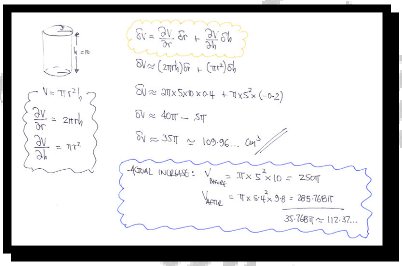
# PARTIAL DIFFERENTIATION INTRODUCTION

**Question 1** (**)

A right circular cylinder has radius 5 cm and height 10 cm.

Use a differential approximation to find an approximate increase in the volume of this cylinder if the radius increases by 0.4 cm and its height decreases by 0.2 cm.

$35\pi \approx 109.96...\text{cm}^3$

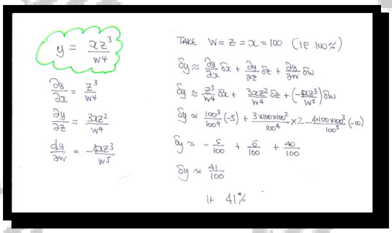


**Question 2** (**)

$$y = \frac{xz^3}{w^4}, \ w \neq 0.$$

Determine an approximate percentage increase in $y$, if $x$ decreases by $5\%$, $z$ increases by $2\%$ and $w$ decreases by $10\%$.

$\approx 41\%$



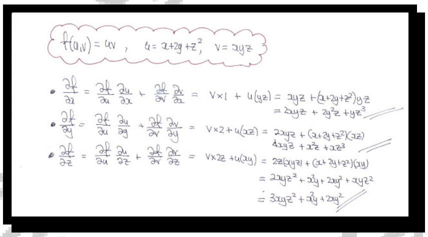
**Question 3** (**)

The function $f$ depends on $u$ and $v$ so that

$$f\left[u(x,y,z),v(x,y,z)\right]=uv\,,\quad u=x+2y+z^2 \quad \text{and} \quad v=xyz\,.$$

Find simplified expressions for $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, in terms of $x$, $y$ and $z$.

$$\boxed{\frac{\partial f}{\partial x}=2xyz+2y^2z+yz^3}\,,\quad \boxed{\frac{\partial f}{\partial y}=4xyz+x^2z+xz^3}\,,\quad \boxed{\frac{\partial f}{\partial z}=3xyz^2+x^2y+2xy^2}$$




**Question 4** (**)

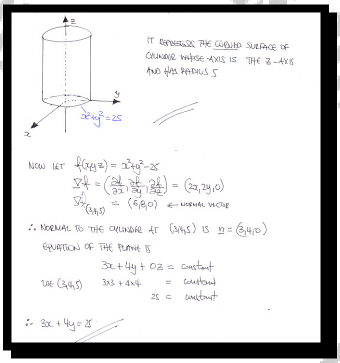
A surface $S$ is defined by the Cartesian equation

$$x^2+y^2=25\,.$$

**a)** Draw a sketch of $S$, and describe it geometrically.

**b)** Determine an equation of the tangent plane on $S$ at the point with Cartesian coordinates $(3,4,5)$.

$$\boxed{3x+4y=25}$$



**Question 5** (**)

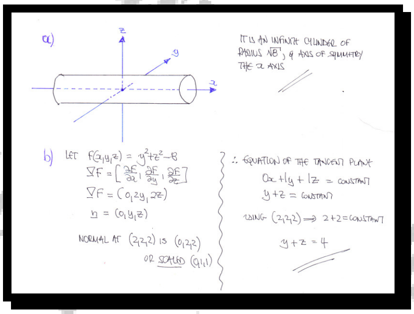
A surface $S$ is defined by the Cartesian equation

$$y^2 + z^2 = 8.$$

**a)** Draw a sketch of $S$, and describe it geometrically.

**b)** Determine an equation of the tangent plane on $S$ at the point with Cartesian coordinates $(2,2,2)$.

$$\boxed{y + z = 4}$$



**Question 6 (**)**

The function $\varphi$ depends on $u$, $v$ and $w$ so that

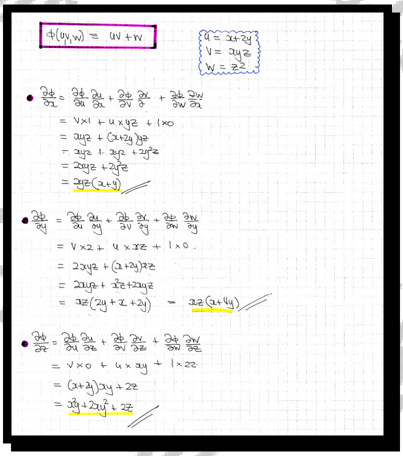
$$\varphi\left[u(x,y,z),v(x,y,z),w(x,y,z)\right]=uv+w.$$

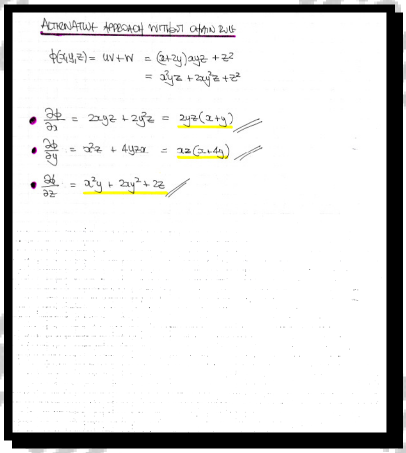
It is further given that

$$u=x+2y,\qquad v=xyz\quad\text{and}\quad w=z^2.$$

By using **the chain rule for partial differentiation** find simplified expressions for $\dfrac{\partial\varphi}{\partial x}$, $\dfrac{\partial\varphi}{\partial y}$ and $\dfrac{\partial\varphi}{\partial z}$, in terms of $x$, $y$ and $z$.

$$\boxed{\frac{\partial\varphi}{\partial x}=2yz(x+y)},\ \boxed{\frac{\partial\varphi}{\partial y}=xz(x+4y)},\ \boxed{\frac{\partial\varphi}{\partial z}=x^2y+2xy^2+2z}$$



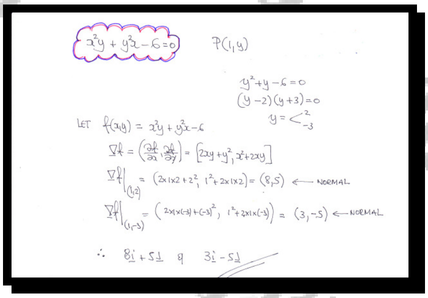


**Question 7** **(**)**

The point $P(1, y)$ lies on the contour with equation $x^2y + y^2x - 6 = 0$.

Determine the possible normal vectors at $P$

$\boxed{8\mathbf{i}+5\mathbf{j}}$, $\boxed{3\mathbf{i}-5\mathbf{j}}$

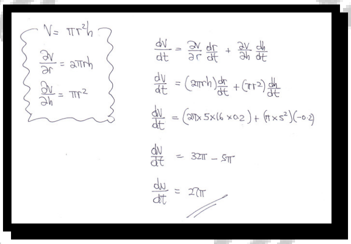


**Question 8** **(**)**

The radius of a right circular cylinder is increasing at the constant rate of $0.2 \text{ cms}^{-1}$ and its height is decreasing at the constant rate of $0.2 \text{ cms}^{-1}$.

Determine the rate at which the volume of this cylinder is increasing when the radius is $5 \text{ cm}$ and its height is $16 \text{ cm}$.

$\boxed{27\pi \approx 84.82...\text{cm}^3\text{s}^{-1}}$



**Question 9 (**)**

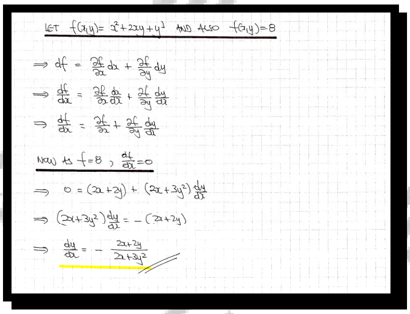
A curve has implicit equation

$$x^2 + 2xy + y^3 = 8.$$

Use partial differentiation to find an expression for $\dfrac{dy}{dx}$ .

***No credit will be given for obtaining the answer with alternative methods***

$$\frac{dy}{dx} = -\frac{2x+2y}{2x+3y^2}$$



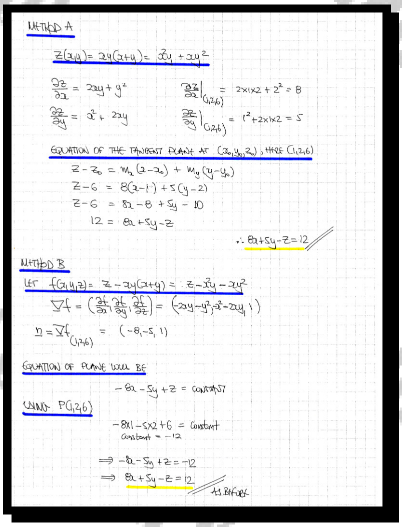
**Question 10** (**)

A surface $S$ is defined by the Cartesian equation

$$z = xy(x+y).$$

Find an equation of the tangent plane on $S$ at the point $(1,2,6)$.

$\boxed{\phantom{xx}}$, $\boxed{8x+5y-z=12}$


METHOD A

$z(x,y) = xy(x+y) = x^2y + xy^2$

$\frac{\partial z}{\partial x} = 2xy + y^2 \qquad \left.\frac{\partial z}{\partial x}\right|_{(1,2,6)} = 2\times1\times2 + 2^2 = 8$

$\frac{\partial z}{\partial y} = x^2 + 2xy \qquad \left.\frac{\partial z}{\partial y}\right|_{(1,2,6)} = 1^2 + 2\times1\times2 = 5$

EQUATION OF THE TANGENT PLANE AT $(x_0, y_0, z_0)$, HERE $(1,2,6)$

$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$

$z - 6 = 8(x-1) + 5(y-2)$

$z - 6 = 8x - 8 + 5y - 10$

$12 = 8x + 5y - z$

$\therefore 8x + 5y - z = 12$

METHOD B

LET $f(x,y,z) = z - xy(x+y) = z - x^2y - xy^2$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (-2xy - y^2, -x^2 - 2xy, 1)$

$\underline{n} = \nabla f_{(1,2,6)} = (-8, -5, 1)$

EQUATION OF PLANE WILL BE

$-8x - 5y + z = \text{CONSTANT}$

USING $P(1,2,6)$

$-8\times1 - 5\times2 + 6 = \text{Constant}$

Constant $= -12$

$\Rightarrow -8x - 5y + z = -12$

$\Rightarrow 8x + 5y - z = 12$

AS BEFORE

**Question 11 (**)**

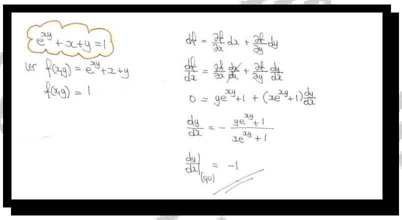
A curve has implicit equation

$$e^{xy} + x + y = 1.$$

Use partial differentiation to find the value of $\frac{dy}{dx}$ at $(0,0)$.

***No credit will be given for obtaining the answer with alternative methods***

$$\left.\frac{dy}{dx}\right|_{(0,0)} = -1$$


$e^{xy} + x + y = 1$

Let $f(x,y) = e^{xy} + x + y$

$f(x,y) = 1$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

$0 = ye^{xy} + 1 + (xe^{xy} + 1)\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{ye^{xy} + 1}{xe^{xy} + 1}$

$\left.\frac{dy}{dx}\right|_{(0,0)} = -1$

**Question 12** (**+)

The function $f$ is defined as

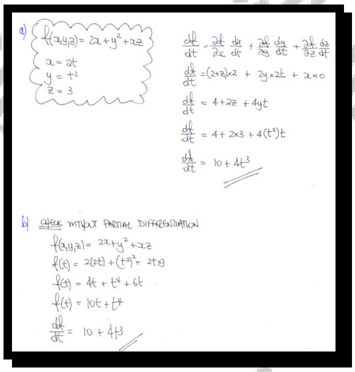
$$f(x,y,z) \equiv 2x + y^2 + xz,$$

where $x = 2t$, $y = t^2$ and $z = 3$.

**a)** Use partial differentiation to find an expression for $\frac{df}{dt}$, in terms of $t$.

**b)** Verify the answer obtained in part **(a)** by a method **not** involving partial differentiation.

$$\boxed{\frac{df}{dt} = 4t^3 + 10}$$



a) $f(x,y,z) = 2x + y^2 + xz$
$x = 2t$
$y = t^2$
$z = 3$

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

$\frac{df}{dt} = (2+z)\times 2 + 2y \times 2t + x \times 0$

$\frac{df}{dt} = 4 + 2z + 4yt$

$\frac{df}{dt} = 4 + 2\times 3 + 4(t^2)t$

$\frac{df}{dt} = 10 + 4t^3$

b) CHECK WITHOUT PARTIAL DIFFERENTIATION

$f(x,y,z) = 2x + y^2 + xz$

$f(t) = 2(2t) + (t^2)^2 + 2t \times 3$

$f(t) = 4t + t^4 + 6t$

$f(t) = 10t + t^4$

$\frac{df}{dt} = 10 + 4t^3$

**Question 13 (**+)**

The function $\varphi$ is defined as

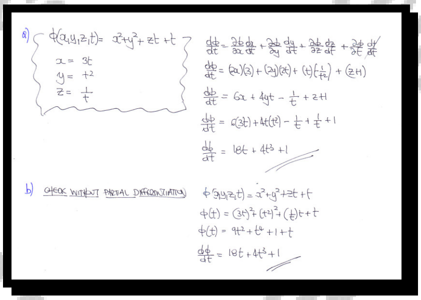
$$\varphi(x, y, z) \equiv x^2 + y^2 + tz + t \, , \quad t \neq 0 \, ,$$

where $x = 3t$, $y = t^2$ and $z = \frac{1}{t}$.

**a)** Use partial differentiation to find an expression for $\frac{d\varphi}{dt}$, in terms of $t$.

**b)** Verify the answer obtained in part **(a)** by a method not involving partial differentiation.

$$\boxed{\frac{d\varphi}{dt} = 4t^3 + 18t + 1}$$



**Question 14** **(**+)**

Plane Cartesian coordinates $(x.y)$ are related to plane polar coordinates $(r,\theta)$ by the transformation equations

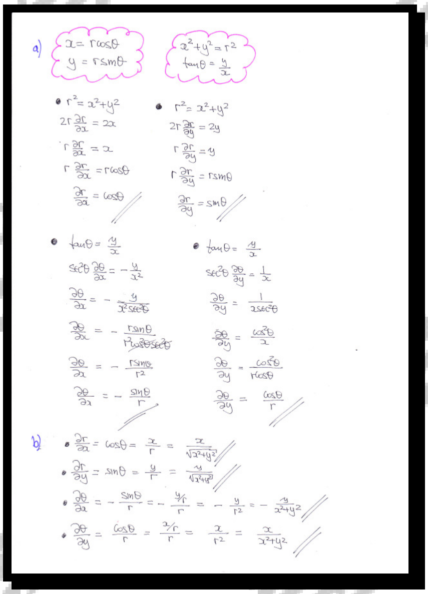
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta .$$

**a)** Find simplified expressions for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$, in terms of $r$ and $\theta$.

**b)** Deduce simplified expressions for $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$, in terms of $x$ and $y$.

$$\boxed{\frac{\partial r}{\partial x} = \cos\theta}, \quad \boxed{\frac{\partial r}{\partial y} = \sin\theta}, \quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}}, \quad \boxed{\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}}$$

$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}}, \quad \boxed{\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}}, \quad \boxed{\frac{\partial \theta}{\partial x} = -\frac{y}{x^2+y^2}}, \quad \boxed{\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}}$$



a) $x = r\cos\theta$, $y = r\sin\theta$ ; $x^2+y^2 = r^2$, $\tan\theta = \frac{y}{x}$

- $r^2 = x^2+y^2$
  $2r\frac{\partial r}{\partial x} = 2x$
  $r\frac{\partial r}{\partial x} = x$
  $r\frac{\partial r}{\partial x} = r\cos\theta$
  $\frac{\partial r}{\partial x} = \cos\theta$
- $r^2 = x^2+y^2$
  $2r\frac{\partial r}{\partial y} = 2y$
  $r\frac{\partial r}{\partial y} = y$
  $r\frac{\partial r}{\partial y} = r\sin\theta$
  $\frac{\partial r}{\partial y} = \sin\theta$
- $\tan\theta = \frac{y}{x}$
  $\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$
  $\frac{\partial \theta}{\partial x} = -\frac{y}{x^2\sec^2\theta}$
  $\frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2\cos^2\theta\sec^2\theta}$
  $\frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2}$
  $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$
- $\tan\theta = \frac{y}{x}$
  $\sec^2\theta\frac{\partial \theta}{\partial y} = \frac{1}{x}$
  $\frac{\partial \theta}{\partial y} = \frac{1}{x\sec^2\theta}$
  $\frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{x}$
  $\frac{\partial \theta}{\partial y} = \frac{\cos^2\theta}{r\cos\theta}$
  $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$

b)
- $\frac{\partial r}{\partial x} = \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$
- $\frac{\partial r}{\partial y} = \sin\theta = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$
- $\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} = -\frac{y/r}{r} = -\frac{y}{r^2} = -\frac{y}{x^2+y^2}$
- $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} = \frac{x/r}{r} = \frac{x}{r^2} = \frac{x}{x^2+y^2}$

**Question 15 (**+)**

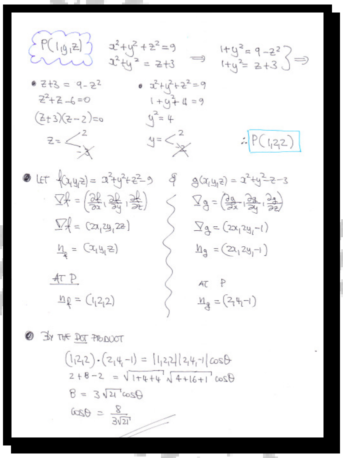
The point $P(1, y_0, z_0)$ lies on both surfaces with Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad z = x^2 + y^2 - 3.$$

At the point $P$, the two surfaces intersect each other at an angle $\theta$.

Given further that $P$ lies in the first octant, determine the exact value of $\cos\theta$.

$$\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$



$P(1,y,z)$ $\quad x^2+y^2+z^2=9$, $x^2+y^2=z+3$ $\Rightarrow$ $1+y^2=9-z^2$, $1+y^2=z+3$ $\Rightarrow$

- $z+3=9-z^2$, $z^2+z-6=0$, $(z+3)(z-2)=0$, $z = 2$ ~~$-3$~~
- $x^2+y^2+z^2=9$, $1+y^2+4=9$, $y^2=4$, $y = 2$ ~~$-2$~~

$\therefore P(1,2,2)$

- LET $f(x,y,z)=x^2+y^2+z^2-9$ & $g(x,y,z)=x^2+y^2-z-3$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$, $\nabla f = (2x,2y,2z)$, $n_f = (x,y,z)$

AT P: $n_f = (1,2,2)$

$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$, $\nabla g = (2x,2y,-1)$, $n_g = (2x,2y,-1)$

AT P: $n_g = (2,4,-1)$

- BY THE DOT PRODUCT

$(1,2,2)\cdot(2,4,-1) = |1,2,2||2,4,-1|\cos\theta$

$2+8-2 = \sqrt{1+4+4}\sqrt{4+16+1}\cos\theta$

$8 = 3\sqrt{21}\cos\theta$

$\cos\theta = \frac{8}{3\sqrt{21}}$

**Question 16 (**+)**

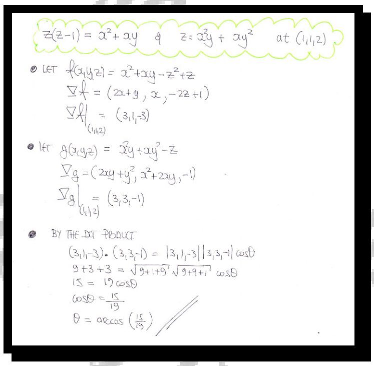
The point $P(1,1,2)$ lies on both surfaces with Cartesian equations

$$z(z-1) = x^2 + xy \quad \text{and} \quad z = x^2y + xy^2 .$$

At the point $P$, the two surfaces intersect each other at an angle $\theta$.

Determine the exact value of $\theta$.

$$\boxed{\theta = \arccos\left(\frac{15}{19}\right)}$$



$z(z-1) = x^2 + xy$ & $z = x^2y + xy^2$ at $(1,1,2)$

- LET $f(x,y,z) = x^2 + xy - z^2 + z$

  $\nabla f = (2x+y, x, -2z+1)$

  $\nabla f \big|_{(1,1,2)} = (3,1,-3)$

- LET $g(x,y,z) = x^2y + xy^2 - z$

  $\nabla g = (2xy + y^2, x^2 + 2xy, -1)$

  $\nabla g \big|_{(1,1,2)} = (3,3,-1)$

- BY THE DOT PRODUCT

  $(3,1,-3) \cdot (3,3,-1) = |3,1,-3||3,3,-1| \cos\theta$

  $9 + 3 + 3 = \sqrt{9+1+9}\sqrt{9+9+1} \cos\theta$

  $15 = 19 \cos\theta$

  $\cos\theta = \frac{15}{19}$

  $\theta = \arccos\left(\frac{15}{19}\right)$

**Question 17** (***)

The point $P(-1,1,3)$ lies on both surfaces with Cartesian equations

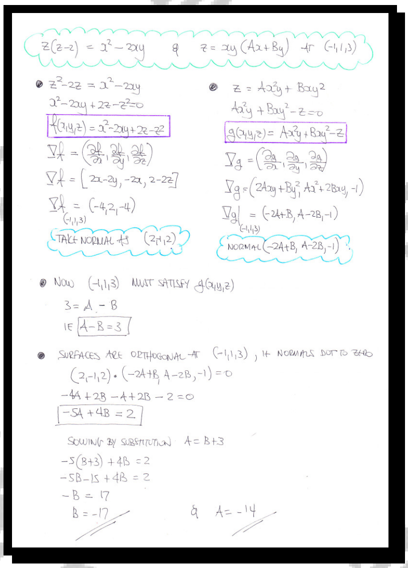
$$z(z-2) = x^2 - 2xy \quad \text{and} \quad z = xy(Ax+By),$$

where $A$ and $B$ are non zero constants.

The two surfaces intersect each other orthogonally at the point $P$.

Determine the value of $A$ and the value of $B$.

$\boxed{A=-14}$ , $\boxed{B=-17}$



**Question 18** (***)

The function $f$ depends on $u$, $v$ and $t$ so that

$$f\{u[x(t),y(t),z(t)],v[x(t),y(t),z(t)],t\}=u^2+v+2t .$$

It is further given that

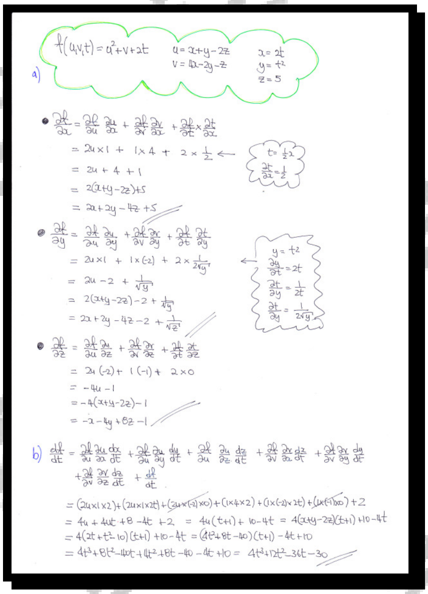
$$u=x+y-2z, \quad v=4x-2y-z \qquad \text{and} \qquad x=2t, \quad y=t^2, \quad z=5 .$$

**a)** Find simplified expressions for $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, in terms of $x$, $y$ and $z$.

**b)** Determine an expression for $\frac{df}{dt}$, in terms of $t$.

$\boxed{\frac{\partial f}{\partial x}=2x+2y-4z+5}$, $\boxed{\frac{\partial f}{\partial y}=2x+2y-4z+\frac{1}{\sqrt{z}}-2}$, $\boxed{\frac{\partial f}{\partial z}=-x-4y+8z-1}$,

$\boxed{\frac{df}{dt}=4t^3+12t^2-36t-30}$



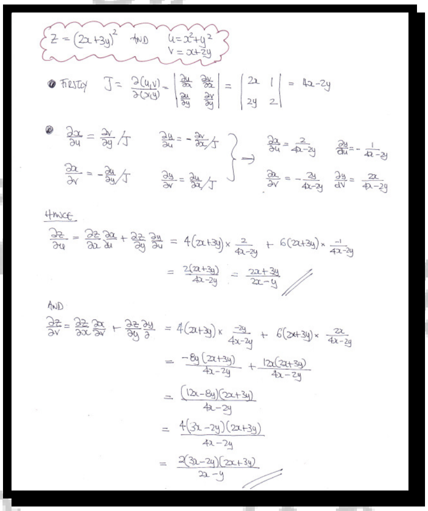
**Question 19 (***)**

The function $z$ depends on $u$ and $v$ so that

$$z=(2x+3y)^2,\quad u=x^2+y^2 \quad \text{and} \quad v=x+2y.$$

Find simplified expressions for $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$, in terms of $x$ and $y$.

$$\boxed{\frac{\partial z}{\partial u}=\frac{2x+3y}{2x-y}},\quad \boxed{\frac{\partial z}{\partial v}=\frac{2(3x-2y)(2x+3y)}{2x-y}}$$



$z=(2x+3y)^2$ AND $u=x^2+y^2$, $v=x+2y$

• FIRSTLY $J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & 1\\ 2y & 2\end{vmatrix}=4x-2y$

• $\frac{\partial x}{\partial u}=\frac{\partial v}{\partial y}/J$, $\frac{\partial y}{\partial u}=-\frac{\partial v}{\partial x}/J$, $\frac{\partial x}{\partial v}=-\frac{\partial u}{\partial y}/J$, $\frac{\partial y}{\partial v}=\frac{\partial u}{\partial x}/J$ $\Rightarrow$ $\frac{\partial x}{\partial u}=\frac{2}{4x-2y}$, $\frac{\partial y}{\partial u}=-\frac{1}{4x-2y}$, $\frac{\partial x}{\partial v}=-\frac{2y}{4x-2y}$, $\frac{\partial y}{\partial v}=\frac{2x}{4x-2y}$

HENCE

$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}=4(2x+3y)\times\frac{2}{4x-2y}+6(2x+3y)\times\frac{-1}{4x-2y}$

$=\frac{2(2x+3y)}{4x-2y}=\frac{2x+3y}{2x-y}$

AND

$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=4(2x+3y)\times\frac{-2y}{4x-2y}+6(2x+3y)\times\frac{2x}{4x-2y}$

$=\frac{-8y(2x+3y)}{4x-2y}+\frac{12x(2x+3y)}{4x-2y}$

$=\frac{(12x-8y)(2x+3y)}{4x-2y}$

$=\frac{4(3x-2y)(2x+3y)}{4x-2y}$

$=\frac{2(3x-2y)(2x+3y)}{2x-y}$

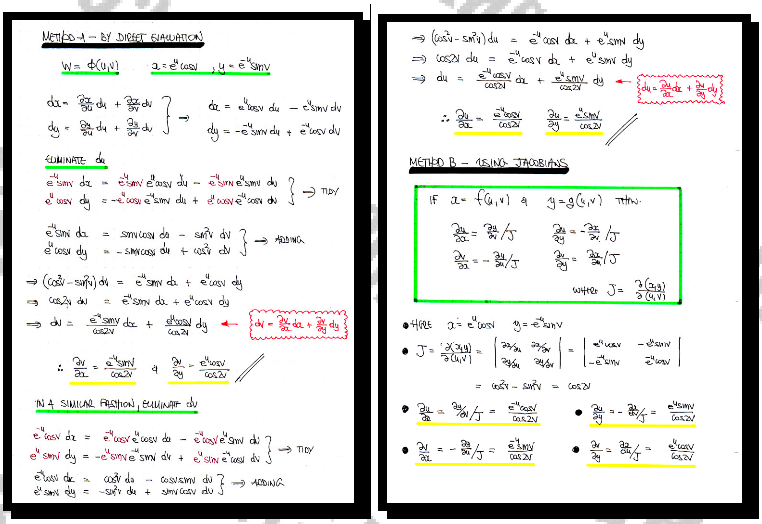
**Question 20 (***)**

The function $w = \varphi\left[u(x,y), v(x,y)\right]$ satisfies

$$x = \mathrm{e}^{u}\cos v \quad \text{and} \quad y = \mathrm{e}^{-u}\sin v .$$

Determine simplified expressions for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial v}{\partial y}$, in terms of $u$ and $v$.

$\boxed{\phantom{xx}}$, $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\mathrm{e}^{-u}\cos v}{\cos 2v}}$, $\boxed{\dfrac{\partial u}{\partial y} = \dfrac{\mathrm{e}^{u}\sin v}{\cos 2v}}$, $\boxed{\dfrac{\partial v}{\partial x} = \dfrac{\mathrm{e}^{-u}\sin v}{\cos 2v}}$, $\boxed{\dfrac{\partial v}{\partial y} = \dfrac{\mathrm{e}^{u}\cos v}{\cos 2v}}$



**Question 21** (***)

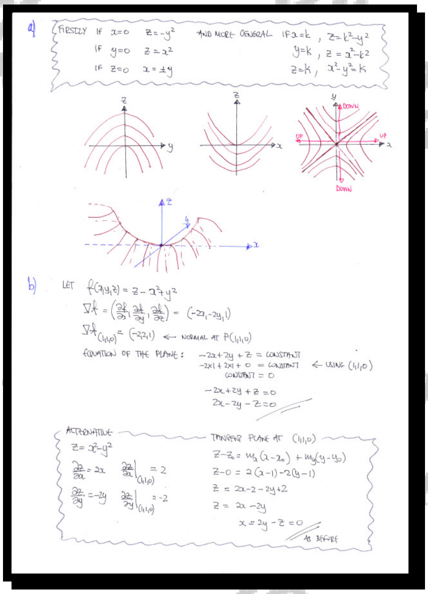
A surface $S$ has Cartesian equation

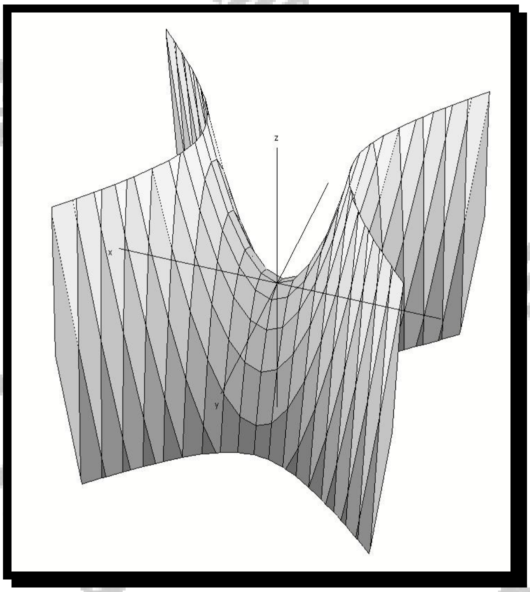
$$z = x^2 - y^2 .$$

**a)** Sketch profiles of $S$ parallel to the $y$-$z$ plane, parallel to the $x$-$z$ plane, and parallel to the $x$-$y$ plane.

**b)** Find an equation of the tangent plane on $S$, at the point $P(1,1,0)$.

$$\boxed{2x - 2y - z = 0}$$





**Question 22** (***)

A surface $S$ is given parametrically by

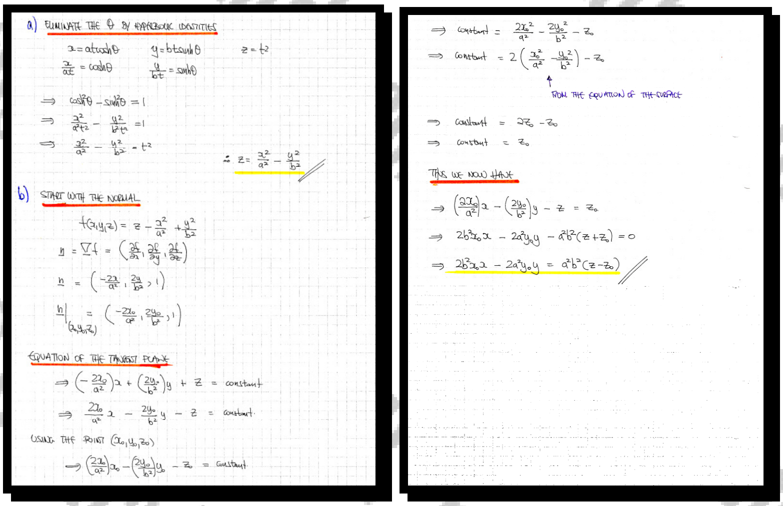
$$x = at\cosh\theta\,,\ x = bt\sinh\theta\,,\ z = t^2\,,$$

where $t$ and $\theta$ are real parameters, and $a$ and $b$ are non zero constants.

**a)** Find a Cartesian equation for $S$.

**b)** Determine an equation of the tangent plane on $S$ at the point with Cartesian coordinates $(x_0, y_0, z_0)$.

$\boxed{\phantom{xx}}$, $\boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, $\boxed{2b^2 x x_0 - 2a^2 y y_0 = a^2 b^2 (z + z_0)}$


a) ELIMINATE THE θ BY HYPERBOLIC IDENTITIES

$x = at\cosh\theta \quad y = bt\sinh\theta \quad z = t^2$

$\frac{x}{at} = \cosh\theta \quad \frac{y}{bt} = \sinh\theta$

$\Rightarrow \cosh^2\theta - \sinh^2\theta = 1$

$\Rightarrow \frac{x^2}{a^2t^2} - \frac{y^2}{b^2t^2} = 1$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2$

$\therefore z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

b) START WITH THE NORMAL

$f(x,y,z) = z - \frac{x^2}{a^2} + \frac{y^2}{b^2}$

$\underline{n} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\underline{n} = \left(-\frac{2x}{a^2}, \frac{2y}{b^2}, 1\right)$

$\underline{n}\big|_{(x_0,y_0,z_0)} = \left(-\frac{2x_0}{a^2}, \frac{2y_0}{b^2}, 1\right)$

EQUATION OF THE TANGENT PLANE

$\Rightarrow \left(-\frac{2x_0}{a^2}\right)x + \left(\frac{2y_0}{b^2}\right)y + z = \text{constant}$

$\Rightarrow \frac{2x_0}{a^2}x - \frac{2y_0}{b^2}y - z = \text{constant}$

USING THE POINT $(x_0, y_0, z_0)$

$\Rightarrow \left(\frac{2x_0}{a^2}\right)x_0 - \left(\frac{2y_0}{b^2}\right)y_0 - z_0 = \text{constant}$

$\Rightarrow \text{constant} = \frac{2x_0^2}{a^2} - \frac{2y_0^2}{b^2} - z_0$

$\Rightarrow \text{constant} = 2\left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}\right) - z_0$

FROM THE EQUATION OF THE SURFACE

$\Rightarrow \text{constant} = 2z_0 - z_0$

$\Rightarrow \text{constant} = z_0$

THUS WE NOW HAVE

$\Rightarrow \left(\frac{2x_0}{a^2}\right)x - \left(\frac{2y_0}{b^2}\right)y - z = z_0$

$\Rightarrow 2b^2x_0x - 2a^2y_0y - a^2b^2(z+z_0) = 0$

$\Rightarrow 2b^2x_0x - 2a^2y_0y = a^2b^2(z-z_0)$

**Question 23** **(***)**

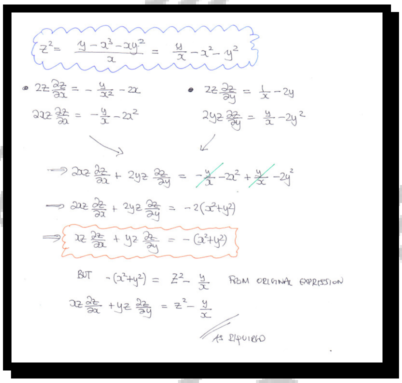
The function $z$ depends on $x$ and $y$ so that

$$z^2(x,y) = \frac{y - x^3 - xy^2}{x}, \quad x \neq 0 .$$

Show that

$$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = z^2 - \frac{y}{x} .$$

proof



$z^2 = \frac{y - x^3 - xy^2}{x} = \frac{y}{x} - x^2 - y^2$

- $2z\frac{\partial z}{\partial x} = -\frac{y}{x^2} - 2x$

  $2xz\frac{\partial z}{\partial x} = -\frac{y}{x} - 2x^2$

- $2z\frac{\partial z}{\partial y} = \frac{1}{x} - 2y$

  $2yz\frac{\partial z}{\partial y} = \frac{y}{x} - 2y^2$

$\Rightarrow 2xz\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial y} = -\frac{y}{x} - 2x^2 + \frac{y}{x} - 2y^2$

$\Rightarrow 2xz\frac{\partial z}{\partial x} + 2yz\frac{\partial z}{\partial y} = -2(x^2+y^2)$

$\Rightarrow xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -(x^2+y^2)$

BUT $-(x^2+y^2) = z^2 - \frac{y}{x}$ FROM ORIGINAL EXPRESSION

$xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = z^2 - \frac{y}{x}$

// AS REQUIRED

**Question 24 (***)**

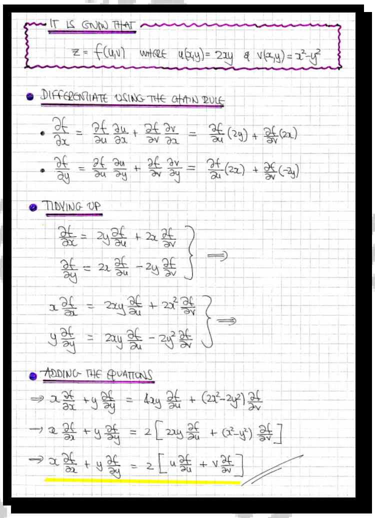
The function $f$ depends on $u$ and $v$ where

$$u = 2xy \qquad \text{and} \qquad v = x^2 - y^2.$$

Assuming $x \neq y$, $x \neq 0$ and $y \neq 0$, show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left[u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right].$$

[ ], [proof]


IT IS GIVEN THAT

$z = f(u,v)$ WHERE $u(x,y) = 2xy$ & $v(x,y) = x^2 - y^2$

- DIFFERENTIATE USING THE CHAIN RULE

- $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(2x)$
- $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-2y)$

- TIDYING UP

$\frac{\partial f}{\partial x} = 2y\frac{\partial f}{\partial u} + 2x\frac{\partial f}{\partial v}$, $\frac{\partial f}{\partial y} = 2x\frac{\partial f}{\partial u} - 2y\frac{\partial f}{\partial v}$ $\Longrightarrow$

$x\frac{\partial f}{\partial x} = 2xy\frac{\partial f}{\partial u} + 2x^2\frac{\partial f}{\partial v}$, $y\frac{\partial f}{\partial y} = 2xy\frac{\partial f}{\partial u} - 2y^2\frac{\partial f}{\partial v}$ $\Longrightarrow$

- ADDING THE EQUATIONS

$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 4xy\frac{\partial f}{\partial u} + (2x^2 - 2y^2)\frac{\partial f}{\partial v}$

$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left[2xy\frac{\partial f}{\partial u} + (x^2 - y^2)\frac{\partial f}{\partial v}\right]$

$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2\left[u\frac{\partial f}{\partial u} + v\frac{\partial f}{\partial v}\right]$

**Question 25** (***)

The functions $F$ and $G$ satisfy

$$G(x,y) \equiv F\left[u(x,y),\ v(x,y)\right],$$

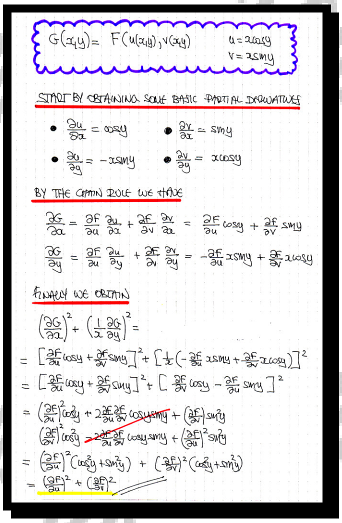
where $u$ and $v$ satisfy the following transformation equations.

$$u = x\cos y, \quad v = x\sin y.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial x}\right]^2 + \left[\frac{1}{x}\frac{\partial G}{\partial y}\right]^2 = \left[\frac{\partial F}{\partial u}\right]^2 + \left[\frac{\partial F}{\partial v}\right]^2.$$

, proof



$G(x,y) = F(u(x,y), v(x,y))$ $\quad u = x\cos y$, $v = x\sin y$

START BY OBTAINING SOME BASIC PARTIAL DERIVATIVES

- $\frac{\partial u}{\partial x} = \cos y$
- $\frac{\partial v}{\partial x} = \sin y$
- $\frac{\partial u}{\partial y} = -x\sin y$
- $\frac{\partial v}{\partial y} = x\cos y$

BY THE CHAIN RULE WE HAVE

$$\frac{\partial G}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y$$

$$\frac{\partial G}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial F}{\partial u}x\sin y + \frac{\partial F}{\partial v}x\cos y$$

FINALLY WE OBTAIN

$$\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{1}{x}\frac{\partial G}{\partial y}\right)^2 =$$

$$= \left[\frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y\right]^2 + \left[\frac{1}{x}\left(-\frac{\partial F}{\partial u}x\sin y + \frac{\partial F}{\partial v}x\cos y\right)\right]^2$$

$$= \left[\frac{\partial F}{\partial u}\cos y + \frac{\partial F}{\partial v}\sin y\right]^2 + \left[\frac{\partial F}{\partial v}\cos y - \frac{\partial F}{\partial u}\sin y\right]^2$$

$$= \left(\frac{\partial F}{\partial u}\right)^2\cos^2 y + 2\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}\cos y\sin y + \left(\frac{\partial F}{\partial v}\right)\sin^2 y$$
$$\left(\frac{\partial F}{\partial v}\right)^2\cos^2 y - 2\frac{\partial F}{\partial u}\frac{\partial F}{\partial v}\cos y\sin y + \left(\frac{\partial F}{\partial u}\right)^2\sin^2 y$$

$$= \left(\frac{\partial F}{\partial u}\right)^2(\cos^2 y + \sin^2 y) + \left(\frac{\partial F}{\partial v}\right)^2(\cos^2 y + \sin^2 y)$$

$$= \left(\frac{\partial F}{\partial u}\right)^2 + \left(\frac{\partial F}{\partial v}\right)^2$$

**Question 26** (***+)

The function $f$ is defined as

$$f(x, y, z) \equiv x^3 - 75x + 3z(y-1)^2 + z^3 .$$

The point $Q$ lies on $f$.

The derivatives at $Q$ in the directions $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}-\mathbf{k}$, are equal.

**a)** Show that $Q$ must lie on the surface of a sphere $S$.

The point $P(1,3,a)$ lies on $S$.

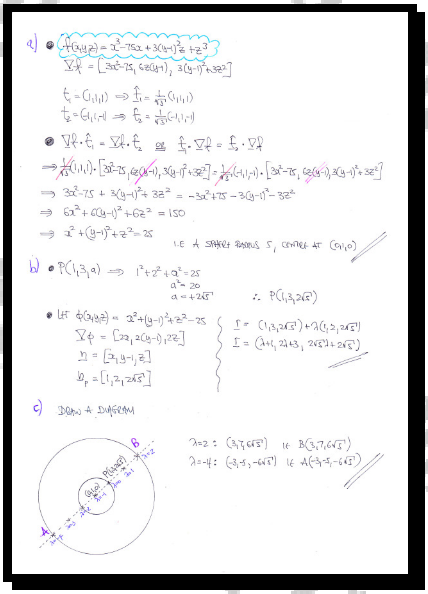
**b)** Find a vector equation of the normal line to $S$ at $P$.

A sphere $T$ is concentric to $S$ and has radius three times as large as that of $S$.

The normal line to $S$ at $P$ intersects the surface of $T$ at the points $A$ and $B$.

**c)** Determine the coordinates of $A$ and $B$.

$$\boxed{(x, y, z) = \left[\lambda+1, 2\lambda+3, 2(\lambda+1)\sqrt{5}\right]}, \quad \boxed{A\left(3,7,6\sqrt{5}\right) \;\; B\left(-3,-5,-6\sqrt{5}\right)}$$



**Question 27** (***+)

The function $z$ depends on $x$ and $y$ so that

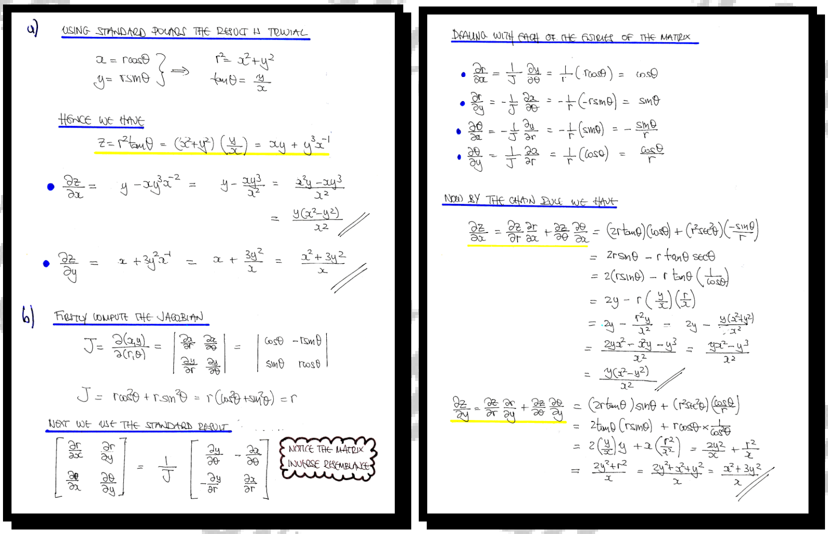
$$z = r^2 \tan\theta, \quad x = r\cos\theta \quad \text{and} \quad y = r\sin\theta .$$

**a)** Express $r$ and $\theta$ in terms of $x$ and $y$ and hence determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, in terms of $x$ and $y$.

*Give each of the answers as a single simplified fraction*

**b)** Verify the answer to part **(a)** by implicit differentiation using Jacobians

$$\frac{\partial z}{\partial x} = \frac{y\left(x^2 - y^2\right)}{x^2}, \quad \frac{\partial z}{\partial y} = \frac{x^2 + 3y^2}{x}$$



a) USING STANDARD POLARS THE RESULT IS TRIVIAL

$x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$ $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$

HENCE WE HAVE

$z = r^2\tan\theta = (x^2+y^2)\left(\frac{y}{x}\right) = xy + y^3x^{-1}$

- $\frac{\partial z}{\partial x} = y - xy^3x^{-2} = y - \frac{xy^3}{x^2} = \frac{x^2y - xy^3}{x^2} = \frac{y(x^2-y^2)}{x^2}$
- $\frac{\partial z}{\partial y} = x + 3y^2x^{-1} = x + \frac{3y^2}{x} = \frac{x^2+3y^2}{x}$

b) FIRSTLY COMPUTE THE JACOBIAN

$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$J = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$

NEXT WE USE THE STANDARD RESULT

$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} \frac{\partial y}{\partial \theta} & -\frac{\partial x}{\partial \theta} \\ -\frac{\partial y}{\partial r} & \frac{\partial x}{\partial r} \end{bmatrix}$

NOTICE THE MATRIX INVERSE RESEMBLANCE

DEALING WITH EACH OF THE ENTRIES OF THE MATRIX

- $\frac{\partial r}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \theta} = \frac{1}{r}(r\cos\theta) = \cos\theta$
- $\frac{\partial r}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \theta} = -\frac{1}{r}(-r\sin\theta) = \sin\theta$
- $\frac{\partial \theta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial r} = -\frac{1}{r}(\sin\theta) = -\frac{\sin\theta}{r}$
- $\frac{\partial \theta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial r} = \frac{1}{r}(\cos\theta) = \frac{\cos\theta}{r}$

NOW BY THE CHAIN RULE WE HAVE

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial x} = (2r\tan\theta)(\cos\theta) + (r^2\sec^2\theta)\left(-\frac{\sin\theta}{r}\right)$

$= 2r\sin\theta - r\tan\theta\sec\theta$

$= 2(r\sin\theta) - r\tan\theta\left(\frac{1}{\cos\theta}\right)$

$= 2y - r\left(\frac{y}{x}\right)\left(\frac{r}{x}\right)$

$= 2y - \frac{r^2y}{x^2} = 2y - \frac{y(x^2+y^2)}{x^2}$

$= \frac{2yx^2 - x^2y - y^3}{x^2} = \frac{yx^2 - y^3}{x^2}$

$= \frac{y(x^2-y^2)}{x^2}$

$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = (2r\tan\theta)\sin\theta + (r^2\sec^2\theta)\left(\frac{\cos\theta}{r}\right)$

$= 2\tan\theta(r\sin\theta) + r\cos\theta \times \frac{1}{\cos^2\theta}$

$= 2\left(\frac{y}{x}\right)y + x\left(\frac{r^2}{x^2}\right) = \frac{2y^2}{x} + \frac{r^2}{x}$

$= \frac{2y^2+r^2}{x} = \frac{2y^2+x^2+y^2}{x} = \frac{x^2+3y^2}{x}$

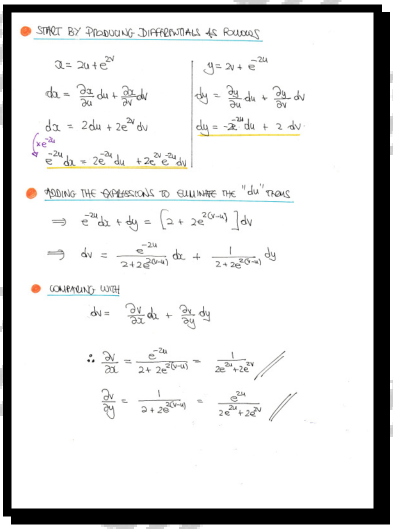
**Question 28** (***+)

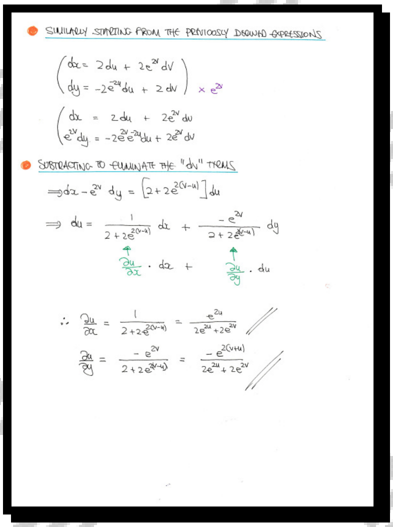
The function $\varphi$ depends on $u$ and $v$ so that

$$x = 2u + e^{2v} \quad \text{and} \quad y = 2v + e^{-2u}.$$

Without using standard results involving Jacobians, determine simplified expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, in terms of $u$ and $v$.

$\boxed{\phantom{xx}}$, $\boxed{\frac{\partial u}{\partial x} = \frac{1}{2+2e^{2(v-u)}}}$, $\boxed{\frac{\partial u}{\partial y} = -\frac{e^{2v}}{2+2e^{2(v-u)}}}$, $\boxed{\frac{\partial v}{\partial x} = \frac{e^{-2u}}{2+2e^{2(v-u)}}}$, $\boxed{\frac{\partial v}{\partial y} = \frac{1}{2+2e^{2(v-u)}}}$





**Question 29 (***+)**

A hill is modelled by the equation

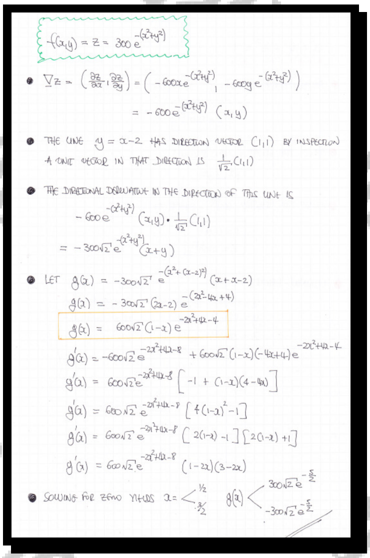
$$f(x,y) = 300\,\mathrm{e}^{-\left(x^2+y^2\right)}, \quad x \in \mathbb{R}, \ y \in \mathbb{R}.$$

A railway runs along the straight line with equation

$$y = x - 2.$$

Determine the steepest slope that the train needs to climb.

$$\boxed{\pm\, 300\sqrt{2}\,\mathrm{e}^{-\frac{5}{2}}}$$


$f(x,y) = z = 300\,e^{-(x^2+y^2)}$

- $\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(-600x\,e^{-(x^2+y^2)}, -600y\,e^{-(x^2+y^2)}\right)$
  $= -600\,e^{-(x^2+y^2)}\,(x,y)$
- THE LINE $y = x-2$ HAS DIRECTION VECTOR $(1,1)$ BY INSPECTION
  A UNIT VECTOR IN THAT DIRECTION IS $\frac{1}{\sqrt{2}}(1,1)$
- THE DIRECTIONAL DERIVATIVE IN THE DIRECTION OF THIS LINE IS
  $-600\,e^{-(x^2+y^2)}(x,y) \cdot \frac{1}{\sqrt{2}}(1,1)$
  $= -300\sqrt{2}\,e^{-(x^2+y^2)}(x+y)$
- LET $g(x) = -300\sqrt{2}\,e^{-(x^2+(x-2)^2)}(x+x-2)$
  $g(x) = -300\sqrt{2}\,(2x-2)\,e^{-(2x^2-4x+4)}$
  $g(x) = 600\sqrt{2}\,(1-x)\,e^{-2x^2+4x-4}$
  $g'(x) = -600\sqrt{2}\,e^{-2x^2+4x-8} + 600\sqrt{2}\,(1-x)(-4x+4)\,e^{-2x^2+4x-4}$
  $g'(x) = 600\sqrt{2}\,e^{-2x^2+4x-8}\left[-1 + (1-x)(4-4x)\right]$
  $g'(x) = 600\sqrt{2}\,e^{-2x^2+4x-8}\left[4(1-x)^2-1\right]$
  $g'(x) = 600\sqrt{2}\,e^{-2x^2+4x-8}\left[2(1-x)-1\right]\left[2(1-x)+1\right]$
  $g'(x) = 600\sqrt{2}\,e^{-2x^2+4x-8}(1-2x)(3-2x)$
- SOLVING FOR ZERO YIELDS $x = \frac{1}{2}, \frac{3}{2}$ $\quad g(x) = 300\sqrt{2}\,e^{-\frac{5}{2}}, \ -300\sqrt{2}\,e^{-\frac{5}{2}}$

**Question 30** (***+)

The functions $F$ and $G$ satisfy

$$G(u,v) \equiv F\left[x(u,v),\ y(u,v)\right],$$

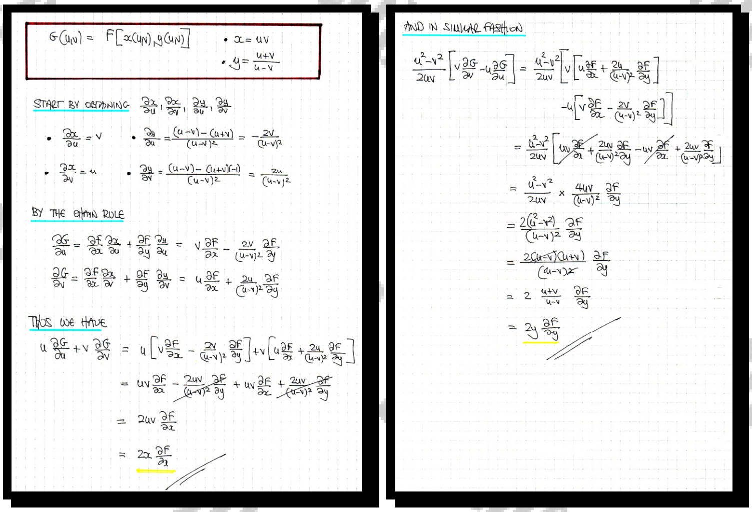
where $x$ and $y$ satisfy the following transformation equations.

$$x = uv, \quad y = \frac{u+v}{u-v}.$$

Use the chain rule for partial derivatives to show that

$$u\frac{\partial G}{\partial u} + v\frac{\partial G}{\partial v} = 2x\frac{\partial F}{\partial x} \quad \text{and} \quad \frac{u^2-v^2}{2uv}\left[v\frac{\partial G}{\partial v} - u\frac{\partial G}{\partial u}\right] = 2y\frac{\partial F}{\partial y}.$$

[ ], [proof]


$G(u,v) = F[x(u,v), y(u,v)]$ • $x = uv$ • $y = \frac{u+v}{u-v}$

START BY OBTAINING $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$

• $\frac{\partial x}{\partial u} = v$ • $\frac{\partial y}{\partial u} = \frac{(u-v)-(u+v)}{(u-v)^2} = -\frac{2v}{(u-v)^2}$

• $\frac{\partial x}{\partial v} = u$ • $\frac{\partial y}{\partial v} = \frac{(u-v)-(u+v)(-1)}{(u-v)^2} = \frac{2u}{(u-v)^2}$

BY THE CHAIN RULE

$\frac{\partial G}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} = v\frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2}\frac{\partial F}{\partial y}$

$\frac{\partial G}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} = u\frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2}\frac{\partial F}{\partial y}$

THUS WE HAVE

$u\frac{\partial G}{\partial u} + v\frac{\partial G}{\partial v} = u\left[v\frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2}\frac{\partial F}{\partial y}\right] + v\left[u\frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2}\frac{\partial F}{\partial y}\right]$

$= uv\frac{\partial F}{\partial x} - \frac{2uv}{(u-v)^2}\frac{\partial F}{\partial y} + uv\frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2}\frac{\partial F}{\partial y}$

$= 2uv\frac{\partial F}{\partial x}$

$= 2x\frac{\partial F}{\partial x}$

AND IN SIMILAR FASHION

$\frac{u^2-v^2}{2uv}\left[v\frac{\partial G}{\partial v} - u\frac{\partial G}{\partial u}\right] = \frac{u^2-v^2}{2uv}\left[v\left[u\frac{\partial F}{\partial x} + \frac{2u}{(u-v)^2}\frac{\partial F}{\partial y}\right] - u\left[v\frac{\partial F}{\partial x} - \frac{2v}{(u-v)^2}\frac{\partial F}{\partial y}\right]\right]$

$= \frac{u^2-v^2}{2uv}\left[uv\frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2}\frac{\partial F}{\partial y} - uv\frac{\partial F}{\partial x} + \frac{2uv}{(u-v)^2}\frac{\partial F}{\partial y}\right]$

$= \frac{u^2-v^2}{2uv} \times \frac{4uv}{(u-v)^2}\frac{\partial F}{\partial y}$

$= \frac{2(u^2-v^2)}{(u-v)^2}\frac{\partial F}{\partial y}$

$= \frac{2(u-v)(u+v)}{(u-v)^2}\frac{\partial F}{\partial y}$

$= 2\,\frac{u+v}{u-v}\frac{\partial F}{\partial y}$

$= 2y\frac{\partial F}{\partial y}$

**Question 31** (****)

The function $z$ depends on $x$ and $y$ so that

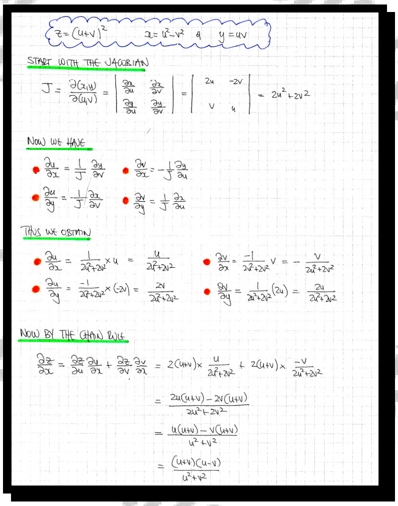
$$z = (u+v)^2, \quad x = u^2 - v^2 \quad \text{and} \quad y = uv.$$

Show clearly that …

**i.** … $\dfrac{\partial z}{\partial x} = \dfrac{x}{z-2y}$.

**ii.** … $\dfrac{\partial z}{\partial y} = \dfrac{2z}{z-2y}$.

proof



$z = (u+v)^2 \qquad x = u^2 - v^2 \quad \& \quad y = uv$

START WITH THE JACOBIAN

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$$

NOW WE HAVE

- $\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v}$ • $\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u}$
- $\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v}$ • $\frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}$

THUS WE OBTAIN

- $\frac{\partial u}{\partial x} = \frac{1}{2u^2+2v^2} \times u = \frac{u}{2u^2+2v^2}$ • $\frac{\partial v}{\partial x} = \frac{-1}{2u^2+2v^2} v = -\frac{v}{2u^2+2v^2}$
- $\frac{\partial u}{\partial y} = \frac{-1}{2u^2+2v^2} \times (-2v) = \frac{2v}{2u^2+2v^2}$ • $\frac{\partial v}{\partial y} = \frac{1}{2u^2+2v^2}(2u) = \frac{2u}{2u^2+2v^2}$

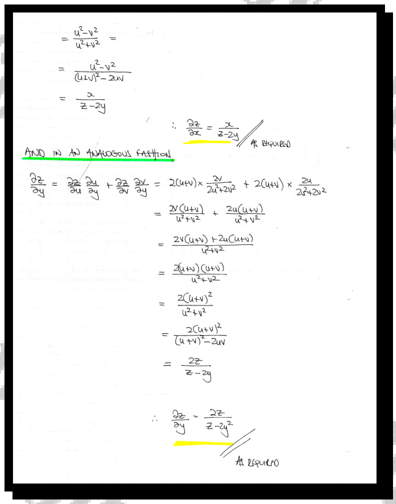
NOW BY THE CHAIN RULE

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 2(u+v) \times \frac{u}{2u^2+2v^2} + 2(u+v) \times \frac{-v}{2u^2+2v^2}$$

$$= \frac{2u(u+v) - 2v(u+v)}{2u^2+2v^2}$$

$$= \frac{u(u+v) - v(u+v)}{u^2+v^2}$$

$$= \frac{(u+v)(u-v)}{u^2+v^2}$$



$$= \frac{u^2-v^2}{u^2+v^2} =$$

$$= \frac{u^2-v^2}{(u+v)^2 - 2uv}$$

$$= \frac{x}{z-2y}$$

$\therefore \frac{\partial z}{\partial x} = \frac{x}{z-2y}$ // AS REQUIRED

AND IN AN ANALOGOUS FASHION

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 2(u+v) \times \frac{2v}{2u^2+2v^2} + 2(u+v) \times \frac{2u}{2u^2+2v^2}$$

$$= \frac{2v(u+v)}{u^2+v^2} + \frac{2u(u+v)}{u^2+v^2}$$

$$= \frac{2v(u+v) + 2u(u+v)}{u^2+v^2}$$

$$= \frac{2(u+v)(u+v)}{u^2+v^2}$$

$$= \frac{2(u+v)^2}{u^2+v^2}$$

$$= \frac{2(u+v)^2}{(u+v)^2 - 2uv}$$

$$= \frac{2z}{z-2y}$$

$\therefore \frac{\partial z}{\partial y} = \frac{2z}{z-2y^2}$ // AS REQUIRED

**Question 32** (****)

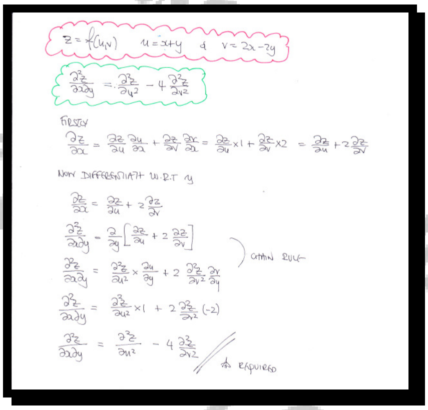
The function $z$ depends on $x$ and $y$ so that

$$z = f(u,v), \quad u = x + y \quad \text{and} \quad v = 2x - 2y.$$

Show clearly that

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4\frac{\partial^2 z}{\partial v^2}.$$

proof


$z = f(u,v)$ $u = x+y$ & $v = 2x-2y$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4\frac{\partial^2 z}{\partial v^2}$

FIRSTLY

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \times 1 + \frac{\partial z}{\partial v} \times 2 = \frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$

NOW DIFFERENTIATE W.R.T $y$

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial u} + 2\frac{\partial z}{\partial v}\right]$ CHAIN RULE

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} \times \frac{\partial u}{\partial y} + 2\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} \times 1 + 2\frac{\partial^2 z}{\partial v^2}(-2)$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2} - 4\frac{\partial^2 z}{\partial v^2}$ // AS REQUIRED

**Question 33** **(****)**

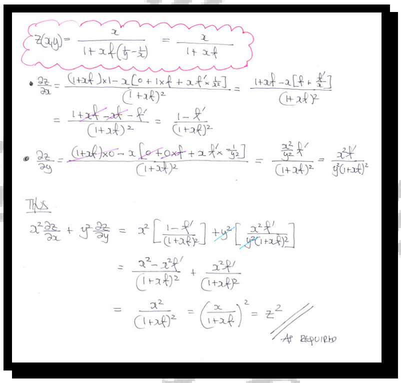
The function $z$ depends on $x$ and $y$ so that

$$z = \frac{x}{1 + x\,f} \qquad \text{where} \qquad f = f\left(\frac{1}{y} - \frac{1}{x}\right).$$

Show clearly that

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2 .$$

proof



**Question 34** (****)

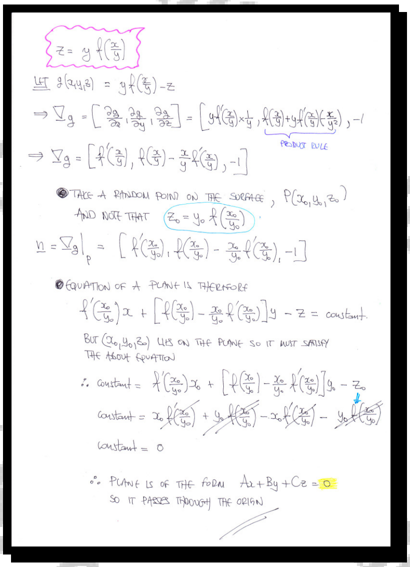
The surface $S$ has equation

$$z = y\,f\left(\frac{x}{y}\right),$$

where the function $f\left(\frac{x}{y}\right)$ is differentiable.

Show that the tangent plane at any point on $S$ passes through the origin $O$

proof



$z = y f\left(\frac{x}{y}\right)$

LET $g(x,y,z) = y f\left(\frac{x}{y}\right) - z$

$\Rightarrow \nabla g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right] = \left[y f'\left(\frac{x}{y}\right)\times\frac{1}{y}, f\left(\frac{x}{y}\right) + y f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right), -1\right]$ PRODUCT RULE

$\Rightarrow \nabla g = \left[f'\left(\frac{x}{y}\right), f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right), -1\right]$

- TAKE A RANDOM POINT ON THE SURFACE, $P(x_0, y_0, z_0)$ AND NOTE THAT $z_0 = y_0 f\left(\frac{x_0}{y_0}\right)$

$\underline{n} = \nabla g\big|_P = \left[f'\left(\frac{x_0}{y_0}\right), f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right), -1\right]$

- EQUATION OF A PLANE IS THEREFORE

$f'\left(\frac{x_0}{y_0}\right)x + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right]y - z = \text{constant}$

BUT $(x_0, y_0, z_0)$ LIES ON THE PLANE SO IT MUST SATISFY THE ABOVE EQUATION

$\therefore \text{constant} = f'\left(\frac{x_0}{y_0}\right)x_0 + \left[f\left(\frac{x_0}{y_0}\right) - \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right]y_0 - z_0$

$\text{constant} = x_0 f'\left(\frac{x_0}{y_0}\right) + y_0 f\left(\frac{x_0}{y_0}\right) - x_0 f'\left(\frac{x_0}{y_0}\right) - y_0 f\left(\frac{x_0}{y_0}\right)$

$\text{constant} = 0$

$\therefore$ PLANE IS OF THE FORM $Ax + By + Cz = 0$ SO IT PASSES THROUGH THE ORIGIN

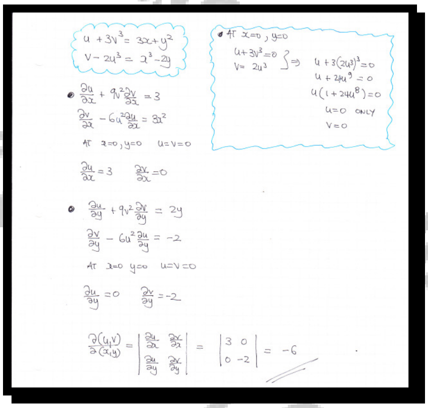
**Question 35** (****)

The functions $u = u(x, y)$ and $v = v(x, y)$ satisfy

$$u + 3v^3 = 3x + y^2 \quad \text{and} \quad v - 2u^3 = x^3 - 2y .$$

Determine the value of $\frac{\partial(u,v)}{\partial(x,y)}$ at $(x,y) = (0,0)$.

$$\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{(0,0)} = -6$$



**Question 36** (****)

A surface $S$ has equation $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + 2xy - 4x + 2y^2 + 2yz - 8y - z^2 + 4z.$$

**a)** Show that there is no point on $S$ where the normal to $S$ is parallel to the $z$ axis and hence state the geometric significance of this result with reference to the stationary points of $S$.

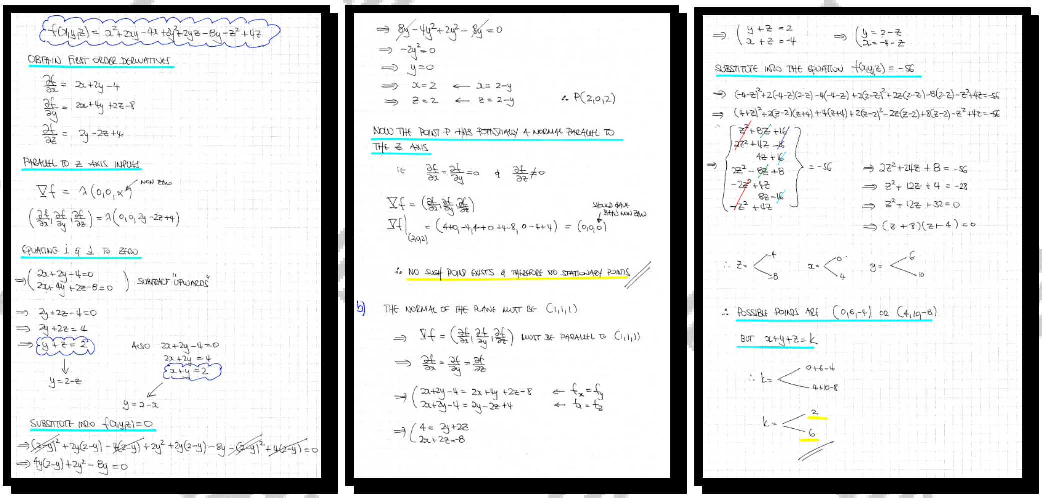
$S$ is translated to give a new surface $T$ with equation

$$f(x, y, z) = -56.$$

The plane with equation $x + y + z = k$, where $k$ is a constant, is a tangent plane to $T$.

**b)** Determine the two possible values of $k$.

[ ] , $k = 2 \cup k = 6$



**Question 37 (****)**

A surface $S$ has equation $f(x, y, z) = 0$, where

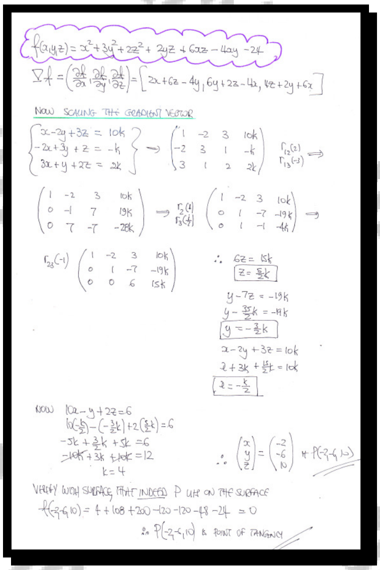
$$f(x, y, z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24.$$

Show that the plane with equation

$$10x - y + 2z = 6,$$

is a tangent plane to $S$, and find the coordinates of the point of tangency.

$$\boxed{(-2, -6, 10)}$$



$f(x,y,z) = x^2 + 3y^2 + 2z^2 + 2yz + 6xz - 4xy - 24$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = [2x + 6z - 4y, 6y + 2z - 4x, 4z + 2y + 6x]$

NOW SCALING THE GRADIENT VECTOR

$\begin{cases} x - 2y + 3z = 10k \\ -2x + 3y + z = -k \\ 3x + y + 2z = 2k \end{cases} \Rightarrow \begin{pmatrix} 1 & -2 & 3 & 10k \\ -2 & 3 & 1 & -k \\ 3 & 1 & 2 & 2k \end{pmatrix} \begin{matrix} r_{12}(2) \\ r_{13}(-3) \end{matrix} \Rightarrow$

$\begin{pmatrix} 1 & -2 & 3 & 10k \\ 0 & -1 & 7 & 19k \\ 0 & 7 & -7 & -28k \end{pmatrix} \Rightarrow \begin{matrix} r_2(-1) \\ r_3(\frac{1}{7}) \end{matrix} \begin{pmatrix} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -19k \\ 0 & 1 & -1 & -4k \end{pmatrix} \Rightarrow$

$r_{23}(-1) \begin{pmatrix} 1 & -2 & 3 & 10k \\ 0 & 1 & -7 & -19k \\ 0 & 0 & 6 & 15k \end{pmatrix}$

$\therefore 6z = 15k$, $\boxed{z = \frac{5}{2}k}$

$y - 7z = -19k$

$y - \frac{35}{2}k = -19k$

$\boxed{y = -\frac{3}{2}k}$

$x - 2y + 3z = 10k$

$x + 3k + \frac{15}{2}k = 10k$

$\boxed{x = -\frac{k}{2}}$

NOW $10x - y + 2z = 6$

$10\left(-\frac{k}{2}\right) - \left(-\frac{3}{2}k\right) + 2\left(\frac{5}{2}k\right) = 6$

$-5k + \frac{3}{2}k + 5k = 6$

$-10k + 3k + 10k = 12$

$k = 4$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -6 \\ 10 \end{pmatrix}$ & $P(-2,-6,10)$

VERIFY WITH SURFACE THAT INDEED P LIES ON THE SURFACE

$f(-2,-6,10) = 4 + 108 + 200 - 120 - 120 - 48 - 24 = 0$

$\therefore P(-2,-6,10)$ IS POINT OF TANGENCY

**Question 38** (****)

It is given that $g$ is a twice differentiable function of one variable, with domain all real numbers.

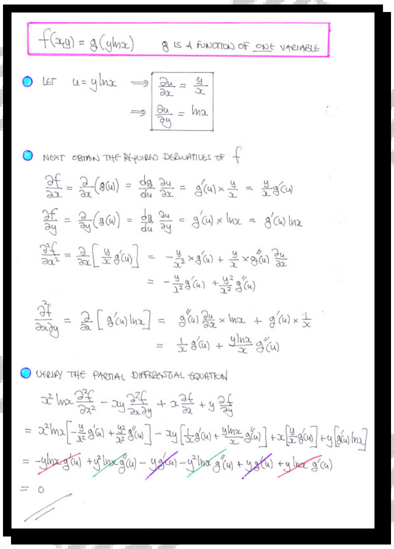
It is further given that for $x > 0$

$$f(x, y) = g(y \ln x).$$

Show that

$$x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy \frac{\partial^2 f}{\partial x \partial y} + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 0.$$

proof



$f(x,y) = g(y\ln x)$ g IS A FUNCTION OF ONE VARIABLE

LET $u = y\ln x \implies \frac{\partial u}{\partial x} = \frac{y}{x}$, $\implies \frac{\partial u}{\partial y} = \ln x$

NEXT OBTAIN THE REQUIRED DERIVATIVES OF $f$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(g(u)) = \frac{dg}{du}\frac{\partial u}{\partial x} = g'(u) \times \frac{y}{x} = \frac{y}{x}g'(u)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(g(u)) = \frac{dg}{du}\frac{\partial u}{\partial y} = g'(u) \times \ln x = g'(u)\ln x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{y}{x}g'(u)\right] = -\frac{y}{x^2} \times g'(u) + \frac{y}{x} \times g''(u)\frac{\partial u}{\partial x} = -\frac{y}{x^2}g'(u) + \frac{y^2}{x^2}g''(u)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[g'(u)\ln x\right] = g''(u)\frac{\partial u}{\partial x} \times \ln x + g'(u) \times \frac{1}{x} = \frac{1}{x}g'(u) + \frac{y\ln x}{x}g''(u)$$

VERIFY THE PARTIAL DIFFERENTIAL EQUATION

$$x^2 \ln x \frac{\partial^2 f}{\partial x^2} - xy\frac{\partial^2 f}{\partial x \partial y} + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$

$$= x^2\ln x\left[-\frac{y}{x^2}g'(u) + \frac{y^2}{x^2}g''(u)\right] - xy\left[\frac{1}{x}g'(u) + \frac{y\ln x}{x}g''(u)\right] + x\left[\frac{y}{x}g'(u)\right] + y\left[g'(u)\ln x\right]$$

$$= -y\ln x\, g'(u) + y^2\ln x\, g''(u) - yg'(u) - y^2\ln x\, g''(u) + yg'(u) + y\ln x\, g'(u)$$

$$= 0$$

**Question 39 (****)**

The function $w$ depends on $x$ and $y$ so that

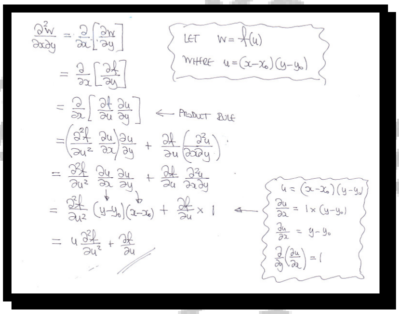
$$w = f(u), \quad \text{and} \quad u = (x - x_0)(y - y_0),$$

where $x_0$ and $y_0$ are constants.

Show clearly that

$$\frac{\partial^2 w}{\partial x \partial y} = u\frac{\partial^2 f}{\partial u^2} + \frac{\partial f}{\partial u}.$$

proof



**Question 40** (****)

The functions $f$ and $G$ satisfy

$$G(r,\theta,\varphi) \equiv f\left[x(r,\theta,\varphi),\ y(r,\theta,\varphi),\ z(r,\theta,\varphi)\right],$$

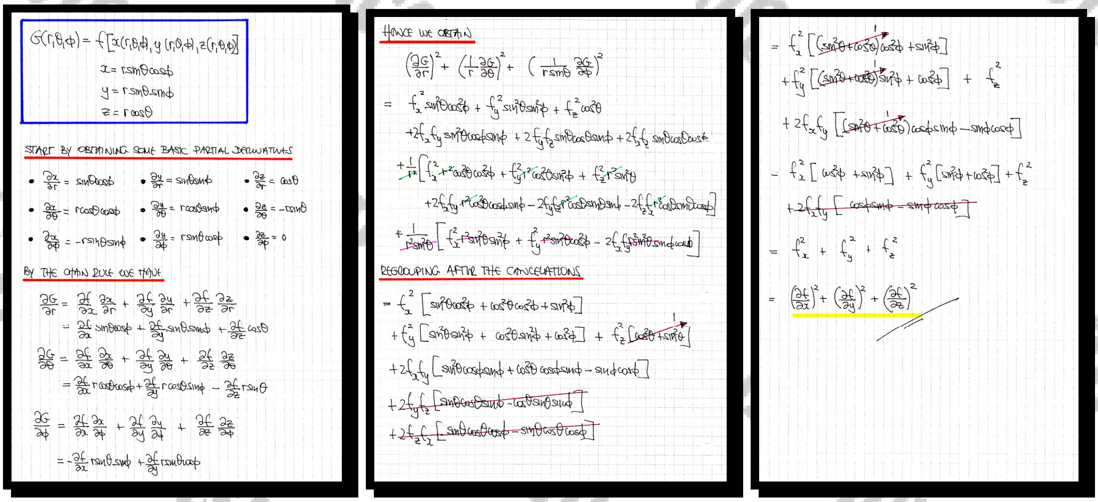
where $x$, $y$ and $z$ satisfy the standard Spherical Polar Coordinates transformation relationships

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

Use the chain rule for partial derivatives to show that

$$\left[\frac{\partial G}{\partial r}\right]^2 + \left[\frac{1}{r}\frac{\partial G}{\partial \theta}\right]^2 + \left[\frac{1}{r\sin\theta}\frac{\partial G}{\partial \varphi}\right]^2 = \left[\frac{\partial f}{\partial x}\right]^2 + \left[\frac{\partial f}{\partial y}\right]^2 + \left[\frac{\partial f}{\partial z}\right]^2.$$

[ ], proof



**Question 41** (****+)

It is given that

$$z(x,y) = f(u,v),$$

so that

$$u = x^3 + y^3 \quad \text{and} \quad v = \frac{y}{x}.$$

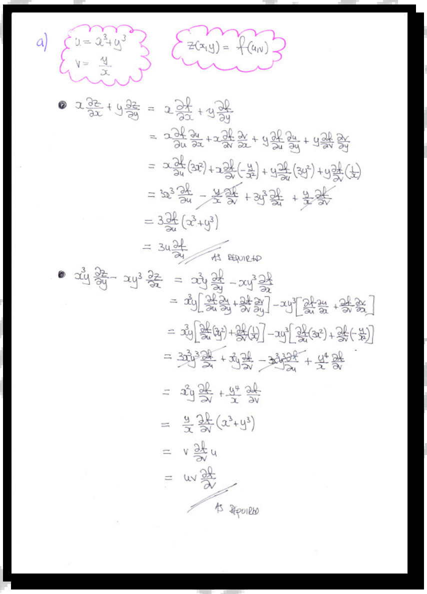
**a)** Use the chain rule to show that

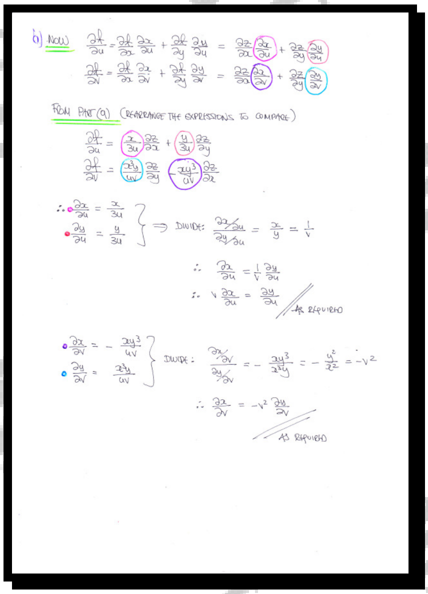
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3u\frac{\partial f}{\partial u} \quad \text{and} \quad yx^3\frac{\partial z}{\partial y} - xy^3\frac{\partial z}{\partial x} = uv\frac{\partial f}{\partial v}.$$

**b)** Hence show further that

$$v\frac{\partial x}{\partial u} = \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial x}{\partial v} = -v^2\frac{\partial y}{\partial v}.$$

proof





**Question 42** (****+)

It is given that $f$ and $g$ are differentiable functions of one variable, with domain all real numbers.

It is further given that for $x > 0$

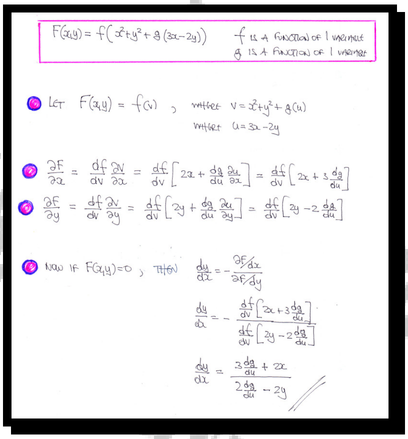
$$F(x, y) = f\left[x^2 + y^2 + g(3x - 2y)\right].$$

If the function $y = y(x)$ is a rearrangement of $F(x, y) = 0$, show that

$$\frac{dy}{dx} = \frac{3\frac{dg}{du} + 2x}{2\frac{dg}{du} - 2y},$$

where $u = 3x - 2y$

proof



$F(x,y) = f\left(x^2+y^2+g(3x-2y)\right)$ f IS A FUNCTION OF 1 VARIABLE, g IS A FUNCTION OF 1 VARIABLE

- LET $F(x,y) = f(v)$, WHERE $v = x^2+y^2+g(u)$, WHERE $u = 3x-2y$
- $\frac{\partial F}{\partial x} = \frac{df}{dv}\frac{\partial v}{\partial x} = \frac{df}{dv}\left[2x + \frac{dg}{du}\frac{\partial u}{\partial x}\right] = \frac{df}{dv}\left[2x + 3\frac{dg}{du}\right]$
- $\frac{\partial F}{\partial y} = \frac{df}{dv}\frac{\partial v}{\partial y} = \frac{df}{dv}\left[2y + \frac{dg}{du}\frac{\partial u}{\partial y}\right] = \frac{df}{dv}\left[2y - 2\frac{dg}{du}\right]$
- NOW IF $F(x,y)=0$, THEN $\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}$

$$\frac{dy}{dx} = -\frac{\frac{df}{dv}\left[2x+3\frac{dg}{du}\right]}{\frac{df}{dv}\left[2y-2\frac{dg}{du}\right]}$$

$$\frac{dy}{dx} = \frac{3\frac{dg}{du} + 2x}{2\frac{dg}{du} - 2y}$$

**Question 43** (****+)

The surface $S$ has Cartesian equation

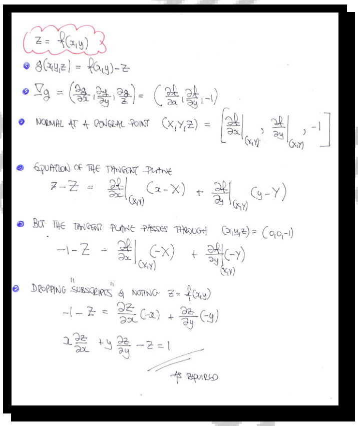
$$z = f(x, y).$$

The tangent plane at any point on $S$ passes through the point $(0,0,-1)$.

Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 1.$$

proof


$z = f(x,y)$

- $g(x,y,z) = f(x,y) - z$
- $\underline{\nabla} g = \left(\frac{\partial g}{\partial x}, \frac{\partial y}{\partial y}, \frac{\partial g}{z}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$
- NORMAL AT A GENERAL POINT $(X,Y,Z) = \left[\frac{\partial f}{\partial x}\Big|_{(X,Y)}, \frac{\partial f}{\partial y}\Big|_{(X,Y)}, -1\right]$
- EQUATION OF THE TANGENT PLANE
  $z - Z = \frac{\partial f}{\partial x}\Big|_{(X,Y)}(x - X) + \frac{\partial f}{\partial y}\Big|_{(X,Y)}(y - Y)$
- BUT THE TANGENT PLANE PASSES THROUGH $(x,y,z) = (0,0,-1)$
  $-1 - Z = \frac{\partial f}{\partial x}\Big|_{(X,Y)}(-X) + \frac{\partial f}{\partial y}\Big|_{(X,Y)}(-Y)$
- DROPPING "SUBSCRIPTS" & NOTING $z = f(x,y)$
  $-1 - z = \frac{\partial z}{\partial x}(-x) + \frac{\partial z}{\partial y}(-y)$
  $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 1$

AS REQUIRED

**Question 44** (*****)

It is given that the function $f$ depends on $x$ and $y$, and the function $g$ depends on $u$ and $v$, so that

$$f(x,y) = g(u,v), \quad u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

**a)** Show that

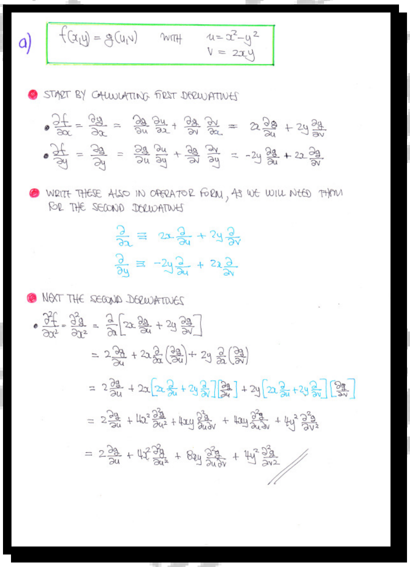
$$\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial g}{\partial u} + 4x^2\frac{\partial^2 g}{\partial u^2} + 8xy\frac{\partial^2 g}{\partial u \partial v} + 4y^2\frac{\partial^2 g}{\partial v^2},$$

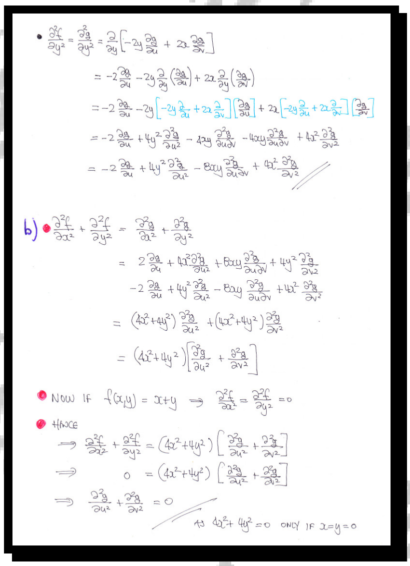
and find a similar expression for $\frac{\partial^2 f}{\partial y^2}$.

**b)** Deduce that if $f(x,y) = x + y$

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

proof





**Question 45** (*****)

The function $z$ depends on $x$ and $y$ so that

$$z = f(u,v), \quad u = x - 2\sqrt{y} \quad \text{and} \quad v = x + 2\sqrt{y}.$$

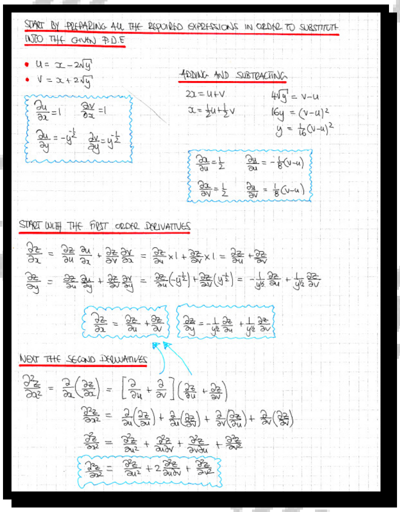
Show that the partial differential equation

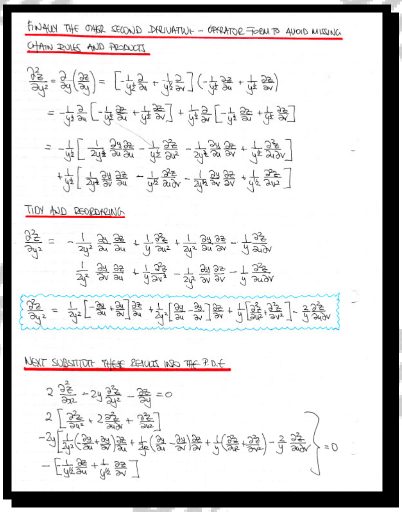
$$2\frac{\partial^2 z}{\partial x^2} - 2y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0,$$

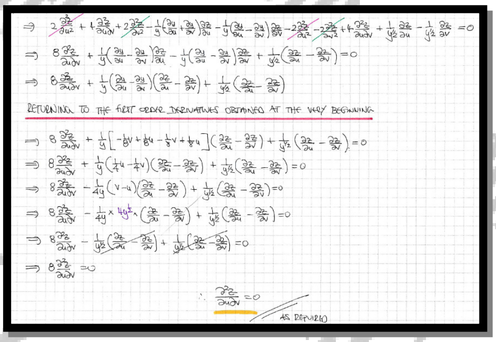
can be simplified to

$$\frac{\partial^2 z}{\partial u \partial v} = 0.$$

[ ], proof







**Question 46** (*****)

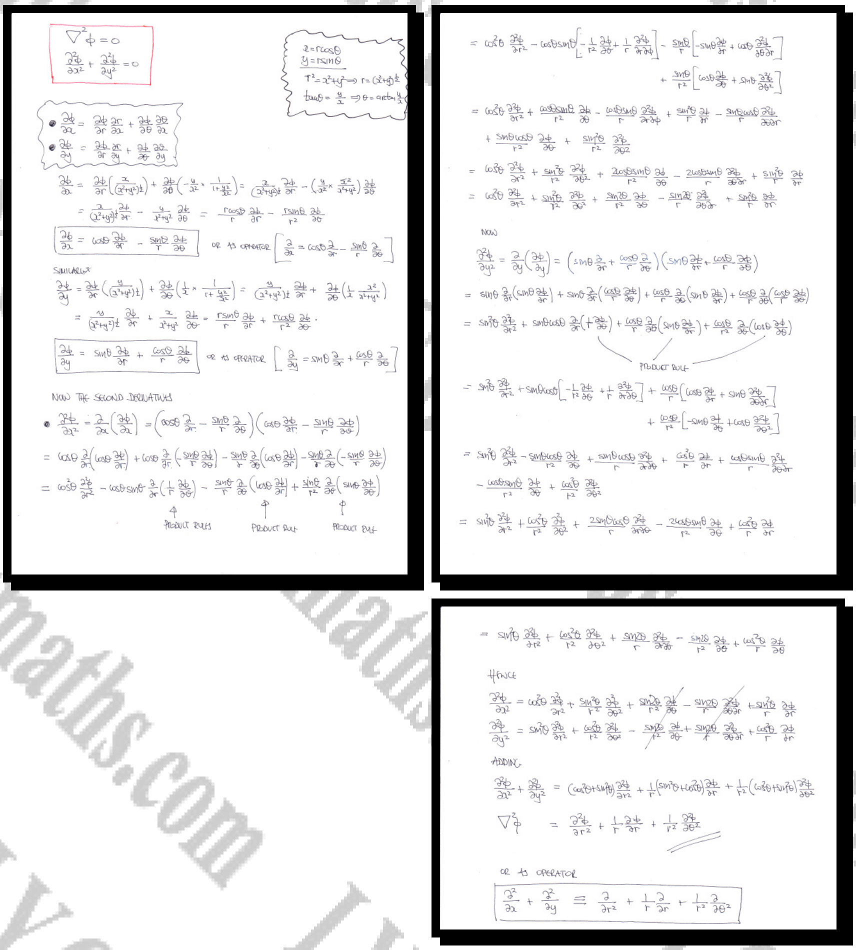
$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 .$$

The above partial differential equation is Laplace's equation in a two dimensional Cartesian system of coordinates.

Show clearly that Laplace's equation in the standard two dimensional Polar system of coordinates is given by

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0 .$$

proof



**Question 47** (*****)

It is given that for $\varphi = \varphi(x, y)$ and $\psi = \psi(x, y)$

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Show that the above pair of coupled partial differential equations transform in plane polar coordinates to

$$\frac{\partial \varphi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad \frac{\partial \psi}{\partial r} = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta}.$$

proof