LAPLACE'S EQUATION

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$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad \Phi = \Phi(x, y)$$

Two Dimensional in Cartesian

Question 1

The temperature distribution, $T(x, y)$, in a rectangular plate for which $0 \le x \le 2$ and $0 \le y \le 1$, satisfies Laplace's Equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

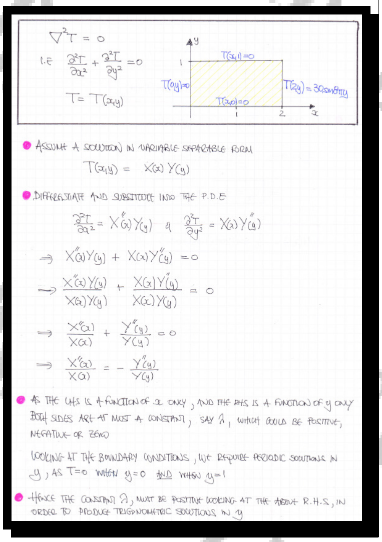
The edges of the rectangular plate are maintained at the following temperatures

- $T(x,0) = 0, \quad x \in [0,2]$
- $T(x,1) = 0, \quad x \in [0,2]$
- $T(0,y) = 0, \quad y \in [0,1]$
- $T(2,y) = 30\sin(8\pi y), \quad y \in [0,1]$.

Determine the temperature distribution of the plate

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{T(x,y) = \frac{80\sinh(8\pi x)\sin(8\pi y)}{\sinh 16\pi}}$$


$\nabla^2 T = 0$

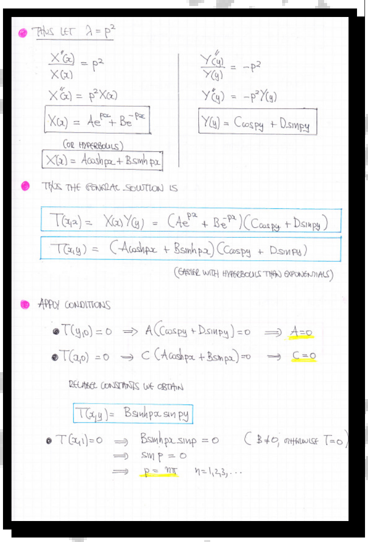
I.E $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

$T = T(x,y)$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM
$$T(x,y) = X(x)Y(y)$$
- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E
$$\frac{\partial^2 T}{\partial x^2} = X''(x)Y(y) \quad \& \quad \frac{\partial^2 T}{\partial y^2} = X(x)Y''(y)$$
$$\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$$
$$\Rightarrow \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X(x)Y''(y)}{X(x)Y(y)} = 0$$
$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$
$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$
- AS THE LHS IS A FUNCTION OF x ONLY, AND THE RHS IS A FUNCTION OF y ONLY BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH COULD BE POSITIVE, NEGATIVE OR ZERO

LOOKING AT THE BOUNDARY CONDITIONS, WE REQUIRE PERIODIC SOLUTIONS IN y, AS $T=0$ WHEN $y=0$ AND WHEN $y=1$

- HENCE THE CONSTANT λ, MUST BE POSITIVE LOOKING AT THE ABOVE R.H.S, IN ORDER TO PRODUCE TRIGONOMETRIC SOLUTIONS IN y


- THUS LET $\lambda = p^2$

$\frac{X''(x)}{X(x)} = p^2$

$X''(x) = p^2 X(x)$

$\boxed{X(x) = Ae^{px} + Be^{-px}}$

(OR HYPERBOLICS)

$\boxed{X(x) = A\cosh px + B\sinh px}$

$\frac{Y''(y)}{Y(y)} = -p^2$

$Y''(y) = -p^2 Y(y)$

$\boxed{Y(y) = C\cos py + D\sin py}$

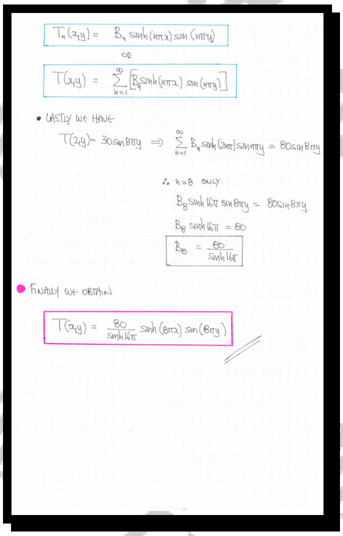
- THUS THE GENERAL SOLUTION IS
$$\boxed{T(x,y) = X(x)Y(y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py)}$$
$$\boxed{T(x,y) = (A\cosh px + B\sinh px)(C\cos py + D\sin py)}$$
(EASIER WITH HYPERBOLICS THAN EXPONENTIALS)

- APPLY CONDITIONS
  - $T(y,0) = 0 \Rightarrow A(C\cos py + D\sin py) = 0 \Rightarrow A = 0$
  - $T(x,0) = 0 \Rightarrow C(A\cosh px + B\sinh px) = 0 \Rightarrow C = 0$

  RELABEL CONSTANTS WE OBTAIN

  $\boxed{T(x,y) = B\sinh px \sin py}$

  - $T(x,1) = 0 \Rightarrow B\sinh px \sin p = 0 \quad (B \ne 0, \text{ OTHERWISE } T=0)$
  $\Rightarrow \sin p = 0$
  $\Rightarrow p = n\pi \quad n = 1, 2, 3, \dots$


$$\boxed{T_n(x,y) = B_n \sinh(n\pi x)\sin(n\pi y)}$$
OR
$$\boxed{T(x,y) = \sum_{n=1}^{\infty}\left[B_n \sinh(n\pi x)\sin(n\pi y)\right]}$$

- LASTLY WE HAVE

$T(2,y) = 30\sin 8\pi y \Rightarrow \sum_{n=1}^{\infty} B_n \sinh(2n\pi)\sin n\pi y = 80\sin 8\pi y$

$\therefore n = 8$ ONLY

$B_8 \sinh 16\pi \sin 8\pi y = 80\sin 8\pi y$

$B_8 \sinh 16\pi = 80$

$\boxed{B_8 = \frac{80}{\sinh 16\pi}}$

- FINALLY WE OBTAIN
$$\boxed{T(x,y) = \frac{80}{\sinh 16\pi}\sinh(8\pi x)\sin(8\pi y)}$$

Question 2

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

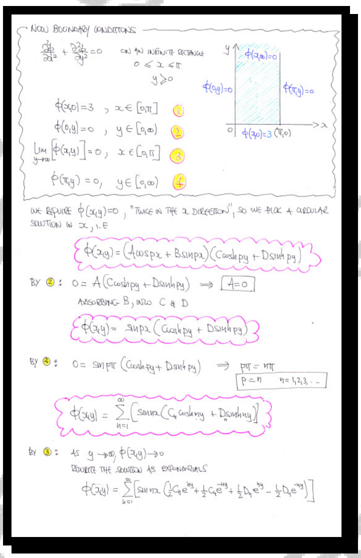
where $0 \le x \le \pi$ and $y > 0$.

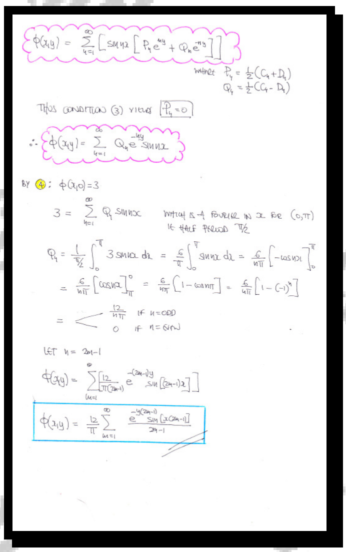
Determine an expression for $\varphi(x, y)$, given further that

- $\varphi(x, 0) = 3, \quad x \in [0, \pi]$
- $\varphi(0, y) = 0, \quad y \in [0, \infty)$
- $\lim\limits_{y \to \infty} [\varphi(x, y)] = 0, \quad x \in [0, \pi]$
- $\varphi(\pi, y) = 0, \quad y \in [0, \infty)$.

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{\varphi(x, y) = \frac{12}{\pi} \sum_{n=1}^{\infty} \frac{\exp[-y(2n-1)]\sin[x(2n-1)]}{(2n-1)^3}}$$





Question 3

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

where $0 \le x \le a$ and $0 \le y \le b$, with a and b positive constants.

Determine an expression for $\varphi(x, y)$, given further that

- $\varphi(x,0) = 0, \quad x \in [0,a]$
- $\varphi(x,b) = 0, \quad x \in [0,a]$
- $\varphi(0, y) = 0, \quad y \in [0,b]$
- $\varphi(a, y) = by - y^2, \quad y \in [0,b]$.

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$$\boxed{\varphi(x, y) = \frac{8b^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sinh\left[\frac{\pi x}{b}(2n-1)\right] \sin\left[\frac{\pi y}{b}(2n-1)\right]}{(2n-1)^3 \sinh\left[\frac{\pi a}{b}(2n-1)\right]}}$$

[solution overleaf]

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- LOOK FOR A SOLUTION IN VARIABLE SEPARABLE FORM $\phi(x,y) = X(x)Y(y)$

$$\frac{\partial^2 \phi}{\partial x^2} = X''(x)Y(y)$$
$$\frac{\partial^2 \phi}{\partial y^2} = X(x)Y''(y)$$

- SUB INTO THE PDE

$$X''(x)Y(y) + X(x)Y''(y) = 0$$
$$X''(x)Y(y) = -X(x)Y''(y)$$
$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

AS THE LHS IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF y ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

- $\boxed{\text{IF } \lambda = 0}$ $\quad X''(x) = 0$, $\quad X(x) = Ax + B$; $\quad Y''(y) = 0$, $\quad Y(y) = Cy + D$

$$\therefore \phi(x,y) = (Ax+B)(Cy+D)$$

- $\boxed{\text{IF } \lambda > 0, \text{ SAY } \lambda = p^2}$ $\quad X''(x) = p^2 X(x)$, $\quad X(x) = A\cosh px + B\sinh px$ (OR EXPONENTIALS); $\quad Y''(y) = -p^2 Y(y)$, $\quad Y(y) = C\cos py + D\sin py$

$$\therefore \phi(x,y) = (A\cosh px + B\sinh px)(C\cos py + D\sin py)$$

- $\boxed{\text{IF } \lambda < 0, \text{ SAY } \lambda = -p^2}$ $\quad X''(x) = -p^2 X(x)$, $\quad X(x) = A\cos px + B\sin px$; $\quad Y''(y) = p^2 Y(y)$, $\quad Y(y) = C\cosh py + D\sinh py$

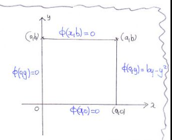
$$\therefore \phi(x,y) = (A\cos px + B\sin px)(C\cosh py + D\sinh py)$$

THE CHOICE OF SOLUTION NOW DEPENDS ON THE BOUNDARY CONDITIONS

NOW BOUNDARY CONDITIONS

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{ON A RECTANGLE} \quad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b$$

1. $\phi(x,0) = 0, \quad x \in [0,a]$
2. $\phi(x,b) = 0, \quad x \in [0,a]$
3. $\phi(0,y) = 0, \quad y \in [0,b]$
4. $\phi(a,y) = by - y^2, \quad y \in [0,b]$



WE REQUIRE $\phi(x,y) = 0$ "TWICE IN THE y DIRECTION", SO WE PICK A CIRCULAR SOLUTION IN y

I.E. $\phi(x,y) = (A\cosh px + B\sinh px)(C\cos py + D\sin py)$

BY ①: $\quad 0 = (A\cosh px + B\sinh px) \times C \implies \boxed{C = 0}$

ABSORBING D INTO A & B

$$\phi(x,y) = \sin py\,(A\cosh px + B\sinh px)$$

BY ②: $\quad 0 = \sin(pb)(A\cosh px + B\sinh px) \implies pb = n\pi \implies \boxed{p = \frac{n\pi}{b} \quad n = 1,2,3,\ldots}$

$$\phi(x,y) = \sum_{n=1}^{\infty}\left[\sin\left(\frac{n\pi y}{b}\right)\left[A_n\cosh\frac{n\pi x}{b} + B_n\sinh\frac{n\pi x}{b}\right]\right]$$

BY ③: $\quad 0 = \sum_{n=1}^{\infty}\left[A_n \sin\frac{n\pi y}{b}\right] \implies \boxed{A_n = 0}$

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

BY ④: $\quad by - y^2 = \sum_{n=1}^{\infty} \underbrace{B_n\sinh\frac{n\pi a}{b}}_{D_n}\sin\frac{n\pi y}{b}$

THIS IS A FOURIER SERIES IN y, PERIOD b

$$\implies D_n = \frac{1}{b/2}\int_0^b (by - y^2)\sin\frac{n\pi y}{b}\,dy$$
$$\implies D_n = \frac{2}{b}\int_0^b by\sin\frac{n\pi y}{b}\,dy - \frac{2}{b}\int y^2\sin\frac{n\pi y}{b}\,dy$$
$$\implies D_n = 2\int_0^b y\sin\frac{n\pi y}{b}\,dy - \frac{2}{b}\int_0^b y^2\sin\frac{n\pi y}{b}\,dy$$

CARRY OUT INTEGRATIONS BY PARTS

- $\displaystyle\int_0^b y\sin\frac{n\pi y}{b}\,dy = \left[-\frac{by}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b + \int_0^b \frac{b}{n\pi}\cos\frac{n\pi y}{b}\,dy$

y	1
$-\frac{b}{n\pi}\cos\frac{n\pi y}{b}$	$\sin\frac{n\pi y}{b}$

$$= -\frac{b^2}{n\pi}\cos n\pi + \frac{b^2}{n^2\pi^2}\cancel{\left[\sin\frac{n\pi y}{b}\right]_0^b}$$
$$= -\frac{b^2}{n\pi}(-1)^n$$

- $\displaystyle\int_0^b y^2\sin\frac{n\pi y}{b}\,dy = \left[-\frac{b}{n\pi}y^2\cos\frac{n\pi y}{b}\right]_0^b + \int_0^b \frac{2by}{n\pi}\cos\frac{n\pi y}{b}\,dy$

y^2	$2y$
$-\frac{b}{n\pi}\cos\frac{n\pi y}{b}$	$\sin\frac{n\pi y}{b}$

$$= -\frac{b^3}{n\pi}\cos n\pi + \frac{2b}{n\pi}\int_0^b y\cos\frac{n\pi y}{b}$$

BY PARTS AGAIN

y	1
$\frac{b}{n\pi}\sin\frac{n\pi y}{b}$	$\cos\frac{n\pi y}{b}$

$$= -\frac{b^3}{n\pi}(-1)^n + \frac{2b}{n\pi}\left\{\cancel{\left[\frac{by}{n\pi}\sin\frac{n\pi y}{b}\right]_0^b} - \frac{b}{n\pi}\int_0^b \sin\frac{n\pi y}{b}\,dy\right\}$$
$$= -\frac{b^3}{n\pi}(-1)^n - \frac{2b^2}{n^2\pi^2}\left[-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right]_0^b$$
$$= -\frac{b^3}{n\pi}(-1)^n + \frac{2b^3}{n^3\pi^3}\left[\cos\frac{n\pi y}{b}\right]_0^b$$
$$= -\frac{b^3}{n\pi}(-1)^n + \frac{2b^3}{n^3\pi^3}\left[\cos n\pi - 1\right]$$
$$= -\frac{b^3}{n\pi}(-1)^n + \frac{2b^3}{n^3\pi^3}\left[(-1)^n - 1\right]$$

THUS $\quad D_n = 2\left[-\frac{b^2}{n\pi}(-1)^n\right] - \frac{2}{b}\left[-\frac{b^3}{n\pi}(-1)^n + \frac{2b^3}{n^3\pi^3}\left[(-1)^n - 1\right]\right]$

HENCE

$$B_n\sinh\frac{n\pi a}{b} = \cancel{-\frac{2b^2}{n\pi}(-1)^n} + \cancel{\frac{2b^2}{n\pi}(-1)^n} - \frac{4b^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$\boxed{B_n\sinh\frac{n\pi a}{b} = \frac{4b^2}{n^3\pi^3}\left(1 - (-1)^n\right)}$$

IF n = ODD $\quad B_n = \frac{8b^2}{n^3\pi^3}$

IF n = EVEN $\quad B_n = 0$

- REWRITE $n = 2m-1 \quad m = 1,2,3,$

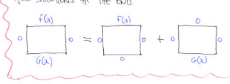
$$B_{2m-1}\sinh\frac{(2m-1)\pi a}{b} = \frac{8b^2}{(2m-1)^3\pi^3} \quad m = 1,2,3,\ldots$$

$$\boxed{B_{2m-1} = \frac{8b^2}{(2m-1)^3\pi^3\sinh\left[\frac{(2m-1)\pi a}{b}\right]}}$$

THUS

$$\phi(x,y) = \frac{8b^2}{\pi^3}\sum_{m=1}^{\infty}\left[\frac{\sinh\left(\frac{\pi x}{b}(2m-1)\right)\sin\left(\frac{\pi y}{b}(2m-1)\right)}{(2m-1)^3\sinh\left[\frac{(2m-1)\pi a}{b}\right]}\right]$$

NOTE IF $\phi(x,y) \neq 0$ ON MORE THAN ONE SIDE, THEN USE LINEARITY & ADD SOLUTIONS AT THE END



Question 4

The function $u = u(x,y)$ satisfies Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le 2, \; 0 \le y \le 1.$$

Determine an expression for $u = u(x,y)$, given further that

$$u(0,y) = 0, \quad u(2,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} \left[\frac{8\sin\left(\frac{1}{2}n\pi\right)}{n^2\pi^2 \sinh\left(\frac{1}{2}n\pi\right)} \sin\left(\frac{1}{2}n\pi x\right) \sinh\left(\frac{1}{2}n\pi y\right) \right]}$$

ASSUME A SOLUTION IN VARIABLE FORM, DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$u(x,y) = X(x)Y(y) \implies \frac{\partial^2 u}{\partial x^2} = X''(x)Y(y), \quad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\implies X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\implies \frac{X''(x)Y(y)}{X(x)Y(y)} + \frac{X(x)Y''(y)}{X(x)Y(y)} = \frac{0}{X(x)Y(y)}$$

$$\implies \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$\implies \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

AS THE L.H.S OF THE ABOVE EQUATION IS A FUNCTION OF x ONLY & THE R.H.S. IS A FUNCTION OF y ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ – LOOKING AT THE BOUNDARY CONDITIONS

(1) $u(0,y) = 0$
(2) $u(2,y) = 0$
(3) $u(x,0) = 0$
(4) $u(x,1) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$

LOOKING AT (1) & (2), WE OBSERVE THAT WE NEED A CIRCULAR SOLUTION IN x (AS $u(0,y) = u(2,y)$), SO THE CONSTANT λ MUST BE NEGATIVE, $\lambda = -p^2$

$$\implies \frac{X''(x)}{X(x)} = -p^2 \qquad \implies -\frac{Y''(y)}{Y(y)} = -p^2$$

$$\implies X''(x) = -p^2 X(x) \qquad \implies Y''(y) = p^2 Y(y)$$

$$\implies X(x) = A\cos px + B\sin px \qquad \implies Y(y) = C\cosh py + D\sinh py$$

(OR EXPONENTIALS)

THUS THE GENERAL SOLUTION IS

$$\boxed{u(x,y) = X(x)Y(y) = [A\cos px + B\sin px][C\cosh py + D\sinh py]}$$

- APPLY (1), $u(0,y) = 0$

$$\implies 0 = A(C\cosh py + D\sinh py), \quad 0 \le y \le 1$$
$$\implies A = 0$$

ABSORBING A INTO C AND D, YIELDS

$$\boxed{u(x,y) = (\sin px)(C\cosh py + D\sinh py)}$$

- APPLY (3), $u(x,0) = 0$

$$\implies 0 = (\sin px)C$$
$$\implies C = 0$$
$$\implies \boxed{u(x,y) = D\sin px \sinh py}$$

- APPLY (2), $u(2,y) = 0$

$$\implies 0 = D\sin 2p \sinh py$$
$$\implies D \ne 0, \text{ OTHERWISE WE HAVE A TRIVIAL SOLUTION}$$
$$\implies \sin 2p = 0$$
$$\implies 2p = n\pi \qquad n = 0,1,2,3,\dots$$
$$\implies p = \frac{n\pi}{2} \qquad n = 0,1,2,3,\dots$$

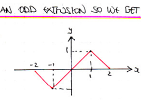
$$\therefore \boxed{\begin{aligned} u_n(x,y) &= D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right) \\ u(x,y) &= \sum_{n=1}^{\infty}\left[D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)\right] \end{aligned}}$$

- FINALLY APPLY (4) $u(x,1) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \end{cases}$

$$\implies u(x,1) = \sum_{n=1}^{\infty}\left[D_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi}{2}\right)\right]$$

$$\implies u(x,1) = \sum_{n=1}^{\infty}\left[\left[D_n \sinh\frac{n\pi}{2}\right]\sin\left(\frac{n\pi x}{2}\right)\right]$$

THIS IS A FOURIER SERIES IN x – BUILT AN ODD EXTENSION SO WE GET AN EVEN INTEGRAND WITH PERIOD 4



$$\implies D_n \sinh\frac{n\pi}{2} = \frac{1}{L}\int_{-L}^{L} \overset{\text{ODD}}{f(x)} \overset{\text{ODD}}{\sin\frac{n\pi x}{L}}\, dx$$

$$\implies D_n \sinh\frac{n\pi}{2} = \frac{1}{2}\int_{-2}^{2} u(x,1)\sin\frac{n\pi x}{2}\, dx$$

$$\implies D_n \sinh\frac{n\pi}{2} = \int_{0}^{2} u(x,1)\sin\frac{n\pi x}{2}\, dx$$

$$\implies D_n \sinh\frac{n\pi}{2} = \int_0^1 x\sin\frac{n\pi x}{2}\, dx + \int_1^2 (2-x)\sin\frac{n\pi x}{2}\, dx$$

INTEGRATE EACH SEPARATELY (BY PARTS)

x	1
$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$	$\sin\left(\frac{n\pi x}{2}\right)$

$2-x$	-1
$-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$	$\sin\left(\frac{n\pi x}{2}\right)$

$$\implies D_n \sinh\left(\frac{n\pi}{2}\right) = \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \left[\frac{2(x-2)}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_1^2 - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$\implies D_n \sinh\left(\frac{n\pi}{2}\right) = \cancel{-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \cancel{\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)} - \frac{2}{n\pi}\int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$\implies D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\left[\sinh\left(\frac{n\pi x}{2}\right)\right]_0^1 - \frac{4}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_1^2$$

$$\implies D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{4}{n^2\pi^2}\sinh\left(\frac{n\pi}{2}\right) - \frac{4}{n^2\pi^2}\left[-\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\implies D_n \sinh\left(\frac{n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\implies D_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2\sinh\left(\frac{n\pi}{2}\right)}$$

FINALLY WE HAVE A GENERAL SOLUTION

$$\therefore \boxed{u(x,y) = \sum_{n=1}^{\infty}\left[\frac{8}{n^2\pi^2}\frac{\sin\left(\frac{n\pi}{2}\right)}{\sinh\left(\frac{n\pi}{2}\right)}\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)\right]}$$

Question 5

A square plate of unit length has three of its sides kept at temperature $0\ ^\circ\text{C}$, while the fourth side is kept at temperature $50\ ^\circ\text{C}$.

In a steady state the temperature, $\theta(x, y)$ satisfies Laplace's Equation

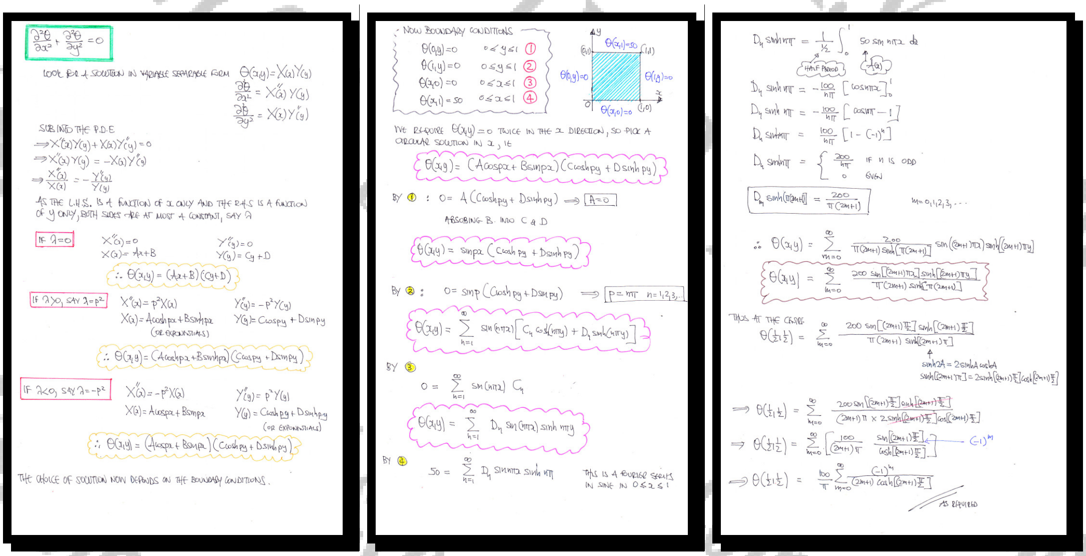
$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0,$$

Solve the equation and hence show that

$$\theta\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\frac{100}{\pi}\sum_{n=0}^{\infty}\left\{\frac{(-1)^n}{(2n+1)\cosh\left[(2n+1)\frac{\pi}{2}\right]}\right\}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

proof



Question 6

The function $\varphi = \varphi(x, y)$ satisfies Laplace's Equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

in a rectangular region where $0 \le x \le 3$ and $0 \le y \le 2$.

It is further given that

- $\varphi(x,0) = 0, \quad x \in [0,3]$
- $\varphi(x,2) = 0, \quad x \in [0,3]$
- $\varphi(0, y) = 0, \quad y \in [0,2]$
- $\varphi(3, y) = \begin{cases} y & y \in [0,1] \\ 2-y & y \in (1,2] \end{cases}.$

Solve the equation and hence show that

$$\varphi\left(\frac{3}{2},1\right) = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\operatorname{sech}\left[\frac{3\pi}{4}(2n-1)\right]}{(2n-1)^2}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$\boxed{\text{proof}}$

[solution overleaf]

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\phi(x,y) = X(x)Y(y) \implies \frac{\partial^2 \phi}{\partial x^2} = X''(x)Y(y)$$
$$\frac{\partial^2 \phi}{\partial y^2} = X(x)Y''(y)$$

- SUB INTO THE PDE

$X''Y + XY'' = 0$ DIVIDE BY XY

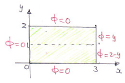
$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

- AS THE LHS IS A FUNCTION OF x AND THE RHS IS A FUNCTION OF y ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

λ CAN BE POSITIVE NEGATIVE, POSITIVE OR ZERO

INSPECTING THE BOUNDARY CONDITION WE REQUIRE CIRCULAR SOLUTIONS IN y (SEE DIAGRAM) SO WE CAN GET ZERO TWICE



AS THERE IS A MINUS ALREADY IN THE RHS, $\lambda>0$ FOR CIRCULAR SOLUTION

$\therefore$ SAY $\lambda = p^2$

HENCE $X''(x) = p^2X(x)$ $\quad$ $Y''(y) = -p^2Y(y)$

$X(x) = A\cosh px + B\sinh px$ & $Y(y) = D\cos py + E\sin py$

(OR EXPONENTIALS)

HENCE AN APPROPRIATE GENERAL SOLUTION WILL BE

$$\boxed{\phi(x,y) = X(x)Y(y) = (A\cosh px + B\sinh px)(D\cos py + E\sin py)}$$

BOUNDARY CONDITIONS

① $\phi(x,0)=0 \quad 0 \leq x \leq 3$
② $\phi(x,2)=0 \quad 0 \leq x \leq 3$
③ $\phi(0,y)=0 \quad 0 \leq y \leq 2$
④ $\phi(3,y)= \begin{cases} y & 0 \leq y \leq 1 \\ 2-y & 1 < y \leq 2 \end{cases}$

- BY ③ $\implies 0 = A(D\cos py + E\sin py)$

$\boxed{A=0}$

RENAMING THE CONSTANTS

$$\therefore \phi(x,y) = \sinh px\left[D\cos py + E\sin py\right]$$

- BY ① $\implies 0 = \sinh px\,[D]$

$\boxed{D=0}$

RENAMING THE CONSTANTS

$$\therefore \phi(x,y) = E\sinh px \sin py$$

- BY ② $\implies 0 = E\sinh px \sin 2p$

$E \neq 0 \quad \sinh px \neq 0$ FOR ALL $x \implies \sin 2p = 0$

$\implies 2p = n\pi$

$\boxed{p = \frac{n\pi}{2}} \quad n=1,2,3,\ldots$

NOTE THAT MINUSES CAN BE ABSORBED IN THE CONSTANTS $n=0$, $\sin 0 = 0$

$$\therefore \phi(x,y) = \sum_{n=1}^{\infty}\left[E_n \sinh\left(\frac{n\pi x}{2}\right)\sin\left(\frac{n\pi y}{2}\right)\right]$$

- BY ④ $f(y) = \sum_{n=1}^{\infty}\left[\underbrace{E_n\sinh\left(\frac{3n\pi}{2}\right)}_{a_n}\sin\left(\frac{n\pi y}{2}\right)\right]$

WHICH IS A FOURIER SERIES IN y, $0 \leq y \leq 2$
I.E PERIOD 2 (HALF PERIOD $L=1$)

THUS

$$\overbrace{E_n\sinh\left(\frac{3n\pi}{2}\right)}^{a_n} = \frac{1}{1}\int_0^2 f(y)\sin\left(\frac{n\pi y}{2}\right)dy$$

$$a_n = \int_0^1 y\sin\left(\frac{n\pi y}{2}\right)dy + \int_1^2 (2-y)\sin\left(\frac{n\pi y}{2}\right)dy$$

$$a_n = \int_0^1 y\sin\left(\frac{n\pi y}{2}\right)dy + \int_1^2 2\sin\left(\frac{n\pi y}{2}\right)dy - \int_1^2 y\sin\left(\frac{n\pi y}{2}\right)dy$$

COMPUTE $\int y\sin\frac{n\pi y}{2}\,dy$ ... BY PARTS

y	1
$-\frac{2}{n\pi}\cos\frac{n\pi y}{2}$	$\sin\frac{n\pi y}{2}$

$$= -\frac{2}{n\pi}y\cos\frac{n\pi y}{2} + \frac{2}{n\pi}\int\cos\frac{n\pi y}{2}\,dy$$

$$= -\frac{2}{n\pi}y\cos\frac{n\pi y}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi y}{2} + C$$

THUS

$$\int_0^1 y\sin\left(\frac{n\pi y}{2}\right)dy = \left[-\frac{2}{n\pi}y\cos\frac{n\pi y}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi y}{2}\right]_0^1$$
$$= \left(-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - (0+0)$$
$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$-\int_1^2 y\sin\left(\frac{n\pi y}{2}\right)dy = \left[\frac{2}{n\pi}y\cos\left(\frac{n\pi y}{2}\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi y}{2}\right)\right]_1^2$$
$$= \left(\frac{4}{n\pi}\cos n\pi - \cancel{\frac{4}{n^2\pi^2}\sin n\pi}\right) - \left(\frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$
$$= \frac{4}{n\pi}\cos n\pi - \frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

LASTLY

$$\int_1^2 2\sin\left(\frac{n\pi y}{2}\right)dy = \left[-\frac{4}{n\pi}\cos\frac{n\pi y}{2}\right]_1^2 = \left[\frac{4}{n\pi}\cos\frac{n\pi y}{2}\right]_2^1$$
$$= \frac{4}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] = \frac{4}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n\pi}\cos n\pi$$

HENCE

$$a_n = \cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$
$$\cancel{-\frac{2}{n\pi}\cos\frac{n\pi}{2}} + \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{4}{n\pi}\cos n\pi}$$
$$\cancel{\frac{4}{n\pi}\cos\frac{n\pi}{2}} \qquad \cancel{-\frac{4}{n\pi}\cos n\pi}$$

$$a_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$E_n\sin\left(\frac{3n\pi}{2}\right) = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

n	1	2	3	4	5	6	7	8	ETC
$\sin\frac{n\pi}{2}$	1	0	-1	0	1	0	-1	0	

THUS $\sin\left[\frac{\pi}{2}(2m-1)\right] = (-1)^{m-1} \quad m=1,2,3,4,\ldots$

$$\therefore E_{2m-1} = \frac{8}{(2m-1)^2\pi^2}(-1)^{m-1} \times \frac{1}{\sin\left[\frac{3\pi}{2}(2m-1)\right]}$$

THUS THE SOLUTION IS

$$\phi(x,y) = \sum_{m=1}^{\infty}\left[\frac{8(-1)^{m-1}}{\pi^2(2m-1)^2}\,\frac{\sinh\left[\frac{\pi x}{2}(2m-1)\right]\sin\left[\frac{\pi y}{2}(2m-1)\right]}{\sinh\left[\frac{3\pi}{2}(2m-1)\right]}\right]$$

AT THE CENTRE OF THE PLATE $\left(\frac{3}{2},1\right)$

$$\phi\left(\frac{3}{2},1\right) = \sum_{m=1}^{\infty}\left[\frac{8(-1)^{m-1}}{\pi^2(2m-1)^2}\,\frac{\sinh\left[\frac{3\pi}{4}(2m-1)\right]\sin\left[\frac{\pi}{2}(2m-1)\right]}{\sinh\left[\frac{3\pi}{2}(2m-1)\right]}\right]$$

$$= \sum_{m=1}^{\infty}\left[\frac{8(-1)^{m-1}}{\pi^2(2m-1)^2}(-1)^{m-1} \times \frac{\cancel{\sinh\left[\frac{3\pi}{4}(2m-1)\right]}}{2\cancel{\sinh\left[\frac{3\pi}{4}(2m-1)\right]}\cosh\left[\frac{3\pi}{4}(2m-1)\right]}\right]$$

$$= \sum_{m=1}^{\infty}\frac{8}{\pi^2}\,\frac{\cancel{(-1)^{2(m-1)}}}{2\cosh\left[\frac{3\pi}{4}(2m-1)\right]}$$

$$= \frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\operatorname{sech}\left[\frac{3\pi}{4}(2m-1)\right]}{(2m-1)^2}$$

OR $\quad \frac{4}{\pi^2}\sum_{m=0}^{\infty}\frac{\operatorname{sech}\left[\frac{3\pi}{4}(2m+1)\right]}{(2m+1)^2}$

Question 7

The temperature $\theta = \theta(x, y)$ for a **steady** two-dimensional heat flow in the semi-infinite region for which $y \geq 0$ satisfies Laplace's Equation

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0,$$

subject to the boundary conditions

- $\dfrac{\partial \theta}{\partial x}(0, y) = 0, \quad y \in [0, \infty)$
- $\dfrac{\partial \theta}{\partial x}(L, y) = 0, \quad y \in [L, \infty)$
- $\lim\limits_{y \to \infty} \left[\theta(x, y)\right] \leq |M|, \ M \in \mathbb{R}, \quad x \in (-\infty, \infty), \ y \in [x, \infty)$
- $\theta(x, 0) = \dfrac{x(L - x)}{L^2}, \quad x \in (-\infty, \infty), \quad f(-x) = f(x), \quad f(x) = f(x + 2L)$

Solve the equation and hence show that

$$\theta\left(\tfrac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \exp\left[-\frac{2n\pi y}{L}\right]}{n^2}.$$

[*You must derive the standard Cartesian solution of Laplace's equation in variable separate form*]

$\boxed{}$, $\boxed{\text{proof}}$

[solution overleaf]

LOOK FOR A SOLUTION IN VARIABLE SEPARABLE FORM

$$\theta(x,y) = X(x)Y(y)$$

$$\frac{\partial^2\theta}{\partial x^2}(x,y) = X''(x)Y(y) \quad \& \quad \frac{\partial^2\theta}{\partial y^2} = X(x)Y''(y)$$

SUBSTITUTE INTO THE P.D.E AND REARRANGE

$$\Rightarrow \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

$$\Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\Rightarrow X''(x)Y(y) = -X(x)Y''(y) = 0$$

$$\Rightarrow \frac{X''(x)Y(y)}{X(x)Y(y)} = -\frac{X(x)Y''(y)}{X(x)Y(y)}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

AS THE L.H.S IS A FUNCTION OF x ONLY & THE RHS IS A FUNCTION OF y ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, CALL IT λ, WHICH CAN BE POSITIVE, ZERO OR NEGATIVE — CONSIDER SOLUTIONS FOR ALL CASES

IF $\lambda = 0$

- $X''(x) = 0$
  $X(y) = Ax + B$
- $Y''(y) = 0$
  $Y(y) = Cy + D$

$$\boxed{\theta(x,y) = (Ax+B)(Cy+D)}$$

IF $\lambda > 0$, SAY $\lambda = p^2$

- $\dfrac{X''(x)}{X(x)} = p^2$
  $X''(x) = p^2X(x)$
  $X(x) = A\cosh px + B\sinh px$
  (OR EXPONENTIALS)
- $\dfrac{Y''(y)}{Y(y)} = -p^2$
  $Y''(y) = -p^2Y(y)$
  $Y(y) = C\cos py + D\sin py$

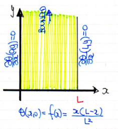
$$\boxed{\theta(x,y) = (A\cosh px + B\sinh px)(C\cos py + D\sin py)}$$

IF $\lambda < 0$, SAY $\lambda = -p^2$

- $\dfrac{X''(x)}{X(x)} = -p^2$
  $X''(x) = -p^2X(x)$
  $X(x) = A\cos px + B\sin px$
- $\dfrac{Y''(y)}{Y(y)} = +p^2$
  $Y''(y) = p^2Y(y)$
  $Y(y) = C\cosh py + D\sinh py$
  (OR EXPONENTIALS)

$$\boxed{\theta(x,y) = (A\cos px + B\sin px)(C\cosh py + D\sinh py)}$$

THE SOLUTION NEEDED IS DICTATED BY THE CONDITIONS OF THIS PROBLEM



① $\dfrac{\partial\theta}{\partial x}(0,y) = 0, \; y \geqslant 0$

② $\dfrac{\partial\theta}{\partial x}(L,y) = 0, \; y \geqslant 0$

③ $\theta(x,0) = f(x) = \dfrac{x(L-x)}{L^2}, \; 0 \leqslant x \leqslant L$

④ $\theta(x,y)$ IS BOUNDED AS $y \to \infty$

FROM THE BOUNDARY CONDITIONS ① & ② WE REQUIRE A CIRCULAR SOLUTION IN x, AS WE REQUIRE $\frac{\partial\theta}{\partial x} = 0$ AT TWO DIFFERENT VALUES OF x.

HENCE THE ONLY PLAUSIBLE SOLUTION IS THAT PRODUCED BY $\lambda = -p^2$ (OR THE CONSTANT TERM PRODUCED BY $\lambda = 0$)

$$\boxed{\theta(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) + \cancel{E}}$$

USED EXPONENTIALS INSTEAD OF HYPERBOLICS

ALREADY INCLUDED

BY ④ $C = 0$, SINCE e^{py} IS UNBOUNDED AS $y \to \infty$ $(p>0)$

$$\theta(x,y) = De^{-py}(A\cos px + B\sin px)$$

$$\boxed{\theta(x,y) = e^{-py}(A\cos px + B\sin px)}$$

$$\frac{\partial\theta}{\partial x}(x,y) = e^{-py}(-Ap\sin px + Bp\cos px)$$

BY ① $\frac{\partial\theta}{\partial x}(0,y) = 0$

$$\Rightarrow 0 = e^{-py}Bp$$

$$\Rightarrow B = 0 \quad (p \neq 0)$$

$$\Rightarrow \boxed{\theta(x,y) = Ae^{-py}\cos px}$$

$$\Rightarrow \frac{\partial\theta}{\partial x}(x,y) = -Ape^{-py}\sin px$$

USING CONDITION ②, $\frac{\partial\theta}{\partial x}(L,y) = 0$

$$\Rightarrow 0 = -Ape^{-py}\sin pL$$

$$\Rightarrow pL = n\pi, \; n \in \mathbb{Z}$$

$$\Rightarrow p = \frac{n\pi}{L}$$

$$\theta(x,y) = \sum_{n=0}^{\infty}\left[A_n e^{-\frac{n\pi y}{L}}\cos\frac{n\pi x}{L}\right]$$

$$\boxed{\theta(x,y) = A_0 + \sum_{n=1}^{\infty}\left[A_n e^{-\frac{n\pi y}{L}}\cos\frac{n\pi x}{L}\right]}$$

USING CONDITION ③

$$\Rightarrow \frac{x(L-x)}{L^2} = A_0 + \sum_{n=1}^{\infty}\left[A_n\cos\frac{n\pi x}{L}\right]$$

$$\Rightarrow \frac{1}{L}x - \frac{x^2}{L^2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{L}\right]$$

BUILDING EVEN EXTENSION – HALF PERIOD IS L

- $a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\,dx = \ldots \text{EVEN} \ldots \frac{2}{L}\int_0^L \frac{1}{L}x - \frac{x^2}{L^2}\,dx$
  $= \dfrac{2}{L}\left[\dfrac{1}{2L}x^2 - \dfrac{x^3}{3L^2}\right]_0^L = \dfrac{2}{L}\left[\dfrac{1}{2}L - \dfrac{1}{3}L\right] = \dfrac{1}{3}$
- $a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \ldots \text{EVEN} \ldots$
  $= \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L\left(\frac{1}{L}x - \frac{x^2}{L^2}\right)\cos\frac{n\pi x}{L}\,dx$
  $= \dfrac{2}{L^3}\displaystyle\int_0^L (Lx - x^2)\cos\frac{n\pi x}{L}\,dx = \ldots \text{BY PARTS}$

$Lx - x^2$	$L - 2x$
$\frac{L}{n\pi}\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$

$$\ldots = \frac{2}{L^3}\left\{\cancel{\left[(Lx-x^2)\frac{L}{n\pi}\sin\frac{n\pi x}{L}\right]_0^L} - \frac{L}{n\pi}\int_0^L (L-2x)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= -\frac{2}{n\pi L^2}\int_0^L (L-2x)\sin\frac{n\pi x}{L}\,dx$$

BY PARTS AGAIN

$L - 2x$	-2
$-\frac{L}{n\pi}\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$

$$\ldots = -\frac{2}{n\pi L^2}\left\{\left[\frac{L}{n\pi}(2x-L)\cos\frac{n\pi x}{L}\right]_0^L + \frac{2L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\,dx\right\}$$

$$= \left[\frac{2}{n^2\pi^2L}(L-2x)\cos\frac{n\pi x}{L}\right]_0^L - \frac{4}{n^2\pi^2L}\times\frac{L}{n\pi}\cancel{\left[\sin\frac{n\pi x}{L}\right]_0^L}$$

$$= \frac{2}{n^2\pi^2L}\left[-L\cos n\pi - L\right] = -\frac{2}{n^2\pi^2}\left[1 + \cos n\pi\right]$$

$$= -\frac{2}{n^2\pi^2}\left[1 + (-1)^n\right] = \begin{cases} -\dfrac{4}{n^2\pi^2} & \text{IF } n \text{ IS EVEN} \\ 0 & \text{IF } n \text{ IS ODD} \end{cases}$$

AND LETTING $n = 2m$

$$a_m = -\frac{4}{(2m)^2\pi^2} = -\frac{1}{m^2\pi^2}$$

$$\Rightarrow \theta(x,y) = \frac{1/3}{2} + \sum_{m=1}^{\infty}\left[\frac{-1}{m^2\pi^2}e^{-\frac{2m\pi y}{L}}\cos\left(\frac{2m\pi x}{L}\right)\right]$$

$$\Rightarrow \boxed{\theta(x,y) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi y}{L}}\cos\left(\frac{2n\pi x}{L}\right)\right]}$$

FINALLY ALONG THE LINE $x = \frac{1}{2}L$

$$\theta\left(\tfrac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{\frac{-2n\pi y}{L}}\cos\left(\frac{2n\pi(\frac{1}{2}L)}{L}\right)\right]$$

$$\theta\left(\tfrac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}e^{-\frac{2n\pi y}{L}}\cos(n\pi)\right]$$

$$\theta\left(\tfrac{1}{2}L, y\right) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}e^{-\frac{2n\pi y}{L}}\right]$$

AS REQUIRED

LAPLACE'S EQUATION

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0, \quad \Phi = \Phi(r,\theta)$$

Two Dimensional in Plane Polars

Question 1

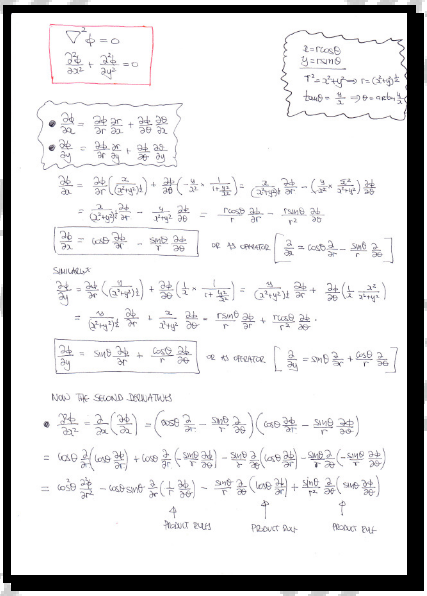
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

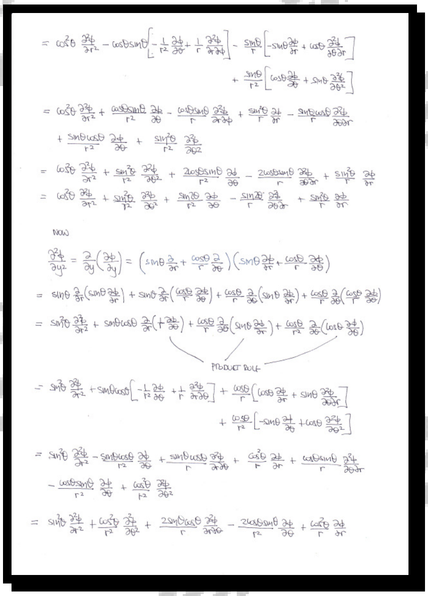
The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

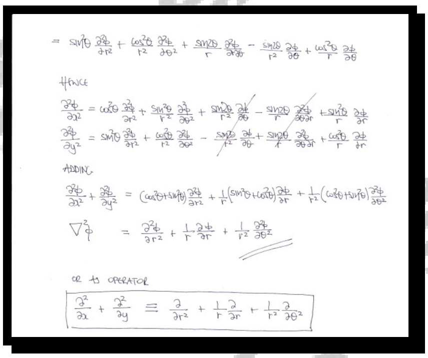
Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0 .$$

$\boxed{\text{proof}}$







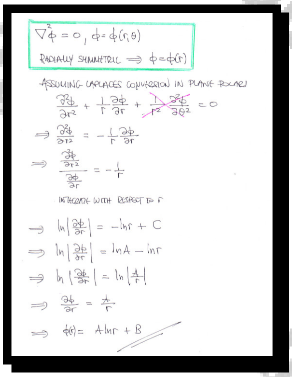
Question 2

In a two dimensional universe the gravitational Φ would satisfy the two dimensional Laplace's equation

$$\nabla^2 \Phi = 0 .$$

Find a general circularly symmetric solution for Φ.

$$\boxed{\Phi(r) = A \ln r + B}$$



$\boxed{\nabla^2 \phi = 0, \ \phi = \phi(r, \theta) \\ \text{RADIALLY SYMMETRIC} \Rightarrow \phi = \phi(r)}$

ASSUMING LAPLACES CONVERSION IN PLANE POLARS

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}} = 0$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial r^2} = -\frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$\Rightarrow \frac{\frac{\partial^2 \phi}{\partial r^2}}{\frac{\partial \phi}{\partial r}} = -\frac{1}{r}$$

INTEGRATE WITH RESPECT TO r

$$\Rightarrow \ln\left|\frac{\partial \phi}{\partial r}\right| = -\ln r + C$$

$$\Rightarrow \ln\left|\frac{\partial \phi}{\partial r}\right| = \ln A - \ln r$$

$$\Rightarrow \ln\left|\frac{\partial \phi}{\partial r}\right| = \ln\left|\frac{A}{r}\right|$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{A}{r}$$

$$\Rightarrow \phi(r) = A \ln r + B$$

Question 3

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

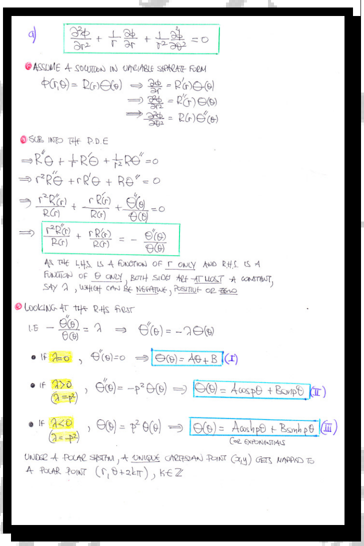
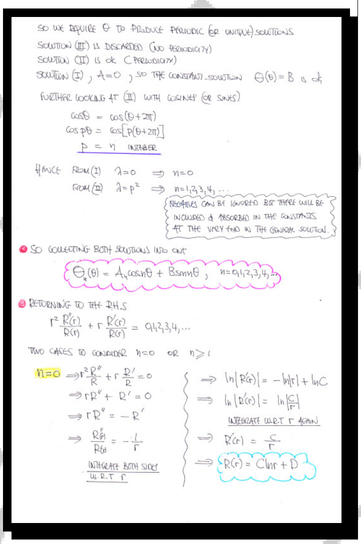
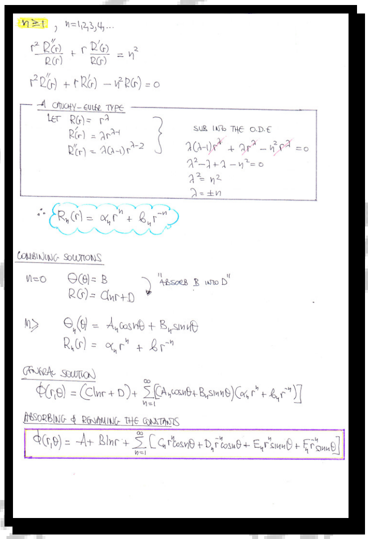
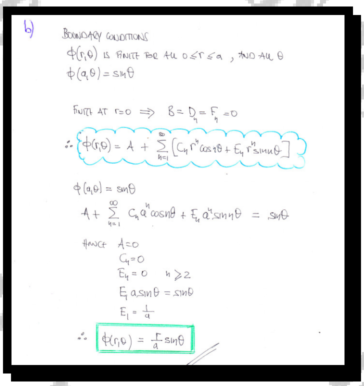
b) Hence find a specific solution subject to the conditions

i. $\Phi(0,\theta) = 0$

ii. $\Phi(r,\theta)$ is finite for $0 \le r \le a$ and for all θ.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi(r,\theta) = \frac{r}{a}\sin\theta}$$

Question 4

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

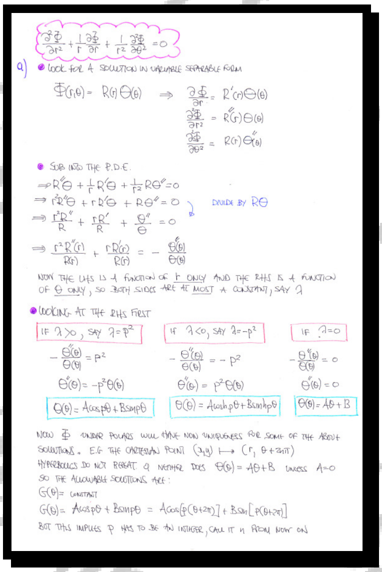
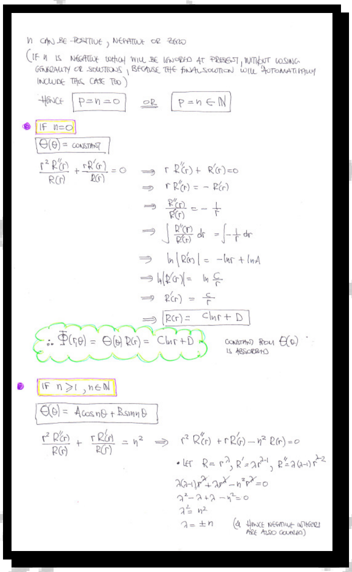
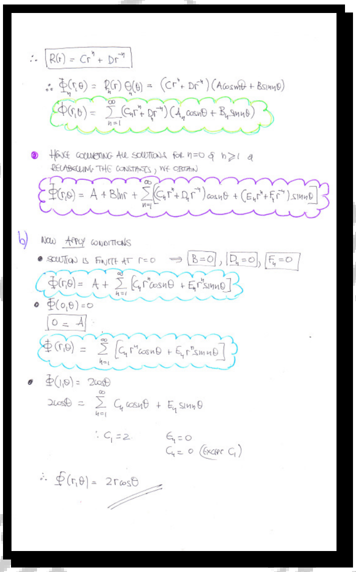
b) Hence find a specific solution subject to the conditions

i. $\Phi(0,\theta) = 0$

ii. $\Phi(1,\theta) = 2\cos\theta$.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi(r,\theta) = 2r\cos\theta}$$

Question 5

The steady state temperature distribution $\Phi = \Phi(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

Given further that $\Phi(1,\theta) = \sin 2\theta$, determine a simplified expression for $\Phi(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$\boxed{}$, $\boxed{\Phi(r,\theta) = r^2 \sin 2\theta}$

ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM – DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

- $\Phi(r,\theta) = R(r)\Theta(\theta) \Rightarrow \frac{\partial \Phi}{\partial r} = R'(r)\Theta(\theta)$

$\Rightarrow \frac{\partial^2 \Phi}{\partial r^2} = R''(r)\Theta(\theta)$

$\Rightarrow \frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\Theta''(\theta)$

- $\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$

$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$

$\Rightarrow \frac{R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r}\frac{R'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r^2}\frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = 0$

$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0$

$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$

$\Rightarrow \boxed{r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$

AS THE L.H.S. OF THE ABOVE EXPRESSION IS A FUNCTION OF r ONLY AND THE R.H.S IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

LOOKING AT THE R.H.S OF THE ABOVE EXPRESSION, WE REQUIRE CIRCULAR SOLUTIONS IN θ, DUE TO THE POLAR SYSTEM REPEATING EVERY 2π – GIVEN THAT THERE IS A MINUS IN THE R.H.S, WE DEDUCE THAT $\lambda > 0$, SAY p^2

$\Rightarrow -\frac{\Theta''(\theta)}{\Theta(\theta)} = p^2$

$\Rightarrow \Theta''(\theta) = -p^2\Theta(\theta)$

$\Rightarrow \boxed{\Theta(\theta) = A\cos(p\theta) + B\sin(p\theta)}$

LOOKING AT THE "θ-SOLUTION" WITH SAY SINES (OR COSINES)

$\sin\theta = \sin(\theta + 2\pi) \Rightarrow \sin(p\theta) = \sin(p(\theta+2\pi))$

$= \sin(p\theta + 2p\pi)$

$\Rightarrow p = n = \text{INTEGER} = 0,1,2,3,4,\dots$

NOTES

$n=0$ IS O.K. AS IT PRODUCES A CONSTANT SOLUTION

$n<0$ NEED NOT BE CONSIDERED SEPARATELY AS THEY WILL BE ABSORBED IN THE CONSTANTS AT THIS STAGE, BUT THEY WILL ACTUALLY APPEAR AGAIN AT THE END

$\therefore \boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \quad n = 0,1,2,3,4,\dots}$

NEXT WE RETURN TO THE L.H.S (FUNCTION OF r ONLY), WITH n INTEGER

$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n = 0,1,2,3,4,\dots$

IF $n=0$

$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0$

$\Rightarrow rR''(r) + R'(r) = 0$

$\Rightarrow rR''(r) = -R'(r)$

$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$

INTEGRATE W.R.T r

$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$

$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$

$\Rightarrow R'(r) = \frac{C}{r}$

INTEGRATE W.R.T r

$\Rightarrow R(r) = C\ln r + D$

IF $n = 1,2,3,4,\dots$

$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$

$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r)$

THIS IS A CAUCHY–EULER O.D.E.

LET $R(r) = r^k$, $R'(r) = kr^{k-1}$, $R''(r) = k(k-1)r^{k-2}$ $\Big\}$ SUB INTO THE O.D.E.

$\Rightarrow k(k-1)\cancel{r^k} + k\cancel{r^k} - n^2\cancel{r^k} = 0$

$\Rightarrow k^2 - k + k - n^2 = 0$

$\Rightarrow k^2 = n^2$

$\Rightarrow k = \pm n$

$\Rightarrow R_n(r) = \alpha_n r^n + \beta_n r^{-n}$

COLLECTING ALL THE RESULTS

IF $n=0$

$\Theta_0(\theta) = A_0$

$R_0(r) = C\ln r + D$ $\Big\}$ ABSORBING CONSTANTS

$\boxed{\Phi_0(r,\theta) = \Theta_0(\theta)R_0(r) = C\ln r + D}$

IF $n = 1,2,3,4,\dots$

$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$

$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$

FINALLY WE HAVE A GENERAL SOLUTION

$$\Phi(r,\theta) = [C\ln r + D] + \sum_{n=1}^{\infty}\Big[[A_n\cos n\theta + B_n\sin n\theta][\alpha_n r^n + \beta_n r^{-n}]\Big]$$

ABSORBING & RELABELLING THE CONSTANTS

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

NEXT WE APPLY CONDITIONS, NOTING THAT $\Phi(r,\theta)$ MUST BE FINITE IN ALL PARTS OF THE DISC AND IN PARTICULAR AT $r=0$

$\therefore B = 0,\ D_n = 0,\ F_n = 0$

$$\boxed{\Phi(r,\theta) = A + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + E_n r^n\sin n\theta\right]}$$

ALSO $\Phi(1,\theta) = \sin 2\theta$

$\Rightarrow \sin 2\theta = A + \sum_{n=1}^{\infty}\left[C_n\cos n\theta + E_n\sin n\theta\right]$

$\therefore A = 0,\ C_n = 0,\ E_2 = 1,\ E_n = 0 \quad n \neq 2$

$\Rightarrow \Phi(r,\theta) = r^2\sin 2\theta$

$= r^2(2\sin\theta\cos\theta)$

$= 2(r\sin\theta)(r\cos\theta)$

$= 2xy$

Question 6

The steady state temperature distribution $\Phi = \Phi(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

Given further that

$$2\Phi(1,\theta) + \frac{\partial \Phi}{\partial r}(1,\theta) = 100 - 2\cos 2\theta ,$$

determine a simplified expression for $\Phi(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{\Phi(r,\theta) = 50 - \frac{1}{2}r^2 \cos 2\theta}$$


$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \quad \text{FOR} \quad \phi = \phi(r,\theta), \quad 0 \leqslant r \leqslant 1, \quad 0 \leqslant \theta < 2\pi$$

- ASSUME A SOLUTION IN VARIABLE FORM

$$\phi(r,\theta) = R(r)\Theta(\theta)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\frac{\partial\phi}{\partial r} = R'(r)\Theta(\theta) \qquad \frac{\partial^2\phi}{\partial r^2} = R''(r)\Theta(\theta) \qquad \frac{\partial^2\phi}{\partial\theta^2} = R(r)\Theta''(\theta)$$

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow \frac{R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r}\frac{R'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r^2}\frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = \frac{0}{R(r)\Theta(\theta)}$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow \boxed{r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

- LOOKING AT THE ABOVE EQUATION BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, AS THE LHS IS A FUNCTION OF r ONLY & THE R.H.S IS A FUNCTION OF θ ONLY
- AS THE POLAR SYSTEM REPEATS IN θ EVERY 2π, WE REQUIRE PERIODIC SOLUTIONS IN θ, SO LOOKING AT THE R.H.S ABOVE, WHICH ALREADY CONTAINS A MINUS, IMPLIES THAT THE CONSTANT λ MUST BE POSITIVE
- THUS THE RHS. OF THE ABOVE EQUATION YIELDS

$$\Rightarrow -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda = p^2$$

$$\Rightarrow \Theta''(\theta) = -p^2\Theta(\theta)$$

$$\Rightarrow \Theta(\theta) = A\cos p\theta + B\sin p\theta$$

LOOKING FURTHER AT THE ABOVE SOLUTION WITH SINES (OR COSINES)

$$\sin\theta = \sin(\theta + 2\pi) \Rightarrow \sin p\theta = \sin[p(\theta+2\pi)] = \sin(p\theta + 2p\pi)$$

HENCE $p = n$ = INTEGER $= 0,1,2,3,4,5,\ldots$

- NOTE HERE THAT
  - (I) $n=0$ IS OK AS IT PRODUCES A CONSTANT SOLUTION WHICH DOES NOT VIOLATE THE REQUIRED PERIODICITY IN θ
  - (II) NEGATIVE INTEGERS NEED NOT BE CONSIDERED SEPARATELY AT THIS STAGE AS WE CAN ABSORB THEM IN THE CONSTANTS, AND WILL ACTUALLY APPEAR AT THE VERY END

$$\therefore \boxed{\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \quad n = 0,1,2,3,\ldots}$$

- RETURNING TO THE LHS NOW THAT n IS AN INTEGER

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0,1,2,3,4,\ldots$$

THERE ARE TWO CASES TO CONSIDER — EITHER $n=0$ OR $n \geqslant 1$

IF $n=0$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = 0$$

$$\Rightarrow rR''(r) + R'(r) = 0$$

$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$$

$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE W.R.T r AGAIN

$$\Rightarrow R_0(r) = C\ln r + D$$

IF $n \geqslant 1$ $(n = 1,2,3,4,\ldots)$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2$$

$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

THIS IS A CAUCHY-EULER ODE

LET $R(r) = r^k$, $R'(r) = kr^{k-1}$, $R''(r) = k(k-1)r^{k-2}$ SUB INTO THE O.D.E

$$k(k-1)r^k + kr^k - n^2r^k = 0$$

$$k^2 - k + k - n^2 = 0$$

$$k^2 = n^2$$

$$k = \pm n$$

$$\Rightarrow R_n(r) = \alpha_n r^n + b_n r^{-n}$$

- COLLECTING ALL THE RESULTS NOW

$n=0$

$$\Theta_0(\theta) = A_0$$

$$R_0(r) = C\ln r + D$$

ABSORBING CONSTANTS

$$\phi_0(r,\theta) = C\ln r + D$$

$n = 1,2,3,4,\ldots$

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$$

$$R_n(\theta) = \alpha_n r^n + b_n r^{-n}$$

$$\phi_n(r,\theta) = (A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + b_n r^{-n})$$

- THE GENERAL SOLUTION OF THE P.D.E IS

$$\phi(r,\theta) = [C\ln r + D] + \sum_{n=1}^{\infty}\left[[A_n\cos n\theta + B_n\sin n\theta][\alpha_n r^n + b_n r^{-n}]\right]$$

FINALLY ABSORBING AND RELABELLING THE CONSTANTS YIELDS

$$\boxed{\phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

- NEXT WE ARE READY TO APPLY CONDITIONS — BEFORE THAT WE OBSERVE THAT $\phi(r,\theta)$ MUST BE FINITE (BOUNDED) EVERYWHERE, AND IN PARTICULAR AT $r=0$

HENCE $B = 0$, $D_n = 0$, $F_n = 0$

$$\boxed{\phi(r,\theta) = A + \sum_{n=1}^{\infty}\left[r^n(C_n\cos n\theta + E_n\sin n\theta)\right]}$$

- DIFFERENTIATE W.R.T r & APPLY THE CONDITION

$$\frac{\partial\phi}{\partial r} = \sum_{n=1}^{\infty}\left[nr^{n-1}(C_n\cos n\theta + E_n\sin n\theta)\right]$$

$$\Rightarrow 2\phi(1,\theta) + \frac{\partial\phi}{\partial r}(1,\theta) = 100 - 2\cos 2\theta$$

$$\Rightarrow 2A + \sum_{n=1}^{\infty}\left[2[C_n\cos n\theta + E_n\sin n\theta]\right] + \sum_{n=1}^{\infty}\left[n[C_n\cos n\theta + E_n\sin n\theta]\right] = 100 - 2\cos 2\theta$$

$$\Rightarrow 2A + \sum_{n=1}^{\infty}\left[(2+n)(C_n\cos n\theta + E_n\sin n\theta)\right] = 100 - 2\cos 2\theta$$

$$\therefore A = 50 \qquad (2+\ )C_2\cos 2\theta = -2\cos 2\theta \qquad \text{\& EVERYTHING ELSE ZERO}$$

$$C_2 = -\frac{1}{2}$$

$$\therefore \boxed{\phi(r,\theta) = 50 - \frac{1}{2}r^2\cos 2\theta}$$

Question 7

The steady state temperature distribution $u = u(r,\theta)$ in a circular thin metal disc of radius a, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 .$$

The top half of the circumference of the disc, $0 \le \theta \le \pi$, is maintained at $100\ °\text{C}$, and the bottom half of the circumference of the disc, $\pi < \theta < 2\pi$, is maintained at $0\ °\text{C}$.

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta) = 50 + \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin n\theta}{n}\left(\frac{r}{a}\right)^n\right]}$$

$$\boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad u = u(r,\theta),\ 0 \le r \le a,\ 0 \le \theta \le 2\pi}$$

CONDITIONS $u(a,\theta) = 100 \quad 0 \le \theta \le \pi$

$u(a,\theta) = 0 \quad \pi < \theta < 2\pi$

$u(0,\theta) = 0$ (FINITE)

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$u(r,\theta) = R(r)\Theta(\theta)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$$\frac{\partial u}{\partial r} = R'(r)\Theta(\theta), \quad \frac{\partial^2 u}{\partial r^2} = R''(r)\Theta(\theta), \quad \frac{\partial^2 u}{\partial \theta^2} = R(r)\Theta''(\theta)$$

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\Rightarrow \frac{R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r}\frac{R'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{1}{r^2}\frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = \frac{0}{R(r)\Theta(\theta)}$$

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

- BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, AS THE L.H.S. IS A FUNCTION OF r ONLY, AND THE R.H.S IS A FUNCTION OF θ ONLY
- DUE TO THE POLAR SYSTEM, WE REQUIRE PERIODIC SOLUTIONS IN θ, SO THE CONSTANT λ MUST BE POSITIVE (GIVEN THERE IS A MINUS IN THE RHS)

- HENCE THE R.H.S YIELDS

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda = p^2$$

$$\Theta''(\theta) = -p^2\Theta(\theta)$$

$$\Theta(\theta) = A\cos p\theta + B\sin p\theta$$

- LOOKING FURTHER AT THE ABOVE SOLUTION WITH COSINES (OR SINES)

$$\cos\theta = \cos(\theta + 2\pi)$$

$$\cos p\theta = \cos(p(\theta + 2\pi))$$

$$\therefore \underline{\underline{p = n = \text{INTEGER}}}$$

- COLLECTING THE RESULTS SO FAR

$$\boxed{\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 0,1,2,3,\dots}$$

NOTE $n=0$ IS O.K. AS ALTHOUGH IS NOT PERIODIC SOLUTION IT IS STILL OK AS IT PRODUCES A CONSTANT SOLUTION

NEGATIVE INTEGERS ARE ABSORBED INTO THE CONSTANTS

- NOW RETURNING TO THE LHS

$$\frac{r^2R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0,1,2,3,\dots$$

- TWO CASES TO CONSIDER – EITHER $n=0$ OR $n \ge 1$

IF $n=0$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0$$

$$\Rightarrow rR''(r) + R'(r) = 0$$

$$\Rightarrow rR''(r) = -R'(r) = 0$$

$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE BOTH SIDES W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln r + \ln C$$

$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE W.R.T r AGAIN

$$\Rightarrow \underline{\underline{R(r) = C\ln r + D}}$$

IF $n \ge 1$, $n = 1,2,3,4,\dots$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$

$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

THIS IS A CAUCHY-EULER TYPE

LET $R(r) = r^\lambda$

$R'(r) = \lambda r^{\lambda-1}$

$R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

$\Rightarrow$ SUB INTO THE O.D.E

$$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda\cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$$

$$\lambda^2 - \lambda + \lambda - n^2 = 0$$

$$\lambda^2 = n^2$$

$$\lambda = \pm n$$

$$\Rightarrow \underline{\underline{R_n(r) = \alpha_n r^n + \beta_n r^{-n}}}$$

- COLLECTING ALL THE SOLUTIONS

$n=0$ $\quad \Theta_0(\theta) = A_0$, $\quad R(r) = C\ln r + D$ $\rightarrow$ ABSORBING A_0 INTO C & D

$$\boxed{u_0(r,\theta) = C\ln r + D}$$

$n \ge 1$ $\quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$

$R_n(r) = \alpha_n r^n + \beta r^{-n}$

- FINALLY THE GENERAL SOLUTION IS

$$u(r,\theta) = (C\ln r + D) + \sum_{n=1}^{\infty}\left[\left[A_n\cos n\theta + B_n\sin n\theta\right]\left[\alpha_n r^n + \beta_n r^{-n}\right]\right]$$

ABSORBING & RELABELLING THE CONSTANTS

$$\boxed{u(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

- NOW CONDITIONS

THE TEMPERATURE MUST BE FINITE AT THE CENTRE OF THE DISC, IE WHERE $r=0$

$$\therefore B = 0,\ D_n = 0,\ F_n = 0$$

$$\therefore \boxed{u(r,\theta) = A + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + E_n r^n\sin n\theta\right]}$$

- NEXT APPLY $u(a,\theta) = f(\theta) = \begin{cases} 100 & 0 \le \theta \le \pi \\ 0 & \pi < \theta < 2\pi \end{cases}$

$$f(\theta) = A + \sum_{n=1}^{\infty}\left[C_n a^n\cos n\theta + E_n a^n\sin n\theta\right]$$

THIS IS A FOURIER SERIES IN THE INTERVAL $0 \le \theta < 2\pi$ (HALF PERIOD π)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} 100\,d\theta = \frac{1}{\pi}\times 100\pi = 100$$

$$\therefore A = \frac{a_0}{2} = 50$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi} 100\cos n\theta\,d\theta$$

$$= \frac{1}{\pi}\times\frac{100}{n}\Big[\sin n\theta\Big]_0^{\pi} = 0 \qquad \therefore C_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi} 100\sin n\theta\,d\theta = \frac{100}{\pi n}\Big[-\cos n\theta\Big]_0^{\pi}$$

$$= \frac{100}{\pi n}\Big[\cos n\theta\Big]_{\pi}^{0} = \frac{100}{\pi n}(1+1) = \frac{200}{\pi n}.$$

$$\therefore E_n a^n = b_n = \frac{200}{\pi n}$$

$$E_n = \frac{200}{\pi n a^n}$$

$$\therefore \boxed{u(r,\theta) = 50 + \sum_{n=1}^{\infty}\left[\frac{200}{\pi n}\left(\frac{r}{a}\right)^n\sin n\theta\right]}$$

Question 8

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2 \Phi_1 = 0 ,\ 1 < r < 2$$

$$\nabla^2 \Phi_2 = 0 ,\ r \geq 2 .$$

It is further given that

$$\Phi_1(1,\theta) = 0, \quad \Phi_1(2,\theta) = 1, \quad \Phi_2(2,\theta) = 1 \quad \text{and} \quad \lim_{r\to\infty}\left[\Phi_2(r,\theta) - r\cos\theta\right] = 1 .$$

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_1(r,\theta) = \frac{\ln r}{\ln 2}}, \quad \boxed{\Phi_2(r,\theta) = 1 + r\cos\theta - \frac{4}{r}\cos\theta}$$

[solution overleaf]

a) $$\boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0}$$

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$\phi(r,\theta) = R(r)\Theta(\theta) \Rightarrow \frac{\partial \phi}{\partial r} = R'(r)\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \phi}{\partial r^2} = R''(r)\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \phi}{\partial \theta^2} = R(r)\Theta''(\theta)$$

- SUB INTO THE P.D.E

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$
$$\Rightarrow r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$
$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$
$$\Rightarrow \boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

AS THE L.H.S. IS A FUNCTION OF r ONLY AND R.H.S. IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH CAN BE NEGATIVE, POSITIVE OR ZERO

- LOOKING AT THE R.H.S FIRST

I.E $-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \Rightarrow \Theta''(\theta) = -\lambda\Theta(\theta)$

- IF $\lambda = 0$, $\Theta''(\theta) = 0 \Rightarrow \boxed{\Theta(\theta) = A\theta + B}$ (I)
- IF $\lambda > 0$ $(\lambda = p^2)$, $\Theta''(\theta) = -p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cos p\theta + B\sin p\theta}$ (II)
- IF $\lambda < 0$ $(\lambda = -p^2)$, $\Theta''(\theta) = p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cosh p\theta + B\sinh p\theta}$ (III)
  (OR EXPONENTIALS)

UNDER A POLAR SYSTEM, A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT $(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$

SO WE REQUIRE Θ TO PRODUCE PERIODIC (OR UNIQUE) SOLUTIONS

SOLUTION (III) IS DISCARDED (NO PERIODICITY)

SOLUTION (II) IS OK (PERIODICITY)

SOLUTION (I), $A=0$, SO THE CONSTANT SOLUTION $\Theta(\theta) = B$ IS OK

FURTHER LOOKING AT (II) WITH COSINES (OR SINES)

$$\cos\theta = \cos(\theta + 2\pi)$$
$$\cos p\theta = \cos[p(\theta + 2\pi)]$$
$$p = n \quad \text{INTEGER}$$

HENCE FROM (I) $\lambda = 0 \Rightarrow n = 0$

FROM (II) $\lambda = p^2 \Rightarrow n = 1,2,3,4,\dots$

NEGATIVES CAN BE IGNORED BUT THERE WILL BE INCLUDED & ABSORBED IN THE CONSTANTS AT THE VERY END IN THE GENERAL SOLUTION.

- SO COLLECTING BOTH SOLUTIONS INTO ONE

$$\Theta_n(\theta) = A_n\cos n\theta + B\sin n\theta, \quad n = 0,1,2,3,4,\dots$$

- RETURNING TO THE R.H.S

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0,1,2,3,4,\dots$$

TWO CASES TO CONSIDER $n=0$ OR $n \geq 1$

$n = 0$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = 0$$
$$\Rightarrow rR'' + R' = 0$$
$$\Rightarrow rR'' = -R'$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE BOTH SIDES W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

INTEGRATE W.R.T r AGAIN

$$\Rightarrow R'(r) = \frac{C}{r}$$
$$\Rightarrow R(r) = C\ln r + D$$

$n \geq 1$, $n = 1,2,3,4\dots$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$
$$r^2R''(r) + rR'(r) - n^2R(r) = 0$$

A CAUCHY–EULER TYPE

LET $R(r) = r^\lambda$
$R'(r) = \lambda r^{\lambda-1}$
$R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

SUB INTO THE O.D.E

$$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda \cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$$
$$\lambda^2 - \lambda + \lambda - n^2 = 0$$
$$\lambda^2 = n^2$$
$$\lambda = \pm n$$

$$\therefore R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

COMBINING SOLUTIONS

$n=0$ $\quad \Theta(\theta) = B$
$\quad R(r) = C\ln r + D$ "ABSORB B INTO D"

$n \geq 1$ $\quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$
$\quad R_n(r) = \alpha_n r^n + \beta r^{-n}$

GENERAL SOLUTION

$$\phi(r,\theta) = (C\ln r + D) + \sum_{n=1}^{\infty}\left[(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING & RENAMING THE CONSTANTS

$$\boxed{\phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b)
$\nabla^2\phi_1 = 0 \quad 1 < r < 2$ $\qquad \phi_1(1,\theta) = 0$ ①
$\nabla^2\phi_2 = 0 \quad r > 2$ $\qquad \phi_1(2,\theta) = \phi_2(2,\theta) = 1$ ②③
$\qquad \phi_2(r,\theta) - r\cos\theta \to 1$ ④ AS $r \to \infty$

USING GENERAL SOLUTION OF PART (a)

$$\phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$$
$$\phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + L_n r^{-n}\cos n\theta + M_n r^n\sin n\theta + P_n r^{-n}\sin n\theta\right]$$

- BY ④: $\phi_2(r,\theta) - r\cos\theta \to 1$, AS $r \to \infty$

$$\left[G + H\ln r + \sum_{n=1}^{\infty}K_n r^n\cos n\theta + L_n r^{-n}\cos n\theta + M_n r^n\sin n\theta + P_n r^{-n}\sin n\theta\right] - r\cos\theta \to 1$$
AS $r \to \infty$

- $G = 1$
- $H = 0$
- $K_n = 0$ $n \neq 1$, $K_1 = 1$
- $M_n = 0$
- L_n = UNDETERMINED
- P_n = UNDETERMINED

$$\therefore \phi_2(r,\theta) = 1 + r\cos\theta + \sum_{n=1}^{\infty}\left[L_n r^{-n}\cos n\theta + P_n r^{-n}\sin n\theta\right]$$

- BY ③ $\phi_2(2,\theta) = 1$

$$1 = 1 + 2\cos\theta + \sum_{n=1}^{\infty}\left[L_n \times 2^{-n}\cos n\theta + P_n \times 2^{-n}\sin n\theta\right]$$
$$\sum_{n=1}^{\infty}\left[\frac{L_n}{2^n}\cos n\theta + \frac{P_n}{2^n}\sin n\theta\right] = -2\cos\theta$$

$\therefore P_n = 0$

$L_n = 0$ EXCEPT $\frac{L_1}{2}\cos\theta = -2\cos\theta \Rightarrow L_1 = -4$

$$\therefore \phi_2(r,\theta) = 1 + r\cos\theta - \frac{4}{r}\cos\theta$$

- BY ① $\phi_1(1,\theta) = 0$

$$A + \cancel{B\ln 1} + \sum_{n=1}^{\infty}\left[C_n\cos n\theta + D_n\cos n\theta + E_n\sin n\theta + F_n\sin n\theta = 0\right]$$

$\therefore A = 0$ $\quad B$ = UNDETERMINED $\quad C_n + D_n = 0 \quad \forall n \geq 1$
$\quad E_n + F_n = 0 \quad \forall n \geq 0$

- BY ② $\phi_1(2,\theta) = 1$

$$B\ln 2 + \sum_{n=1}^{\infty}\left[C_n 2^n\cos n\theta + D_n 2^{-n}\cos n\theta + E_n 2^n\sin n\theta + F_n 2^{-n}\sin n\theta\right] = 1$$

$\therefore B = \frac{1}{\ln 2}$ $\quad 2^nC_n + 2^{-n}D_n = 0 \quad \forall n$
$\quad 2^nE_n + 2^{-n}F_n = 0 \quad \forall n$

HENCE

$$\left.\begin{aligned} C_n + D_n &= 0 \\ 2^nC_n + \frac{1}{2^n}D_n &= 0 \end{aligned}\right\} \Rightarrow C_n = -D_n$$
$$\Rightarrow -2^nD_n + \frac{1}{2^n}D_n = 0$$
$$\Rightarrow D_n\left(\frac{1}{2^n} - 2^n\right) = 0$$
$\forall n \geq 1$

$\therefore D_n = 0$
$C_n = 0$

& SIMILARLY AND IN ANALOGY
$E_n = F_n = 0$

$$\therefore \phi_1(r,\theta) = \frac{\ln r}{\ln 2}$$

Question 9

The steady state temperature distribution $u = u(r,\theta)$ in a thin metal disc in the shape of a circular sector of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 .$$

It is further given that $u(r,0) = u\left(r,\frac{1}{2}\pi\right) = 0$ and $\frac{\partial u}{\partial r}(\theta,1) = \theta$.

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta) = \sum_{k=1}^{\infty}\left[\frac{(-1)^{k+1} r^{2k} \sin(2k\theta)}{2k^2}\right]}$$

Question 10

The steady state temperature distribution $u = u(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 .$$

It is further given that $u(1,\theta) = \begin{cases} \theta & -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi \\ 0 & \text{otherwise} \end{cases}$

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty}\left[\left[\frac{1}{n}\cos\left(\frac{1}{2}n\pi\right) + \frac{2}{\pi n^2}\sin\left(\frac{1}{2}n\pi\right)\right] r^n \sin n\theta\right]}$$

Question 11

The steady state temperature distribution $u = u(r,\theta)$ in a circular thin metal disc of radius 1, satisfies Laplace's Equation in plane polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 .$$

It is further given that $u(1,\theta) = \begin{cases} \pi - \theta & 0 \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases}$

Determine a simplified expression for $u(r,\theta)$.

[*You are expected to derive the general solution of the partial equation in variable separate form*]

$$\boxed{u(r,\theta) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{r^n}{n}\left[\sin n\theta + \frac{1-(-1)^n}{\pi n}\cos n\theta\right]\right]}$$

Question 12

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0.$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0, \ 0 < r < 1$$

$$\nabla^2\Phi_2 = 0, \ r \geq 1.$$

It is further given that

- $|\Phi_1(0,\theta)| \leq M$, $M \in \mathbb{R}$.
- $\lim_{r\to\infty}\left[\Phi_2(r,\theta) - r\cos\theta\right] = 0$.
- $\Phi_1(1,\theta) = \Phi_2(1,\theta)$.
- $\frac{\partial\Phi_1}{\partial r}(1,\theta) + 3\frac{\partial\Phi_2}{\partial r}(1,\theta) = 0$.

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_1(r,\theta) = 3r\cos\theta}, \ \boxed{\Phi_2(r,\theta) = r\cos\theta + \frac{2}{r}\cos\theta}$$

[solution overleaf]

a)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$$\phi(r,\theta) = R(r)\Theta(\theta) \Rightarrow \frac{\partial \phi}{\partial r} = R'(r)\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \phi}{\partial r^2} = R''(r)\Theta(\theta)$$
$$\Rightarrow \frac{\partial^2 \phi}{\partial \theta^2} = R(r)\Theta''(\theta)$$

- SUB INTO THE P.D.E

$$\Rightarrow R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$
$$\Rightarrow r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$
$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$
$$\Rightarrow \boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

AS THE L.H.S. IS A FUNCTION OF r ONLY AND R.H.S. IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH CAN BE NEGATIVE, POSITIVE OR ZERO

- LOOKING AT THE R.H.S FIRST

I.E $-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \Rightarrow \Theta''(\theta) = -\lambda\Theta(\theta)$

- IF $\lambda = 0$, $\Theta''(\theta) = 0 \Rightarrow \boxed{\Theta(\theta) = A\theta + B}$ (I)
- IF $\lambda > 0$ $(\lambda = p^2)$, $\Theta''(\theta) = -p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cos p\theta + B\sin p\theta}$ (II)
- IF $\lambda < 0$ $(\lambda = -p^2)$, $\Theta''(\theta) = p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cosh p\theta + B\sinh p\theta}$ (III)

(OR EXPONENTIALS)

UNDER A POLAR SYSTEM, A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT $(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$

SO WE REQUIRE Θ TO PRODUCE PERIODIC (OR UNIQUE) SOLUTIONS

SOLUTION (III) IS DISCARDED (NO PERIODICITY)

SOLUTION (II) IS OK (PERIODICITY)

SOLUTION (I), $A = 0$, SO THE CONSTANT SOLUTION $\Theta(\theta) = B$ IS OK

FURTHER LOOKING AT (II) WITH COSINES (OR SINES)

$$\cos\theta = \cos(\theta + 2\pi)$$
$$\cos p\theta = \cos[p(\theta + 2\pi)]$$
$$p = n \quad \text{INTEGER}$$

HENCE FROM (I) $\lambda = 0 \Rightarrow n = 0$

FROM (II) $\lambda = p^2 \Rightarrow n = 1,2,3,4,\dots$

NEGATIVES CAN BE IGNORED BUT THERE WILL BE INCLUDED & ABSORBED IN THE CONSTANTS AT THE VERY END IN THE GENERAL SOLUTION.

- SO COLLECTING BOTH SOLUTIONS INTO ONE

$$\Theta_n(\theta) = A_n\cos n\theta + B\sin n\theta, \quad n = 0,1,2,3,4,\dots$$

- RETURNING TO THE L.H.S

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0,1,2,3,4,\dots$$

TWO CASES TO CONSIDER $n = 0$ OR $n \geqslant 1$

$n = 0$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} = 0$$
$$\Rightarrow rR'' + R' = 0$$
$$\Rightarrow rR'' = -R'$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE BOTH SIDES W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

INTEGRATE W.R.T r AGAIN

$$\Rightarrow R'(r) = \frac{C}{r}$$
$$\Rightarrow R(r) = C\ln r + D$$

$n \geqslant 1$, $n = 1,2,3,4\dots$

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$
$$r^2R''(r) + rR'(r) - n^2R(r) = 0$$

A CAUCHY-EULER TYPE

LET $R(r) = r^\lambda$, $R'(r) = \lambda r^{\lambda-1}$, $R''(r) = \lambda(\lambda-1)r^{\lambda-2}$

SUB INTO THE O.D.E

$$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda\cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$$
$$\lambda^2 - \lambda + \lambda - n^2 = 0$$
$$\lambda^2 = n^2$$
$$\lambda = \pm n$$

$$\therefore R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

COMBINING SOLUTIONS

$n = 0$: $\Theta(\theta) = B$, $R(r) = C\ln r + D$ — "ABSORB B INTO D"

$n \geqslant 1$: $\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$, $R_n(r) = \alpha_n r^n + \beta_n r^{-n}$

GENERAL SOLUTION

$$\phi(r,\theta) = (C\ln r + D) + \sum_{n=1}^{\infty}\left[(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING & RENAMING THE CONSTANTS

$$\boxed{\phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_nr^n\cos n\theta + D_nr^{-n}\cos n\theta + E_nr^n\sin n\theta + F_nr^{-n}\sin n\theta\right]}$$

NOTE THAT n BEING A NEGATIVE INTEGER IS INCLUDED HERE

b)

$\nabla^2\phi_1 = 0 \quad r < 1$

$\nabla^2\phi_2 = 0 \quad r \geqslant 1$

$|\phi_1(0,\theta)| \leqslant M$ (1)

$\lim_{r\to\infty}[\phi_2(r,\theta) - r\cos\theta] = 0$ (2)

$\phi_1(1,\theta) = \phi_2(1,\theta)$ (3)

$\frac{\partial\phi_1}{\partial r}(1,\theta) + 3\frac{\partial\phi_2}{\partial r}(1,\theta)$ (4)

USING THE ANSWER OF PART (a) FOR ϕ_1 & ϕ_2

- $\phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_nr^n\cos n\theta + D_nr^{-n}\cos n\theta + E_nr^n\sin n\theta + F_nr^{-n}\sin n\theta\right]$
- $\phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[K_nr^n\cos n\theta + L_nr^{-n}\cos n\theta + M_nr^n\sin n\theta + P_nr^{-n}\sin n\theta\right]$

- BY (1) $\phi_1(r,\theta)$ IS FINITE AT $r = 0$

$B = 0$, $D_n = 0$, $F_n = 0$

$$\phi_1(r,\theta) = A + \sum_{n=1}^{\infty}\left[C_nr^n\cos n\theta + E_nr^n\sin n\theta\right]$$

- BY (2) AS $r \to \infty$

$$r\cos\theta = G + H\ln r + \sum_{n=1}^{\infty}\left[K_nr^n\cos n\theta + L_nr^{-n}\cos n\theta + M_nr^n\sin n\theta + P_nr^{-n}\sin n\theta\right]$$

$G = 0$, $H = 0$, $L_n = P_n =$ UNDETERMINED, $M_n = 0$, $K_n = 0$ EXCEPT $K_1 = 1$

$$\phi_2(r,\theta) = r\cos\theta + \sum_{n=1}^{\infty}\left[L_nr^{-n}\cos n\theta + P_nr^{-n}\sin n\theta\right]$$

- BY (3) $\phi_1(1,\theta) = \phi_2(1,\theta)$

$$A + \sum_{n=1}^{\infty}\left[C_n\cos n\theta + E_n\sin n\theta\right] = \cos\theta + \sum_{n=1}^{\infty}\left[L_n\cos n\theta + P_n\sin n\theta\right]$$

$\therefore \boxed{A = 0}$

$\boxed{E_n = P_n}$

$\boxed{C_n = L_n}$ $n = 2,3,4,5,\dots$

$\boxed{C_1 = 1 + L_1}$

- $\frac{\partial\phi_1}{\partial r} = \sum_{n=1}^{\infty}\left[nC_nr^{n-1}\cos n\theta\right] + \sum_{n=1}^{\infty}\left[nE_nr^{n-1}\sin n\theta\right]$

$\frac{\partial\phi_2}{\partial r} = \cos\theta + \sum_{n=1}^{\infty}\left[-nL_nr^{-n-1}\cos n\theta\right] + \sum_{n=1}^{\infty}\left[-nP_nr^{-n-1}\sin n\theta\right]$

APPLY (4) $\frac{\partial\phi_1}{\partial r}(1,\theta) + 3\frac{\partial\phi_2}{\partial r}(1,\theta) = 0$

$$\sum_{n=1}^{\infty}\left[nC_n\cos n\theta\right] + \sum_{n=1}^{\infty}\left[nE_n\sin n\theta\right] + 3\cos\theta - 3\sum_{n=1}^{\infty}nL_n\cos n\theta - 3\sum_{n=1}^{\infty}nP_n\sin n\theta = 0$$

$$3\cos\theta + \sum_{n=1}^{\infty}(nC_n - 3nL_n)\cos n\theta + \sum_{n=1}^{\infty}(nE_n - 3nP_n)\sin n\theta = 0$$

$$3\cos\theta + \sum_{n=1}^{\infty}\left[n(C_n - 3L_n)\cos n\theta\right] + \sum_{n=1}^{\infty}\left[n[E_n - 3P_n]\sin n\theta\right] = 0$$

$n = 1$: $3 + 1(C_1 - 3L_1) = 0$

$\boxed{C_1 = 3L_1 - 3}$

$n \geqslant 2$: $\boxed{C_n = 3L_n}$

$\boxed{E_n = 3P_n}$

COMBINING RESULTS FROM (3) & (4)

- $E_n = P_n$, $E_n = 3P_n$, $n \geqslant 1$

$\therefore \boxed{E_n = P_n = 0}$

- $C_n = L_n$, $C_n = 3L_n$, $n \geqslant 2$

$C_n = L_n = 0 \quad n \geqslant 2$

- $n = 1$

$C_1 = L_1 + 1$, $C_1 = 3L_1 - 3$ $\Rightarrow 3L_1 - 3 = L_1 + 1$

$2L_1 = 4$

$\boxed{L_1 = 2}$

& $\boxed{C_1 = 3}$

$$\therefore \phi_1(r,\theta) = 3r\cos\theta$$
$$\phi_2(r,\theta) = r\cos\theta + \frac{2}{r}\cos\theta$$

Question 13

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0.$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$ and $\Phi_2 = \Phi_2(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0,\ r > 1$$

$$\nabla^2\Phi_2 = 0,\ 0 < r < 1.$$

It is further given that

- $\lim\limits_{r\to\infty}\left[\Phi_1(r,\theta) - r\cos\theta\right] = 2.$
- $\Phi_1(1,\theta) = \Phi_2(1,\theta).$
- $1 + \frac{\partial\Phi_1}{\partial r}(1,\theta) = \frac{\partial\Phi_2}{\partial r}(1,\theta).$
- $\lim\limits_{r\to 0}\left[r\Phi_2(r,\theta) - \cos\theta\right] = 0.$

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{},\ \boxed{\Phi_1(r,\theta) = 2 + \left(r + \tfrac{1}{r}\right)\cos\theta},\ \boxed{\Phi_2(r,\theta) = 2 + \ln r + \left(r + \tfrac{1}{r}\right)\cos\theta}$$

[solution overleaf]

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\phi(r,\theta) = R(r)\,\Theta(\theta)$$
$$\frac{\partial \phi}{\partial r} = R'(r)\,\Theta(\theta)$$
$$\frac{\partial^2 \phi}{\partial r^2} = R''(r)\,\Theta(\theta)$$
$$\frac{\partial^2 \phi}{\partial \theta^2} = R(r)\,\Theta''(\theta)$$

- SUBSTITUTE THESE EXPRESSIONS INTO THE P.D.E

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0}$$
$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$
$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) = -R(r)\Theta''(\theta)$$
$$\Rightarrow \frac{r^2R''(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} + \frac{rR'(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} = -\frac{\cancel{R(r)}\Theta''(\theta)}{\cancel{R(r)}\Theta(\theta)}$$
$$\Rightarrow \boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

- AS THE LHS OF THE ABOVE EXPRESSION IS A FUNCTION OF r ONLY AND THE RHS IS A FUNCTION OF θ ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH MAY BE POSITIVE, NEGATIVE OR ZERO

- LOOKING AT THE R.H.S FIRST

$$\boxed{-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda} \iff \boxed{\Theta''(\theta) = -\lambda\Theta(\theta)}$$

- IF $\lambda = 0$ $\quad \Theta''(\theta) = 0 \Rightarrow \boxed{\Theta(\theta) = A\theta + B}$ – (I)
- IF $\lambda > 0, \lambda = p^2$ $\quad \Theta''(\theta) = -p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cos p\theta + B\sin p\theta}$ – (II)
- IF $\lambda < 0, \lambda = -p^2$ $\quad \Theta''(\theta) = p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cosh p\theta + B\sinh p\theta}$ – (III)

(OR EXPONENTIALS)

- UNDER A SYSTEM OF POLAR CO-ORDINATES (r,θ), A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT $(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$

HENCE WE REQUIRE Θ TO PRODUCE PERIODIC (OR CONSTANT) SOLUTIONS

- SOLUTION (III) IS DISCARDED AS IT HAS NO PERIODICITY
- SOLUTION (II) IS FINE AS IT IS PERIODIC
- SOLUTION (I) IS O.K IF $A = 0$, BUT THEN IT IS CONSTANT AND IT CAN BE ABSORBED INTO (II)

- LOOKING FURTHER INTO (II) WITH SINES (OR COSINES)

$$\sin\theta = \sin(\theta + 2\pi)$$
$$\sin p\theta = \sin[p(\theta + 2\pi)]$$

$\therefore$ $p = n$, n = INTEGER

- HENCE WE CONCLUDE AT THIS STAGE

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \quad n = 0,1,2,3,4,\dots$$

AS WE SHALL SEE LATER NEGATIVE INTEGERS WILL BE INCLUDED IN THE FINAL SOLUTION, DUE TO ITS NATURE

- RETURNING TO THE L.H.S OF THE P.D.E (IN SEPARATED FORM)

$$\boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = \lambda = 0,1,2,3,4,\dots}$$

IF $\lambda = n = 0$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = 0$$
$$\Rightarrow rR''(r) + R'(r) = 0$$
$$\Rightarrow rR''(r) = -R'(r)$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$
$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE W.R.T r

$$\Rightarrow R(r) = C\ln|r| + D$$

IF $\lambda = n = 1,2,3,4,5,\dots$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2 \leftarrow \text{(II)}$$
$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

CAUCHY – EULER O.D.E

LET $R(r) = r^\lambda$, $R'(r) = \lambda r^{\lambda-1}$, $R''(r) = \lambda(\lambda-1)r^{\lambda-2}$ — SUB INTO ABOVE O.D.E

$$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda\cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$$
$$\lambda^2 - \lambda + \lambda - n^2 = 0$$
$$\lambda^2 = n^2$$
$$\lambda = \pm n$$

HENCE

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n} \qquad n = 1,2,3,4,\dots$$

AND NOTE THAT NEGATIVE INTEGERS ARE NOW INCLUDED

- FINALLY WE CAN COMBINE ALL THE SOLUTIONS INTO A GENERAL SOLUTION

- $n = 0$ $\quad \Theta_0(\theta) = B$, $\quad R_0(r) = C\ln r + D$ — ABSORB B INTO D
- $n = 1,2,3,4,\dots$ $\quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$, $\quad R_n(r) = \alpha r^n + \beta r^{-n}$

$$\Phi(r,\theta) = [C\ln r + D] + \sum_{n=1}^{\infty}\left[(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING & RELABELLING SOME OF THESE CONSTANTS, WE OBTAIN

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b) - APPLYING THE CONDITIONS

- $\nabla^2\phi_1 = 0 \quad r \geqslant 1$
- $\nabla^2\phi_2 = 0 \quad 0 < r < 1$

(1) $\lim_{r \to \infty}\left[\phi_1 - r\cos\theta\right] = 2$

(2) $\phi_1(1,\theta) = \phi_2(1,\theta)$

(3) $1 + \frac{\partial \phi_1}{\partial r}(1,\theta) = \frac{\partial \phi_2}{\partial r}(1,\theta)$

(4) $\lim_{r \to 0}\left[r\phi_2 - \cos\theta\right] = 0$

- LET ϕ_1 & ϕ_2 BE

- $\phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$
- $\phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + L_n r^{-n}\cos n\theta + M_n r^n\sin n\theta + P_n r^{-n}\sin n\theta\right]$

- BY CONDITION (1)

AS $r \to \infty$, $\phi_1(r,\theta) \to 2 + r\cos\theta$

$$\Rightarrow A = 2$$
$$\Rightarrow B = 0$$
$$\Rightarrow E_n = 0$$
$$\Rightarrow C_1 = 1, \; C_n = 0, \; n \neq 1$$
$$\Rightarrow D_n, F_n \text{ UNDETERMINED}$$

$$\therefore \phi_1(r,\theta) = 2 + r\cos\theta + \sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta\right]$$

- BY CONDITION (4)

AS $r \to 0$ $\quad r\phi_2(r,\theta) \to \cos\theta$

$$\Rightarrow G, H, K_n, M_n \text{ UNDETERMINED}$$
$$\Rightarrow P_n = 0$$
$$\Rightarrow L_1 = 1, \; L_n = 0 \; n \geqslant 2$$

$$\therefore \phi_2(r,\theta) = G + H\ln r + \frac{1}{r}\cos\theta + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + M_n r^n\sin n\theta\right]$$

- BY CONTINUITY AT $r = 1$, CONDITION (2)

$\phi_1(1,\theta) = \phi_2(1,\theta)$

$$2 + \cancel{\cos\theta} + \sum_{n=1}^{\infty}\left[D_n\cos n\theta + F_n\sin n\theta\right] = G + \cancel{\cos\theta} + \sum_{n=1}^{\infty}\left[K_n\cos n\theta + M_n\sin n\theta\right]$$

$G = 2 \qquad D_n = K_n, \; \forall n \qquad F_n = M_n, \; \forall n$

- DIFFERENTIATE TO APPLY (3)

$$\frac{\partial \phi_1}{\partial r} = \cos\theta + \sum_{n=1}^{\infty}\left[-nD_n r^{-n-1}\cos n\theta - nF_n r^{-n-1}\sin n\theta\right]$$
$$\frac{\partial \phi_2}{\partial r} = \frac{H}{r} - \frac{1}{r^2}\cos\theta + \sum_{n=1}^{\infty}\left[nK_n r^{n-1}\cos n\theta + nM_n r^{n}\sin n\theta\right]$$

$1 + \frac{\partial \phi_1}{\partial r}(1,\theta) = \frac{\partial \phi_2}{\partial r}(1,\theta)$

$$1 + \cos\theta + \sum_{n=1}^{\infty}\left[-nD_n\cos n\theta - nF_n\sin n\theta\right] = H - \cos\theta + \sum_{n=1}^{\infty}\left[nK_n\cos n\theta + nM_n\sin n\theta\right]$$

$\therefore H = 1$ & BY TIDYING UP THE REST

$$2\cos\theta = \sum_{n=1}^{\infty}\left[n[D_n + K_n]\cos n\theta + n[F_n + M_n]\sin n\theta\right]$$

- $n = 1,2,3,4,\dots$: $n[F_n + M_n] = 0$ & FROM EARLIER $F_n = M_n, \forall n$ $\quad \therefore F_n = M_n = 0$
- $n = 1$: $2\cos\theta = (D_1 + K_1)\cos\theta$, $D_1 + K_1 = 2$ BUT FROM EARLIER $D_n = K_n, \forall n$ $\quad \therefore D_1 = K_1 = 1$
- $n \geqslant 2$: $n[D_n + K_n] = 0$ & FROM EARLIER $D_n = K_n, \forall n$ $\quad \therefore D_n = K_n = 0$, $n \geqslant 2$

- FINALLY WE HAVE EXPRESSIONS FOR BOTH ϕ_1 & ϕ_2

$$\phi_1(r,\theta) = 2 + r\cos\theta + \frac{1}{r}\cos\theta$$
$$\phi_2(r,\theta) = 2 + \ln r + \frac{1}{r}\cos\theta + r\cos\theta$$

$$\phi_1(r,\theta) = 2 + \left(r + \frac{1}{r}\right)\cos\theta \qquad r > 1$$
$$\phi_2(r,\theta) = 2 + \ln r + \left(r + \frac{1}{r}\right)\cos\theta \qquad 0 < r < 1$$

Question 14

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_0 = \Phi_0(r,\theta)$ and $\Phi_1 = \Phi_1(r,\theta)$ satisfy

$$\nabla^2 \Phi_0 = 0, \; r > 1$$

$$\nabla^2 \Phi_1 = 0, \; 0 \le r < 1 .$$

It is further given that

- $\lim_{r \to \infty} \left[\Phi_0(r,\theta) - 2r\cos\theta \right] = 0$.
- $\lim_{r \to 0} \left[r\Phi_1(r,\theta) \right] = 0$
- $\Phi_0(1,\theta) + \Phi_1(1,\theta) = 3$.
- $\frac{\partial \Phi_0}{\partial r}(1,\theta) = 3\frac{\partial \Phi_1}{\partial r}(1,\theta)$.
- .

b) Determine expressions for Φ_1 and Φ_2.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta \right]},$$

$$\boxed{\Phi_0(r,\theta) = \left(2r - \frac{4}{r}\right)\cos\theta}, \; \boxed{\Phi_1(r,\theta) = 3 + 2r\cos\theta}$$

[solution overleaf]

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\Phi(r,\theta) = R(r)\Theta(\theta)$$
$$\frac{\partial \Phi}{\partial r} = R'(r)\Theta(\theta)$$
$$\frac{\partial^2 \Phi}{\partial r^2} = R''(r)\Theta(\theta)$$
$$\frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\Theta''(\theta)$$

- SUB INTO THE P.D.E

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$
$$\Rightarrow \frac{r^2R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{rR'(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = 0$$
$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0$$
$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

- NOW THE LHS IS A FUNCTION OF r ONLY AND THE RHS IS A FUNCTION OF θ ONLY.

HENCE BOTH SIDES MUST BE AT MOST A CONSTANT, SAY λ, WHICH MAY BE NEGATIVE, POSITIVE OR ZERO

- LOOKING AT THE RHS

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

IF $\lambda = 0 \Rightarrow \Theta''(\theta) = 0$
$$\Rightarrow \boxed{\Theta(\theta) = A\theta + B} \text{ — I}$$

IF $\lambda < 0,\ \lambda = -p^2 \Rightarrow \Theta''(\theta) = p^2\Theta(\theta)$
$$\Rightarrow \boxed{\Theta(\theta) = A\cosh p\theta + B\sinh p\theta} \text{ — II}$$
(OR EXPONENTIALS)

IF $\lambda > 0,\ \lambda = p^2 \Rightarrow \Theta''(\theta) = -p^2\Theta(\theta)$
$$\Rightarrow \boxed{\Theta(\theta) = A\cos p\theta + B\sin p\theta} \text{ — III}$$

- NOW IN PLANE POLARS, A UNIQUE CARTESIAN POINT (x,y) TO A POLAR POINT $(r, \theta + 2k\pi)$, $k \in \mathbb{Z}$

SO WE REQUIRE PERIODIC, OR CONSTANT.

SOLUTION (II) IS TO BE DISCARDED, AS IT NEITHER PERIODIC NOR CONSTANT.

SOLUTION (I) WE MAY ONLY KEEP THE CONSTANT TERM

SOLUTION (III) IS FINE AS IT IS PERIODIC

- LOOKING AT (III) IN SINES OR COSINES

$$\cos\theta = \cos(\theta + 2\pi)$$
$$\cos p\theta = \cos(p\theta + 2p\pi)$$
$$\therefore p = n, \text{ INTEGER}$$

- COLLECTING ALL THE RESULTS

FROM (III) $\lambda = p^2$, $p = n = 0, 1, 2, 3, 4, \ldots$

THUS $\lambda = 0$ IS ALREADY INCLUDED IN HERE

NOTE THAT NEGATIVE INTEGERS WILL ALSO BE INCLUDED AT THE END, DUE TO THE NATURE OF THE FINAL SOLUTION

HENCE

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta, \quad n = 0, 1, 2, 3, 4, \ldots$$

- LOOKING NOW AT THE LHS OF THE O.D.E

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2 \qquad n = 0, 1, 2, 3, \ldots$$

IF $\boxed{n = 0}$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = 0$$
$$\Rightarrow rR''(r) + R'(r) = 0$$
$$\Rightarrow rR''(r) = -R'(r)$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

- INTEGRATE BOTH SIDES W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + k$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$
$$\Rightarrow R'(r) = \frac{C}{r}$$

- INTEGRATE AGAIN W.R.T r

$$\Rightarrow R(r) = C\ln r + D$$

- IF $n = 1, 2, 3, 4, \ldots$

$$\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2$$
$$r^2R''(r) + rR'(r) - n^2R(r) = 0$$

- THIS IS A CAUCHY-EULER EQUATION

LET
$$\left.\begin{aligned} R(r) &= r^\mu \\ R'(r) &= \mu r^{\mu-1} \\ R''(r) &= \mu(\mu-1)r^{\mu-2} \end{aligned}\right\}$$

SUB INTO THE O.D.E

$$r^2\mu(\mu-1)r^{\mu-2} + r\mu r^{\mu-1} - n^2r^\mu = 0$$
$$\mu(\mu-1)r^\mu + \mu r^\mu - n^2r^\mu = 0$$
$$\mu(\mu-1) + \mu - n^2 = 0$$
$$\mu^2 = n^2$$
$$\mu = \pm n$$

$$\therefore R_n = \alpha_n r^n + \beta_n r^{-n}, \quad n = 1, 2, 3, 4, 5, \ldots$$

- COLLECTING THE RESULTS

- $n = 0$: $\Theta(\theta) = B$, $R(r) = C\ln r + D$ — ABSORB B INTO D
- $n \geqslant 1$: $\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$, $R_n(r) = \alpha r^n + \beta r^{-n}$

NOTE THAT NEGATIVE INTEGERS ARE NOW INCLUDED

$$\Phi(r,\theta) = \left[C\ln r + D\right] + \sum_{n=1}^{\infty}\left(A_n\cos n\theta + B_n\sin n\theta\right)\left(\alpha_n r^n + \beta_n r^{-n}\right)$$

ABSORBING AND RELABELLING THE CONSTANTS GIVES

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b)

$$\nabla^2\Phi_0 = 0 \qquad r > 1$$
$$\nabla^2\Phi_1 = 0 \qquad 0 \leqslant r < 1$$

CONDITIONS

$$\lim_{r\to\infty}\left[\Phi_0 - 2r\cos\theta\right] = 0 \quad (1)$$
$$\lim_{r\to 0}\left[r\Phi_1\right] = 0 \quad (2)$$
$$\Phi_0(1,\theta) + \Phi_1(1,\theta) = 3 \quad (3)$$
$$\frac{\partial\Phi_0}{\partial r}(1,\theta) = 3\frac{\partial\Phi_1}{\partial r}(1,\theta) \quad (4)$$

LET
$$\Phi_0(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$$
$$\Phi_1(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + L_n r^{-n}\cos n\theta + M_n r^n\sin n\theta + P_n r^{-n}\sin n\theta\right]$$

- BY (1)

$A = 0$

$B = 0$

$C = 2$, $C_n = 0$ $n \geqslant 2$

$E_n = 0$

D_n = UNDETERMINED

F_n = UNDETERMINED

$$\therefore \Phi_0(r,\theta) = 2r\cos\theta + \sum_{n=1}^{\infty} D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta$$

- APPLYING CONDITION (2)

G = UNDETERMINED

H = UNDETERMINED

K_n = UNDETERMINED

M_n = UNDETERMINED

$L_n = 0$

$P_n = 0$

$$\therefore \Phi_1(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty} k_n r^n\cos n\theta + M_n r^n\sin n\theta$$

- APPLY (3)

$$2\cos\theta + \sum_{n=1}^{\infty}\left[D_n\cos n\theta + F_n\sin n\theta\right] + G + \sum_{n=1}^{\infty} k_n\cos n\theta + M_n\sin n\theta = 3$$

$$\therefore \boxed{G = 3}$$

$$2\cos\theta + D_1\cos\theta + k_1\cos\theta = 0 \Rightarrow \boxed{D_1 + k_1 = -2}$$
$$D_n + k_n = 0 \quad n \geqslant 2 \Rightarrow \boxed{D_n = -k_n \quad n \geqslant 2}$$
$$F_n + M_n = 0 \Rightarrow \boxed{F_n = -M_n}$$

- DIFFERENTIATE TO APPLY THE LAST CONDITION

$$\Phi_0 = 2r\cos\theta + \sum_{n=1}^{\infty} D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta$$
$$\frac{\partial\Phi_0}{\partial r} = 2\cos\theta + \sum_{n=1}^{\infty} -nD_n r^{-n-1}\cos n\theta - nF_n r^{-n-1}\sin n\theta$$
$$\frac{\partial\Phi_1}{\partial r} = \frac{H}{r} + \sum_{n=1}^{\infty} nk_n r^{n-1}\cos n\theta + nM_n r^{n-1}\sin n\theta$$

- APPLY (4)

$$2\cos\theta + \sum_{n=1}^{\infty} -nD_n\cos n\theta - nF_n\sin n\theta = 3H + \sum 3nk_n\cos n\theta + 3nM_n\sin n\theta$$

$$\boxed{H = 0}$$

- $2\cos\theta - D_1\cos\theta = 3k_1\cos\theta$

$$\left.\begin{aligned} &\boxed{2 - D_1 = 3k_1} \\ &\text{\& FROM EARLIER} \\ &\boxed{D_1 + k_1 = -2} \end{aligned}\right\} \Rightarrow 2 + k_1 = 3k_1 - 2$$
$$4 = 2k_1$$
$$\boxed{k_1 = 2} \ \& \ \boxed{D_1 = -4}$$

- $-nD_n\cos n\theta = 3nk_n\cos n\theta$, $n \geqslant 2$

$$-nD_n = 3nk_n$$
$$\left.\begin{aligned} &\boxed{-D_n = 3k_n} \\ &\boxed{D_n = -k_n} \\ &\text{(FROM EARLIER)} \end{aligned}\right\} \boxed{k_n = D_n = 0} \quad n \geqslant 2$$

- $-nF_n\sin n\theta = 3nM_n\sin n\theta$

$$\left.\begin{aligned} &\boxed{-F_n = 3M_n} \\ &\text{\& FROM EARLIER} \\ &\boxed{F_n = -M_n} \end{aligned}\right\} \boxed{M_n = F_n = 0}$$

HENCE
$$\Phi_0(r,\theta) = 2r\cos\theta - \frac{4}{r}\cos\theta$$
$$\Phi_1(r,\theta) = 3 + 2r\cos\theta$$

Question 15

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0.$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_0 = \Phi_0(r,\theta)$ and $\Phi_1 = \Phi_1(r,\theta)$ satisfy

$$\nabla^2\Phi_0 = 0, \; r > 1$$

$$\nabla^2\Phi_1 = 0, \; 0 \le r \le 1.$$

It is further given that

- $\lim\limits_{r\to\infty}\left[\Phi_0(r,\theta) - r\cos\theta\right] = 0$.
- $\Phi_1(0,\theta) \le M$, $M \in \mathbb{R}$.
- $2\Phi_0(1,\theta) + \Phi_1(1,\theta) = 2\pi$.
- $\frac{\partial\Phi_0}{\partial r}(1,\theta) = \frac{\partial\Phi_1}{\partial r}(1,\theta)$.

b) Determine expressions for Φ_0 and Φ_1.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_0(r,\theta) = \left(r - \frac{3}{r}\right)\cos\theta}, \quad \boxed{\Phi_1(r,\theta) = 2\pi + 4r\cos\theta}$$

[solution overleaf]

a) $$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\phi(r,\theta) = R(r)\Theta(\theta)$$
$$\frac{\partial \phi}{\partial r} = R'(r)\Theta(\theta)$$
$$\frac{\partial^2 \phi}{\partial r^2} = R''(r)\Theta(\theta)$$
$$\frac{\partial^2 \phi}{\partial \theta^2} = R(r)\Theta''(\theta)$$

- SUBSTITUTE INTO THE P.D.E.

$$\Rightarrow R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$
$$\Rightarrow r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$
$$\Rightarrow -R(r)\Theta''(\theta) = r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta)$$
$$\Rightarrow -\frac{R(r)\Theta''(\theta)}{R(r)\Theta(\theta)} = \frac{r^2R''(r)\Theta(\theta)}{R(r)\Theta(\theta)} + \frac{rR'(r)\Theta(\theta)}{R(r)\Theta(\theta)}$$
$$\Rightarrow \boxed{-\frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)}}$$

AS THE L.H.S IS A FUNCTION OF θ ONLY AND THE R.H.S IS A FUNCTION OF r ONLY, BOTH SIDES ARE AT MOST A CONSTANT. THIS CONSTANT, SAY λ, CAN BE POSITIVE, NEGATIVE OR ZERO.

- LOOKING AT THE L.H.S FIRST

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda \Rightarrow \boxed{\Theta''(\theta) = -\lambda\Theta(\theta)}$$

IF $\lambda = 0$: $\Theta''(\theta) = 0 \Rightarrow \boxed{\Theta(\theta) = A\theta + B}$ (I)

IF $\lambda > 0$, $\lambda = p^2$: $\Theta''(\theta) = -p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cos p\theta + B\sin p\theta}$ (II)

IF $\lambda < 0$, $\lambda = -p^2$: $\Theta''(\theta) = +p^2\Theta(\theta) \Rightarrow \boxed{\Theta(\theta) = A\cosh p\theta + B\sinh p\theta}$ (III) (OR EXPONENTIALS)

- UNDER A POLAR CO-ORDINATE SYSTEM, A UNIQUE CARTESIAN POINT (x,y) GETS MAPPED TO A POLAR POINT $(r, \theta + 2k\pi)$, WHERE k IS AN INTEGER. HENCE WE REQUIRE θ TO PRODUCE PERIODIC (OR CONSTANT) SOLUTIONS
  - SOLUTION III IS DISCARDED (NO PERIODICITY IN θ)
  - SOLUTION II IS FINE (FULLY PERIODIC)
  - SOLUTION I IS OK IF $A=0$, IE $\Theta = B$ = CONSTANT

- LOOKING IN SOLUTION (II) IN MORE DETAIL WITH EITHER SINES OR COSINES

$$\sin\theta = \sin(\theta + 2\pi)$$
$$\sin(p\theta) = \sin(p(\theta + 2\pi)) = \sin(p\theta + 2p\pi)$$

$\therefore 2p\pi = 2n\pi \quad n \in \mathbb{N}$

$\therefore \boxed{p = n \text{ IS AN INTEGER}}$

HENCE (I), $\lambda = 0 \Rightarrow n = 0$

(II), $\lambda = p^2 \Rightarrow n = 0, \pm1, \pm2, \pm3, \ldots$

AT THIS STAGE WE SHALL NOT INCLUDE NEGATIVE INTEGERS FOR THE SAKE OF SIMPLICITY, BUT AS WE SHALL SEE THESE WILL "AUTOMATICALLY" BE INCLUDED IN THE FINAL ANSWER (DUE TO ITS NATURE)

THUS, SO FAR

$$\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta \qquad n = 0,1,2,3,4,\ldots$$

- RETURNING TO THE R.H.S OF THE O.D.E

$$\boxed{\frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = \lambda = n = 0,1,2,3,4,\ldots}$$

IF $n=0$ $\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = 0$

$$\Rightarrow rR''(r) + R'(r) = 0$$
$$\Rightarrow rR''(r) = -R'(r)$$
$$\Rightarrow \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE BOTH SIDES W.R.T r

$$\Rightarrow \ln|R'(r)| = -\ln|r| + \ln C$$
$$\Rightarrow \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$
$$\Rightarrow R'(r) = \frac{C}{r}$$

INTEGRATE AGAIN W.R.T r

$$\Rightarrow \boxed{R(r) = C\ln r + D}$$

IF $n = 1,2,3,4,\ldots$ $\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = n^2$

$$\Rightarrow r^2R''(r) + rR'(r) - n^2R(r) = 0$$

CAUCHY–EULER TYPE

LET $R(r) = r^\lambda$, $R'(r) = \lambda r^{\lambda-1}$, $R''(r) = \lambda(\lambda-1)r^{\lambda-2}$ } SUB INTO THE O.D.E.

$$\lambda(\lambda-1)\cancel{r^\lambda} + \lambda\cancel{r^\lambda} - n^2\cancel{r^\lambda} = 0$$
$$\lambda^2 - \lambda + \lambda - n^2 = 0$$
$$\lambda^2 = n^2$$
$$\lambda = \pm n$$

THUS $$R_n(r) = \alpha_n r^n + \beta_n r^{-n} \qquad n = 1,2,3,4,\ldots$$

AND NOTE THAT NEGATIVE INTEGERS ARE NOW INCLUDED

- COMBINING THE SOLUTIONS

$n = 0$: $\Theta(\theta) = B$, $R(r) = C\ln r + D$ } ABSORB B INTO D

$n = 1,2,3,4,\ldots$: $\Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$, $R_n(\theta) = \alpha_n r^n + \beta_n r^{-n}$

GENERAL SOLUTIONS

$$\phi(r,\theta) = [C\ln r + D] + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})$$

(NOTE $n=0$, PRODUCES A CONSTANT WHICH IS ALREADY INCLUDED IN THE FIRST PART)

REWRITING, ABSORBING & RELABELLING CONSTANTS, WE OBTAIN

$$\boxed{\phi(r,\theta) = A\ln r + B + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b) $\nabla^2\phi_0 = 0 \quad r > 0$

$\nabla^2\phi_1 = 0 \quad 0 \le r < 1$

$\lim_{r\to\infty}[\phi_0 - r\cos\theta] = 0$ (1)

$\phi_1(0,\theta) \le M,\ M \in \mathbb{R}$ (2)

$\phi_1(1,\theta) + 2\phi_0(1,\theta) = 2\pi$ (3)

$\frac{\partial\phi_1}{\partial r}(1,\theta) = \frac{\partial\phi_0}{\partial r}(1,\theta)$ (4)

- LET $\phi_0 = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$

$\phi_1 = G + H\ln r + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + L_n r^{-n}\cos n\theta + M_n r^n\sin n\theta + P_n r^{-n}\sin n\theta\right]$

- BY (1) AS $r \to \infty$ $\phi_0 \to r\cos\theta$

$\therefore A = 0$, $B = 0$; D_n = UNDETERMINED, F_n = UNDETERMINED; $E_n = 0$, $C_1 = 1$, $C_n = 0$ $(n \ge 2)$

$$\therefore \boxed{\phi_0(r,\theta) = r\cos\theta + \sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta\right]}$$

- BY (2) $\phi_1(0,\theta) \le M,\ M \in \mathbb{R}$, IE ϕ_1 IS BOUNDED AS $r \to 0$

$\therefore H = 0$, $L_n = 0$, $P_n = 0$; G = UNDETERMINED, K_n = UNDETERMINED, M_n = UNDETERMINED

$$\boxed{\phi_1(r,\theta) = G + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + M_n r^n\sin n\theta\right]}$$

- BY (3) $\phi_1(1,\theta) + 2\phi_0(1,\theta) = 2\pi$

$$G + \sum_{n=1}^{\infty}[K_n\cos n\theta + M_n\sin n\theta] + 2\cos\theta + \sum_{n=1}^{\infty}2D_n\cos n\theta + 2F_n\sin n\theta = 2\pi$$

THUS $G = 2\pi$; $K_1 + 2 + 2D_1 = 0$; $K_n + 2D_n = 0$ $(n = 2,3,4,\ldots)$; $M_n + 2F_n = 0$ $(n = 1,2,3,4,\ldots)$

- $\phi_0(r,\theta) = r\cos\theta + \sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta\right]$

$\phi_1(r,\theta) = 2\pi + \sum_{n=1}^{\infty}\left[K_n r^n\cos n\theta + M_n r^n\sin n\theta\right]$ } $\Rightarrow$

$\frac{\partial\phi_0}{\partial r}(r,\theta) = \cos\theta + \sum_{n=1}^{\infty}\left[-nD_n r^{-n-1}\cos n\theta - nF_n r^{-n-1}\sin n\theta\right]$

$\frac{\partial\phi_1}{\partial r}(r,\theta) = \sum_{n=1}^{\infty}\left[nK_n r^{n-1}\cos n\theta + nM_n r^{n-1}\sin n\theta\right]$ } $\Rightarrow$

BY (4) $\frac{\partial\phi_1}{\partial r}(1,\theta) = \frac{\partial\phi_2}{\partial r}(1,\theta)$

$$\cos\theta + \sum_{n=1}^{\infty}[-nD_n\cos n\theta - nF_n\sin n\theta] = \sum_{n=1}^{\infty}(nK_n\cos n\theta + nM_n\sin n\theta]$$

$1 - D_1 = K_1$

$-nD_n = nK_n \quad n = 2,3,4,5,\ldots$

$-nF_n = nM_n \quad n = 1,2,3,4,\ldots$

$K_1 + 2 + 2D_1 = 0$, $K_1 = 1 - D_1$ } $\Rightarrow 1 - D_1 + 2 + 2D_1 = 0$

$\boxed{D_1 = -3}$ $\boxed{K_1 = 4}$

$K_n + 2D_n = 0$, $nK_n = -nD_n$ } $n = 2,3,4,5,\ldots$

$K_n + 2D_n = 0$

$K_n + D_n = 0$

$\therefore K_n = D_n = 0 \quad n = 2,3,4,5,\ldots$

$M_n + 2F_n = 0$, $nM_n = -nF_n$ } $n = 1,2,3,4,\ldots$

$M_n + 2F_n = 0$

$M_n + F_n = 0$

$\therefore M_n = F_n = 0 \quad n = 1,2,3,\ldots$

THUS

$$\boxed{\begin{aligned}\phi_0(r,\theta) &= r\cos\theta - \frac{3}{r}\cos\theta = \left(r - \frac{3}{r}\right)\cos\theta \\ \phi_1(r,\theta) &= 2\pi + 4r\cos\theta\end{aligned}}$$

Question 16

The function $\Phi = \Phi(r,\theta)$ satisfies Laplace's Equation in plane polar coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0 .$$

a) Derive the general solution of the above equation, in variable separable form.

The functions $\Phi_1 = \Phi_1(r,\theta)$, $\Phi_2 = \Phi_2(r,\theta)$ and $\Phi_3 = \Phi_3(r,\theta)$ satisfy

$$\nabla^2\Phi_1 = 0 ,\ r > 2$$

$$\nabla^2\Phi_2 = 0 ,\ 1 < r < 2$$

$$\nabla^2\Phi_3 = 0 ,\ 0 < r < 1 .$$

It is further given that

- $\displaystyle\lim_{r\to\infty}\left[\Phi_1(r,\theta) - r\cos\theta\right] = 0$.
- $\displaystyle\frac{\partial\Phi_1}{\partial r}(2,\theta) = \frac{\partial\Phi_2}{\partial r}(2,\theta) = 2\cos\theta$.
- $\Phi_2(1,\theta) = \Phi_3(1,\theta) = \cos\theta$.
- $\displaystyle\lim_{r\to 0}\left[r\frac{\partial\Phi_3}{\partial r}(r,\theta)\right] = 1$.

b) Determine expressions for Φ_1, Φ_2 and Φ_3.

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[\left(C_n r^n + D_n r^{-n}\right)\cos n\theta + \left(E_n r^n + F_n r^{-n}\right)\sin n\theta\right]},$$

$$\boxed{\Phi_1(r,\theta) = r\cos\theta - \frac{4}{r}\cos\theta},\ \boxed{\Phi_2(r,\theta) = \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta},\ \boxed{\Phi_3(r,\theta) = \ln r + r\cos\theta}$$

[solution overleaf]

ASSUME A SOLUTION IN VARIABLE SEPARATE FORM – DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$\boxed{\Phi(r,\theta) = R(r)\,\Theta(\theta) \implies \frac{\partial \Phi}{\partial r} = R'(r)\Theta(\theta) \implies \frac{\partial^2 \Phi}{\partial r^2} = R''(r)\Theta(\theta) \implies \frac{\partial^2 \Phi}{\partial \theta^2} = R(r)\Theta''(\theta)}$$

$$\implies \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

$$\implies R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

$$\implies r^2R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0$$

$$\implies \frac{r^2R''(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} + \frac{rR'(r)\cancel{\Theta(\theta)}}{R(r)\cancel{\Theta(\theta)}} + \frac{\cancel{R(r)}\Theta''(\theta)}{\cancel{R(r)}\Theta(\theta)} = \frac{0}{R(r)\Theta(\theta)}$$

$$\implies \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)}$$

AS THE L.H.S IS A FUNCTION OF r ONLY AND THE R.H.S IS A FUNCTION OF θ ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

LOOKING AT THE R.H.S OF THE ABOVE EXPRESSION, WE REQUIRE TRIGONOMETRIC SOLUTIONS IN θ [DUE TO THE POLAR SYSTEM REPEATING EVERY 2π]

GIVEN FURTHER THAT THERE IS ALREADY A MINUS SIGN IN THE R.H.S, WE DEDUCE $\lambda > 0$, SAY p^2

$$\implies -\frac{\Theta''(\theta)}{\Theta(\theta)} = p^2$$

$$\implies \Theta''(\theta) = -p^2\Theta(\theta)$$

$$\implies \boxed{\Theta(\theta) = A\cos(p\theta) + B\sin(p\theta)}$$

EXAMINING FURTHER THE "θ SOLUTION" WITH SAY SINES (OR COSINES)

$$\sin\theta = \sin(\theta + 2\pi) \implies \sin(p\theta) = \sin[p(\theta + 2\pi)] = \sin(p\theta + 2p\pi)$$

$$\implies p = n = \text{INTEGER} = 0, 1, 2, 3, 4, \dots$$

NOTES

- $n = 0$ IS O.K AS IT PRODUCES A CONSTANT SOLUTION
- $n < 0$ NEED NOT TO BE CONSIDERED, AS AT THIS STAGE WILL BE ABSORBED INTO THE CONSTANTS, BUT THEY WILL ACTUALLY APPEAR AT THE END

$$\therefore \boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)}$$

RETURNING TO THE L.H.S (FUNCTION OF r ONLY) WITH $\lambda = n = \text{INTEGER}$

$$\implies r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2, \quad n = 0, 1, 2, 3, 4, \dots$$

- IF $n = 0$

$$\implies r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = 0$$

$$\implies rR''(r) + R'(r) = 0$$

$$\implies rR''(r) = -R'(r)$$

$$\implies \frac{R''(r)}{R'(r)} = -\frac{1}{r}$$

INTEGRATE W.R.T r

$$\implies \ln|R'(r)| = -\ln r + \ln C$$

$$\implies \ln|R'(r)| = \ln\left|\frac{C}{r}\right|$$

$$\implies R'(r) = \frac{C}{r}$$

$$\implies R(r) = C\ln r + D$$

IF $n = 1, 2, 3, 4, \dots$

$$\implies r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = n^2$$

$$\implies r^2R''(r) + rR'(r) - n^2R(r) = 0$$

THIS IS A CAUCHY EULER O.D.E.

LET $R(r) = r^{\lambda}$

$R'(r) = \lambda r^{\lambda - 1}$

$R''(r) = \lambda(\lambda - 1)r^{\lambda - 2}$

SUBSTITUTE INTO THE O.D.E.

$\lambda(\lambda-1)\cancel{r^{\lambda}} + \lambda\cancel{r^{\lambda}} - n^2\cancel{r^{\lambda}} = 0$

$\lambda^2 - \lambda + \lambda - n^2 = 0$

$\lambda^2 = n^2$

$\lambda = \pm n$

$R_n(r) = \alpha_n r^n + \beta_n r^{-n}, \quad n = 1, 2, 3, 4, \dots$

COMBINING RESULTS

$n = 0 \quad \Theta(\theta) = B$

$\qquad\quad R(r) = C\ln r + D$

$n = 1, 2, 3, 4, \dots \quad \Theta_n(\theta) = A_n\cos n\theta + B_n\sin n\theta$

$\qquad\qquad\qquad\quad R_n(r) = \alpha_n r^n + \beta_n r^{-n}$

HENCE THE GENERAL SOLUTION IS

$$\Phi(r,\theta) = [C\ln r + D] + \sum_{n=1}^{\infty}\left[(A_n\cos n\theta + B_n\sin n\theta)(\alpha_n r^n + \beta_n r^{-n})\right]$$

ABSORBING AND RELABELLING CONSTANTS WE OBTAIN

$$\boxed{\Phi(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]}$$

b) NOW WE HAVE THREE FUNCTIONS IN THE 3 DIFFERENT REGIONS OF r, AND CONDITIONS TO BE APPLIED

$$\Phi_1(r,\theta) = A + B\ln r + \sum_{n=1}^{\infty}\left[C_n r^n\cos n\theta + D_n r^{-n}\cos n\theta + E_n r^n\sin n\theta + F_n r^{-n}\sin n\theta\right]$$

$$\Phi_2(r,\theta) = G + H\ln r + \sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta + K_n r^{-n}\cos n\theta + L_n r^n\sin n\theta + M_n r^{-n}\sin n\theta\right]$$

$$\Phi_3(r,\theta) = N + P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$

APPLY CONDITION AS $r \to \infty \quad \Phi_1(r,\theta) \to r\cos\theta$

HENCE $A = B = 0$

$E_n = 0$

$C_1 = 1, \quad C_n = 0 \quad n \geqslant 2$

D_n, F_n ARE UNDETERMINED

$$\boxed{\Phi_1(r,\theta) = r\cos\theta + \sum_{n=1}^{\infty}\left[D_n r^{-n}\cos n\theta + F_n r^{-n}\sin n\theta\right]}$$

DIFFERENTIATE W.R.T r AND APPLY $\frac{\partial \Phi_1}{\partial r} = 2\cos\theta$ AT $r = 2$

$$\frac{\partial \Phi_1}{\partial r}(r,\theta) = \cos\theta - \sum_{n=1}^{\infty}\left[nD_n r^{-n-1}\cos n\theta + nF_n r^{-n-1}\sin n\theta\right]$$

$$2\cos\theta = \cos\theta - \sum_{n=1}^{\infty}\left[nD_n 2^{-n-1}\cos n\theta + nF_n 2^{-n-1}\sin n\theta\right]$$

HENCE $F_n = 0$

$D_n = 0 \quad n = 2, 3, 4, 5, \dots$

BUT IF $n = 1$

$2\cos\theta = \cos\theta - 1 \times D_1 \times 2\cos\theta$

$2\cos\theta = \cos\theta - \frac{1}{4}D_1\cos\theta$

$2 = 1 - \frac{1}{4}D_1$

$\frac{1}{4}D_1 = -1$

$D_1 = -4$

$$\therefore \Phi_1(r,\theta) = r\cos\theta - \frac{4}{r}\cos\theta$$

APPLY CONDITION $\Phi_2(r,\theta) = \cos\theta$ AT $r = 1$

$$\implies \cos\theta = G + \cancel{H\ln 1} + \sum_{n=1}^{\infty}\left[J_n\cos n\theta + K_n\cos n\theta + L_n\sin n\theta + M_n\sin n\theta\right]$$

$$\implies \cos\theta = G + \sum_{n=1}^{\infty}\left[(J_n + K_n)\cos n\theta + (L_n + M_n)\sin n\theta\right]$$

HENCE $G = 0$

H IS UNDETERMINED

$J_1 + K_1 = 1$

$J_n + K_n = 0 \quad n = 2, 3, 4, 5, \dots$

$L_n + M_n = 0 \quad n = 1, 2, 3, 4, \dots$

$$\boxed{\Phi_2(r,\theta) = H\ln r + \sum_{n=1}^{\infty}\left[J_n r^n\cos n\theta + K_n r^{-n}\cos n\theta + L_n r^n\sin n\theta + M_n r^{-n}\sin n\theta\right]}$$

DIFFERENTIATE Φ_2 W.R.T r AND APPLY $\frac{\partial \Phi_2}{\partial r} = 2\cos\theta$ AT $r = 2$

$$\implies \frac{\partial \Phi_2}{\partial r} = \frac{H}{r} + \sum_{n=1}^{\infty}\left[nJ_n r^{n-1}\cos n\theta - nK_n r^{-n-1}\cos n\theta + nL_n r^{n-1}\sin n\theta - nM_n r^{-n-1}\sin n\theta\right]$$

$$\implies 2\cos\theta = \frac{H}{2} + \sum_{n=1}^{\infty}\left[nJ_n 2^{n-1}\cos n\theta - \frac{nK_n}{2^{n+1}}\cos n\theta + nL_n 2^{n-1}\sin n\theta - \frac{nM_n}{2^{n+1}}\sin n\theta\right]$$

HENCE $H = 0$

IF $n = 1 \quad 2\cos\theta = J_1\cos\theta - \frac{K_1}{4}\cos\theta$

$\qquad\qquad 2 = J_1 - \frac{1}{4}K_1$

IF $n = 2, 3, 4, 5, \dots \quad 0 = nJ_n 2^{n-1}\cos n\theta - \frac{nK_n}{2^{n+1}}\cos n\theta$

$\qquad\qquad 0 = J_n 2^{n-1} - \frac{K_n}{2^{n+1}}$

IF $n = 1, 2, 3, 4, \dots \quad 0 = nL_n 2^{n-1}\sin n\theta - \frac{nM_n}{2^{n+1}}\sin n\theta$

$\qquad\qquad 0 = L_n 2^{n-1} - \frac{M_n}{2^{n+1}}$

FROM EARLIER $J_1 + K_1 = 1$

$$\left.\begin{aligned} J_1 + K_1 &= 1 \\ J_1 - \tfrac{1}{4}K_1 &= 2 \end{aligned}\right\} \implies \frac{5}{4}K_1 = -1 \implies K_1 = -\frac{4}{5} \quad \& \quad J_1 = \frac{9}{5}$$

ALSO FROM EARLIER $J_n + K_n = 0, \quad n = 2, 3, 4, 5, \dots$

$$\left.\begin{aligned} J_n + K_n &= 0 \\ J_n 2^{n-1} - \tfrac{K_n}{2^{n+1}} &= 0 \end{aligned}\right\} \implies J_n = -K_n \implies -K_n 2^{n-1} - \frac{K_n}{2^{n+1}} = 0 \implies -K_n\left(2^{n-1} - \frac{1}{2^{n+1}}\right) = 0$$

$$\therefore K_n = 0 \quad \& \quad J_n = 0 \qquad n = 2, 3, 4, 5, \dots$$

AND IN ANALOGY $L_n = M_n = 0, \quad n = 1, 2, 3, 4, \dots$

$$\implies \Phi_2(r,\theta) = \frac{9}{5}r\cos\theta - \frac{4}{5r}\cos\theta$$

$$\implies \Phi_2(r,\theta) = \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta$$

NOW $\Phi_3(r,\theta) = \cos\theta$ AT $r = 1$

$$\implies \cos\theta = N + \cancel{P\ln 1} + \sum_{n=1}^{\infty}\left[Q_n\cos n\theta + R_n\cos n\theta + T_n\sin n\theta + W_n\sin n\theta\right]$$

$$\implies \cos\theta = N = \sum_{n=1}^{\infty}\left[(Q_n + R_n)\cos n\theta + (T_n + W_n)\sin n\theta\right]$$

HENCE $N = 0$

P IS UNDETERMINED

$T_n + W_n = 0, \quad n = 1, 2, 3, 4, 5, \dots$

$Q_1 + R_1 = 1$

$Q_n + R_n = 0, \quad n = 2, 3, 4, 5, \dots$

DIFFERENTIATE Φ_3 W.R.T r AND APPLY $r\frac{\partial \Phi_3}{\partial r} \to 1$ AS $r \to 0$

$$\Phi_3(r,\theta) = P\ln r + \sum_{n=1}^{\infty}\left[Q_n r^n\cos n\theta + R_n r^{-n}\cos n\theta + T_n r^n\sin n\theta + W_n r^{-n}\sin n\theta\right]$$

$$\frac{\partial \Phi_3}{\partial r}(r,\theta) = \frac{P}{r} + \sum_{n=1}^{\infty}\left[nQ_n r^{n-1}\cos n\theta - nR_n r^{-n-1}\cos n\theta + nT_n r^{n-1}\sin n\theta - nW_n r^{-n-1}\sin n\theta\right]$$

$$r\frac{\partial \Phi_3}{\partial r}(r,\theta) = P + \sum_{n=1}^{\infty}\left[nQ_n r^n\cos n\theta - nR_n r^{-n}\cos n\theta + nT_n r^n\sin n\theta - nW_n r^{-n}\sin n\theta\right]$$

NOW AS $r \to 0 \quad r\frac{\partial \Phi_3}{\partial r} \to 1$

$P = 1 \quad R_n = W_n = 0, \quad n = 1, 2, 3, 4, \dots$

BUT FROM EARLIER

$T_n + W_n = 0 \quad \& \quad \text{NOW } W_n = 0 \implies T_n = 0 \quad (n = 1, 2, 3, 4, \dots)$

$Q_n + R_n = 0 \quad \& \quad \text{NOW } R_n = 0 \implies Q_n = 0 \quad (n = 2, 3, 4, 5, \dots)$

$Q_1 + R_1 = 1 \quad \& \quad \text{NOW } R_1 = 0 \implies Q_1 = 1$

$$\therefore \Phi_3(r,\theta) = \ln r + r\cos\theta$$

COLLECTING ALL 3 RESULTS WE HAVE

$$\boxed{\Phi(r,\theta) = \begin{cases} \left(r - \frac{4}{r}\right)\cos\theta & r > 2 \\ \frac{1}{5}\left(9r - \frac{4}{r}\right)\cos\theta & 1 < r < 2 \\ \ln r + r\cos\theta & 0 < r < 1 \end{cases}}$$

LAPLACE'S EQUATION

$$\frac{\partial^2 \Phi}{\partial r^2}+\frac{1}{r}\frac{\partial \Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}+\frac{\partial^2 \Phi}{\partial z^2}=0, \quad \Phi=\Phi(r,\theta,z)$$

Three Dimensional in Cylindrical Polars

Question 1

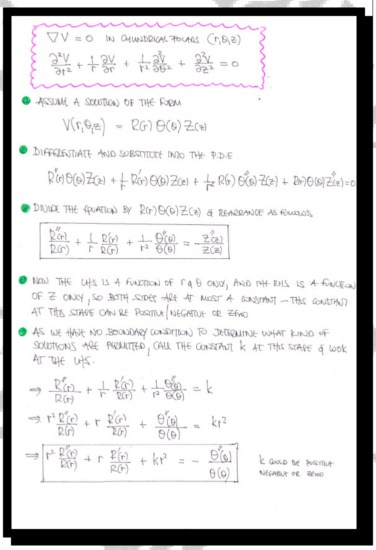
The potential function $V = V(r,\theta,z)$ satisfies Laplace's equation in cylindrical polar coordinates, shown below.

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 .$$

Use separation of variables to show that the radial part of the general solution of Laplace's equation in cylindrical polar coordinates, satisfies Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0, \quad n = 0, 1, 2, 3, \ldots$$

proof



$\nabla V = 0$ IN CYLINDRICAL POLARS (r,θ,z)

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- ASSUME A SOLUTION OF THE FORM

$$V(r,\theta,z) = R(r)\Theta(\theta)Z(z)$$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$$R''(r)\Theta(\theta)Z(z) + \frac{1}{r}R'(r)\Theta(\theta)Z(z) + \frac{1}{r^2}R(r)\Theta''(\theta)Z(z) + R(r)\Theta(\theta)Z''(z) = 0$$

- DIVIDE THE EQUATION BY $R(r)\Theta(\theta)Z(z)$ & REARRANGE AS FOLLOWS

$$\boxed{\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = -\frac{Z''(z)}{Z(z)}}$$

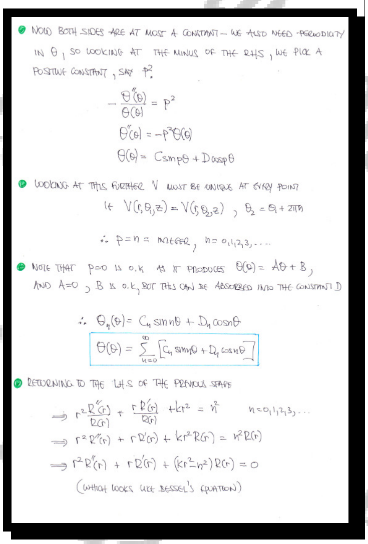
- NOW THE LHS IS A FUNCTION OF r & θ ONLY, AND THE RHS IS A FUNCTION OF z ONLY, SO BOTH SIDES ARE AT MOST A CONSTANT – THIS CONSTANT AT THIS STAGE CAN BE POSITIVE/NEGATIVE OR ZERO
- AS WE HAVE NO BOUNDARY CONDITION TO DETERMINE WHAT KIND OF SOLUTIONS ARE PERMITTED, CALL THE CONSTANT k AT THIS STAGE & LOOK AT THE LHS.

$$\Rightarrow \frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} = k$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + \frac{\Theta''(\theta)}{\Theta(\theta)} = kr^2$$

$$\Rightarrow \boxed{r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} + kr^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}}$$

k COULD BE POSITIVE NEGATIVE OR ZERO



- NOW BOTH SIDES ARE AT MOST A CONSTANT – WE ALSO NEED PERIODICITY IN θ, SO LOOKING AT THE MINUS OF THE RHS, WE PICK A POSITIVE CONSTANT, SAY p^2.

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = p^2$$

$$\Theta''(\theta) = -p^2\Theta(\theta)$$

$$\Theta(\theta) = C\sin p\theta + D\cos p\theta$$

- LOOKING AT THIS FURTHER V MUST BE UNIQUE AT EVERY POINT

IE $V(r,\theta_1,z) = V(r,\theta_2,z)$, $\theta_2 = \theta_1 + 2\pi n$

$\therefore$ $p = n =$ INTEGER, $n = 0,1,2,3,\ldots$

- NOTE THAT $p=0$ IS O.K AS IT PRODUCES $\Theta(\theta) = A\theta + B$, AND $A=0$, B IS O.K, BUT THIS CAN BE ABSORBED INTO THE CONSTANT D

$$\therefore \Theta_n(\theta) = C_n \sin n\theta + D_n \cos n\theta$$

$$\boxed{\Theta(\theta) = \sum_{n=0}^{\infty}\left[C_n \sin n\theta + D_n \cos n\theta\right]}$$

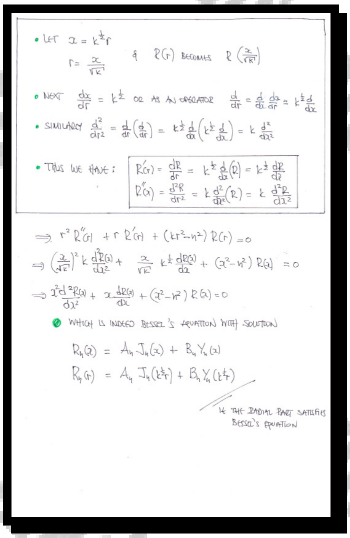
- RETURNING TO THE LHS OF THE PREVIOUS STAGE

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + kr^2 = n^2 \qquad n = 0,1,2,3,\ldots$$

$$\Rightarrow r^2R''(r) + rR'(r) + kr^2R(r) = n^2R(r)$$

$$\Rightarrow r^2R''(r) + rR'(r) + (kr^2 - n^2)R(r) = 0$$

(WHICH LOOKS LIKE BESSEL'S EQUATION)



- LET $x = k^{\frac{1}{2}}r$

$r = \frac{x}{\sqrt{k}}$ & $R(r)$ BECOMES $R\left(\frac{x}{\sqrt{k}}\right)$

- NEXT $\frac{dx}{dr} = k^{\frac{1}{2}}$ OR AS AN OPERATOR $\frac{d}{dr} = \frac{d}{dx}\frac{dx}{dr} = k^{\frac{1}{2}}\frac{d}{dx}$
- SIMILARLY $\frac{d^2}{dr^2} = \frac{d}{dr}\left(\frac{d}{dr}\right) = k^{\frac{1}{2}}\frac{d}{dx}\left(k^{\frac{1}{2}}\frac{d}{dx}\right) = k\frac{d^2}{dx^2}$
- THUS WE HAVE:

$$\boxed{\begin{aligned} R'(r) &= \frac{dR}{dr} = k^{\frac{1}{2}}\frac{d}{dx}(R) = k^{\frac{1}{2}}\frac{dR}{dx} \\ R''(x) &= \frac{d^2R}{dr^2} = k\frac{d^2}{dx^2}(R) = k\frac{d^2R}{dx^2} \end{aligned}}$$

$$\Rightarrow r^2R''(r) + rR'(r) + (kr^2 - n^2)R(r) = 0$$

$$\Rightarrow \left(\frac{x}{\sqrt{k}}\right)^2 k\frac{d^2R(x)}{dx^2} + \frac{x}{\sqrt{k}}k^{\frac{1}{2}}\frac{dR(x)}{dx} + (x^2 - n^2)R(x) = 0$$

$$\Rightarrow x^2\frac{d^2R(x)}{dx^2} + x\frac{dR(x)}{dx} + (x^2 - n^2)R(x) = 0$$

- WHICH IS INDEED BESSEL'S EQUATION WITH SOLUTION

$$R_n(x) = A_nJ_n(x) + B_nY_n(x)$$

$$R_n(r) = A_nJ_n(k^{\frac{1}{2}}r) + B_nY_n(k^{\frac{1}{2}}r)$$

IE THE RADIAL PART SATISFIES BESSEL'S EQUATION