HEAT EQUATION

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$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad \theta = \theta(x,t)$$

One Dimensional

Question 1

A thin rod of length 2 m has temperature $z = 20$ °C throughout its length.

At time $t = 0$, the temperature z is suddenly dropped to $z = 0$ °C at both its ends at $x = 0$, and at $x = 2$.

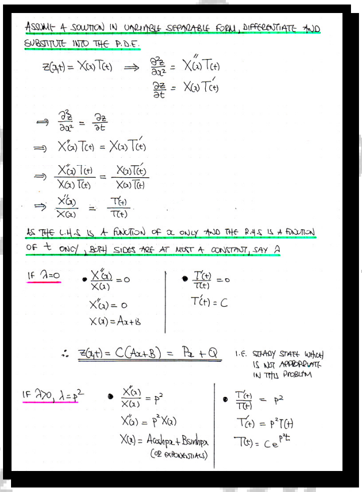
The temperature distribution along the rod $z(x,t)$, satisfies the standard heat equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}, \quad 0 \le x \le 2, \quad t \ge 0.$$

Assuming the rod is insulated along its length, determine an expression for $z(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$z(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{80}{\pi(2n-1)} \exp\left[-\frac{\pi^2 (2n-1)^2 t}{4} \right] \sin\left[\frac{(2n-1)\pi x}{2} \right] \right\}$$



ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM, DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E.

$z(x,t) = X(x)T(t) \implies \frac{\partial^2 z}{\partial x^2} = X''(x)T(t)$, $\frac{\partial z}{\partial t} = X(x)T'(t)$

$\implies \frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$

$\implies X''(x)T(t) = X(x)T'(t)$

$\implies \frac{X''(x)T(t)}{X(x)T(t)} = \frac{X(x)T'(t)}{X(x)T(t)}$

$\implies \frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$

AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

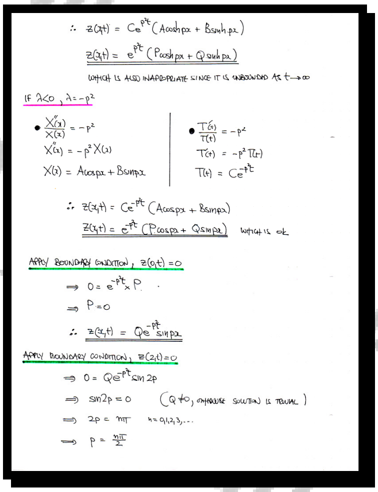
IF $\lambda = 0$

- $\frac{X''(x)}{X(x)} = 0$, $X''(x) = 0$, $X(x) = Ax + B$
- $\frac{T'(t)}{T(t)} = 0$, $T'(t) = C$

$\therefore z(x,t) = C(Ax+B) = Px + Q$ I.E. STEADY STATE WHICH IS NOT APPROPRIATE IN THIS PROBLEM

IF $\lambda > 0$, $\lambda = p^2$

- $\frac{X''(x)}{X(x)} = p^2$, $X''(x) = p^2 X(x)$, $X(x) = A\cosh px + B\sinh px$ (OR EXPONENTIALS)
- $\frac{T'(t)}{T(t)} = p^2$, $T'(t) = p^2 T(t)$, $T(t) = Ce^{p^2 t}$



$\therefore z(x,t) = Ce^{p^2 t}(A\cosh px + B\sinh px)$

$z(x,t) = e^{p^2 t}(P\cosh px + Q\sinh px)$

WHICH IS ALSO INAPPROPRIATE SINCE IT IS UNBOUNDED AS $t \to \infty$

IF $\lambda < 0$, $\lambda = -p^2$

- $\frac{X''(x)}{X(x)} = -p^2$, $X''(x) = -p^2 X(x)$, $X(x) = A\cos px + B\sin px$
- $\frac{T'(t)}{T(t)} = -p^2$, $T'(t) = -p^2 T(t)$, $T(t) = Ce^{-p^2 t}$

$\therefore z(x,t) = Ce^{-p^2 t}(A\cos px + B\sin px)$

$z(x,t) = e^{-p^2 t}(P\cos px + Q\sin px)$ WHICH IS OK

APPLY BOUNDARY CONDITION, $z(0,t) = 0$

$\implies 0 = e^{-p^2 t} \times P$

$\implies P = 0$

$\therefore z(x,t) = Qe^{-p^2 t}\sin px$

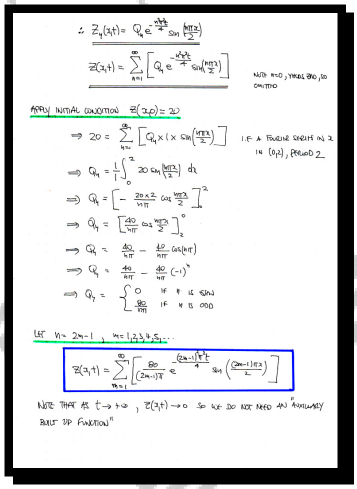
APPLY BOUNDARY CONDITION, $z(2,t) = 0$

$\implies 0 = Qe^{-p^2 t}\sin 2p$

$\implies \sin 2p = 0$ ($Q \neq 0$, OTHERWISE SOLUTION IS TRIVIAL)

$\implies 2p = n\pi$ $n = 0,1,2,3,\dots$

$\implies p = \frac{n\pi}{2}$



$\therefore z_n(x,t) = Q_n e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right)$

$z(x,t) = \sum_{n=1}^{\infty} \left[Q_n e^{-\frac{n^2\pi^2 t}{4}} \sin\left(\frac{n\pi x}{2}\right) \right]$ NOTE $n=0$, YIELDS ZERO, SO OMITTED

APPLY INITIAL CONDITION $z(x,0) = 20$

$\implies 20 = \sum_{n=1}^{\infty} \left[Q_n \times 1 \times \sin\left(\frac{n\pi x}{2}\right) \right]$ I.E A FOURIER SERIES IN x IN $(0,2)$, PERIOD 2

$\implies Q_n = \frac{1}{1}\int_0^2 20\sin\left(\frac{n\pi x}{2}\right)\, dx$

$\implies Q_n = \left[-\frac{20 \times 2}{n\pi} \cos\frac{n\pi x}{2} \right]_0^2$

$\implies Q_n = \left[\frac{40}{n\pi} \cos\frac{n\pi x}{2} \right]_2^0$

$\implies Q_n = \frac{40}{n\pi} - \frac{40}{n\pi}\cos(n\pi)$

$\implies Q_n = \frac{40}{n\pi} - \frac{40}{n\pi}(-1)^n$

$\implies Q_n = \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ \frac{80}{n\pi} & \text{IF } n \text{ IS ODD} \end{cases}$

LET $n = 2m-1$, $m = 1,2,3,4,5,\dots$

$$z(x,t) = \sum_{m=1}^{\infty} \left[\frac{80}{(2m-1)\pi} e^{-\frac{(2m-1)^2\pi^2 t}{4}} \sin\left(\frac{(2m-1)\pi x}{2}\right) \right]$$

NOTE THAT AS $t \to +\infty$, $z(x,t) \to 0$ SO WE DO NOT NEED AN "AUXILIARY BUILT UP FUNCTION"

Question 2

At time $t<0$, a long thin rod of length l has temperature distribution $\theta(x)$ given by

$$\theta(x) = xl - x^2 .$$

At time $t=0$ the temperature is suddenly dropped to 0 °C at both ends of the rod, and maintained at 0 °C for $t \geq 0$.

The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq l, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$ and hence show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32} .$$

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{\theta(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{8l^2}{(2n-1)^3 \pi^3} \exp\left[-\frac{\alpha^2 \pi^2 (2n-1)^2 t}{l^2} \right] \sin\left[\frac{(2n-1)\pi x}{l} \right] \right\}}$$

[solution overleaf]

a) $\dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial \theta}{\partial t}$

- LOOK FOR A SOLUTION IN VARIABLE SEPARATE FORM

$\theta(x,t) = X(x)T(t)$

$\dfrac{\partial^2 \theta}{\partial x^2}(x,t) = X''(x)T(t)$

$\dfrac{\partial \theta}{\partial t}(x,t) = X(x)T'(t)$

- SUB INTO THE P.D.E YIELDS

$\Rightarrow X''(x)T(t) = \dfrac{1}{\alpha^2}X(x)T'(t)$

$\Rightarrow \dfrac{X''(x)T(t)}{X(x)T(t)} = \dfrac{1}{\alpha^2}\dfrac{X(x)T'(t)}{X(x)T(t)}$

$\Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)}$

- AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT, SAY λ, WHICH CAN BE POSITIVE, NEGATIVE OR ZERO.

HENCE $\dfrac{X''(x)}{X(x)} = \lambda$ $\qquad$ $\dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)} = \lambda$

$X''(x) = \lambda X(x)$ $\qquad$ $T'(t) = \lambda\alpha^2 T(t)$

- IF $\lambda=0$: $X''(x)=0 \Rightarrow X(x)=Ax+B$; $T'(t)=0 \Rightarrow T(t)=C$ $\Big\}$ $\theta(x,t) = Ax+B$ (I) (STEADY STATE)

IF $\lambda<0$, $\lambda=-p^2$: $X''(x) = -p^2X(x) \Rightarrow X(x) = A\cos px + B\sin px$; $T'(t) = -p^2\alpha^2T(t) \Rightarrow T(t) = Ce^{-\alpha^2p^2t}$

$\theta(x,t) = e^{-\alpha^2p^2t}(A\cos px + B\sin px)$ (II)

IF $\lambda>0$, $\lambda=p^2$: $X''(x) = p^2X(x) \Rightarrow X(x) = A\cosh px + B\sinh px$; $T'(t) = p^2\alpha^2T(t) \Rightarrow T(t) = Ce^{\alpha^2p^2t}$

$\theta(x,t) = e^{\alpha^2p^2t}(A\cosh px + B\sinh px)$ (III)

- NOW SOLUTION (I) IS NOT SUITABLE AS IT HAS NO TIME-DEPENDENCE, I.E IT IS A STEADY STATE TRIVIAL SOLUTION

SOLUTION (III) HAS TO BE DISCARDED, AS IT PRODUCES UNBOUNDED SOLUTIONS AS $t \to \infty$

$\therefore \theta(x,t) = e^{-\alpha^2p^2t}(A\cos px + B\sin px)$

- BOUNDARY AND INITIAL CONDITIONS

$\theta(0,t)=0$ — ①

$\theta(l,t)=0$ — ②

$\theta(x,0)=xl-x^2$ — ③

- BY ① $\Rightarrow 0 = e^{-\alpha^2p^2t}(A) \Rightarrow A=0$

$\therefore \theta(x,t) = Be^{-\alpha^2p^2t}\sin px$

- BY ② $\Rightarrow 0 = Be^{-\alpha^2p^2t}\sin pl \Rightarrow \sin pl = 0 \Rightarrow pl = n\pi,\ n=1,2,3,\dots$

$B \neq 0,\ e^{-\alpha^2p^2t} \neq 0$

$\therefore \theta_n(x,t) = B_n e^{-\frac{\alpha^2n^2\pi^2}{l^2}t}\sin\left(\dfrac{n\pi x}{l}\right)$

OR

$\theta(x,t) = \displaystyle\sum_{n=1}^{\infty}\left[B_n e^{-\left(\frac{\alpha n\pi}{l}\right)^2 t}\sin\left(\frac{n\pi x}{l}\right)\right]$

- BY ③ $\Rightarrow xl - x^2 = \displaystyle\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{l}\right)$

THIS IS A FOURIER EXPANSION IN x, $0 \le x \le l$.

$\therefore B_n = \dfrac{1}{l/2}\displaystyle\int_0^l (xl-x^2)\sin\frac{n\pi x}{l}\,dx$

BY PARTS

$xl-x^2$	$l-2x$
$-\frac{l}{n\pi}\cos\frac{n\pi x}{l}$	$\sin\frac{n\pi x}{l}$

$\Rightarrow B_n = \dfrac{2}{l}\left\{\left[-[xl-x^2]\times\dfrac{l}{n\pi}\cos\dfrac{n\pi x}{l}\right]_0^l + \dfrac{l}{n\pi}\displaystyle\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx\right\}$

$\Rightarrow B_n = \dfrac{2}{l}\times\dfrac{l}{n\pi}\displaystyle\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx$

BY PARTS AGAIN

$l-2x$	-2
$\frac{l}{n\pi}\sin\frac{n\pi x}{l}$	$\cos\frac{n\pi x}{l}$

$\Rightarrow B_n = \dfrac{2}{n\pi}\left\{\left[(l-2x)\times\dfrac{l}{n\pi}\sin\dfrac{n\pi x}{l}\right]_0^l + \dfrac{2l}{n\pi}\displaystyle\int_0^l \sin\frac{n\pi x}{l}\,dx\right\}$

$\Rightarrow B_n = \dfrac{2}{n\pi}\times\dfrac{2l}{n\pi}\displaystyle\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$

$\Rightarrow B_n = \dfrac{4l}{n^2\pi^2}\left[-\dfrac{l}{n\pi}\cos\dfrac{n\pi x}{l}\right]_0^l$

$\Rightarrow B_n = \dfrac{4l^2}{n^3\pi^3}\left[\cos\dfrac{n\pi x}{l}\right]_l^0$

$\Rightarrow B_n = \dfrac{4l^2}{n^3\pi^3}\left[1-\cos n\pi\right]$

$\Rightarrow B_n = \dfrac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right] = \begin{cases}\dfrac{8l^2}{n^3\pi^3} & \text{IF } n \text{ IS ODD}\\ 0 & \text{IF } n \text{ IS EVEN}\end{cases}$

$\therefore B_{2m-1} = \dfrac{8l^2}{(2m-1)^3\pi^3}$

$\therefore \theta(x,t) = \displaystyle\sum_{m=1}^{\infty}\left[\frac{8l^2}{(2m-1)^3\pi^3}e^{-\left[\frac{(2m-1)\pi\alpha}{l}\right]^2t}\sin\left[\frac{(2m-1)\pi x}{l}\right]\right]$

b) CONSIDER $\theta\left(\frac{l}{2},0\right)$ IE THE INITIAL TEMPERATURE AT THE MIDPOINT

$\Rightarrow \dfrac{l}{2}\left(l-\dfrac{l}{2}\right) = \displaystyle\sum_{m=1}^{\infty}\left[\frac{8l^2}{(2m-1)^3\pi^3}\times 1\times\sin\left[\frac{(2m-1)\pi}{2}\right]\right]$, where $\sin\left[\frac{(2m-1)\pi}{2}\right] = (-1)^{m-1}$

$\Rightarrow \dfrac{l^2}{4} = \displaystyle\sum_{m=1}^{\infty}\left[\frac{8l^2}{(2m-1)^3\pi^3}\times(-1)^{m-1}\right]$

$\Rightarrow \dfrac{l^2}{4} = \dfrac{8l^2}{\pi^3}\displaystyle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^3}$

$\Rightarrow \dfrac{\pi^3}{32} = \displaystyle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^3}$

OR STARTING FROM $m=0$

$\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32}$

Question 3

The temperature distribution $\theta(x,t)$ along a thin bar of length 2 m satisfies the partial differential equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{9}\frac{\partial\theta}{\partial t},\quad 0\le x\le 2,\quad t\ge 0.$$

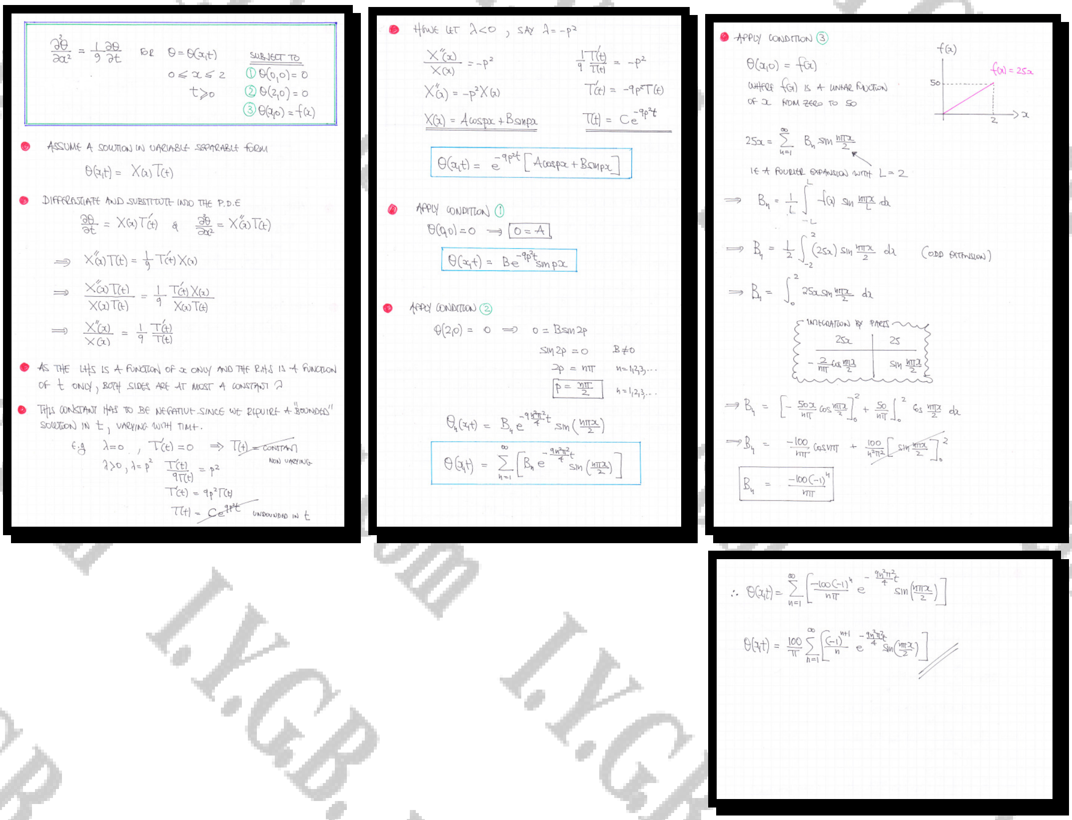
Initially the bar has a linear temperature distribution, with temperature 0 °C at one end of the bar where $x=0$ m, and temperature 50 °C at the other end where $x=2$ m.

At time $t=0$ the temperature is suddenly dropped to 0 °C at both ends of the rod, and maintained at 0 °C for $t\ge 0$.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$ and hence show that

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{\theta(x,t)=\frac{100}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n}\exp\left[-\frac{9n^2\pi^2 t}{4}\right]\sin\left[\frac{n\pi x}{2}\right]\right\}}$$



Question 4

The temperature $\Theta(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 \Theta}{\partial x^2} = 4\frac{\partial \Theta}{\partial t},$$

where x is a spatial coordinate and t is time, with $t \geq 0$.

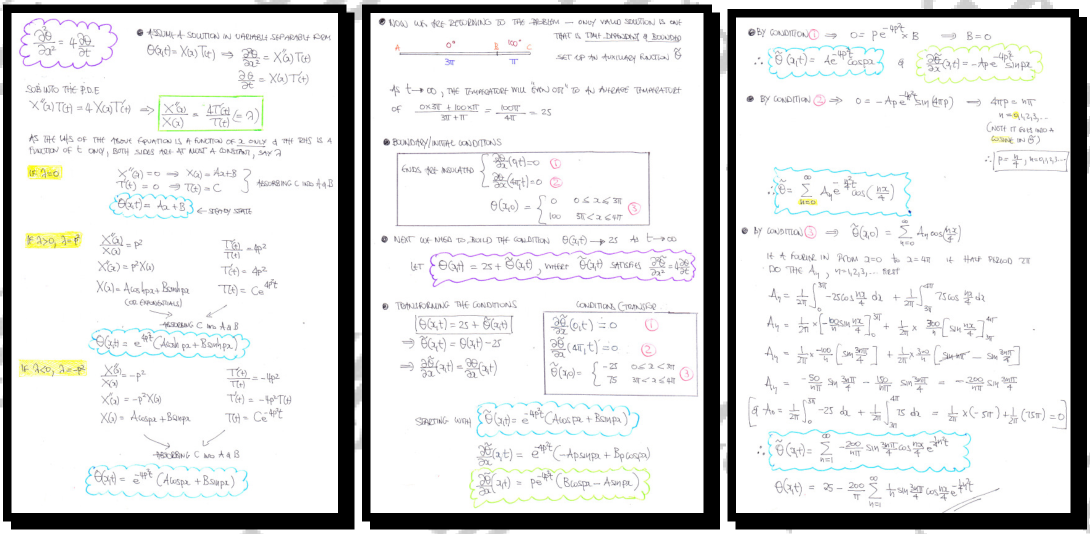
For $t < 0$, two thin rods, of lengths 3π and π, have temperatures 0 °C and 100 °C, respectively. At time $t = 0$ the two rods are joined end to end into a single rod of length 4π.

The rods are made of the same material, have perfect thermal contact and are insulated along their length.

Determine an expression for $\Theta(x,t)$, $t \geq 0$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\Theta(x,t) = 25 - \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n}e^{-\frac{1}{4}n^2 t}\sin\left(\frac{3}{4}n\pi\right)\cos\left(\frac{1}{4}nx\right)\right]$$



Question 5

Solve the heat equation for $u = u(x,t)$

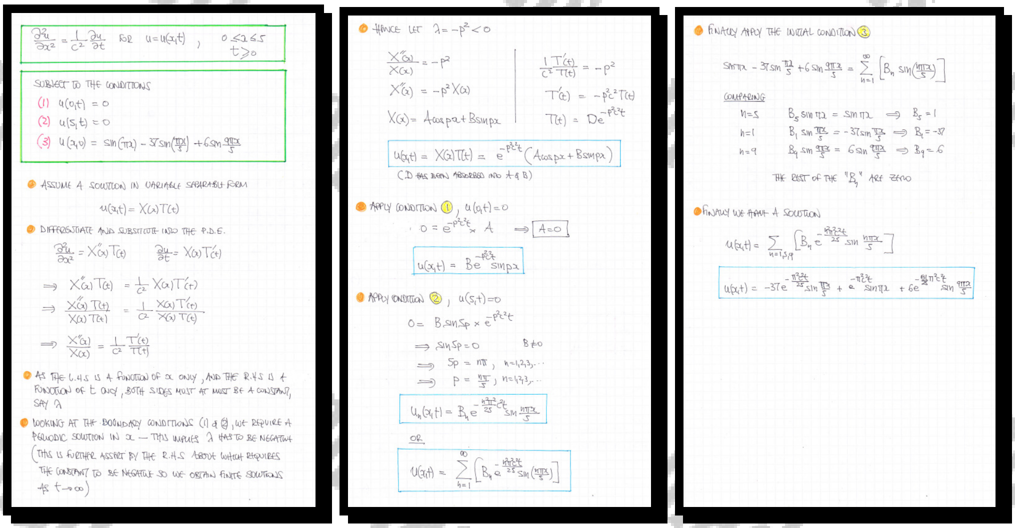
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}, \quad 0 \le x \le 5, \quad t \ge 0,$$

subject to the conditions

$$u(0,t) = 0, \qquad u(5,t) = 0 \qquad \text{and} \qquad u(x,0) = \sin \pi x - 37\sin\left(\tfrac{1}{5}\pi x\right) + 6\sin\left(\tfrac{9}{5}\pi x\right).$$

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{u(x,t) = -37\,e^{-\frac{1}{25}\pi^2 c^2 t}\sin\left(\tfrac{1}{5}\pi x\right) + e^{-\pi^2 c^2 t}\sin(\pi x) + 6\,e^{-\frac{81}{25}\pi^2 c^2 t}\sin\left(\tfrac{9}{5}\pi x\right)}$$



Question 6

A long thin rod of length L has temperature $\theta = 0$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_1 at both ends of the rod, at $x = 0$ and at $x = L$.

Both ends of the rod are maintained at temperature T_1 for $t \geq 0$.

The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\theta(x,t) = T_1\left[1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)}\exp\left[-\frac{\alpha^2\pi^2(2n-1)^2 t}{L^2}\right]\sin\left[\frac{(2n-1)\pi x}{L}\right]\right\}\right]$$

SOLVING THE HEAT EQUATION BY SEPARATION OF VARIABLES AND IGNORING ANY CONDITIONS AT THIS STAGE

LET $\theta(x,t) = X(x)T(t)$

$\frac{\partial^2\theta}{\partial x^2}(x,t) = X''(x)T(t)$

$\frac{\partial\theta}{\partial t}(x,t) = X(x)T'(t)$

SUBSTITUTE INTO THE P.D.E

$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} \Longrightarrow X''(x)T(t) = \frac{1}{\alpha^2}X(x)T'(t)$

$\Longrightarrow \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \lambda$

BOTH SIDES OF THE ABOVE EQUATION MUST AT MOST BE A CONSTANT λ AS THE L.H.S IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY. THIS CONSTANT MAY BE ZERO, POSITIVE OR NEGATIVE

- IF $\lambda = 0$: $X''(x) = 0 \Rightarrow X(x) = Ax + B$; $T'(t) = 0 \Rightarrow T(t) = C$ $\Rightarrow \theta(x,t) = (Ax+B)\times C$, $\theta(x,t) = Ax + B$

I.E STEADY FLOW WITHOUT TIME DEPENDENCY

- IF $\lambda > 0$, SAY p^2

$\Rightarrow \frac{X''(x)}{X(x)} = p^2$ & $\Rightarrow \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = p^2$

$\Rightarrow X''(x) = p^2 X(x)$ $\quad \Rightarrow T'(t) = \alpha^2 p^2 T(t)$

$\Rightarrow X(x) = Ae^{px} + Be^{-px}$ $\quad \Rightarrow T(t) = Ce^{\alpha^2 p^2 t}$

$\therefore \theta(x,t) = Ce^{\alpha^2p^2t}(Ae^{px} + Be^{-px})$

THIS SOLUTION IS NOT APPLICABLE AS IT PRODUCES UNBOUNDED TEMPERATURE $\theta(x,t)$ AS $t \to \infty$

- IF $\lambda < 0$, SAY $\lambda = -p^2$

$\Rightarrow \frac{X''(x)}{X(x)} = -p^2$ $\quad \Rightarrow \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -p^2$

$\Rightarrow X''(x) = -p^2X(x)$ $\quad \Rightarrow T'(t) = -\alpha^2p^2T(t)$

$\Rightarrow X(x) = A\cos px + B\sin px$ $\quad \Rightarrow T(t) = Ce^{-\alpha^2p^2t}$

$\therefore \theta(x,t) = Ce^{-\alpha^2p^2t}(A\cos px + B\sin px)$

$\theta(x,t) = e^{-\alpha^2p^2t}(A\cos px + B\sin px)$

THIS SOLUTION IS "ACCEPTABLE" TO THIS TYPE OF PROBLEM AS $\theta(x,t)$ IS BOUNDED AS $t \to \infty$

NOW THE INITIAL & BOUNDARY CONDITIONS NEED TO BE BUILT IN

LET $\theta(x,t) = T_1 + \tilde{\theta}(x,t)$ WHERE $\tilde{\theta}(x,t)$ SATISFIES $\frac{\partial^2\tilde{\theta}}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\tilde{\theta}}{\partial t}$

SO THAT $\tilde{\theta}(x,t) \to 0$ AS $t \to \infty$ AND HENCE $\theta(x,t) \to T_1$ AS $t \to \infty$

- AT THE END $x = 0$, $\theta(0,t) = T_1 \Rightarrow \tilde{\theta}(0,t) = 0$, $t > 0$ — I
- AT THE END $x = L$, $\theta(L,t) = T_1 \Rightarrow \tilde{\theta}(L,t) = 0$, $t > 0$ — II
- INITIAL TEMPERATURE IS ZERO, $\theta(x,0) = 0 \Rightarrow \tilde{\theta}(x,0) = -T_1$, $0 < x < L$ — III

APPLYING EACH OF THESE CONDITIONS IN TURN

$\tilde{\theta}(x,t) = e^{-p^2\alpha^2t}(A\cos px + B\sin px)$

(I) $0 = e^{-p^2\alpha^2t}(A)$ FOR ALL $t > 0$ $\quad \therefore A = 0$

$\tilde{\theta}(x,t) = Be^{-p^2\alpha^2t}\sin px$

(II) $0 = Be^{-p^2\alpha^2t}\sin pL$ FOR ALL $t > 0$ $\quad \therefore pL = n\pi$, $n \in \mathbb{Z}$ $\quad \therefore p = \frac{n\pi}{L}$

$\tilde{\theta}(x,t) = \sum_{n=1}^{\infty}\left[B_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\sin\frac{n\pi x}{L}\right]$ NEGATIVE n ABSORBED INTO B_n

(III) $-T_1 = \sum_{n=1}^{\infty}\left[B_n\sin\frac{n\pi x}{L}\right]$ WHICH IS A SIMPLE FOURIER IN $(0,L)$ WITH $f(x) = -T_1$

$\therefore B_n = \frac{1}{L/2}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L -T_1\sin\frac{n\pi x}{L}\,dx$

$= \frac{2}{L}\left[\frac{LT_1}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L = \frac{2T_1}{n\pi}\left[\cos n\pi - 1\right] = \frac{2T_1}{n\pi}\left[(-1)^n - 1\right]$

$= \begin{cases} -2 & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$

$\therefore B_{2m-1} = \frac{-4T_1}{(2m-1)\pi}$ $\quad m = 1, 2, 3, \ldots$

$\tilde{\theta}(x,t) = \sum_{m=1}^{\infty}\left[\frac{-4T_1}{\pi(2m-1)}e^{-\frac{\alpha^2\pi^2(2m-1)^2t}{L^2}}\sin\left[\frac{(2m-1)\pi x}{L}\right]\right]$

$\therefore \theta(x,t) = T_1 - \frac{4T_1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{2n-1}e^{-\frac{\alpha^2\pi^2(2n-1)^2t}{L^2}}\sin\left[\frac{(2n-1)\pi x}{L}\right]\right]$

Question 7

A long thin rod of length L has temperature $\theta = T_1$ throughout its length.

At time $t = 0$ the temperature is suddenly raised to T_2 at one of its ends at $x = 0$, and is maintained at T_2 for $t \geq 0$.

The temperature distribution along the rod $\theta(x,t)$, satisfies the standard heat equation

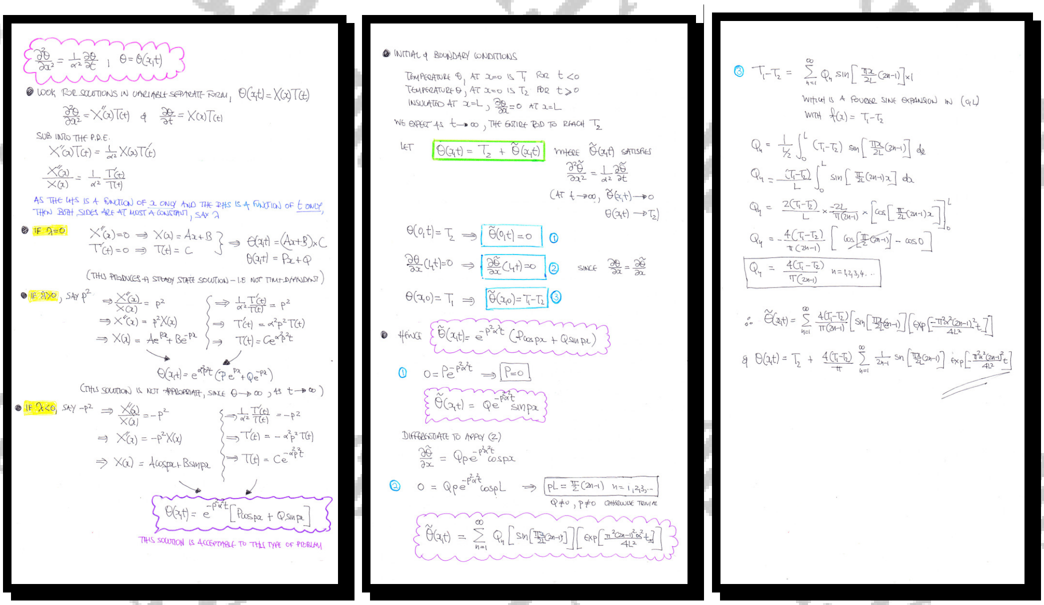
$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

where α is a positive constant.

Assuming the rod is insulated along its length, determine an expression for $\theta(x,t)$.

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\theta(x,t) = T_2 + \frac{4(T_1 - T2)}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{(2n-1)}\exp\left[-\frac{\alpha^2\pi^2(2n-1)^2 t}{4L^2}\right]\sin\left[\frac{(2n-1)\pi x}{2L}\right]\right\}$$



Question 8

The temperature $u(x,t)$ satisfies the one dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{9}\frac{\partial u}{\partial t}, \quad t \ge 0, \ 0 \le x \le 2$$

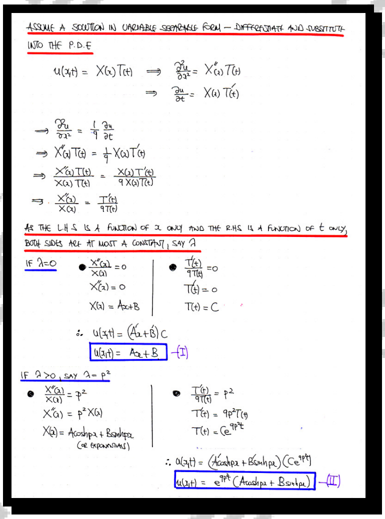
where x is a spatial coordinate and t is time.

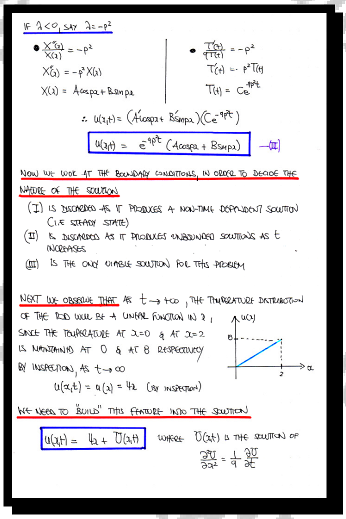
It is further given that

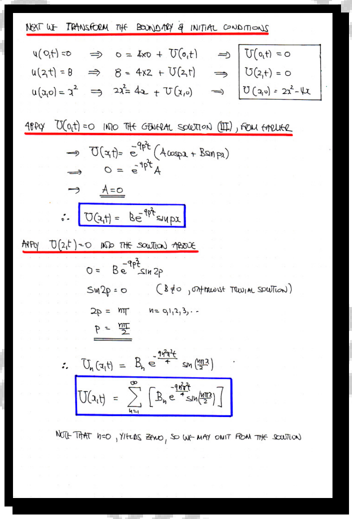
$$u(0,t) = 0, \qquad u(2,t) = 8, \qquad u(x,0) = 2x^2$$

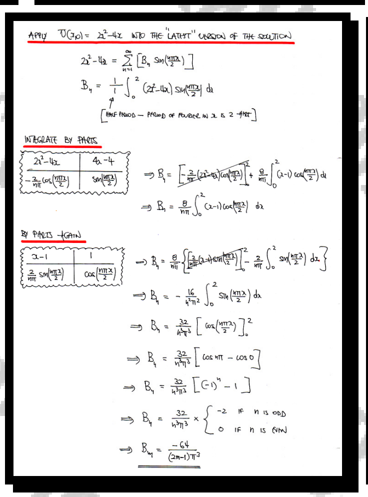
Determine an expression for $u(x,t)$.

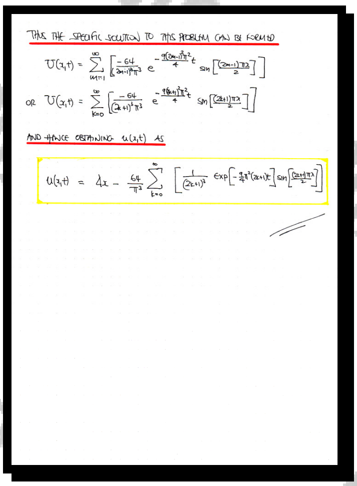
$$u(x,t) = 4x - \sum_{k=0}^{\infty}\left[\left[\frac{64}{(2k+1)^3\pi^3}\right]\exp\left[-\frac{9(2k+1)^2 t}{4}\right]\sin\left[\frac{(2k+1)\pi x}{2}\right]\right]$$











Question 9

The temperature $u(x,t)$ in a thin rod of length π satisfies the heat equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}, \quad 0 \le x \le \pi, \quad t \ge 0,$$

where c is a positive constant.

The initial temperature distribution of the rod is

$$u(x,0) = \tfrac{1}{2}\cos(4x), \ 0 \le x \le \pi.$$

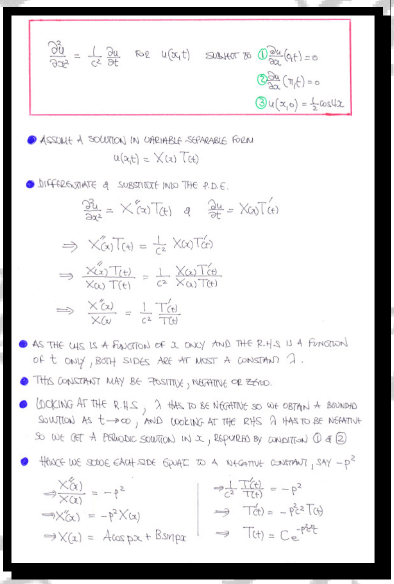
For $t \ge 0$, heat is allowed to flow freely along the rod, with the rod including its endpoints insulated.

Show that

$$u(x,t) = \frac{1}{2}\mathrm{e}^{-16c^2 t}\cos 4x.$$

[*You must derive the standard solution of the heat equation in variable separate form*]

proof



$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ FOR $u(x,t)$ SUBJECT TO ① $\frac{\partial u}{\partial x}(0,t)=0$

② $\frac{\partial u}{\partial x}(\pi,t)=0$

③ $u(x,0)=\frac{1}{2}\cos 4x$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$u(x,t) = X(x)T(t)$

- DIFFERENTIATE & SUBSTITUTE INTO THE P.D.E.

$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$ & $\frac{\partial u}{\partial t} = X(x)T'(t)$

$\Rightarrow X''(x)T(t) = \frac{1}{c^2}X(x)T'(t)$

$\Rightarrow \frac{X''(x)T(t)}{X(x)T(t)} = \frac{1}{c^2}\frac{X(x)T'(t)}{X(x)T(t)}$

$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T'(t)}{T(t)}$

- AS THE LHS IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, BOTH SIDES ARE AT MOST A CONSTANT λ.
- THIS CONSTANT MAY BE POSITIVE, NEGATIVE OR ZERO.
- LOOKING AT THE R.H.S, λ HAS TO BE NEGATIVE SO WE OBTAIN A BOUNDED SOLUTION AS $t\to\infty$, AND LOOKING AT THE RHS λ HAS TO BE NEGATIVE SO WE GET A PERIODIC SOLUTION IN x, REQUIRED BY CONDITION ① & ②
- HENCE WE SOLVE EACH SIDE EQUAL TO A NEGATIVE CONSTANT, SAY $-p^2$

$\Rightarrow \frac{X''(x)}{X(x)} = -p^2$

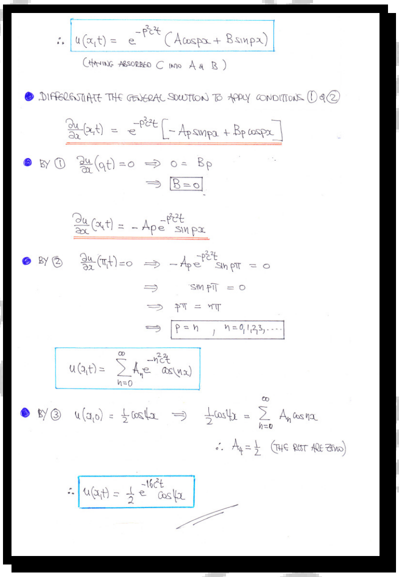
$\Rightarrow X''(x) = -p^2 X(x)$

$\Rightarrow X(x) = A\cos px + B\sin px$

$\Rightarrow \frac{1}{c^2}\frac{T'(t)}{T(t)} = -p^2$

$\Rightarrow T'(t) = -p^2c^2 T(t)$

$\Rightarrow T(t) = Ce^{-p^2c^2t}$



$\therefore u(x,t) = e^{-p^2c^2t}(A\cos px + B\sin px)$

(HAVING ABSORBED C INTO A & B)

- DIFFERENTIATE THE GENERAL SOLUTION TO APPLY CONDITIONS ① & ②

$\frac{\partial u}{\partial x}(x,t) = e^{-p^2c^2t}\left[-Ap\sin px + Bp\cos px\right]$

- BY ① $\frac{\partial u}{\partial x}(0,t)=0 \Rightarrow 0 = Bp$

$\Rightarrow B=0$

$\frac{\partial u}{\partial x}(x,t) = -Ape^{-p^2c^2t}\sin px$

- BY ② $\frac{\partial u}{\partial x}(\pi,t)=0 \Rightarrow -Ape^{-p^2c^2t}\sin p\pi = 0$

$\Rightarrow \sin p\pi = 0$

$\Rightarrow p\pi = n\pi$

$\Rightarrow p = n, \quad n = 0,1,2,3,\dots$

$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-n^2c^2t}\cos(nx)$

- BY ③ $u(x,0) = \frac{1}{2}\cos 4x \Rightarrow \frac{1}{2}\cos 4x = \sum_{n=0}^{\infty} A_n\cos nx$

$\therefore A_4 = \frac{1}{2}$ (THE REST ARE ZERO)

$\therefore u(x,t) = \frac{1}{2}e^{-16c^2t}\cos 4x$

Question 10

The temperature $\theta(x,t)$ in a thin rod of length L satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}, \quad 0 \le x \le L, \quad t \ge 0.$$

The initial temperature distribution is

$$\theta(x,0) = \begin{cases} \dfrac{\theta_0 x}{L} & 0 \le x < \dfrac{L}{2} \\ 0 & \dfrac{L}{2} < x \le L \end{cases}$$

where θ_0 is a constant.

The endpoints of the rod are maintained at zero temperature for $t \ge 0$.

a) Assuming the rod is insulated along its length, find an expression for $\theta(x,t)$.

b) By considering the initial temperature at $x = \dfrac{L}{2}$, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

[*You must derive the standard solution of the heat equation in variable separate form*]

$$\boxed{\theta(x,t) = \frac{\theta_0}{\pi^2} \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \left[2\sin\left(\frac{n\pi}{2}\right) - n\pi\cos\left(\frac{n\pi}{2}\right) \right] \exp\left[-\frac{n^2\pi^2 t}{L^2} \right] \sin\left[\frac{n\pi x}{L} \right] \right\}}$$

[solution overleaf]

a) $$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}, \quad \theta = \theta(x,t)$$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\theta(x,t) = X(x)T(t) \implies \frac{\partial \theta}{\partial t} = X(x)T'(t), \quad \frac{\partial^2 \theta}{\partial x^2} = X''(x)T(t)$$

- SUB INTO THE PDE.

$$X''(x)T(t) = X(x)T'(t)$$

- DIVIDE BY $X(x)T(t)$

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

AS THE L.H.S. OF THIS EXPRESSION IS A FUNCTION OF x ONLY AND THE R.H.S IS A FUNCTION OF t ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT λ, WHICH MAY BE POSITIVE, NEGATIVE, OR ZERO

- CONSIDER EACH CASE SEPARATELY

$\lambda = 0$

$$X''(x) = 0 \implies X(x) = Ax + B$$
$$T'(t) = 0 \implies T(t) = C$$

ABSORBING CONSTANTS $\theta(x,t) = Ax + B$ (STEADY STATE) — (I)

$\lambda > 0$, SAY $\lambda = p^2$

$$\frac{X''(x)}{X(x)} = p^2 \qquad \frac{T'(t)}{T(t)} = p^2$$
$$X''(x) = p^2 X(x) \qquad T'(t) = p^2 T(t)$$
$$X(x) = A\cosh px + B\sinh px \qquad T(t) = Ce^{p^2 t}$$

(OR EXPONENTIALS)

ABSORBING CONSTANTS $\theta(x,t) = e^{p^2 t}(A\cosh px + B\sinh px)$ — (II)

$\lambda < 0$ SAY $\lambda = -p^2$

$$\frac{X''(x)}{X(x)} = -p^2 \qquad \frac{T'(t)}{T(t)} = -p^2$$
$$X''(x) = -p^2 X(x) \qquad T'(t) = -p^2 T(t)$$
$$X(x) = A\cos px + B\sin px \qquad T(t) = Ce^{-p^2 t}$$

ABSORBING CONSTANTS $\theta(x,t) = e^{-p^2 t}(A\cos px + B\sin px)$ — (III)

EXAMINING THE 3 POSSIBILITIES AND THE NATURE OF THE PROBLEM, SUGGEST THAT IS ONLY (III) THAT COULD MODEL THE PROBLEM SINCE

- AS $t \to \infty \quad \theta \to 0$ (IN FACT THIS IS PERFECT SO WE DO NOT HAVE TO BUILT THE "USUAL" $\tilde{\theta}(x,t)$)
- WE REQUIRE A SOLUTION IN x, WHICH PRODUCES 0 TWICE IE AT THE ENDPOINTS OF THE ROD

BOUNDARY CONDITIONS

$\theta(0,t) = 0$ ①

$\theta(L,t) = 0$ ②

INITIAL CONDITION

③ $$\theta(x,0) = \begin{cases} \frac{\theta_0 x}{L} & 0 \le x < \frac{L}{2} \\ 0 & \frac{L}{2} < x \le L \end{cases}$$

APPLY ①

$$0 = e^{-p^2 t} \times A \implies A = 0$$

$$\therefore \theta(x,t) = Be^{-p^2 t}\sin px$$

APPLY ②

$$0 = Be^{-p^2 t}\sin pL \qquad pL = n\pi, \ n = 1,2,3\ldots, \ B \neq 0$$
$$p = \frac{n\pi}{L}$$

$$\therefore \theta(x,t) = \sum_{n=1}^{\infty}\left[B_n e^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L}\right]$$

APPLY ③

$$\theta(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

WHICH IS A FOURIER WITH PERIOD L

$$B_n = \frac{1}{L/2}\int_0^L \theta(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{L}\int_0^{\frac{L}{2}} \frac{\theta_0 x}{L}\sin\left(\frac{n\pi x}{L}\right)dx \quad \left(\text{NOTE } \theta(x,0) = 0 \text{ FOR } \tfrac{1}{2}L < x \le L\right)$$

$$B_n = \frac{2\theta_0}{L^2}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)dx \quad \ldots \text{ BY PARTS}$$

x	1
$-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$	$\sin\left(\frac{n\pi x}{L}\right)$

$$B_n = \frac{2\theta_0}{L^2}\left\{\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^{\frac{L}{2}} + \frac{L}{n\pi}\int_0^{\frac{L}{2}}\cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$B_n = \frac{2\theta_0}{L^2}\left\{\left[\frac{Lx}{n\pi}\cos\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{0} + \left[\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^{\frac{L}{2}}\right\}$$

$$B_n = \frac{2\theta_0}{L^2}\left[0 - \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0\right\}$$

$$B_n = \theta_0\left[\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}\cos\frac{n\pi}{2}\right]$$

$$B_n = \frac{\theta_0}{n^2\pi^2}\left[2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right]$$

$$\therefore \theta(x,t) = \sum_{n=1}^{\infty}\left(\frac{\theta_0}{n^2\pi^2}\left[2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right]e^{-\frac{n^2\pi^2}{L^2}t}\sin\frac{n\pi x}{L}\right]$$

b) NOW CONSIDER $\theta\left(\frac{L}{2},0\right)$ IN ABOVE EXPRESSION

$$\sum_{n=1}^{\infty}\frac{\theta_0}{n^2\pi^2}\left[2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right]\sin\frac{n\pi}{2}$$

$$= \sum_{n=1}^{\infty}\frac{\theta_0}{n^2\pi^2}\times\left[2\sin^2\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

$$\sin^2\frac{n\pi}{2} = \begin{cases} 1 & \text{IF } n = \text{ODD} \\ 0 & \text{IF } n = \text{EVEN} \end{cases}$$

$$= \sum_{n=1}^{\infty}\frac{\theta_0}{n^2\pi^2}\left[2(1-(-1)^n)\times\frac{1}{2} - n\pi\times\frac{1}{2}\sin n\pi\right]$$

$$= \sum_{n=1}^{\infty}\frac{\theta_0}{n^2\pi^2}\left(1-(-1)^n\right)$$

n	1	2	3	4	5	6
$1-(-1)^n$	2	0	2	0	2	0

$$= \sum_{m=1}^{\infty}\frac{2\theta_0}{(2m-1)^2\pi^2} \quad \leftarrow 1,3,5,7\ldots \ 2m-1$$

NOW AT THE MIDPOINT $\theta\left(\frac{L}{2},0\right) = \begin{cases} \frac{\theta_0}{2} & \text{TO THE LEFT} \\ 0 & \text{TO THE RIGHT} \end{cases}$

AVERAGE IS $\frac{\theta_0}{4}$

$$\therefore \sum_{m=1}^{\infty}\frac{2\theta_0}{\pi^2(2m-1)^2} = \frac{\theta_0}{4}$$

$$\frac{2\theta_0}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \frac{\theta_0}{4}$$

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \frac{\pi^2}{8} \quad \text{OR} \quad \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$$

Question 11

The temperature $\theta(x,t)$ in a long thin rod of length L satisfies the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t}, \quad 0 \le x \le L, \quad t \ge 0,$$

where α is a positive constant.

The initial temperature distribution of the rod is

$$\theta(x,0) = \sin\left(\frac{\pi x}{L}\right), \; 0 \le x \le L.$$

For $t \ge 0$, heat is allowed to flow freely with the endpoints of the rod insulated.

Show that

$$\theta(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{\exp\left(-\frac{4\alpha^2 n^2 \pi^2 t}{L^2}\right)\cos\left(\frac{2m\pi x}{L}\right)}{4n^2 - 1}\right].$$

[*You must derive the standard solution of the heat equation in variable separate form*]

proof

[solution overleaf]

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}$$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$$\theta(x,t) = X(x)T(t)$$
$$\frac{\partial^2\theta}{\partial x^2} = X''(x)T(t)$$
$$\frac{\partial\theta}{\partial t} = X(x)T'(t)$$

- SUBSTITUTE INTO THE P.D.E

$$\Rightarrow X''(x)T(t) = \frac{1}{\alpha^2}X(x)T'(t)$$
$$\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = \lambda$$

AS THE LHS IS A FUNCTION OF x ONLY, AND THE RIGHT HAND SIDE IS A FUNCTION OF t ONLY, THEN BOTH SIDES ARE AT MOST A CONSTANT, SAY λ

IF $\lambda = 0$

- $X''(x) = 0$, $X(x) = Ax + B$
- $T'(t) = 0$, $T(t) = C$

$$\therefore \theta(x,t) = (Ax+B)C = Ax + B \quad \text{(STEADY STATE)}$$

IF $\lambda > 0$ SAY $\lambda = p^2$

- $X''(x) = p^2X(x)$, $X(x) = A\cosh px + B\sinh px$ or exponentials
- $T'(t) = \alpha^2p^2T(t)$, $T(t) = Ce^{\alpha^2p^2t}$

$$\therefore \theta(x,t) = e^{p^2\alpha^2t}\left[A\cosh px + B\sinh px\right]$$

ABSORBED C INTO A & B

IF $\lambda < 0$ SAY $\lambda = -p^2$

- $X''(x) = -p^2X(x)$, $X(x) = A\cos px + B\sin px$
- $T'(t) = -\alpha^2p^2T(t)$, $T(t) = Ce^{-\alpha^2p^2t}$

$$\theta(x,t) = e^{-\alpha^2p^2t}\left[A\cos px + B\sin px\right]$$

ABSORBED C INTO A & B

- WE REQUIRE A TIME DEPENDANT SOLUTION, WHICH IS BOUNDED AS $t \to \infty$

HENCE

$$\theta(x,t) = e^{-p^2\alpha^2t}\left[A\cos px + B\sin px\right]$$

- BOUNDARY & INITIAL CONDITIONS

$\theta(x,0) = \sin\frac{\pi x}{L}$ [1]

$\frac{\partial\theta}{\partial x}(0,t) = 0$ [2]

$\frac{\partial\theta}{\partial x}(L,t) = 0$ [3]

$$\frac{\partial\theta}{\partial x} = e^{-p^2\alpha^2t}\left[-Ap\sin px + Bp\cos px\right]$$

BY [2]: $0 = e^{-p^2\alpha^2t}\left[Bp\right] \Rightarrow \boxed{B = 0}$

BY [3]: $0 = e^{-p^2\alpha^2t}\left[-Ap\sin pL\right] \Rightarrow pL = n\pi \quad (n = \text{integer})$, $\boxed{p = \frac{n\pi}{L}}$

$$\therefore \theta(x,t) = \sum_{n=0}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right)$$

NOTE THAT THE SUMMATION MUST START FROM ZERO, SINCE $n=0$ PRODUCES A NON ZERO TERM FROM THE COSINES.

BY [1]

$$\sin\frac{\pi x}{L} = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi x}{L}$$
$$\sin\frac{\pi x}{L} = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}$$
$$\sin\frac{\pi x}{L} = \left(\frac{a_0}{2}\right) + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}$$

IE A FOURIER EXPANSION IN x, $0 \leq x \leq L$

- HENCE

$$a_0 = \frac{1}{L/2}\int_0^L \sin\frac{\pi x}{L}\,dx = \frac{2}{L}\times\frac{L}{\pi}\left[-\cos\frac{\pi x}{L}\right]_0^L$$
$$= \frac{2}{\pi}\left[\cos\frac{\pi x}{L}\right]_L^0 = \frac{2}{\pi}\left[1-(-1)\right] = \frac{4}{\pi}$$

$$\therefore \boxed{A_0 = \frac{2}{\pi}} \quad \left(\text{SINCE } A_0 = \frac{a_0}{2}\right)$$

- $$A_n = \frac{1}{L/2}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$\sin(P+Q) = \sin P\cos Q + \cos P\sin Q$$
$$\sin(P-Q) = \sin P\cos Q - \cos P\sin Q$$
$$\sin(P+Q) + \sin(P-Q) = 2\sin P\cos Q$$

$$= \frac{1}{L}\int_0^L \sin\frac{(n+1)\pi x}{L} + \sin\frac{(1-n)\pi x}{L}\,dx$$
$$= \frac{1}{L}\int_0^L \sin\frac{(n+1)\pi x}{L}\,dx - \frac{1}{L}\int_0^L \sin\frac{(n-1)\pi x}{L}\,dx$$

$n=1$ THIS IS ZERO

- SO FOR $n \geq 2$

$$= \frac{1}{L}\times\frac{L}{(n+1)\pi}\left[-\cos\frac{(n+1)\pi x}{L}\right]_0^L - \frac{1}{L}\times\frac{L}{(n-1)\pi}\left[-\cos\frac{(n-1)\pi x}{L}\right]_0^L$$
$$= \frac{1}{(n+1)\pi}\left[\cos\frac{(n+1)\pi x}{L}\right]_L^0 - \frac{1}{(n-1)\pi}\left[\cos\frac{(n-1)\pi x}{L}\right]_L^0$$
$$= \frac{1}{(n+1)\pi}\left[1-\cos[(n+1)\pi]\right] - \frac{1}{(n-1)\pi}\left[1-\cos[(n-1)\pi]\right]$$

$$\cos[(n+1)\pi] = \cos[n\pi+\pi] = \cos n\pi\cos\pi - \sin n\pi\sin\pi = -\cos n\pi$$
$$\cos[(n-1)\pi] = \cos[n\pi-\pi] \quad \cos n\pi\cos\pi + \sin n\pi\sin\pi = -\cos n\pi$$

$$A_n = \frac{1}{(n+1)\pi}\left[1-(-\cos n\pi)\right] - \frac{1}{(n-1)\pi}\left[1-(-\cos n\pi)\right]$$
$$A_n = \frac{1}{(n+1)\pi}\left[1+\cos n\pi\right] - \frac{1}{(n-1)\pi}\left[1+\cos n\pi\right]$$
$$A_n = \frac{1+\cos n\pi}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$
$$A_n = \frac{1+\cos n\pi}{\pi}\left[\frac{(n-1)-(n+1)}{(n+1)(n-1)}\right]$$
$$A_n = \frac{1+\cos n\pi}{\pi}\times\frac{-2}{n^2-1}$$
$$A_n = \frac{1+(-1)^n}{\pi}\times\frac{-2}{n^2-1}$$
$$A_n = -\frac{2}{\pi(n^2-1)}\left(1+(-1)^n\right)$$

- 0 IF n IS ODD
- $-\frac{4}{\pi(n^2-1)}$ IF n IS EVEN

$$\therefore \text{WRITE IT AS } \boxed{A_m = -\frac{4}{\pi(4m^2-1)}} \quad m \geq 1$$

- $$\therefore \theta(x,t) = \frac{2}{\pi} + \sum_{m=1}^{\infty}\left[\frac{-4}{\pi(4m^2-1)}\cos\left(\frac{2m\pi x}{L}\right)\times e^{-\frac{4\alpha^2m^2\pi^2t}{L^2}}\right]$$

$$\theta(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\left[\frac{e^{-\frac{4\alpha^2m^2\pi^2}{L^2}t}}{4m^2-1}\cos\left(\frac{2m\pi x}{L}\right)\right]$$

Question 12

A long thin rod AB, of length L, has constant temperature $\theta = 0$ throughout its length. Another long thin rod CD, also of length L, has constant temperature $\theta = 100$ throughout its length.

At time $t = 0$ the temperature the ends B and C are brought into full contact, while the ends A and D are maintained at respective temperatures $\theta = 0$ and $\theta = 100$.

Show that, for $t \geq 0$, the temperature $\theta(t)$ of the point where the two rods are joined satisfies

$$50 - \frac{100}{\pi} \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{2n+1} \exp\left[-\frac{\alpha^2 \pi^2 (2n+1)^2 t}{4L^2} \right] \right].$$

You may assume

- $ABCD$ is a straight line
- The rods is insulated along their lengths
- The temperature distribution along the rod $\theta(x,t)$ satisfies the standard heat equation

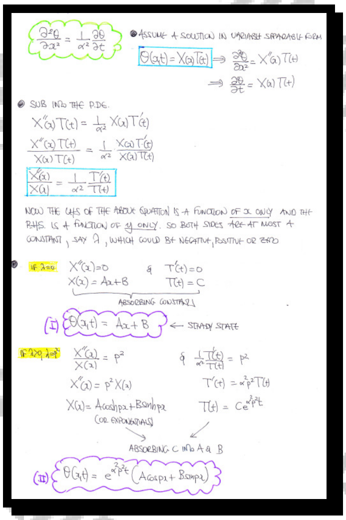
  $$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq L, \quad t \geq 0,$$

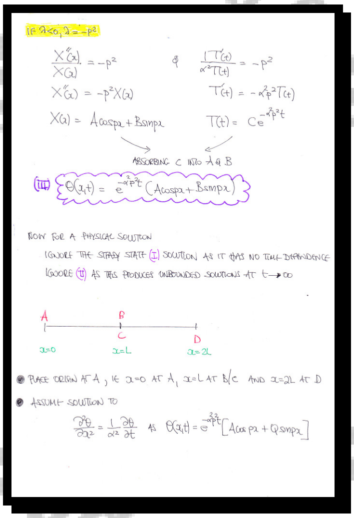
  where α is a positive constant.

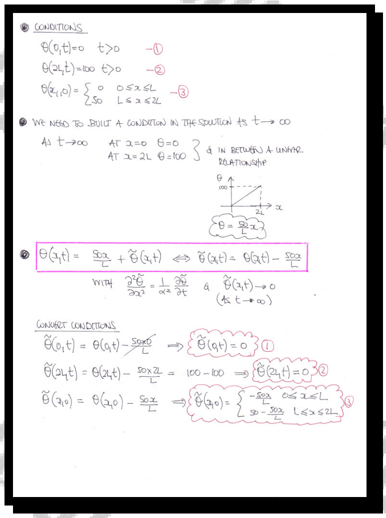
[*You must derive the standard solution of the heat equation in variable separate form*]

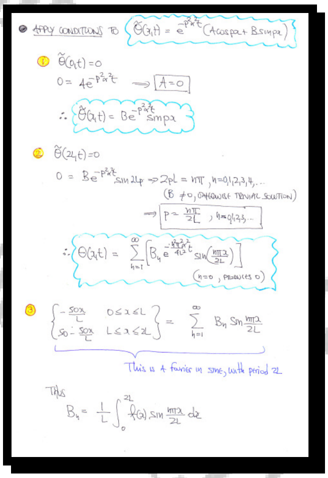
proof

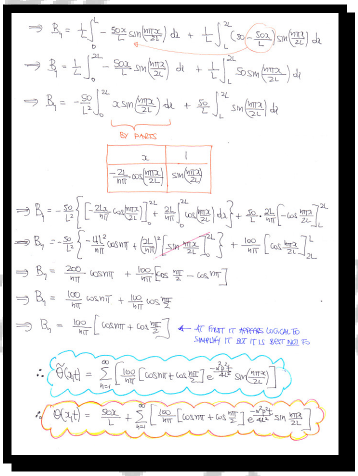
[solution overleaf]

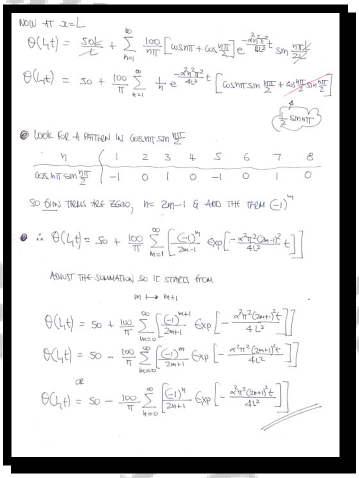












Question 13

The one dimensional heat equation is given by

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}, \quad 0 \le x \le L, \quad t \ge 0,$$

where α is a positive constant, known as the thermal diffusivity.

a) Obtain a general solution of the above equation by trying a solution in variable separable form.

A long thin rod AB, of length $2L$, has its endpoints, at $x=0$ and $x=2L$, maintained at constant temperature $\theta=0$ and its midpoint is maintained at temperature $\theta=100$, until a steady temperature distribution $\Theta(x)$ is reached throughout its length.

b) Show that

$$\Theta(x) = \begin{cases} \dfrac{100}{L}x & 0 \le x < L \\ \dfrac{100}{L}(2L-x) & L < x \le 2L \end{cases}$$

c) Prove that

$$\int_L^{2L} \Theta(x)\sin\left(\frac{n\pi x}{2L}\right)\,dx = (-1)^{n+1}\int_0^L \Theta(x)\sin\left(\frac{n\pi x}{2L}\right)\,dx$$

At $t=0$, the heat source which was maintaining the midpoint of the rod at $\theta=100$ is removed, but its endpoints are still maintained at $\theta=0$. The rod is insulated throughout its length and allowed to cool.

d) Show that for $t \ge 0$, the temperature $\theta(t)$ of the midpoint of the rod satisfies

$$\frac{800}{\pi^2}\sum_{n=0}^{\infty}\left[\frac{1}{(2n+1)^2}\exp\left[-\frac{\alpha^2\pi^2(2n+1)^2 t}{4L^2}\right]\right].$$

proof

[solution overleaf]

a) $\dfrac{\partial^2 \theta}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial \theta}{\partial t}$

- ASSUME A SOLUTION IN VARIABLE SEPARATE FORM

$\theta(x,t) = X(x)T(t)$

$\dfrac{\partial^2 \theta}{\partial x^2}(x,t) = X''(x)T(t)$

$\dfrac{\partial T}{\partial t}(x,t) = X(x)T'(t)$

- SUB INTO THE P.D.E

$\Rightarrow X''(x)T(t) = \dfrac{1}{\alpha^2}X(x)T'(t)$

$\Rightarrow \dfrac{X''(x)T(t)}{X(x)T(t)} = \dfrac{1}{\alpha^2}\dfrac{X(x)T'(t)}{X(x)T(t)}$

$\Rightarrow \dfrac{X''(x)}{X(x)} = \dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)}$

- NOW THE LHS OF THIS EQUATION IS A FUNCTION OF x ONLY AND THE RHS IS A FUNCTION OF t ONLY. HENCE BOTH SIDES ARE AT MOST A CONSTANT λ, WHICH MAY BE NEGATIVE POSITIVE OR ZERO

$\lambda = 0$ $\quad X''(x) = 0 \Rightarrow X(x) = Ax + B$

$T'(t) = 0 \Rightarrow T(t) = C$

$\therefore \boxed{\theta(x,t) = X(x)T(t) = Px + Q}$

IE STEADY STATE (NO TIME DEPENDENCE)

IF $\lambda < 0$, $\lambda = -p^2$ $\quad \dfrac{X''(x)}{X(x)} = -p^2 \Rightarrow X''(x) = -p^2X(x)$

$\Rightarrow X(x) = A\cos px + B\sin px$

$\dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)} = -p^2 \Rightarrow T'(t) = -p^2\alpha^2T(t)$

$\Rightarrow T(t) = Ce^{-p^2\alpha^2 t}$

$\therefore \boxed{\theta(x,t) = X(x)T(t) = e^{-\alpha^2p^2t}(P\cos px + Q\sin px)}$

IF $\lambda > 0$, $\lambda = p^2$ $\quad \dfrac{X''(x)}{X(x)} = p^2 \Rightarrow X''(x) = p^2X(x)$

$\Rightarrow X(x) = A\cosh px + B\sinh px$ (OR EXPONENTIALS)

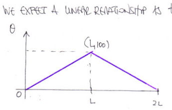
$\dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)} = p^2 \Rightarrow T'(t) = \alpha^2p^2T(t)$

$\Rightarrow T(t) = Ce^{\alpha^2p^2t}$

$\therefore \boxed{\theta(x,t) = X(x)T(t) = e^{\alpha^2p^2t}(P\cosh px + Q\sinh px)}$

This is not an actual physical solution, as it predicts unbounded solutions as $t \to \infty$

b) WE EXPECT A LINEAR RELATIONSHIP AS $t \to \infty$



"1ST SECTION"

GRADIENT $= \dfrac{100}{L}$

$\therefore \boxed{\theta = \dfrac{100}{L}x}$

"2ND SECTION"

GRADIENT $= \dfrac{0-100}{2L-L} = -\dfrac{100}{L}$

THUS $\theta - \theta_0 = -\dfrac{100}{L}(x - x_0)$

$\theta - 0 = -\dfrac{100}{L}(x - 2L)$

$\boxed{\theta = \dfrac{100}{L}(2L - x)}$

$\therefore \theta(x) = \begin{cases} \dfrac{100}{L}x & 0 \leqslant x \leqslant L \\ \dfrac{100}{L}(2L-x) & L \leqslant x \leqslant 2L \end{cases}$

c) $\displaystyle\int_L^{2L} \theta(x)\sin\left(\frac{n\pi x}{2L}\right)dx$ ... BY SUBSTITUTION

$x = 2L - y$
$dx = -dy$
$y = 2L - x$
$x = L, \ y = L$
$x = 2L, \ y = 0$

$= \displaystyle\int_L^0 \theta(y)\sin\left[\frac{n\pi}{2L}(2L-y)\right](-dy)$ (SEE BELOW)

$= \displaystyle\int_0^L \theta(y)\sin\left[n\pi - \frac{n\pi y}{2L}\right]dy$

$= \displaystyle\int_0^L \theta(y)\left[\sin n\pi\cos\frac{n\pi y}{2L} - \cos n\pi\sin\frac{n\pi y}{2L}\right]dy$

$= \displaystyle\int_0^L \theta(y)\left[-(-1)^n\sin\frac{n\pi y}{2L}\right]dy$

$= (-1)^{n+1}\displaystyle\int_0^L \theta(y)\sin\frac{n\pi y}{2L}\,dy$

$= (-1)^{n+1}\displaystyle\int_0^L \theta(x)\sin\frac{n\pi x}{2L}\,dx$

ALSO

$\theta(y) = \begin{cases} \dfrac{100}{L}(2L-y) & L \leqslant y \leqslant 2L \\ \dfrac{100}{L}[2L-(2L-y)] & 0 \leqslant y \leqslant L \end{cases}$

$\theta(y) = \begin{cases} \dfrac{100}{L}y & 0 \leqslant y \leqslant L \\ \dfrac{100}{L}(2L-y) & L \leqslant y \leqslant 2L \end{cases}$

d) NOW APPLY CONDITIONS

$\theta(0,t) = 0$ ---------- (I)

$\theta(2L,t) = 0$ ---------- (II)

$\theta(x,0) = \begin{cases} \dfrac{100}{L}x & 0 \leqslant x \leqslant L \\ \dfrac{100}{L}(2L-x) & L \leqslant x \leqslant 2L \end{cases}$ ---- (III)

START WITH

$\theta(x,t) = e^{-\alpha^2p^2t}(P\cos px + Q\sin px)$

BY (I) $\quad 0 = e^{-\alpha^2p^2t}P \quad \Rightarrow \boxed{P = 0}$

$\theta(x,t) = Qe^{-\alpha^2p^2t}\sin px$

BY (II) $\quad 0 = Qe^{-\alpha^2p^2t}\sin(2pL) \Rightarrow 2pL = n\pi$

$\Rightarrow p = \dfrac{n\pi}{2L} \quad n = 1,2,3,\ldots$

($n = 0$ GIVES ZERO TERM IN THE FOURIER)

$\theta(x,t) = \displaystyle\sum_{n=1}^{\infty} Q_n e^{-\frac{\alpha^2n^2\pi^2}{4L^2}t}\sin\frac{n\pi x}{2L}$

BY (III)

$\theta(x,0) = \displaystyle\sum_{n=1}^{\infty} Q_n\sin\frac{n\pi x}{2L}$ $\quad$ WHICH IS A FOURIER IN SINES WITH HALF PERIOD L

$\Rightarrow Q_n = \dfrac{1}{L}\displaystyle\int_0^{2L}\theta(x)\sin\frac{n\pi x}{L}\,dx$

$\Rightarrow Q_n = \dfrac{1}{L}\displaystyle\int_0^{L}\theta(x)\sin\frac{n\pi x}{2L}\,dx + \frac{1}{L}\int_L^{2L}\theta(x)\sin\frac{n\pi x}{2L}\,dx$

USING PART c

$\Rightarrow Q_n = \dfrac{1}{L}\displaystyle\int_0^{L}\theta(x)\sin\frac{n\pi x}{2L}\,dx + \frac{(-1)^{n+1}}{L}\int_0^{L}\theta(x)\sin\frac{n\pi x}{2L}\,dx$

$\Rightarrow Q_n = \dfrac{1}{L}\left[1 + (-1)^{n+1}\right]\displaystyle\int_0^{L}\theta(x)\sin\frac{n\pi x}{2L}\,dx$

$\Rightarrow Q_n = \dfrac{1}{L}\left[1 + (-1)^{n+1}\right]\displaystyle\int_0^{L}\frac{100}{L}x\sin\frac{n\pi x}{2L}\,dx$

$\Rightarrow Q_n = \dfrac{100}{L^2}\left[1 + (-1)^{n+1}\right]\displaystyle\int_0^{L}x\sin\frac{n\pi x}{2L}\,dx$

BY PARTS

x	1
$-\dfrac{2L}{n\pi}\cos\dfrac{n\pi x}{2L}$	$\sin\dfrac{n\pi x}{L}$

$\Rightarrow Q_n = \dfrac{100}{L^2}\left[1 + (-1)^{n+1}\right]\left\{\dfrac{2L}{n\pi}\left[-x\cos\dfrac{n\pi x}{2L}\right]_0^L - \dfrac{2L}{n\pi}\displaystyle\int_0^L\cos\frac{n\pi x}{2L}\,dx\right\}$

$\Rightarrow Q_n = \dfrac{100}{L^2}\left[1 + (-1)^{n+1}\right]\left\{-\dfrac{2L^2}{n\pi}\cos\dfrac{n\pi}{2} + \dfrac{4L^2}{n^2\pi^2}\left[\sin\dfrac{n\pi x}{2L}\right]_0^L\right\}$

$\Rightarrow Q_n = \dfrac{100}{L^2}\left[1 + (-1)^{n+1}\right]\left\{-\dfrac{2L^2}{n\pi}\cos\dfrac{n\pi}{2} + \dfrac{4L^2}{n^2\pi^2}\sin\dfrac{n\pi}{2}\right\}$

$\Rightarrow Q_n = \dfrac{200}{n^2\pi^2}\left[1 + (-1)^{n+1}\right]\left[2\sin\dfrac{n\pi}{2} - n\pi\cos\dfrac{n\pi}{2}\right]$

THUS

$\theta(x,t) = \displaystyle\sum_{n=1}^{\infty}\frac{200}{n^2\pi^2}\left[1 + (-1)^{n+1}\right]\left[2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right]\exp\left[-\frac{\alpha^2n^2\pi^2}{4L^2}t\right]\sin\frac{n\pi x}{2L}$

NOW LOOKING AT $1 + (-1)^{n+1} = \begin{cases} 0 & \text{IF } n \text{ IS EVEN} \\ 2 & \text{IF } n \text{ IS ODD} \end{cases}$

$\theta(x,t) = \displaystyle\sum_{m=0}^{\infty}\frac{400}{\pi^2(2m+1)^2}\left[2\sin\frac{(2m+1)\pi}{2} - (2m+1)\pi\cos\frac{(2m+1)\pi}{2}\right]e^{-\frac{\alpha^2\pi^2(2m+1)^2}{4L^2}t}\sin\left[\frac{(2m+1)\pi x}{2L}\right]$

AT THE CENTRE

$\theta(L,t) = \displaystyle\sum_{m=0}^{\infty}\frac{400}{\pi^2(2m+1)^2}\left[2\sin\frac{(2m+1)\pi}{2} - (2m+1)\pi\cos\frac{(2m+1)\pi}{2}\right]e^{-\frac{\alpha^2\pi^2(2m+1)^2}{4L^2}t}\sin\left[\frac{(2m+1)\pi}{2}\right]$

NOW $\cos\dfrac{(2m+1)\pi}{2}\sin\dfrac{(2m+1)\pi}{2} = \dfrac{1}{2}\sin(2m+1)\pi = 0$

$\theta(L,t) = \displaystyle\sum_{m=0}^{\infty}\frac{400}{\pi^2(2m+1)^2}\left[2\sin^2\left(\frac{(2m+1)\pi}{2}\right)\right]e^{-\frac{\alpha^2\pi^2(2m+1)^2}{4L^2}t}$

$(\pm 1)^2 = 1$

$\theta(L,t) = \displaystyle\sum_{m=0}^{\infty}\frac{800}{\pi^2(2m+1)^2}e^{-\frac{\alpha^2\pi^2(2m+1)^2}{4L^2}t}$

HEAT EQUATION

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}, \quad \theta=\theta(x,y,t)$$

Two Dimensional

Question 1

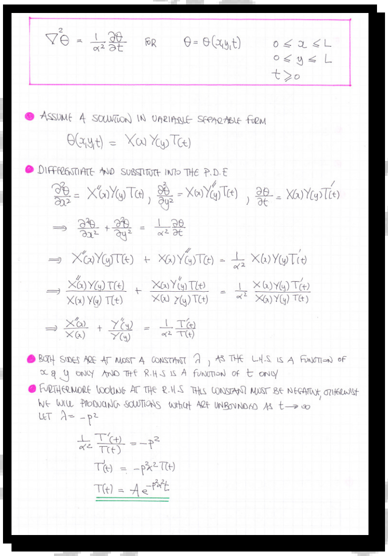
The temperature distribution, $\theta(x, y, t)$, on a square plate satisfies the equation

$$\nabla^2\theta = \frac{1}{\alpha^2}\frac{\partial z}{\partial t}, \quad 0 \le x \le L, \quad 0 \le y \le L, \quad t \ge 0.$$

Find a general solution for $\theta(x, y, t)$, which is periodic in x and in y.

Define any constants used.

$$\boxed{\theta(x, y, t) = e^{-p^2\alpha^2 t}\left[A\cos qx + B\sin qx\right]\left[C\cos ky + D\sin ky\right], \quad p^2 = q^2 + k^2}$$



$\nabla^2\theta = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}$ FOR $\theta = \theta(x,y,t)$, $0 \le x \le L$, $0 \le y \le L$, $t \ge 0$

- ASSUME A SOLUTION IN VARIABLE SEPARABLE FORM

$\theta(x,y,t) = X(x)Y(y)T(t)$

- DIFFERENTIATE AND SUBSTITUTE INTO THE P.D.E

$\frac{\partial^2\theta}{\partial x^2} = X''(x)Y(y)T(t)$, $\frac{\partial^2\theta}{\partial y^2} = X(x)Y''(y)T(t)$, $\frac{\partial\theta}{\partial t} = X(x)Y(y)T'(t)$

$\Rightarrow \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t}$

$\Rightarrow X''(x)Y(y)T(t) + X(x)Y''(y)T(t) = \frac{1}{\alpha^2}X(x)Y(y)T'(t)$

$\Rightarrow \frac{X''(x)Y(y)T(t)}{X(x)Y(y)T(t)} + \frac{X(x)Y''(y)T(t)}{X(x)Y(y)T(t)} = \frac{1}{\alpha^2}\frac{X(x)Y(y)T'(t)}{X(x)Y(y)T(t)}$

$\Rightarrow \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)}$

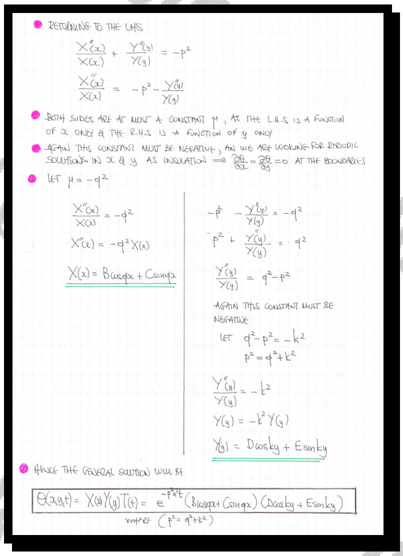
- BOTH SIDES ARE AT MOST A CONSTANT λ, AS THE L.H.S IS A FUNCTION OF x & y ONLY AND THE R.H.S IS A FUNCTION OF t ONLY
- FURTHERMORE LOOKING AT THE R.H.S THIS CONSTANT MUST BE NEGATIVE, OTHERWISE WE WILL PRODUCING SOLUTIONS WHICH ARE UNBOUNDED AS $t \to \infty$

LET $\lambda = -p^2$

$\frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -p^2$

$T'(t) = -p^2\alpha^2 T(t)$

$T(t) = Ae^{-p^2\alpha^2 t}$



- RETURNING TO THE LHS

$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -p^2$

$\frac{X''(x)}{X(x)} = -p^2 - \frac{Y''(y)}{Y(y)}$

- BOTH SIDES ARE AT MOST A CONSTANT μ, AS THE L.H.S IS A FUNCTION OF x ONLY & THE R.H.S IS A FUNCTION OF y ONLY
- AGAIN THIS CONSTANT MUST BE NEGATIVE, AS WE ARE LOOKING FOR PERIODIC SOLUTIONS IN x & y AS INSULATION $\Rightarrow \frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial y} = 0$ AT THE BOUNDARIES
- LET $\mu = -q^2$

$\frac{X''(x)}{X(x)} = -q^2$

$X''(x) = -q^2X(x)$

$X(x) = B\cos qx + C\sin qx$

$-p^2 - \frac{Y''(y)}{Y(y)} = -q^2$

$p^2 + \frac{Y''(y)}{Y(y)} = q^2$

$\frac{Y''(y)}{Y(y)} = q^2 - p^2$

AGAIN THIS CONSTANT MUST BE NEGATIVE

LET $q^2 - p^2 = -k^2$

$p^2 = q^2 + k^2$

$\frac{Y''(y)}{Y(y)} = -k^2$

$Y''(y) = -k^2Y(y)$

$Y(y) = D\cos ky + E\sin ky$

- HENCE THE GENERAL SOLUTION WILL BE

$\theta(x,y,t) = X(x)Y(y)T(t) = e^{-p^2\alpha^2 t}(B\cos qx + C\sin qx)(D\cos ky + E\sin ky)$

WHERE $(p^2 = q^2 + k^2)$

HEAT EQUATION

Miscellaneous Questions

Question 1

The smooth function $u = u(x,t)$ satisfies the diffusion equation

$$\frac{\partial^2 u}{dx^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t},$$

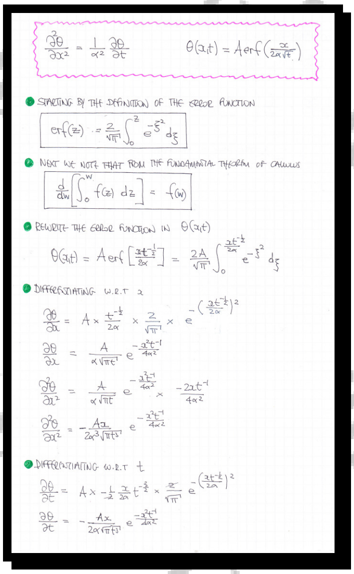
where α is a positive constant.

Show by differentiation that $u(x,t) = A \operatorname{erf}\left(\dfrac{x}{2\alpha\sqrt{t}}\right)$, where A is a non zero constant, satisfies the diffusion equation.

You may assume that

- $\operatorname{erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z \mathrm{e}^{-\xi} \; d\xi$.
- $\dfrac{d}{dw}\left[\displaystyle\int_0^w f(z) \; dz\right] = f(w)$.

proof



$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} \qquad \theta(x,t) = A\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

- STARTING BY THE DEFINITION OF THE ERROR FUNCTION

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}\,d\xi$$

- NEXT WE NOTE THAT FROM THE FUNDAMENTAL THEOREM OF CALCULUS

$$\frac{d}{dw}\left[\int_0^w f(z)\,dz\right] = f(w)$$

- REWRITE THE ERROR FUNCTION IN $\theta(x,t)$

$$\theta(x,t) = A\operatorname{erf}\left[\frac{xt^{-\frac{1}{2}}}{2\alpha}\right] = \frac{2A}{\sqrt{\pi}}\int_0^{\frac{xt^{-\frac{1}{2}}}{2\alpha}} e^{-\xi^2}\,d\xi$$

- DIFFERENTIATING W.R.T x

$$\frac{\partial\theta}{\partial x} = A \times \frac{t^{-\frac{1}{2}}}{2\alpha} \times \frac{2}{\sqrt{\pi}} \times e^{-\left(\frac{xt^{-\frac{1}{2}}}{2\alpha}\right)^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{A}{\alpha\sqrt{\pi t}}e^{-\frac{x^2t^{-1}}{4\alpha^2}}$$

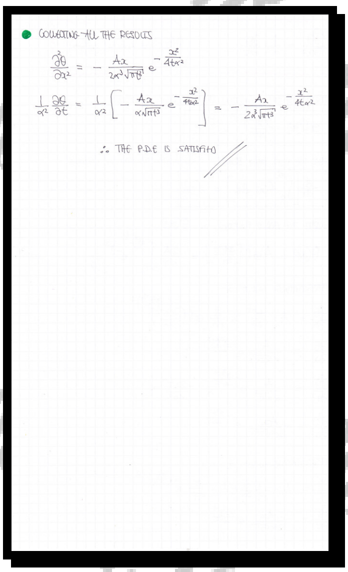
$$\frac{\partial^2\theta}{\partial x^2} = \frac{A}{\alpha\sqrt{\pi t}}e^{-\frac{x^2t^{-1}}{4\alpha^2}} \times \frac{-2xt^{-1}}{4\alpha^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = -\frac{Ax}{2\alpha^3\sqrt{\pi t^3}}e^{-\frac{x^2t^{-1}}{4\alpha^2}}$$

- DIFFERENTIATING W.R.T t

$$\frac{\partial\theta}{\partial t} = A \times -\frac{1}{2}\frac{x}{2\alpha}t^{-\frac{3}{2}} \times \frac{2}{\sqrt{\pi}}e^{-\left(\frac{xt^{-\frac{1}{2}}}{2\alpha}\right)^2}$$

$$\frac{\partial\theta}{\partial t} = -\frac{Ax}{2\alpha\sqrt{\pi t^3}}e^{-\frac{x^2t^{-1}}{4\alpha^2}}$$



- COLLECTING ALL THE RESULTS

$$\frac{\partial^2\theta}{\partial x^2} = -\frac{Ax}{2\alpha^3\sqrt{\pi t^3}}e^{-\frac{x^2}{4t\alpha^2}}$$

$$\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} = \frac{1}{\alpha^2}\left[-\frac{Ax}{\alpha\sqrt{\pi t^3}}e^{-\frac{x^2}{4t\alpha^2}}\right] = -\frac{Ax}{2\alpha^3\sqrt{\pi t^3}}e^{-\frac{x^2}{4t\alpha^2}}$$

$\therefore$ THE P.D.E IS SATISFIED

Question 2

The function $u = u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{dx^2} - 2u = 0,$$

subject to the conditions

$$u(0,t) = 0, \quad u(1,t) = 0 \quad \text{and} \quad u(x,0) = 1.$$

Use the substitution $u(x,t) = e^{kt} w(x,t)$, with a suitable value for the constant k, to find a simplified expression for $u(x,t)$.

$$\boxed{u(x,t) = \frac{2e^{2t}}{\pi} \sum_{n=1}^{\infty} \left[\frac{\exp\left[-(2n-1)^2 \pi^2 t\right] \sin\left[(2n-1)\pi x\right]}{2n-1} \right]}$$