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\frac{(2n-1)\pi c}{2L}} = \frac{4\pi L}{(2n-1)\pi c} \qquad \therefore T_n = \frac{4L}{(2n-1)c}$ //

WAVE EQUATION

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z=z(x,t)$$

Use of Complex Numbers

Question 1

A semi infinite string S_1 of density ρ_1 lies along the x axis for $x<0$ and another semi infinite string S_2 of density ρ_2 lie along the x axis for $x>0$. The two strings are attached to particle P of mass m, at $x=0$.

The mass of the two strings is negligible compared to that of P. The strings and the particle lie undisturbed in an infinite horizontal plane.

A small disturbance z with equation

$$z = \mathrm{Re}\left[A\,\mathrm{e}^{\mathrm{i}(nt-kx)} \right],$$

is propagated from $x<0$ in the direction of x increasing, where n and k are the frequency and wave number, respectively.

Show that the amplitude of reflected wave in the section for which $x<0$, is

$$a\sqrt{\frac{T^2(k-k_2)+m^2n^4}{T^2(k+k_2)+m^2n^4}},$$

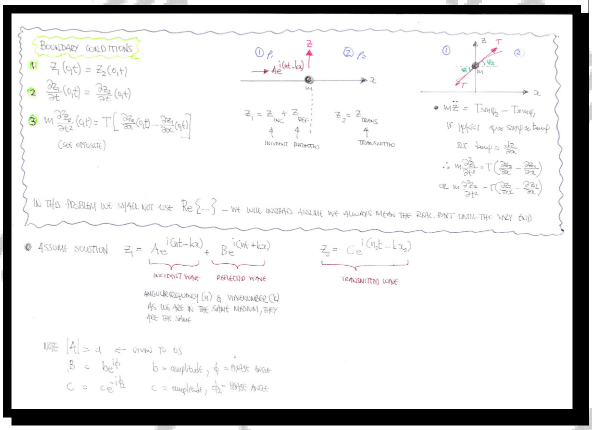
and the amplitude of the transmitted wave in the section for which $x>0$ is

$$\frac{2kTa}{\sqrt{T^2(k+k_2)+m^2n^4}}$$

where T is the tension in the strings and $k_2 = n\sqrt{\frac{\rho_2}{T}}$.

$\boxed{\text{proof}}$

[***solution overleaf***]



BOUNDARY CONDITIONS

① $z_1(0,t) = z_2(0,t)$

② $\frac{\partial z_1}{\partial t}(0,t) = \frac{\partial z_2}{\partial t}(0,t)$

③ $m\frac{\partial^2 z_2}{\partial t^2}(0,t) = T\left[\frac{\partial z_2}{\partial x}(0,t) - \frac{\partial z_1}{\partial x}(0,t)\right]$

(SEE OPPOSITE)

$z_1 = z_{INC} + z_{REF}$ $z_2 = z_{TRANS}$

INCIDENT REFLECTED TRANSMITTED

• $m\ddot{z} = T\sin\psi_2 - T\sin\psi_1$

IF $|\psi| \ll 1$ $\psi \approx \sin\psi \approx \tan\psi$

BUT $\tan\psi = \frac{dz}{dx}$

$\therefore m\frac{\partial^2 z_1}{\partial t^2} = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right)$

OR $m\frac{\partial^2 z_2}{\partial t^2} = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right)$

IN THIS PROBLEM WE SHALL NOT USE $\text{Re}\{\dots\}$ – WE WILL INSTEAD ASSUME WE ALWAYS MEAN THE REAL PART UNTIL THE VERY END

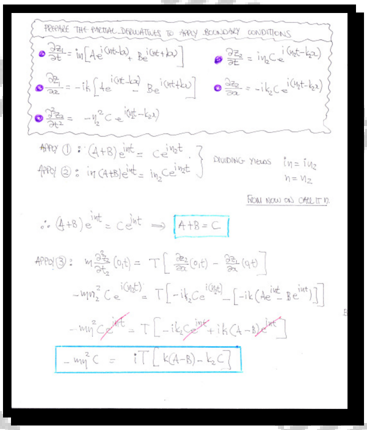
• ASSUME SOLUTION $z_1 = \underbrace{Ae^{i(nt-kx)}}_{\text{INCIDENT WAVE}} + \underbrace{Be^{i(nt+kx)}}_{\text{REFLECTED WAVE}}$ $z_2 = \underbrace{Ce^{i(n_2t-kx_2)}}_{\text{TRANSMITTED WAVE}}$

ANGULAR FREQUENCY (n) & WAVENUMBER (k) AS WE ARE IN THE SAME MEDIUM, THEY ARE THE SAME

NOTE $|A| = a$ ← GIVEN TO US

$B = be^{i\phi}$ b = amplitude, ϕ = PHASE ANGLE

$C = ce^{-i\phi_2}$ c = amplitude, ϕ_2 = PHASE ANGLE



PREPARE THE PARTIAL DERIVATIVES TO APPLY BOUNDARY CONDITIONS

• $\frac{\partial z_1}{\partial t} = in\left[Ae^{i(nt-kx)} + Be^{i(nt+kx)}\right]$ • $\frac{\partial z_2}{\partial t} = in_2Ce^{i(n_2t-k_2x)}$

• $\frac{\partial z_1}{\partial x} = -ik\left[Ae^{i(nt-kx)} - Be^{i(nt+kx)}\right]$ • $\frac{\partial z_2}{\partial x} = -ik_2Ce^{i(n_2t-k_2x)}$

• $\frac{\partial^2 z_2}{\partial t^2} = -n_2^2Ce^{i(n_2t-k_2x)}$

APPLY ① : $(A+B)e^{int} = Ce^{in_2t}$

APPLY ② : $in(A+B)e^{int} = in_2Ce^{in_2t}$

DIVIDING YIELDS $in = in_2$, $n = n_2$

FROM NOW ON CALL IT n.

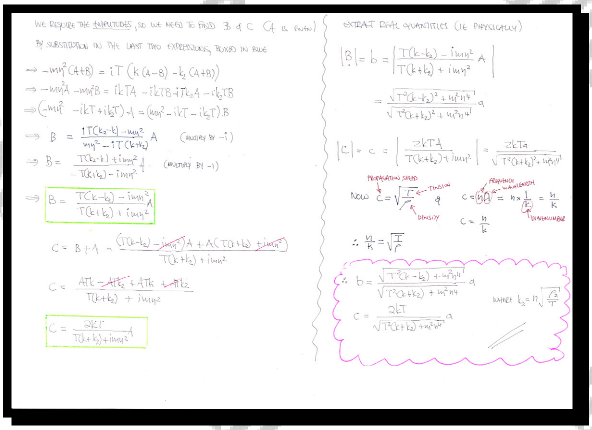
$\therefore (A+B)e^{int} = Ce^{int} \Longrightarrow \boxed{A+B=C}$

APPLY ③ : $m\frac{\partial^2 z_2}{\partial t_2}(0,t) = T\left[\frac{\partial z_2}{\partial x}(0,t) - \frac{\partial z_1}{\partial x}(0,t)\right]$

$-mn_2^2Ce^{i(n_2t)} = T\left[-ik_2Ce^{i(n_2t)} - \left[-ik(Ae^{int} - Be^{int})\right]\right]$

$-mn^2C\cancel{e^{int}} = T\left[-ik_2C\cancel{e^{int}} + ik(A-B)\cancel{e^{int}}\right]$

$\boxed{-mn^2C = iT\left[k(A-B) - k_2C\right]}$



WE REQUIRE THE AMPLITUDES, SO WE NEED TO FIND B & C (A IS GIVEN)

BY SUBSTITUTION IN THE LAST TWO EXPRESSIONS, BOXED IN BLUE

$\Longrightarrow -mn^2(A+B) = iT\left(k(A-B) - k_2(A+B)\right)$

$\Longrightarrow -mn^2A - mn^2B = ikTA - ikTB - iTk_2A - ik_2TB$

$\Longrightarrow (-mn^2 - ikT + ik_2T)A = (mn^2 - ikT - ik_2T)B$

$\Longrightarrow B = \frac{iT(k_2-k) - mn^2}{mn^2 - iT(k+k_2)}A$ (MULTIPLY BY $-i$)

$\Longrightarrow B = \frac{T(k_2-k) + imn^2}{-T(k+k_2) - imn^2}A$ (MULTIPLY BY -1)

$\Longrightarrow \boxed{B = \frac{T(k-k_2) - imn^2}{T(k+k_2) + imn^2}A}$

$C = B + A = \frac{(T(k-k_2) - \cancel{imn^2})A + A(T(k+k_2) \cancel{+imn^2})}{T(k+k_2) + imn^2}$

$C = \frac{ATk - \cancel{ATk_2} + ATk \cancel{+ATk_2}}{T(k+k_2) + imn^2}$

$\boxed{C = \frac{2kT}{T(k+k_2) + imn^2}A}$

EXTRACT REAL QUANTITIES (IE PHYSICALLY)

$|B| = b = \left|\frac{T(k-k_2) - imn^2}{T(k+k_2) + imn^2}A\right|$

$= \frac{\sqrt{T^2(k-k_2)^2 + m^2n^4}}{\sqrt{T^2(k+k_2)^2 + m^2n^4}}a$

$|C| = c = \left|\frac{2kTA}{T(k+k_2) + imn^2}\right| = \frac{2kTa}{\sqrt{T^2(k+k_2)^2 + m^2n^4}}$

NOW $c = \sqrt{\frac{T}{\rho}}$ & $c = n\lambda = n \times \frac{1}{k} = \frac{n}{k}$

$c = \frac{n}{k}$

$\therefore \frac{n}{k} = \sqrt{\frac{T}{\rho}}$

$\therefore b = \frac{\sqrt{T^2(k-k_2) + m^2n^4}}{\sqrt{T^2(k+k_2) + m^2n^4}}a$

$c = \frac{2kT}{\sqrt{T^2(k+k_2) + m^2n^4}}a$

WHERE $k_2 = n\sqrt{\frac{\rho_2}{T}}$

Question 2

Two uniform strings, S_1 and S_2, are joined together at one end and the other two free ends are attached to two fixed points $2L$ apart.

S_1 has length L and density ρ_1 and lies along the x axis for $x < 0$.

S_2 has length L and density ρ_2 and lies along the x axis for $x > 0$.

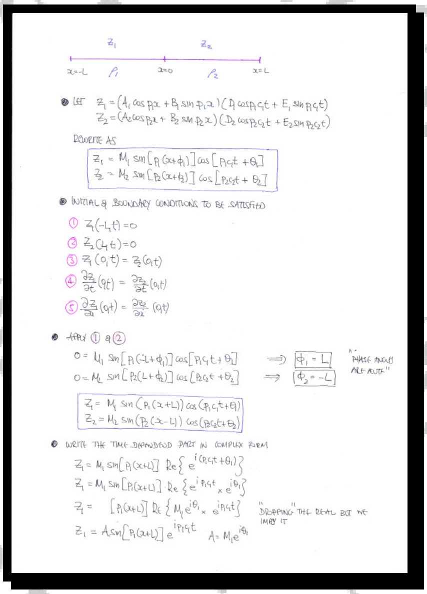
The combined string is taut and the tension is constant throughout.

Given that the combined string performs small amplitude transverse oscillations, show that

$$c_1 \tan\left(\frac{\omega L}{c_1}\right) + c_2 \tan\left(\frac{\omega L}{c_2}\right) = 0,$$

where $\frac{2\pi}{\omega}$ is the period of the normal modes of vibration, and c_1 and c_2 are the respective wave speeds in S_1 and S_2.

proof



z_1 z_2

$x=-L$ ρ_1 $x=0$ ρ_2 $x=L$

- LET $z_1 = (A_1\cos p_1x + B_1\sin p_1x)(D_1\cos p_1c_1t + E_1\sin p_1c_1t)$
  $z_2 = (A_2\cos p_2x + B_2\sin p_2x)(D_2\cos p_2c_2t + E_2\sin p_2c_2t)$

REWRITE AS

$$\boxed{\begin{aligned} z_1 &= M_1\sin[p_1(x+\phi_1)]\cos[p_1c_1t+\theta_1] \\ z_2 &= M_2\sin[p_2(x+\phi_2)]\cos[p_2c_2t+\theta_2] \end{aligned}}$$

- INITIAL & BOUNDARY CONDITIONS TO BE SATISFIED

① $z_1(-L,t)=0$

② $z_2(L,t)=0$

③ $z_1(0,t)=z_2(0,t)$

④ $\frac{\partial z_1}{\partial t}(0,t) = \frac{\partial z_2}{\partial t}(0,t)$

⑤ $\frac{\partial z_1}{\partial x}(0,t) = \frac{\partial z_2}{\partial x}(0,t)$

- APPLY ① & ②

$0 = M_1\sin[p_1(-L+\phi_1)]\cos[p_1c_1t+\theta_1] \implies \boxed{\phi_1 = L}$ "PHASE ANGLES ARE ACUTE"

$0 = M_2\sin[p_2(L+\phi_2)]\cos[p_2c_2t+\theta_2] \implies \boxed{\phi_2 = -L}$

$$\boxed{\begin{aligned} z_1 &= M_1\sin(p_1(x+L))\cos(p_1c_1t+\theta_1) \\ z_2 &= M_2\sin(p_2(x-L))\cos(p_2c_2t+\theta_2) \end{aligned}}$$

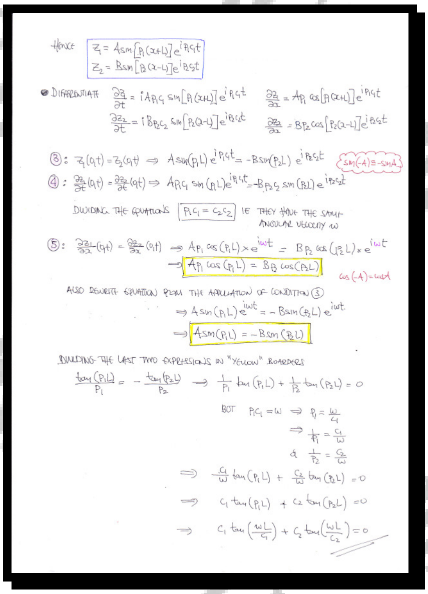
- WRITE THE TIME DEPENDENT PART IN COMPLEX FORM

$z_1 = M_1\sin[p_1(x+L)]\,\text{Re}\{e^{i(p_1c_1t+\theta_1)}\}$

$z_1 = M_1\sin[p_1(x+L)]\cdot\text{Re}\{e^{ip_1c_1t}\times e^{i\theta_1}\}$

$z_1 = [p_1(x+L)]\,\text{Re}\{M_1e^{i\theta_1}\times e^{ip_1c_1t}\}$ "DROPPING" THE REAL BUT WE IMPLY IT

$z_1 = A\sin[p_1(x+L)]e^{ip_1c_1t}$ $A = M_1e^{i\theta_1}$



HENCE $\boxed{\begin{aligned} z_1 &= A\sin[p_1(x+L)]e^{ip_1c_1t} \\ z_2 &= B\sin[p_2(x-L)]e^{ip_2c_2t} \end{aligned}}$

- DIFFERENTIATE $\frac{\partial z_1}{\partial t} = iAp_1c_1\sin[p_1(x+L)]e^{ip_1c_1t}$ $\frac{\partial z_1}{\partial x} = Ap_1\cos[p_1(x+L)]e^{ip_1c_1t}$
  $\frac{\partial z_2}{\partial t} = iBp_2c_2\sin[p_2(x-L)]e^{ip_2c_2t}$ $\frac{\partial z_2}{\partial x} = Bp_2\cos[p_2(x-L)]e^{ip_2c_2t}$

③: $z_1(0,t) = z_2(0,t) \implies A\sin(p_1L)e^{ip_1c_1t} = -B\sin(p_2L)e^{ip_2c_2t}$ $\sin(-A)\equiv-\sin A$

④: $\frac{\partial z_1}{\partial t}(0,t) = \frac{\partial z_2}{\partial t}(0,t) \implies Ap_1c_1\sin(p_1L)e^{ip_1c_1t} = -Bp_2c_2\sin(p_2L)e^{ip_2c_2t}$

DIVIDING THE EQUATIONS $\boxed{p_1c_1 = c_2c_2}$ IE THEY HAVE THE SAME ANGULAR VELOCITY ω

⑤: $\frac{\partial z_1}{\partial x}(0,t) = \frac{\partial z_2}{\partial x}(0,t) \implies Ap_1\cos(p_1L)\times e^{i\omega t} = Bp_2\cos(p_2L)\times e^{i\omega t}$

$\implies \boxed{Ap_1\cos(p_1L) = Bp_2\cos(p_2L)}$ $\cos(-A)=\cos A$

ALSO REWRITE EQUATION FROM THE APPLICATION OF CONDITION ③

$\implies A\sin(p_1L)e^{i\omega t} = -B\sin(p_2L)e^{i\omega t}$

$\implies \boxed{A\sin(p_1L) = -B\sin(p_2L)}$

DIVIDING THE LAST TWO EXPRESSIONS IN "YELLOW" BOARDERS

$\frac{\tan(p_1L)}{p_1} = -\frac{\tan(p_2L)}{p_2} \implies \frac{1}{p_1}\tan(p_1L) + \frac{1}{p_2}\tan(p_2L) = 0$

BUT $p_1c_1 = \omega \implies p_1 = \frac{\omega}{c_1}$

$\implies \frac{1}{p_1} = \frac{c_1}{\omega}$

& $\frac{1}{p_2} = \frac{c_2}{\omega}$

$\implies \frac{c_1}{\omega}\tan(p_1L) + \frac{c_2}{\omega}\tan(p_2L) = 0$

$\implies c_1\tan(p_1L) + c_2\tan(p_2L) = 0$

$\implies c_1\tan\left(\frac{\omega L}{c_1}\right) + c_2\tan\left(\frac{\omega L}{c_2}\right) = 0$

MULTIDIMENSIONAL WAVE EQUATION

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z = z(x, y, t)$$

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \quad z = z(r, \theta, t)$$

Question 1

The two dimensional wave equation for $u = u(x, y, t)$ in a rectangular cartesian region satisfies the following partial differential equation.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \quad 0 \le x \le a \, , \; 0 \le y \le b \, ,$$

where c is a positive constant.

It further given that $u = u(x, y, t)$ satisfies

$$u(0, y, t) = u(a, y, t) = u(x, 0, t) = u(x, b, t) = 0 \, .$$

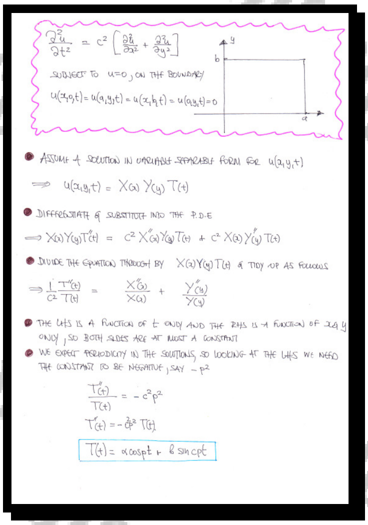
Use separation of variables to show that

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\left[A_{nm}\cos(\lambda_{nm}t) + B_{nm}\sin(\lambda_{nm}t)\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\right],$$

where A_{nm}, B_{nm} and λ_{nm} are constants.

$\boxed{\text{proof}}$

[solution overleaf]



$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

SUBJECT TO $u=0$, ON THE BOUNDARY

$u(x,0,t) = u(a,y,t) = u(x,b,t) = u(0,y,t) = 0$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM FOR $u(x,y,t)$

$$\Rightarrow u(x,y,t) = X(x)\,Y(y)\,T(t)$$

- DIFFERENTIATE & SUBSTITUTE INTO THE P.D.E

$$\Rightarrow X(x)Y(y)T''(t) = c^2X''(x)Y(y)T(t) + c^2X(x)Y''(y)T(t)$$

- DIVIDE THE EQUATION THROUGH BY $X(x)Y(y)T(t)$ & TIDY UP AS FOLLOWS

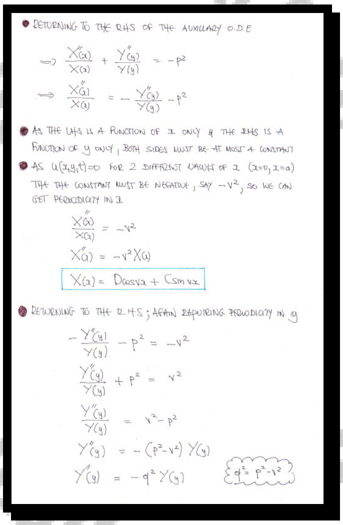
$$\Rightarrow \frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)}$$

- THE LHS IS A FUNCTION OF t ONLY AND THE RHS IS A FUNCTION OF x & y ONLY, SO BOTH SIDES ARE AT MOST A CONSTANT
- WE EXPECT PERIODICITY IN THE SOLUTIONS, SO LOOKING AT THE LHS WE NEED THE CONSTANT TO BE NEGATIVE, SAY $-p^2$

$$\frac{T''(t)}{T(t)} = -c^2p^2$$

$$T''(t) = -c^2p^2\,T(t)$$

$$\boxed{T(t) = \alpha\cos pt + \beta\sin cpt}$$



- RETURNING TO THE RHS OF THE AUXILLARY O.D.E

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -p^2$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - p^2$$

- AS THE LHS IS A FUNCTION OF x ONLY & THE RHS IS A FUNCTION OF y ONLY, BOTH SIDES MUST BE AT MOST A CONSTANT
- AS $u(x,y,t)=0$ FOR 2 DIFFERENT VALUES OF x ($x=0, x=a$) THE THE CONSTANT MUST BE NEGATIVE, SAY $-v^2$, SO WE CAN GET PERIODICITY IN x

$$\frac{X''(x)}{X(x)} = -v^2$$

$$X''(x) = -v^2X(x)$$

$$\boxed{X(x) = D\cos vx + C\sin vx}$$

- RETURNING TO THE R.H.S; AGAIN REQUIRING PERIODICITY IN y

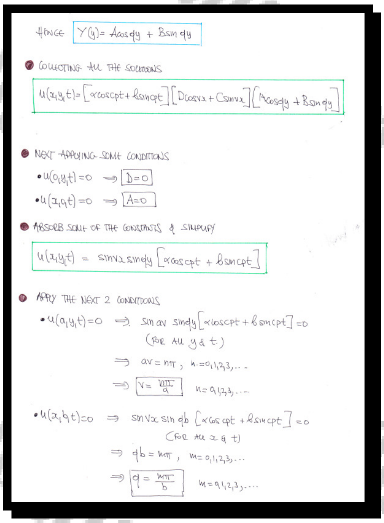
$$-\frac{Y''(y)}{Y(y)} - p^2 = -v^2$$

$$\frac{Y''(y)}{Y(y)} + p^2 = v^2$$

$$\frac{Y''(y)}{Y(y)} = v^2 - p^2$$

$$Y''(y) = -(p^2 - v^2)\,Y(y)$$

$$Y''(y) = -q^2\,Y(y) \qquad q^2 = p^2 - v^2$$



HENCE $\boxed{Y(y) = A\cos qy + B\sin qy}$

- COLLECTING ALL THE SOLUTIONS

$$\boxed{u(x,y,t) = \left[\alpha\cos cpt + \beta\sin cpt\right]\left[D\cos vx + C\sin vx\right]\left[A\cos qy + B\sin qy\right]}$$

- NEXT APPLYING SOME CONDITIONS
  - $u(0,y,t) = 0 \Rightarrow \boxed{D=0}$
  - $u(x,0,t) = 0 \Rightarrow \boxed{A=0}$
- ABSORB SOME OF THE CONSTANTS & SIMPLIFY

$$\boxed{u(x,y,t) = \sin vx \sin qy\left[\alpha\cos cpt + \beta\sin cpt\right]}$$

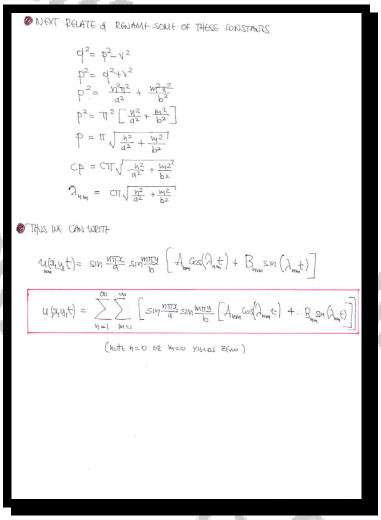
- APPLY THE NEXT 2 CONDITIONS
  - $u(a,y,t) = 0 \Rightarrow \sin av \sin qy\left[\alpha\cos cpt + \beta\sin cpt\right] = 0$ (FOR ALL y & t)

    $\Rightarrow av = n\pi, \quad n = 0,1,2,3,\ldots$

    $\Rightarrow \boxed{v = \frac{n\pi}{a}} \quad n = 0,1,2,3,\ldots$
  - $u(x,b,t) = 0 \Rightarrow \sin vx \sin qb\left[\alpha\cos cpt + \beta\sin cpt\right] = 0$ (FOR ALL x & t)

    $\Rightarrow qb = m\pi, \quad m = 0,1,2,3,\ldots$

    $\Rightarrow \boxed{q = \frac{m\pi}{b}} \quad m = 0,1,2,3,\ldots$



- NEXT RELATE & RENAME SOME OF THESE CONSTANTS

$$q^2 = p^2 - v^2$$

$$p^2 = q^2 + v^2$$

$$p^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$$p^2 = \pi^2\left[\frac{n^2}{a^2} + \frac{m^2}{b^2}\right]$$

$$p = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$cp = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

$$\lambda_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

- THUS WE CAN WRITE

$$u_{nm}(x,y,t) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\left[A_{nm}\cos(\lambda_{nm}t) + B_{nm}\sin(\lambda_{nm}t)\right]$$

$$\boxed{u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\left[A_{nm}\cos(\lambda_{nm}t) + B_{nm}\sin(\lambda_{nm}t)\right]\right]}$$

(NOTE $n=0$ OR $m=0$ YIELDS ZERO)

Question 2

The vertical displacement $z = z(r,\theta,t)$ of a two dimensional standing wave in plane polar coordinates, satisfies the following partial differential equation.

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}.$$

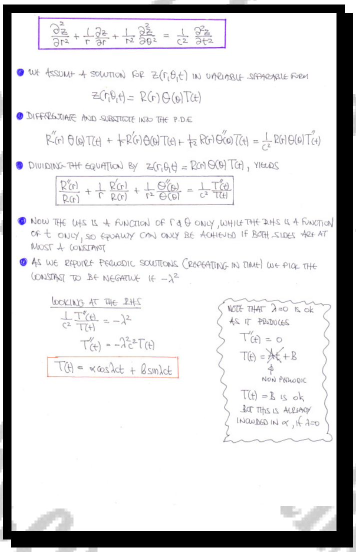
where c is a positive constant.

Use separation of variables to show that the general solution of the above equation can be written as

$$z(r,\theta,t)=[\alpha\cos\lambda ct+\beta\sin\lambda ct]\left[\sum_{n=0}^{\infty}C_n\sin n\theta+D_n\cos n\theta\right]\left[\sum_{n=0}^{\infty}A_nJ_n(\lambda r)+B_nY_n(\lambda r)\right]$$

where α, β, A_n, B_n, C_n and D_n are constants.

$\boxed{\text{proof}}$



$$\boxed{\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}}$$

- WE ASSUME A SOLUTION FOR $z(r,\theta,t)$ IN VARIABLE SEPARABLE FORM

$$z(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$R''(r)\Theta(\theta)T(t)+\frac{1}{r}R'(r)\Theta(\theta)T(t)+\frac{1}{r^2}R(r)\Theta''(\theta)T(t)=\frac{1}{c^2}R(r)\Theta(\theta)T''(t)$$

- DIVIDING THE EQUATION BY $z(r,\theta,t)=R(r)\Theta(\theta)T(t)$, YIELDS

$$\boxed{\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}}$$

- NOW THE LHS IS A FUNCTION OF r & θ ONLY, WHILE THE RHS IS A FUNCTION OF t ONLY, SO EQUALITY CAN ONLY BE ACHIEVED IF BOTH SIDES ARE AT MOST A CONSTANT
- AS WE REQUIRE PERIODIC SOLUTIONS (REPEATING IN TIME) WE PICK THE CONSTANT TO BE NEGATIVE IE $-\lambda^2$

LOOKING AT THE RHS

$$\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\lambda^2$$

$$T''(t)=-\lambda^2c^2T(t)$$

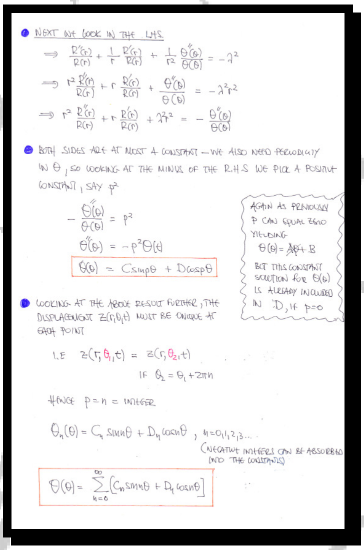
$$\boxed{T(t)=\alpha\cos\lambda ct+\beta\sin\lambda ct}$$

NOTE THAT $\lambda=0$ IS OK AS IT PRODUCES

$$T''(t)=0$$

$$T(t)=\underset{\substack{\uparrow\\ \text{NON PERIODIC}}}{\cancel{At}}+B$$

$T(t)=B$ IS OK BUT THIS IS ALREADY INCLUDED IN α, IF $\lambda=0$



- NEXT WE LOOK IN THE LHS.

$$\Longrightarrow \frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda^2$$

$$\Longrightarrow r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda^2r^2$$

$$\Longrightarrow r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\lambda^2r^2=-\frac{\Theta''(\theta)}{\Theta(\theta)}$$

- BOTH SIDES ARE AT MOST A CONSTANT — WE ALSO NEED PERIODICITY IN θ, SO LOOKING AT THE MINUS OF THE R.H.S WE PICK A POSITIVE CONSTANT, SAY p^2

$$-\frac{\Theta''(\theta)}{\Theta(\theta)}=p^2$$

$$\Theta''(\theta)=-p^2\Theta(t)$$

$$\boxed{\Theta(\theta)=C\sin p\theta+D\cos p\theta}$$

AGAIN AS PREVIOUSLY p CAN EQUAL ZERO YIELDING

$$\Theta(\theta)=\cancel{A\theta}+B$$

BUT THIS CONSTANT SOLUTION FOR $\Theta(\theta)$ IS ALREADY INCLUDED IN D, IF $p=0$

- LOOKING AT THE ABOVE RESULT FURTHER, THE DISPLACEMENT $z(r,\theta,t)$ MUST BE UNIQUE AT EACH POINT

I.E $z(r,\theta_1,t)=z(r,\theta_2,t)$

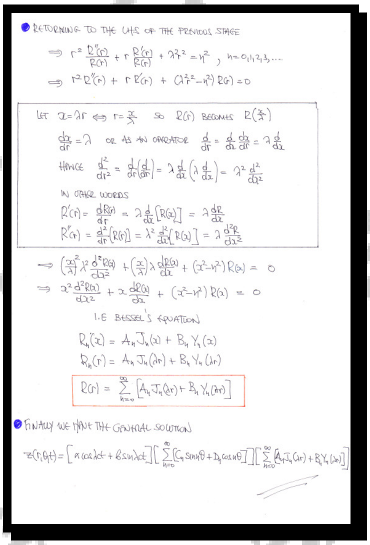
IF $\theta_2=\theta_1+2\pi n$

HENCE $p=n=$ INTEGER

$$\Theta_n(\theta)=C_n\sin n\theta+D_n\cos n\theta\,,\quad n=0,1,2,3\ldots$$

(NEGATIVE INTEGERS CAN BE ABSORBED INTO THE CONSTANTS)

$$\boxed{\Theta(\theta)=\sum_{n=0}^{\infty}\left[C_n\sin n\theta+D_n\cos n\theta\right]}$$



- RETURNING TO THE LHS OF THE PREVIOUS STAGE

$$\Longrightarrow r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\lambda^2r^2=n^2\,,\quad n=0,1,2,3,\ldots$$

$$\Longrightarrow r^2R''(r)+rR'(r)+(\lambda^2r^2-n^2)R(r)=0$$

LET $x=\lambda r \Longleftrightarrow r=\frac{x}{\lambda}$ SO $R(r)$ BECOMES $R\left(\frac{x}{\lambda}\right)$

$\frac{dx}{dr}=\lambda$ OR AS AN OPERATOR $\frac{d}{dr}=\frac{d}{dx}\frac{dx}{dr}=\lambda\frac{d}{dx}$

HENCE $\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{d}{dr}\right)=\lambda\frac{d}{dx}\left(\lambda\frac{d}{dx}\right)=\lambda^2\frac{d^2}{dx^2}$

IN OTHER WORDS

$$R'(r)=\frac{dR(r)}{dr}=\lambda\frac{d}{dx}[R(x)]=\lambda\frac{dR}{dx}$$

$$R''(r)=\frac{d^2}{dr^2}[R(r)]=\lambda^2\frac{d^2}{dx^2}[R(x)]=\lambda\frac{d^2R}{dx^2}$$

$$\Longrightarrow \left(\frac{x}{\lambda}\right)^2\lambda^2\frac{d^2R(x)}{dx^2}+\left(\frac{x}{\lambda}\right)\lambda\frac{dR(x)}{dx}+(x^2-n^2)R(x)=0$$

$$\Longrightarrow x^2\frac{d^2R(x)}{dx^2}+x\frac{dR(x)}{dx}+(x^2-n^2)R(x)=0$$

I.E BESSEL'S EQUATION

$$R_n(x)=A_nJ_n(x)+B_nY_n(x)$$

$$R_n(r)=A_nJ_n(\lambda r)+B_nY_n(\lambda r)$$

$$\boxed{R(r)=\sum_{n=0}^{\infty}\left[A_nJ_n(\lambda r)+B_nY_n(\lambda r)\right]}$$

- FINALLY WE HAVE THE GENERAL SOLUTION

$$z(r,\theta,t)=\left[\alpha\cos\lambda ct+\beta\sin\lambda ct\right]\left[\sum_{n=0}^{\infty}\left[C_n\sin n\theta+D_n\cos n\theta\right]\right]\left[\sum_{n=0}^{\infty}\left[A_nJ_n(\lambda r)+B_nY_n(\lambda r)\right]\right]$$

Question 3

The vertical displacement $z = z(r,\theta,t)$ of a circular drum-skin, secured on a circular rim of radius a, satisfies the wave equation in standard plane polar coordinates

$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}.$$

where c is a positive constant.

The drum-skin is displaced from its equilibrium position and released from rest.

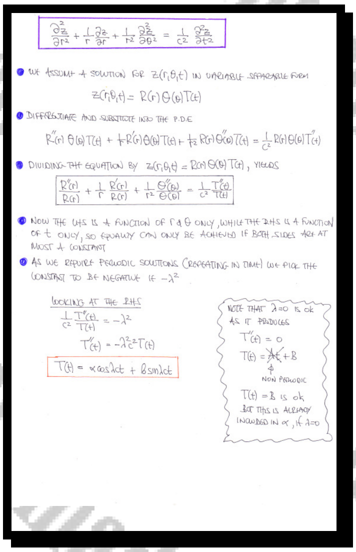
Use separation of variables to show that general solution of the above equation is

$$z(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\left[J_n\left(\frac{r\lambda_{n,m}}{a}\right)\right]\left[\cos\left(\frac{ct\lambda_{n,m}}{a}\right)\right]\left[C_{n,m}\sin n\theta+D_{n,m}\cos n\theta\right]\right]$$

where $C_{n,m}$ and $D_{n,m}$ are constants, and $\lambda_{n,m}$ denotes the m^{th} zero of $J_n(x)$.

$\boxed{\text{proof}}$

[solution overleaf]



$$\boxed{\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}}$$

- WE ASSUME A SOLUTION FOR $z(r,\theta,t)$ IN VARIABLE SEPARABLE FORM

$$z(r,\theta,t)=R(r)\Theta(\theta)T(t)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$R''(r)\Theta(\theta)T(t)+\frac{1}{r}R'(r)\Theta(\theta)T(t)+\frac{1}{r^2}R(r)\Theta''(\theta)T(t)=\frac{1}{c^2}R(r)\Theta(\theta)T''(t)$$

- DIVIDING THE EQUATION BY $z(r,\theta,t)=R(r)\Theta(\theta)T(t)$, YIELDS

$$\boxed{\frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}=\frac{1}{c^2}\frac{T''(t)}{T(t)}}$$

- NOW THE LHS IS A FUNCTION OF r & θ ONLY, WHILE THE RHS IS A FUNCTION OF t ONLY, SO EQUALITY CAN ONLY BE ACHIEVED IF BOTH SIDES ARE AT MOST A CONSTANT
- AS WE REQUIRE PERIODIC SOLUTIONS (REPEATING IN TIME) WE PICK THE CONSTANT TO BE NEGATIVE IE $-\lambda^2$

LOOKING AT THE RHS

$$\frac{1}{c^2}\frac{T''(t)}{T(t)}=-\lambda^2$$

$$T''(t)=-\lambda^2c^2T(t)$$

$$\boxed{T(t)=\alpha\cos\lambda ct+\beta\sin\lambda ct}$$

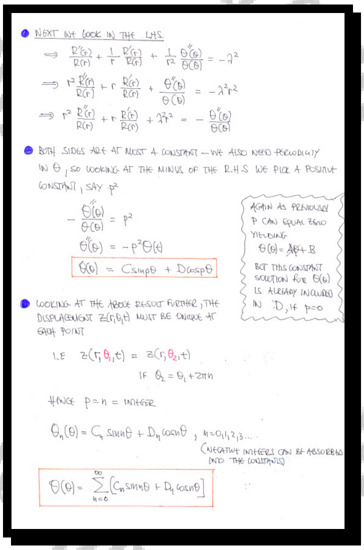
NOTE THAT $\lambda=0$ IS ok AS IT PRODUCES

$$T''(t)=0$$

$$T(t)=\cancel{At}+B$$

NON PERIODIC

$T(t)=B$ IS ok BUT THIS IS ALREADY INCLUDED IN α, IF $\lambda=0$



- NEXT WE LOOK IN THE LHS.

$$\Longrightarrow \frac{R''(r)}{R(r)}+\frac{1}{r}\frac{R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda^2$$

$$\Longrightarrow r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda^2r^2$$

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- BOTH SIDES ARE AT MOST A CONSTANT — WE ALSO NEED PERIODICITY IN θ, SO LOOKING AT THE MINUS OF THE R.H.S WE PICK A POSITIVE CONSTANT, SAY p^2

$$-\frac{\Theta''(\theta)}{\Theta(\theta)}=p^2$$

$$\Theta''(\theta)=-p^2\Theta(t)$$

$$\boxed{\Theta(\theta)=C\sin p\theta+D\cos p\theta}$$

AGAIN AS PREVIOUSLY p CAN EQUAL ZERO YIELDING

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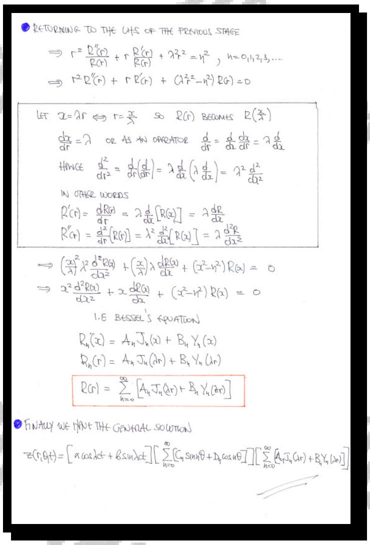
IF $\theta_2=\theta_1+2\pi n$

HENCE $p=n=$ INTEGER

$$\Theta_n(\theta)=C_n\sin n\theta+D_n\cos n\theta\ ,\quad n=0,1,2,3\dots$$

(NEGATIVE INTEGERS CAN BE ABSORBED INTO THE CONSTANTS)

$$\boxed{\Theta(\theta)=\sum_{n=0}^{\infty}\left[C_n\sin n\theta+D_n\cos n\theta\right]}$$



- RETURNING TO THE LHS OF THE PREVIOUS STAGE

$$\Longrightarrow r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\lambda^2r^2=n^2\ ,\quad n=0,1,2,3,\dots$$

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HENCE $\frac{d^2}{dr^2}=\frac{d}{dr}\left(\frac{d}{dr}\right)=\lambda\frac{d}{dx}\left(\lambda\frac{d}{dx}\right)=\lambda^2\frac{d^2}{dx^2}$

IN OTHER WORDS

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$$\Longrightarrow \left(\frac{x}{\lambda}\right)^2\lambda^2\frac{d^2R(x)}{dx^2}+\left(\frac{x}{\lambda}\right)\lambda\frac{dR(x)}{dx}+(x^2-n^2)R(x)=0$$

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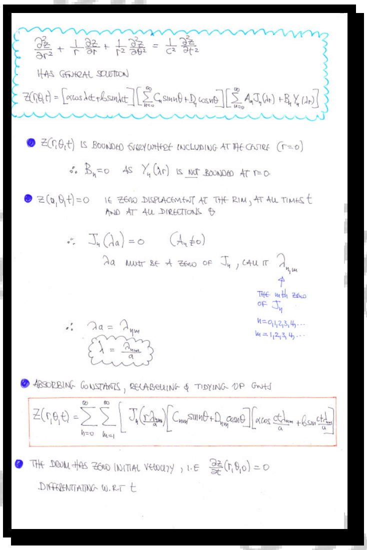
$$z(r,\theta,t)=\left[\alpha\cos\lambda ct+\beta\sin\lambda ct\right]\left[\sum_{n=0}^{\infty}\left[C_n\sin n\theta+D_n\cos n\theta\right]\right]\left[\sum_{n=0}^{\infty}\left[A_nJ_n(\lambda r)+B_nY_n(\lambda r)\right]\right]$$



$$\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

HAS GENERAL SOLUTION

$$z(r,\theta,t)=\left[\alpha\cos\lambda ct+\beta\sin\lambda ct\right]\left[\sum_{n=0}^{\infty}C_n\sin n\theta+D_n\cos n\theta\right]\left[\sum_{n=0}^{\infty}A_nJ_n(\lambda r)+B_nY_n(\lambda r)\right]$$

- $z(r,\theta,t)$ IS BOUNDED EVERYWHERE INCLUDING AT THE CENTRE $(r=0)$

$\therefore B_n=0$ AS $Y_n(\lambda r)$ IS NOT BOUNDED AT $r=0$.

- $z(a,\theta,t)=0$ IE ZERO DISPLACEMENT AT THE RIM, AT ALL TIMES t AND AT ALL DIRECTIONS θ

$\therefore J_n(\lambda a)=0 \quad (A_n\neq0)$

λa MUST BE A ZERO OF J_n, CALL IT $\lambda_{n,m}$

THE mth ZERO OF J_n

$n=0,1,2,3,4,\dots$

$m=1,2,3,4,\dots$

$\therefore \lambda a=\lambda_{nm}$

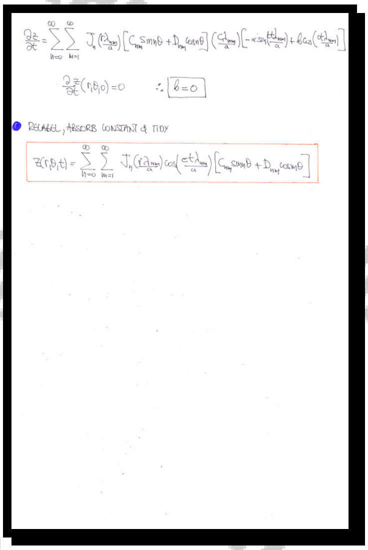
$$\lambda=\frac{\lambda_{nm}}{a}$$

- ABSORBING CONSTANTS, RELABELLING & TIDYING UP GIVES

$$\boxed{z(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[J_n\left(\frac{r\lambda_{nm}}{a}\right)\left[C_{nm}\sin n\theta+D_{nm}\cos n\theta\right]\left[\alpha\cos\frac{ct\lambda_{nm}}{a}+\beta\sin\frac{ct\lambda_{nm}}{a}\right]\right]}$$

- THE DRUM HAS ZERO INITIAL VELOCITY, I.E $\frac{\partial z}{\partial t}(r,\theta,0)=0$

DIFFERENTIATING W.R.T t



$$\frac{\partial z}{\partial t}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}J_n\left(\frac{r\lambda_{nm}}{a}\right)\left[C_{nm}\sin n\theta+D_{nm}\cos n\theta\right]\left(\frac{c\lambda_{nm}}{a}\right)\left[-\alpha\sin\left(\frac{ct\lambda_{nm}}{a}\right)+\beta\cos\left(\frac{ct\lambda_{nm}}{a}\right)\right]$$

$$\frac{\partial z}{\partial t}(r,\theta,0)=0 \qquad \therefore \boxed{\beta=0}$$

- RELABEL, ABSORB CONSTANT & TIDY

$$\boxed{z(r,\theta,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}J_n\left(\frac{r\lambda_{nm}}{a}\right)\cos\left(\frac{ct\lambda_{nm}}{a}\right)\left[C_{nm}\sin n\theta+D_{nm}\cos m\theta\right]}$$