PARTIAL DIFFERENTIATION APPLICATIONS

STATIONARY POINTS

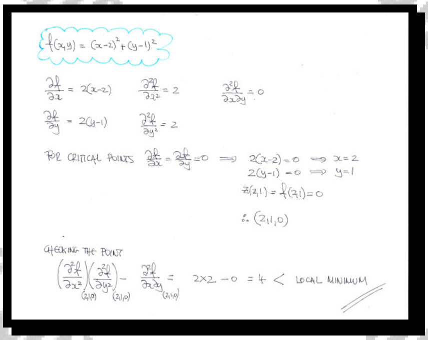
Question 1 (**)

A surface has Cartesian equation $z = f(x,y)$, given by

$$f(x,y) = (x-2)^2 + (y-1)^2 .$$

Investigate the critical points of f.

$\boxed{\text{local minimum at } (2,1,0)}$



$f(x,y) = (x-2)^2 + (y-1)^2$

$\frac{\partial f}{\partial x} = 2(x-2) \qquad \frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$

$\frac{\partial f}{\partial y} = 2(y-1) \qquad \frac{\partial^2 f}{\partial y^2} = 2$

FOR CRITICAL POINTS $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \implies 2(x-2) = 0 \implies x = 2$

$2(y-1) = 0 \implies y = 1$

$z(2,1) = f(2,1) = 0$

$\therefore (2,1,0)$

CHECKING THE POINT

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(2,1,0)} \left(\frac{\partial^2 f}{\partial y^2}\right)_{(2,1,0)} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(2,1,0)} = 2 \times 2 - 0 = 4 <$ LOCAL MINIMUM

Question 2 (**)

$$z = 5xy - 6x^2 - y^2 + 7x - 2y.$$

Investigate the critical points of z.

$\boxed{\text{saddle point at } (-4,-11,-3)}$

$z = 5xy - 6x^2 - y^2 + 7x - 2y$

- OBTAIN FIRST DERIVATIVES

$\frac{\partial z}{\partial x} = 5y - 12x + 7$

$\frac{\partial z}{\partial y} = 5x - 2y - 2$

- SECOND DERIVATIVES (TO CHECK)

$\frac{\partial^2 z}{\partial x^2} = -12$

$\frac{\partial^2 z}{\partial y^2} = -2$ & $\frac{\partial^2 z}{\partial x \partial y} = 5$

(MAXIMUM OR SADDLE)

STATIONARY $\Rightarrow$ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$\left.\begin{array}{l} 5y - 12x + 7 = 0 \\ 5x - 2y - 2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -12x + 5y = -7 \\ 5x - 2y = 2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} -24x + 10y = -14 \\ 25x - 10y = 10 \end{array}\right\} \Rightarrow$ ADD

$\boxed{x = -4}$

$\therefore$ $5y - 12(-4) + 7 = 0$

$5y = -55$

$\boxed{y = -11}$

& $z = 5(-4)(-11) - 6(-4)^2 - (-11)^2 + 7(-4) - 2(-11)$

$z = 220 - 96 - 121 - 28 + 22$

$\boxed{z = -3}$

$\therefore$ STATIONARY AT $(-4,-11,-3)$

$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-12)(-2) - 5^2 = 24 - 25 = -1$ $\therefore$ SADDLE POINT AT $(-4,-11,3)$

Question 3 (**)

A profit function P depends on two variables E and W, as follows.

$$P(E,W) = 9E - 2E^2 - 5EW + 7W - W^2, \; E > 0, \; W > 0.$$

Investigate the critical points of P.

$\boxed{\text{saddle "point" at } (E,W,P) = (1,1,8)}$

$\boxed{P(E,W) = 9E - 2E^2 - 5EW + 7W - W^2}$

- $\frac{\partial P}{\partial E} = 9 - 4E - 5W$
- $\frac{\partial P}{\partial W} = -5E + 7 - 2W$
- $\frac{\partial^2 P}{\partial E^2} = -4 < 0$
- $\frac{\partial^2 P}{\partial W^2} = -2 < 0$
- $\frac{\partial^2 P}{\partial E \partial W} = -5$

BOTH NEGATIVE, SO POSSIBLY A MAXIMUM

- SOLVE $\frac{\partial P}{\partial E} = \frac{\partial P}{\partial W} = 0$

$\left.\begin{array}{l} 9 - 4E - 5W = 0 \\ 7 - 5E - 2W = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4E + 5W = 9 \\ 5E + 2W = 7 \end{array}\right\} \Rightarrow$ SOLVING OR BY INSPECTION

$E = 1, \; W = 1$

- THUS $P(1,1) = 9 - 2 - 5 + 7 - 1 = 8$
- CHECK THE NATURE — THE SECOND DERIVATIVES ARE ALL CONSTANTS SO WE NOT NEED EVALUATE THEM AT $E = W = 1$

$\left(\frac{\partial^2 P}{\partial E^2}\right)\left(\frac{\partial^2 P}{\partial W^2}\right) - \left(\frac{\partial^2 P}{\partial E \partial W}\right)^2 = (-4)(-2) - (-5)^2 = 8 - 25 = -17 < 0$

$\therefore$ A SADDLE

Question 4 (**)

$$f(x,y) = 9 - y^2 - 2x^2 + 4x + y - xy .$$

Investigate the critical points of f.

$\boxed{\text{maximum point at } (1,0,11)}$

$z = f(x,y) = 9 - y^2 - 2x^2 + 4x + y - xy$

- $\frac{\partial f}{\partial x} = -4x + 4 - y \qquad \frac{\partial^2 f}{\partial x^2} = -4 < 0$

  $\frac{\partial f}{\partial y} = -2y + 1 - x \qquad \frac{\partial^2 f}{\partial y^2} = -2 < 0$

  $\frac{\partial^2 f}{\partial x \partial y} = -1$

- FOR LOCAL MINIMUMS, MAXIMUMS, OR SADDLE POINTS SET $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\left.\begin{array}{l} -4x + 4 - y = 0 \\ -2y + 1 - x = 0 \end{array}\right\} \Rightarrow \boxed{y = 4 - 4x}$$

$\Rightarrow -2(4-4x) + 1 - x = 0$

$\Rightarrow -8 + 8x + 1 - x = 0$

$\Rightarrow 7x = 7$

$\Rightarrow x = 1$

$\Rightarrow y = 0$

$\Rightarrow z = 11$

$\therefore P(1,0,11)$ IS POTENTIALLY A MAX AS BOTH $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ ARE NEGATIVE

- CHECK WHETHER IT IS A MAX OR A SADDLE POINT

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_P \left.\frac{\partial^2 f}{\partial y^2}\right|_P - \left(\frac{\partial^2 f}{\partial x \partial y}\right)_P^2 = (-4)(-2) - (-1)^2 = 7 > 0$$

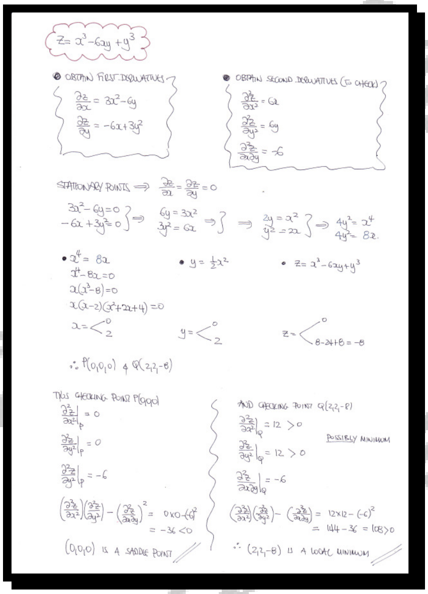
$\therefore P(1,0,11)$ IS A MAX

Question 5 (+)**

$$z = x^3 - 6xy + y^3 .$$

Investigate the critical points of z .

$\boxed{\text{saddle point at } (0,0,0)}$, $\boxed{\text{local minimum at } (2,2,-8)}$



$z = x^3 - 6xy + y^3$

OBTAIN FIRST DERIVATIVES

$\frac{\partial z}{\partial x} = 3x^2 - 6y$

$\frac{\partial z}{\partial y} = -6x + 3y^2$

OBTAIN SECOND DERIVATIVES (TO CHECK)

$\frac{\partial^2 z}{\partial x^2} = 6x$

$\frac{\partial^2 z}{\partial y^2} = 6y$

$\frac{\partial^2 z}{\partial x \partial y} = -6$

STATIONARY POINTS $\Longrightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$\left. \begin{aligned} 3x^2 - 6y &= 0 \\ -6x + 3y^2 &= 0 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} 6y &= 3x^2 \\ 3y^2 &= 6x \end{aligned} \right\} \Rightarrow \left. \begin{aligned} 2y &= x^2 \\ y^2 &= 2x \end{aligned} \right\} \Rightarrow \begin{aligned} 4y^2 &= x^4 \\ 4y^2 &= 8x. \end{aligned}$

- $x^4 = 8x$
  $x^4 - 8x = 0$
  $x(x^3 - 8) = 0$
  $x(x-2)(x^2+2x+4) = 0$
  $x = 0, 2$
- $y = \frac{1}{2}x^2$
  $y = 0, 2$
- $z = x^3 - 6xy + y^3$
  $z = 0, \; 8 - 24 + 8 = -8$

$\therefore P(0,0,0)$ & $Q(2,2,-8)$

THUS CHECKING POINT $P(0,0,0)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_P = 0$

$\left.\frac{\partial^2 z}{\partial y^2}\right|_P = 0$

$\left.\frac{\partial^2 z}{\partial y^2}\right|_P = -6$

$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0 \times 0 - (-6)^2 = -36 < 0$

$(0,0,0)$ IS A SADDLE POINT

AND CHECKING POINT $Q(2,2,-8)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_Q = 12 > 0$

$\left.\frac{\partial^2 z}{\partial y^2}\right|_Q = 12 > 0$

POSSIBLY MINIMUM

$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_Q = -6$

$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right) = 12 \times 12 - (-6)^2 = 144 - 36 = 108 > 0$

$\therefore (2,2,-8)$ IS A LOCAL MINIMUM

Question 6 (**+)

$$z(x, y) = x^4 + y^4 - 4xy .$$

Investigate the critical points of z.

$\boxed{\text{MP1}}$, $\boxed{\text{saddle point at } (0,0,0)}$, $\boxed{\text{local minimum at } (-1,-1,-2)}$, $\boxed{\text{local minimum at } (1,1,-2)}$

$$z = x^4 + y^4 - 4xy \quad x \in \mathbb{R},\ y \in \mathbb{R}$$

- START BY GETTING EXPRESSIONS FOR THE FIRST AND SECOND DERIVATIVES OF z

  - $\frac{\partial z}{\partial x} = 4x^3 - 4y$
  - $\frac{\partial z}{\partial y} = 4y^3 - 4x$
  - $\frac{\partial^2 z}{\partial x^2} = 12x^2$
  - $\frac{\partial^2 z}{\partial y^2} = 12y^2$
  - $\frac{\partial^2 z}{\partial x \partial y} = -4$

- FOR STATIONARY POINTS $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$$\left.\begin{array}{l} 4x^3 - 4y = 0 \Rightarrow y = x^3 \\ 4y^3 - 4x = 0 \Rightarrow x = y^3 \end{array}\right\} \Rightarrow x = (x^3)^3$$

$\Rightarrow x = x^9$

$\Rightarrow x^9 - x = 0$

$\Rightarrow x(x^8 - 1) = 0$

$\Rightarrow x(x^4 - 1)(x^4 + 1) = 0$

$\Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$

$\Rightarrow x(x-1)(x+1)(x^2+1)(x^4+1) = 0$

IRREDUCIBLE OVER THE REALS

- HENCE POTENTIALLY STATIONARY POINTS AT

$$x = \begin{cases} 0 \\ -1 \\ 1 \end{cases} \quad y = \begin{cases} 0 \\ -1 \\ 1 \end{cases} \quad z = \begin{cases} 0 \\ -2 \\ -2 \end{cases}$$

- NOW TO CHECK THE NATURE

$$\begin{vmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} \quad \text{SCALED TO} \quad \begin{vmatrix} 3x^2 & -1 \\ -1 & 3y^2 \end{vmatrix}$$

- CHECKING EACH POINT SEPARATELY

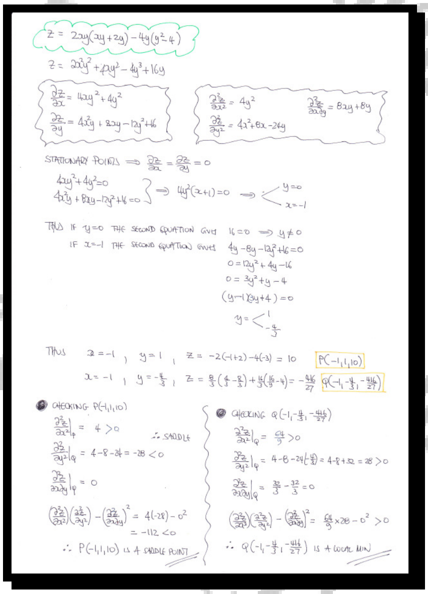
$(0,0,0)$	$(-1,-1,-2)$	$(1,1,-2)$
$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$
$\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0$ $\lambda^2 - 1 = 0$ $\lambda = \pm 1$ MIXED SIGNS IN THE EIGENVALUES $\therefore (0,0,0)$ IS A SADDLE	$\begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0$ $\Rightarrow (3-\lambda)^2 - 1 = 0$ $\Rightarrow (\lambda-3)^2 - 1 = 0$ $\Rightarrow \lambda - 3 = \begin{cases} 1 \\ -1 \end{cases}$ $\Rightarrow \lambda = \begin{cases} 4 \\ 2 \end{cases}$ BOTH EIGENVALUES ARE POSITIVE $\therefore (-1,-1,-2)$ IS A LOCAL MINIMUM	IDENTICAL WORKINGS TO $(-1,-1,-2)$ $\therefore (1,1,-2)$ IS A LOCAL MINIMUM

Question 7 **(***)**

$$z = 2xy(xy + 2y) - 4y\left(y^2 - 4\right).$$

Investigate the critical points of z.

$\boxed{\text{saddle point at } (-1,1,10)}$, $\boxed{\text{local minimum at } \left(-1,-\frac{4}{3},-\frac{416}{27}\right)}$



$z = 2xy(xy+2y) - 4y(y^2-4)$

$z = 2x^2y^2 + 4xy^2 - 4y^3 + 16y$

$\frac{\partial z}{\partial x} = 4xy^2 + 4y^2$

$\frac{\partial z}{\partial y} = 4x^2y + 8xy - 12y^2 + 16$

$\frac{\partial^2 z}{\partial x^2} = 4y^2$ $\quad \frac{\partial^2 z}{\partial x \partial y} = 8xy + 8y$

$\frac{\partial^2 z}{\partial y^2} = 4x^2 + 8x - 24y$

STATIONARY POINTS $\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

$\left.\begin{array}{l} 4xy^2 + 4y^2 = 0 \\ 4x^2y + 8xy - 12y^2 + 16 = 0 \end{array}\right\} \Rightarrow 4y^2(x+1) = 0 \Rightarrow \begin{cases} y = 0 \\ x = -1 \end{cases}$

THUS IF $y=0$ THE SECOND EQUATION GIVES $16 = 0 \Rightarrow y \neq 0$

IF $x=-1$ THE SECOND EQUATION GIVES $4y - 8y - 12y^2 + 16 = 0$

$0 = 12y^2 + 4y - 16$

$0 = 3y^2 + y - 4$

$(y-1)(3y+4) = 0$

$y = \begin{cases} 1 \\ -\frac{4}{3} \end{cases}$

THUS $x = -1$, $y = 1$, $z = -2(-1+2) - 4(-3) = 10$ $\quad \boxed{P(-1,1,10)}$

$x = -1$, $y = -\frac{4}{3}$, $z = \frac{8}{3}\left(\frac{4}{3} - \frac{8}{3}\right) + \frac{16}{3}\left(\frac{16}{9} - 4\right) = -\frac{416}{27}$ $\quad \boxed{Q\left(-1,-\frac{4}{3},-\frac{416}{27}\right)}$

- CHECKING $P(-1,1,10)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_P = 4 > 0$

$\left.\frac{\partial^2 z}{\partial y^2}\right|_P = 4 - 8 - 24 = -28 < 0$ $\quad \therefore$ SADDLE

$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_P = 0$

$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 4(-28) - 0^2 = -112 < 0$

$\therefore P(-1,1,10)$ IS A SADDLE POINT

- CHECKING $Q\left(-1,-\frac{4}{3},-\frac{416}{27}\right)$

$\left.\frac{\partial^2 z}{\partial x^2}\right|_Q = \frac{64}{9} > 0$

$\left.\frac{\partial^2 z}{\partial y^2}\right|_Q = 4 - 8 - 24\left(-\frac{4}{3}\right) = 4 - 8 + 32 = 28 > 0$

$\left.\frac{\partial^2 z}{\partial x \partial y}\right|_Q = \frac{32}{3} - \frac{32}{3} = 0$

$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \frac{64}{9} \times 28 - 0^2 > 0$

$\therefore Q\left(-1,-\frac{4}{3},-\frac{416}{27}\right)$ IS A LOCAL MIN

Question 8 (***)

The function of two variables f is defined as

$$f(x,y) \equiv xy(x+2) - y(y+3), \quad x \in \mathbb{R}, \; y \in \mathbb{R}.$$

Find the coordinates of each of the stationary points of f, where $z = f(x,y)$, and further determine their nature.

$\boxed{}$, $\boxed{\text{saddle point at } (1,0,0)}$, $\boxed{\text{saddle point at } (-3,0,0)}$,

$\boxed{\text{local maximum at } (-1,-2,4)}$

$$\boxed{f(x,y) = xy(x+2) - y(y+3) = x^2y + 2xy - y^2 - 3y}$$

- OBTAIN THE FIRST AND SECOND DERIVATIVES OF f

$\frac{\partial f}{\partial x} = 2xy + 2y \qquad \frac{\partial^2 f}{\partial x^2} = 2y$

$\frac{\partial f}{\partial y} = x^2 + 2x - 2y - 3 \qquad \frac{\partial^2 f}{\partial y^2} = -2$

$\frac{\partial^2 f}{\partial x \partial y} = 2x + 2$

- FIND THE STATIONARY POINTS $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

• $2xy + 2y = 0$
$2y(x+1) = 0$
$y = 0$ OR $x = -1$

• IF $y = 0$
$x^2 + 2x - 3 = 0$
$(x+3)(x-1) = 0$
$x = \begin{cases} 1 \\ -3 \end{cases}$

IF $x = -1$
$1 - 2 - 2y - 3 = 0$
$-4 = 2y$
$y = -2$

$\therefore$ POSSIBLE STATIONARY POINTS

$y = 0 \quad x = 1 \quad z = f(1,0) = 0$
$y = 0 \quad x = -3 \quad z = f(-3,0) = 0$
$x = -1 \quad y = -2 \quad z = f(-1,-2) = 2(1) + 2(1) = 4$

- TO DETERMINE THE NATURE OF EACH OF THESE POINTS

• $(1,0,0)$

$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 4$

$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -16 < 0 \qquad \therefore (1,0,0)$ IS A SADDLE POINT

• $(-3,0,0)$

$\frac{\partial^2 f}{\partial x^2} = 0, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = -4$

$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -16 < 0 \qquad \therefore (-3,0,0)$ IS A SADDLE POINT

• $(-1,-2,4)$

$\frac{\partial^2 f}{\partial x^2} = -4, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$

$\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 8 > 0$ & BOTH $\frac{\partial^2 f}{\partial x^2} < 0$, $\frac{\partial^2 f}{\partial y^2} < 0$

$\therefore (-1,-2,4)$ IS A LOCAL MAXIMUM

Question 9 (***)

The function of two variables f is defined as

$$f(x,y) \equiv 2x^3 + 6xy^2 - 3y^3 - 150x, \quad x \in \mathbb{R}, \ y \in \mathbb{R}.$$

Find the coordinates of each of the stationary points of f, where $z = f(x,y)$, and further determine their nature.

$\boxed{}$, $\boxed{\text{saddle point at } (3,4,-300)}$, $\boxed{\text{saddle point at } (-3,-4,300)}$, $\boxed{\text{local minimum at } (5,0,-500)}$, $\boxed{\text{local maximum at } (-5,0,500)}$

$$\boxed{f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x}$$

- FIND THE FIRST ORDER DERIVATIVES AND SET THEM EQUAL TO ZERO

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 6x^2 + 6y^2 - 150 \\ \frac{\partial f}{\partial y} &= 12xy - 9y^2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 6x^2 + 6y^2 - 150 &= 0 \\ 12xy - 9y^2 &= 0 \end{aligned}\right\} \Rightarrow$$

$$x^2 + y^2 = 25$$
$$3y(4x - 3y) = 0$$

- FROM THE SECOND EQUATION EITHER $y = 0$ OR $y = \frac{4}{3}x$

IF $y = 0$ $\quad x = \begin{cases} 5 \\ -5 \end{cases}$

IF $y = \frac{4}{3}x$

$$x^2 + \frac{16}{9}x^2 = 25$$
$$9x^2 + 16x^2 = 225$$
$$25x^2 = 225$$
$$x^2 = 9$$
$$x = \begin{cases} 3 \\ -3 \end{cases} \qquad y = \begin{cases} 4 \\ -4 \end{cases}$$

- THUS WE HAVE

x	y	$z = f(x,y)$
5	0	−500
−5	0	500
3	4	−300
−3	−4	300

- OBTAIN THE SECOND DERIVATIVES

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x & 12y \\ 12y & 12x - 18y \end{bmatrix}$$

- WHICH CAN BE SCALED TO THE MATRIX $\begin{bmatrix} 2x & 2y \\ 2y & 2x - 3y \end{bmatrix}$

- CHECKING EACH POINT

$(5,0,-500)$

$$\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \Rightarrow \begin{vmatrix} 10-\lambda & 0 \\ 0 & 10-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (10-\lambda)^2 = 0$$
$$\Rightarrow \lambda = 10, 10$$

BOTH EIGENVALUES POSITIVE
$\therefore (5,0,-500)$ IS A LOCAL MIN

$(-5,0,500)$

$$\begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} \Rightarrow \begin{vmatrix} -10-\lambda & 0 \\ 0 & -10-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (-10-\lambda)^2 = 0$$
$$\Rightarrow (\lambda+10)^2 = 0$$
$$\Rightarrow \lambda = -10, -10$$

BOTH EIGENVALUES NEGATIVE
$\therefore (-5,0,500)$ IS A LOCAL MAX

$(3,4,-300)$

$$\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix} \Rightarrow \begin{vmatrix} 6-\lambda & 8 \\ 8 & -6-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (6-\lambda)(-6-\lambda) - 64 = 0$$
$$\Rightarrow (\lambda-6)(\lambda+6) - 64 = 0$$
$$\Rightarrow \lambda^2 - 36 - 64 = 0$$
$$\Rightarrow \lambda^2 = 100$$
$$\Rightarrow \lambda = \pm 10$$

MIXED SIGN EIGENVALUES
$\therefore (3,4,-300)$ IS A "SADDLE"

$(-3,-4,300)$

$$\begin{bmatrix} -6 & -8 \\ -8 & 6 \end{bmatrix} \Rightarrow \begin{vmatrix} -6-\lambda & -8 \\ -8 & 6-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (-6-\lambda)(6-\lambda) - 64 = 0$$
$$\Rightarrow (\lambda+6)(\lambda-6) - 64 = 0$$
$$\Rightarrow \lambda^2 - 36 - 64 = 0$$
$$\Rightarrow \lambda^2 = 100$$
$$\Rightarrow \lambda = \pm 10$$

MIXED SIGN EIGENVALUES
$\therefore (-3,-4,300)$ IS A "SADDLE"

Question 10 (***+)

The function of three variables f is defined as

$$f(x,y,z) \equiv x^2 + y^2 + z^2 + xy - x + y, \quad x \in \mathbb{R}, \ y \in \mathbb{R}, \ z \in \mathbb{R}.$$

Find the stationary value of f, including the triple (x,y,z) which produces this value, further determining the nature of this stationary value.

$\boxed{}$, $\boxed{\text{local minimum of } -1 \text{ at } (1,-1,0)}$

$\boxed{f(x,y,z) = x^2+y^2+z^2+xy-x+y}$

- OBTAIN THE FIRST ORDER PARTIAL DERIVATIVES OF f AND SET THEM EQUAL TO ZERO

$$\frac{\partial f}{\partial x} = 2x+y-1, \quad \frac{\partial f}{\partial y} = 2y+x+1, \quad \frac{\partial f}{\partial z} = 2z$$

$$\left.\begin{array}{l} 2x+y=1 \\ 2y+x=-1 \end{array}\right\} \Rightarrow y = 1-2x$$

$\Rightarrow 2(1-2x)+x=-1$

$\Rightarrow -3x=-3$

$\Rightarrow x=1$

$\Rightarrow y=-1$

$\Rightarrow z=0$

$\therefore f(1,-1,0) = 1+1+0-1-1-1 = -1$

- TO CLASSIFY THE "POINT" WE REQUIRE ALL THE 2ND ORDER DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \quad \frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$\frac{\partial^2 f}{\partial z \partial x} = 0 \quad \frac{\partial^2 f}{\partial z \partial y} = 0 \quad \frac{\partial^2 f}{\partial z^2} = 2$$

- THESE NEED TO BE EVALUATED AT $(1,-1,0)$, BUT THEY ARE ALL CONSTANT

- PROCEED TO FIND THE EIGENVALUES OF THE MATRIX

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

IF ALL 3 ARE POSITIVE $\Rightarrow$ MIN
IF ALL 3 ARE NEGATIVE $\Rightarrow$ MAX
IF MIX OF POSITIVE/NEGATIVE $\Rightarrow$ "SADDLE"

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

- EXPAND BY THE THIRD COLUMN

$$\Rightarrow (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow -(\lambda-2)\left[(2-\lambda)^2-1\right] = 0$

$\Rightarrow (\lambda-2)\left[(\lambda-2)^2-1\right] = 0$

$\Rightarrow (\lambda-2)(\lambda-2-1)(\lambda-2+1) = 0$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-1) = 0$

$\Rightarrow \lambda = 1, 2, 3$

- AS ALL THE EIGENVALUES ARE POSITIVE $(1,-1,0)$ YIELDS A LOCAL MINIMUM OF -1

Question 11 (***+)

The function of three variables f is defined as

$$f(x,y,z) \equiv x^2 + xy + y^2 + 2z^2 + 3x - 2y + z, \quad x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}.$$

Find the stationary value of f, including the triple (x,y,z) which produces this value, further determining the nature of this stationary value.

$\boxed{}$, $\boxed{\text{local minimum of } -\frac{155}{24} \text{ at } \left(\frac{7}{3}, -\frac{8}{3}, -\frac{1}{4}\right)}$

$$\boxed{f(x,y,z) = x^2 + xy + y^2 + 2z^2 + 3x - 2y + z}$$

- FINDING THE FIRST DERIVATIVES AND SET THEM TO ZERO

$$\frac{\partial f}{\partial x} = 2x + y + 3, \quad \frac{\partial f}{\partial y} = x + 2y - 2, \quad \frac{\partial f}{\partial z} = 4z + 1$$

$$\Rightarrow \begin{cases} 2x + y + 3 = 0 \\ x + 2y - 2 = 0 \end{cases} \Rightarrow x = 2 - 2y$$

$$\Rightarrow 2(2-2y) + y + 3 = 0$$
$$\Rightarrow 4 - 4y + y + 3 = 0$$
$$\Rightarrow 7 = 3y$$
$$\Rightarrow y = \frac{7}{3}$$
$$\Rightarrow x = -\frac{8}{3}$$
$$\Rightarrow z = -\frac{1}{4}$$

$$\therefore f\left(-\frac{8}{3}, \frac{7}{3}, \frac{1}{4}\right) = \frac{64}{9} - \frac{56}{9} + \frac{49}{9} - \frac{1}{8} - 8 - \frac{14}{3} - \frac{1}{4} = -\frac{155}{24}$$

- TO FIND THE NATURE WE OBTAIN THE SECOND DERIVATIVES

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = 1 \quad \frac{\partial^2 f}{\partial x \partial z} = 0$$
$$\frac{\partial^2 f}{\partial y \partial x} = 1 \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial y \partial z} = 0$$
$$\frac{\partial^2 f}{\partial z \partial x} = 0 \quad \frac{\partial^2 f}{\partial z \partial y} = 0 \quad \frac{\partial^2 f}{\partial z^2} = 4$$

- NOW THESE DERIVATIVES ARE IN FACT ALL CONSTANTS SO THERE IS NO NEED TO EVALUATE THEM AT $\left(-\frac{8}{3}, \frac{7}{3}, \frac{1}{4}\right)$

- THUS WE HAVE TO FIND THE EIGENVALUES OF THE MATRIX

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

IF ALL 3 ARE POSITIVE $\Rightarrow$ MIN
IF ALL 3 ARE NEGATIVE $\Rightarrow$ MAX
IF MIX OF POSITIVE/NEGATIVE $\Rightarrow$ "SADDLE"

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

EXPAND BY THE BOTTOM ROW

$$\Rightarrow (4-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow -(\lambda-4)\left[(2-\lambda)^2 - 1\right] = 0$$
$$\Rightarrow (\lambda-4)\left[(\lambda-2)^2 - 1\right] = 0$$
$$\Rightarrow (\lambda-4)(\lambda-2-1)(\lambda-2+1) = 0$$
$$\Rightarrow (\lambda-4)(\lambda-3)(\lambda-1) = 0$$
$$\Rightarrow \lambda = 1, 3, 4$$

$\therefore$ ALL POSITIVE $\Rightarrow \left(\frac{7}{3}, -\frac{8}{3}, -\frac{1}{4}\right)$ YIELDS A LOCALLY MINIMUM VALUE OF $-\frac{155}{24}$ FOR $f(x,y,z)$

APPLIED OPTIMIZATION

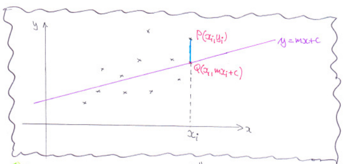
Question 1 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \le n$, is given.

It is required to find a straight line with equation $y = mx + c$, so that the sum of the squares of the vertical distances between P_i and the straight line is least.

Find simplified expressions for each of the constants m and c, in terms of x_i and y_i.

$$\boxed{m = \frac{n\sum_{i=1}^{n}(x_i y_i) - \sum_{i=1}^{n}(x_i)\sum_{i=1}^{n}(y_i)}{n\sum_{i=1}^{n}(x_i)^2 - \sum_{i=1}^{n}(x_i)\sum_{i=1}^{n}(x_i)}}, \quad \boxed{c = \frac{1}{n}\sum_{i=1}^{n}(y_i) - \frac{m}{n}\sum_{i=1}^{n}(x_i)}$$



CONSIDER THE VERTICAL DISTANCE OF THE i^{th} POINT FROM THE LINE OF BEST FIT $y = mx + c$

$|PQ|^2 = (y_i - mx_i - c)^2$

HENCE THE TOTAL OF ALL THE DISTANCES SQUARED (n POINTS)

$T = \sum_{i=1}^{n}(y_i - mx_i - c)^2$

DIFFERENTIATE TO MINIMIZE — NOTE THAT y_i & x_i ARE CONSTANT CO-ORDINATES BUT m & c ARE VARIABLES

$\frac{\partial T}{\partial m} = -2\sum_{i=1}^{n} x_i(y_i - mx_i - c)$

$\frac{\partial T}{\partial c} = -2\sum_{i=1}^{n}(y_i - mx_i - c)$

SOLVING EACH OF THEM FOR ZERO (DIVIDE -2 OUT)

$\sum_{i=1}^{n} x_i y_i - m\sum_{i=1}^{n} x_i^2 - c\sum_{i=1}^{n} x_i = 0$

$\sum_{i=1}^{n} y_i - m\sum_{i=1}^{n} x_i - c\sum_{i=1}^{n} 1 = 0$

DROPPING "DETAILS" ON THE SIGMA NOTATION & TIDY

$\Rightarrow \begin{cases} \sum xy - m\sum x^2 - c\sum x = 0 & \times n \\ \sum y - m\sum x - cn = 0 & \times \sum x \end{cases}$

$\Rightarrow \begin{cases} n\sum xy - mn\sum x^2 - cn\sum x = 0 \\ \sum x\sum y - m\sum x\sum x - cn\sum x = 0 \end{cases}$ SUBTRACT

$\Rightarrow n\sum xy - \sum x\sum y - mn\sum x^2 + m\sum x\sum x = 0$

$\Rightarrow n\sum xy - \sum x\sum y = mn\sum x^2 - m\sum x\sum x$

$\Rightarrow n\sum xy - \sum x\sum y = m\left[n\sum x^2 - \sum x\sum x\right]$

$\Rightarrow m = \frac{n\sum xy - \sum x\sum y}{n\sum x^2 - \sum x\sum x}$

DIVIDE TOP & BOTTOM OF THE FRACTION BY n

$\Rightarrow m = \frac{\sum xy - \frac{\sum x\sum y}{n}}{\sum x^2 - \frac{\sum x\sum x}{n}}$

WHICH IN STATISTICS IS KNOWN AS $\frac{S_{xy}}{S_{xx}}$

& HENCE

$cn = \sum y - m\sum x$

$c = \frac{\sum y}{n} - m\frac{\sum x}{n}$

WHICH IN STATISTICS IS KNOWN AS $\bar{y} - m\bar{x}$ OR $\bar{y} - b\bar{x}$

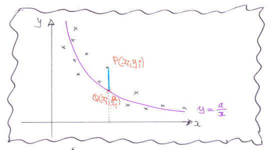
Question 2 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \le n$, is given.

It is required to find a hyperbola with equation $y = \dfrac{a}{x}$, so that the sum of the squares of the vertical distances between P_i and the hyperbola is least.

Find a simplified expression for the constant a, in terms of x_i and y_i.

$$\boxed{a = \dfrac{\displaystyle\sum_{i=1}^{n}\left(\dfrac{y_i}{x_i}\right)}{\displaystyle\sum_{i=1}^{n}\dfrac{1}{(x_i)^2}}}$$



- CONSIDER THE VERTICAL DISTANCE OF THE i^{th} POINT FROM THE "HYPERBOLA OF BEST FIT" $y = \frac{a}{x}$ (MARKED IN LIGHT BLUE, i.e. $|PQ|^2$)
- THE TOTAL OF ALL THE DISTANCES SQUARED, OF n POINTS, IS GIVEN BY

$$T = \sum_{i=1}^{n}\left(y_i - \frac{a}{x_i}\right)^2$$

- DIFFERENTIATE TO MINIMIZE — NOTE x_i & y_i ARE CONSTANTS, BUT a CAN VARY

$$\frac{\partial T}{\partial a} = 2\sum_{i=1}^{n} -\frac{1}{x_i}\left(y_i - \frac{a}{x_i}\right)$$

- SOLVE FOR ZERO, IGNORING SUB/SUPERSCRIPTS ON SIGMA

$$\Rightarrow \cancel{-2}\sum \frac{1}{x}\left(y - \frac{a}{x}\right) = 0$$

$$\Rightarrow \sum \frac{y}{x} - \sum \frac{a}{x^2} = 0$$

$$\Rightarrow \sum \frac{y}{x} = a\sum \frac{1}{x^2}$$

$$\therefore a = \frac{\sum y/x}{\sum 1/x^2}$$

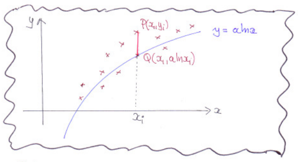
Question 3 (****)

A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \le n$, is given.

It is required to find a curve with equation $y = a \ln x$, so that the sum of the squares of the vertical distances between P_i and the curve is least.

Find a simplified expression for the constant a, in terms of x_i and y_i.

$$\boxed{a = \frac{\sum_{i=1}^{n}\left[y_i \ln(x_i)\right]}{\sum_{i=1}^{n}\left[\ln(x_i)\right]^2}}$$



- CONSIDER THE VERTICAL DISTANCE SQUARED, $|PQ|^2$ WHERE $P(x_i, y_i)$ IS A TYPICAL POINT & $Q(x_i, a\ln x_i)$ LIES ON $y = a\ln x$

$|PQ|^2 = (y_i - a\ln x_i)^2$

- LET T BE THE TOTAL OF ALL THESE DISTANCES (n POINTS)

$$T = \sum_{i=1}^{n}(y_i - a\ln x_i)^2$$

- DIFFERENTIATE TO MINIMIZE, NOTING THAT x_i & y_i ARE CONSTANTS BUT a CAN VARY

$$\frac{\partial T}{\partial a} = \sum_{i=1}^{n}\left[-2\ln x_i\,(y_i - a\ln x_i)\right]$$

- SOLVE FOR ZERO

$$\Rightarrow 0 = \cancel{-2}\sum_{i=1}^{n}\left[y_i \ln x_i - a(\ln x_i)^2\right]$$

$$\Rightarrow a\sum_{i=1}^{n}(\ln x_i)^2 = \sum_{i=1}^{n} y_i \ln x_i$$

$$\Rightarrow a = \frac{\sum_{i=1}^{n} y_i \ln x_i}{\sum_{i=1}^{n}(\ln x_i)^2}$$

Question 4 (****)

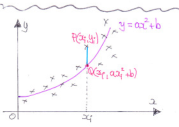
A set of points P_i with Cartesian coordinates (x_i, y_i), $1 < i \le n$, is given.

It is required to find a curve with equation $y = ax^2 + b$, so that the sum of the squares of the vertical distances between P_i and the curve is least.

Find simplified expressions for each of the constants a and b, in terms of x_i and y_i.

$$\boxed{a = \frac{n\sum_{i=1}^{n}\left[x_i^2 y_i\right] - \sum_{i=1}^{n}\left[y_i\right]\sum_{i=1}^{n}\left[x_i^2\right]}{n\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2\right]}},$$

$$\boxed{b = \frac{\sum_{i=1}^{n}\left[y_i\right]\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2 y_i\right]}{n\sum_{i=1}^{n}\left[x_i^4\right] - \sum_{i=1}^{n}\left[x_i^2\right]\sum_{i=1}^{n}\left[x_i^2\right]}}$$



CONSIDER THE VERTICAL DISTANCE OF THE ith POINT FROM THE CURVE OF BEST FIT WITH EQUATION $y = ax^2 + b$

$|PQ| = \sqrt{(y_i - ax_i^2 - b)^2}$

- LET T BE THE TOTAL OF ALL THE VERTICAL DISTANCES SQUARED (n points)

$$T = \sum_{i=1}^{n} (y_i - ax_i^2 - b)^2$$

- DIFFERENTIATE TO MINIMIZE, NOTING THAT x_i & y_i ARE CONSTANT, BUT THE "CONSTANTS" a & b CAN VARY

$$\frac{\partial T}{\partial a} = \sum_{i=1}^{n}\left[-2x_i^2(y_i - ax_i^2 - b)\right]$$
$$\frac{\partial T}{\partial b} = \sum_{i=1}^{n}\left[-2(y_i - ax_i^2 - b)\right]$$

- SOLVE FOR ZERO (DIVIDING THE -2 OUT)

$$\left.\begin{array}{l}\sum_{i=1}^{n}\left[x_i^2(y_i - ax_i^2 - b)\right] = 0 \\ \sum_{i=1}^{n}\left[y_i - ax_i^2 - b\right] = 0\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\sum_{i=1}^{n}(x_i^2 y) - a\sum_{i=1}^{n} x_i^4 - b\sum_{i=1}^{n} x_i^2 = 0 \\ \sum_{i=1}^{n} y_i - a\sum_{i=1}^{n} x_i^2 - b\sum_{i=1}^{n} 1 = 0\end{array}\right.$$

- DROPPING THE SUBSCRIPTS & TIDY

$$\sum x^2 y - a\sum x^4 - b\sum x^2 = 0$$
$$\sum y - a\sum x^2 - bn = 0$$

- MULTIPLY BY THE FIRST EQUATION BY $\sum x^2$ AND THE SECOND BY $\sum x^4$

$$\left\{\begin{array}{l}\sum x^2 \sum x^2 y - a\sum x^2 \sum x^4 - b\sum x^2 \sum x^2 = 0 \\ \sum x^4 \sum y - a\sum x^2 \sum x^4 - bn\sum x^4 = 0\end{array}\right\}$$

SUBTRACT

$$\Rightarrow \sum x^2 \sum x^2 y - \sum y \sum x^4 + bn\sum x^4 - b\sum x^2 \sum x^2 = 0$$
$$\Rightarrow b\left[n\sum x^4 - \sum x^2 \sum x^2\right] = \sum y \sum x^4 - \sum x^2 \sum x^2 y$$
$$\Rightarrow b = \frac{\sum y \sum x^4 - \sum x^2 \sum x^2 y}{n\sum x^4 - \sum x^2 \sum x^2}$$

- SIMILARLY

MULTIPLY THE FIRST EQUATION BY n AND THE SECOND BY $\sum x^2$

$$n\sum x^2 y - an\sum x^4 - bn\sum x^2 = 0$$
$$\sum y \sum x^2 - a\sum x^2 \sum x^2 - bn\sum x^2 = 0$$

SUBTRACT

$$\Rightarrow n\sum x^2 y - \sum y \sum x^2 + a\sum x^2 \sum x^2 - an\sum x^4 = 0$$
$$\Rightarrow n\sum x^2 y - \sum y \sum x^2 = a\left[n\sum x^4 - \sum x^2 \sum x^2\right]$$
$$\Rightarrow a = \frac{n\sum x^2 y - \sum y \sum x^2}{n\sum x^4 - \sum x^2 \sum x^2}$$

Question 5 (****)

The table below shows experimental data connecting two variables x and y.

t	5	10	15	30	70
P	181	158	145	127	107

It is assumed that t and P are related by an equation of the form

$$P = A \times t^k ,$$

where A and k are non zero constants.

By linearizing the above equation, and using partial differentiation to obtain a line of least squares, determine the value of A and the value of k.

$\boxed{A \approx 250}$, $\boxed{k \approx -0.2}$

t	5	10	15	30	70
P	181	158	145	127	107

$P = At^k$

$\ln P = \ln(At^k) = \ln A + \ln t^k$

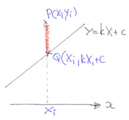
$\ln P = k\ln t + \ln A$

$Y = kX + C$

CONSIDER THE VERTICAL DISTANCE $|PQ|$ FROM THE POINT $P(X_i, Y_i)$, $i = 1,2,3,4,5$ & $Q(X_i, kX_i + C)$

$|PQ| = \sqrt{(Y_i - kX_i - c)^2}$

$|PQ|^2 = (Y_i - kX_i - c)^2$



LET T BE THE TOTAL OF SUCH SQUARED DISTANCES

$T = \sum_{i=1}^{5} (Y_i - kX_i - c)^2$

DIFFERENTIATE FOR MINIMIZING, NOTING X_i & Y_i ARE CONSTANTS

$\frac{\partial T}{\partial k} = \sum_{i=1}^{5} -2X_i(Y_i - kX_i - c)$

$\frac{\partial T}{\partial c} = \sum_{i=1}^{5} -2(Y_i - kX_i - c)$

SOLVE FOR ZERO

$$\Rightarrow \left\{ \begin{array}{l} \cancel{-2}\sum_{i=1}^{5}[X_iY_i - kX_i^2 - cX_i] = 0 \\ \cancel{-2}\sum_{i=1}^{5}[Y_i - kX_i - c] = 0 \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} \sum_{i=1}^{5} X_iY_i - k\sum_{i=1}^{5} X_i^2 - c\sum_{i=1}^{5} X_i = 0 \\ \sum_{i=1}^{5} Y_i - k\sum_{i=1}^{5} X_i - c\underbrace{\sum_{i=1}^{5} 1}_{=5} = 0 \end{array} \right\} \begin{array}{l} \times 5 \\ \times \sum_{i=1}^{5} X_i \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} 5\sum_{i=1}^{5} X_iY_i - 5k\sum_{i=1}^{5} X_i^2 - 5c\sum_{i=1}^{5} X_i = 0 \\ \sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i - k\sum_{i=1}^{5} X_i \sum_{i=1}^{5} X_i - 5c\sum_{i=1}^{5} X_i = 0 \end{array} \right.$$

SUBTRACT

$\Rightarrow 5\sum_{i=1}^{5} X_iY_i - \sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i - 5k\sum_{i=1}^{5} X_i^2 + k\sum_{i=1}^{5} X_i X_i = 0$

$\Rightarrow 5\sum_{i=1}^{5} X_iY_i - \sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i = k\left[5\sum_{i=1}^{5} X_i^2 - \sum_{i=1}^{5} X_i \sum_{i=1}^{5} X_i\right]$

$$\Rightarrow k = \frac{5\sum_{i=1}^{5} X_iY_i - \sum_{i=1}^{5} X_i \sum_{i=1}^{5} Y_i}{5\sum_{i=1}^{5} X_i^2 - \sum_{i=1}^{5} X_i \sum_{i=1}^{5} X_i}$$

& $5c = \sum_{i=1}^{5} Y_i - k\sum_{i=1}^{5} X_i$

$$c = \frac{1}{5}\sum_{i=1}^{5} Y_i - \frac{k}{5}\sum_{i=1}^{5} X_i$$

HENCE

$X = \ln t$	$\ln 5$	$\ln 10$	$\ln 15$	$\ln 30$	$\ln 70$
$Y = \ln P$	$\ln 181$	$\ln 158$	$\ln 145$	$\ln 127$	$\ln 107$

$\sum_{i=1}^{5} X = 14.270$ $\qquad \sum_{i=1}^{5} X^2 = 44.844$

$\sum_{i=1}^{5} Y = 24.755$ $\qquad \sum_{i=1}^{5} XY = 69.829$

$k = \frac{5 \times 69.829 - 14.270 \times 24.755}{5 \times 44.844 - 14.270 \times 14.270} = -0.1996\ldots \simeq \boxed{-0.2}$

$c = \frac{1}{5}(24.755) - \frac{-0.1996\ldots}{5} \times 14.270 = 5.5206\ldots$

$\therefore A = e^{5.5206\ldots}$

$A \simeq 249.79\ldots \simeq \boxed{250}$

$\therefore P = 250 \times t^{-0.2}$

CONSTRAINED OPTIMIZATION

Question 1 (***)

$$f(x,y) = x^2 + y^2, \; x \in \mathbb{R}, \; y \in \mathbb{R}.$$

The region R in the x-y plane is a circle centred at $(-1,1)$ and of radius 1.

Use partial differentiation to determine the maximum and the minimum value of f, whose projection onto the x-y plane is the region R.

$\boxed{}$, $\boxed{f_{\text{max}} = 3 + 2\sqrt{2}}$, $\boxed{f_{\text{min}} = 3 - 2\sqrt{2}}$

USING LAGRANGE'S METHOD, WE HAVE IN THE USUAL NOTATION

- OBJECTIVE FUNCTION

$$f(x,y) = x^2 + y^2$$

- CONSTRAINT

$$(x+1)^2 + (y-1)^2 = 1$$
$$x^2 + 2x + 1 + y^2 - 2y + 1 = 1$$
$$x^2 + y^2 + 2x - 2y + 1 = 0$$
$$\phi(x,y) = x^2 + y^2 + 2x - 2y + 1$$

HENCE WE HAVE THE FOLLOWING EQUATIONS

$$\left.\begin{array}{ll} (\text{I}) & \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \phi}{\partial x} = 0 \\ (\text{II}) & \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \phi}{\partial y} = 0 \\ (\text{III}) & \phi(x,y) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 2x + \lambda(2x+2) = 0 \\ 2y + \lambda(2y-2) = 0 \\ x^2 + y^2 + 2x - 2y + 1 = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} x = -\lambda(x+1) \\ y = -\lambda(y-1) \\ x^2 + y^2 + 2x - 2y + 1 = 0 \end{array}\right\} \Rightarrow$$

DIVIDING THE FIRST TWO EQUATIONS

$$\Rightarrow \frac{x}{y} = \frac{x+1}{y-1}$$
$$\Rightarrow \cancel{xy} - x = \cancel{xy} + y$$
$$\Rightarrow y = -x$$

SUBSTITUTE INTO EQUATION (III)

$$\Rightarrow x^2 + y^2 + 2x - 2y + 1 = 0$$
$$\Rightarrow x^2 + (-x)^2 + 2x - 2(-x) + 1 = 0$$
$$\Rightarrow x^2 + x^2 + 2x + 2x + 1 = 0$$
$$\Rightarrow 2x^2 + 4x + 1 = 0$$
$$\Rightarrow x^2 + 2x + \tfrac{1}{2} = 0$$
$$\Rightarrow x^2 + 2x + 1 = \tfrac{1}{2}$$
$$\Rightarrow (x+1)^2 = \tfrac{1}{2}$$
$$\Rightarrow x + 1 = \begin{cases} +\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{cases}$$
$$\Rightarrow x = \begin{cases} -1 + \frac{\sqrt{2}}{2} = \frac{-2+\sqrt{2}}{2} \\ -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2} \end{cases} \qquad y = \begin{cases} \frac{2-\sqrt{2}}{2} \\ \frac{2+\sqrt{2}}{2} \end{cases}$$

FINALLY WE OBTAIN BY SUBSTITUTING INTO $f(x,y) = x^2 + y^2$

- $f\left[-\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}\right] = \left(\frac{-2+\sqrt{2}}{2}\right)^2 + \left(\frac{2-\sqrt{2}}{2}\right)^2 = \frac{1}{4}\left[4 - 4\sqrt{2} + 2 + 4 - 4\sqrt{2} + 2\right]$
  $= \frac{1}{4}\left[12 - 8\sqrt{2}\right] = 3 - 2\sqrt{2}$

- $f\left[\frac{-2-\sqrt{2}}{2}, \frac{2+\sqrt{2}}{2}\right] = \left(\frac{-2-\sqrt{2}}{2}\right)^2 + \left(\frac{2+\sqrt{2}}{2}\right)^2 = \frac{1}{4}\left[4 + 4\sqrt{2} + 2 + 4 + 4\sqrt{2} + 2\right]$
  $= \frac{1}{4}\left[12 + 8\sqrt{2}\right] = 3 + 2\sqrt{2}$

$$\therefore \; f(x,y)_{\text{MIN}} = 3 - 2\sqrt{2}$$
$$f(x,y)_{\text{MAX}} = 3 + 2\sqrt{2}$$

Question 2 (***)

$$f(x,y) = (x+1)\sqrt{y}\,,\ x \in \mathbb{R}\,,\ y \in \mathbb{R}\,,\ y > 0\,.$$

Find the value of x and the value of y which maximizes value of f, subject to the constraint $x + 2y = 11$.

$\boxed{(7,2)}$

MAXIMIZE $(x+1)\sqrt{y}$ SUBJECT TO $x+2y=11$

LET $f(x,y) = (x+1)y^{\frac{1}{2}}$ & $\phi(x,y) = x+2y-11$

THE LAGRANGIAN $L = f(x,y) - \lambda\phi(x,y)$

$L = (x+1)y^{\frac{1}{2}} - \lambda(x+2y-11)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= y^{\frac{1}{2}} - \lambda \\ \frac{\partial L}{\partial y} &= \frac{1}{2}(x+1)y^{-\frac{1}{2}} - 2\lambda \\ \frac{\partial L}{\partial \lambda} &= -(x+2y-11) \end{aligned}\right\} \text{SOLVE FOR ZERO} \qquad \begin{aligned} \lambda &= y^{\frac{1}{2}} &&\text{—1} \\ \lambda &= \frac{1}{4}(x+1)y^{-\frac{1}{2}} &&\text{—2} \\ x+2y &= 11 &&\text{—3} \end{aligned}$$

EQUATE ① & ②

$y^{\frac{1}{2}} = \frac{1}{4}(x+1)y^{-\frac{1}{2}}$

$y = \frac{1}{4}(x+1)$

$2y = \frac{1}{2}(x+1)$

$11 - x = \frac{1}{2}(x+1)$ BY ③

$22 - 2x = x+1$

$21 = 3x$

$x = 7$ & $y = 2$

TO VERIFY IT IS A MAX, TRY ANOTHER POINT WHICH SATISFIES THE CONSTRAINT SAY $(3,4)$

$(3+1)\sqrt{4} < (7+1)\sqrt{2}$

$8 < 8\sqrt{2}$

Question 3 (***)

The region R in the x-y plane is the ellipse with equation

$$2x^2 + xy = 2y^2 = 15 .$$

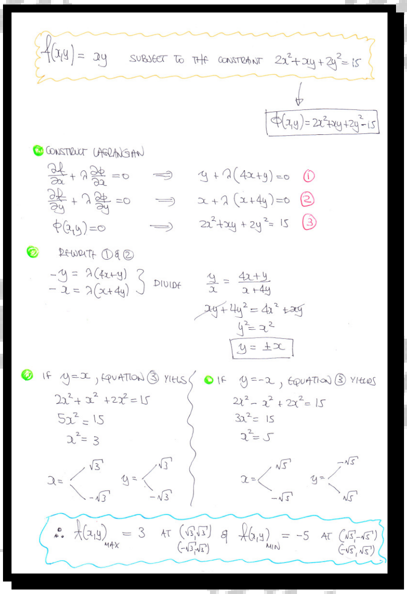
The surface with equation $z = f(x, y)$ is given by

$$f(x, y) = xy , \ x \in \mathbb{R} , \ y \in \mathbb{R} .$$

Determine the maximum value of f and the minimum value of f, whose projection onto the x-y plane is the region R.

Give the corresponding x and y coordinates in each case

$\boxed{f_{\max} = 3 \ \text{ at } \ \left(\sqrt{3}, \sqrt{3}\right) \ \text{ or } \ \left(-\sqrt{3}, -\sqrt{3}\right)}$, $\boxed{f_{\max} = -5 \ \text{ at } \ \left(\sqrt{5}, -\sqrt{5}\right) \ \text{ or } \ \left(-\sqrt{5}, \sqrt{5}\right)}$



$f(x,y) = xy$ SUBJECT TO THE CONSTRAINT $2x^2 + xy + 2y^2 = 15$

$\boxed{\phi(x,y) = 2x^2 + xy + 2y^2 - 15}$

CONSTRUCT LAGRANGIAN

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \implies y + \lambda(4x + y) = 0$ ①

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \implies x + \lambda(x + 4y) = 0$ ②

$\phi(x,y) = 0 \implies 2x^2 + xy + 2y^2 = 15$ ③

REWRITE ① & ②

$\left.\begin{array}{l} -y = \lambda(4x+y) \\ -x = \lambda(x+4y) \end{array}\right\}$ DIVIDE $\quad \frac{y}{x} = \frac{4x+y}{x+4y}$

$\cancel{xy} + 4y^2 = 4x^2 + \cancel{xy}$

$y^2 = x^2$

$\boxed{y = \pm x}$

IF $y = x$, EQUATION ③ YIELDS

$2x^2 + x^2 + 2x^2 = 15$

$5x^2 = 15$

$x^2 = 3$

$x = \sqrt{3}, -\sqrt{3} \quad y = \sqrt{3}, -\sqrt{3}$

IF $y = -x$, EQUATION ③ YIELDS

$2x^2 - x^2 + 2x^2 = 15$

$3x^2 = 15$

$x^2 = 5$

$x = \sqrt{5}, -\sqrt{5} \quad y = -\sqrt{5}, \sqrt{5}$

$\therefore f(x,y)_{MAX} = 3$ AT $(\sqrt{3}, \sqrt{3})$, $(-\sqrt{3}, -\sqrt{3})$ & $f(x,y)_{MIN} = -5$ AT $(\sqrt{5}, -\sqrt{5})$, $(-\sqrt{5}, \sqrt{5})$

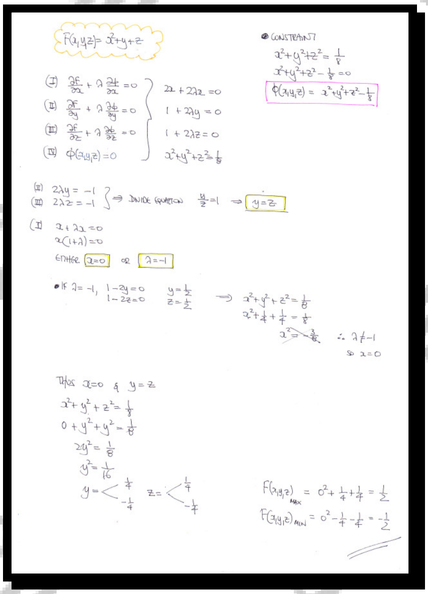
Question 4 (***)

A scalar field F exists on the surface of the sphere with Cartesian equation

$$x^2 + y^2 + z^2 = \frac{1}{8}.$$

Given further that $F = x^2 + y + z$, determine the maximum value of F and the minimum value of F.

$$\boxed{F_{\text{max}} = \frac{1}{2}}, \ \boxed{F_{\text{min}} = -\frac{1}{2}}$$



$F(x,y,z) = x^2+y+z$

CONSTRAINT

$x^2+y^2+z^2 = \frac{1}{8}$

$x^2+y^2+z^2-\frac{1}{8}=0$

$\boxed{\phi(x,y,z) = x^2+y^2+z^2-\frac{1}{8}}$

$$\left.\begin{array}{ll} (\text{I}) & \frac{\partial F}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \\ (\text{II}) & \frac{\partial F}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \\ (\text{III}) & \frac{\partial F}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \\ (\text{IV}) & \phi(x,y,z) = 0 \end{array}\right\} \begin{array}{l} 2x + 2\lambda x = 0 \\ 1 + 2\lambda y = 0 \\ 1 + 2\lambda z = 0 \\ x^2+y^2+z^2 = \frac{1}{8} \end{array}$$

$$\left.\begin{array}{ll} (\text{II}) & 2\lambda y = -1 \\ (\text{III}) & 2\lambda z = -1 \end{array}\right\} \Rightarrow \text{DIVIDE EQUATION} \ \frac{y}{z} = 1 \Rightarrow \boxed{y = z}$$

(I) $x + \lambda x = 0$

$x(1+\lambda) = 0$

EITHER $\boxed{x=0}$ OR $\boxed{\lambda=-1}$

- IF $\lambda = -1$, $1-2y=0$, $1-2z=0$ $\quad y=\frac{1}{2}$, $z=\frac{1}{2}$ $\Rightarrow x^2+y^2+z^2=\frac{1}{8}$

$x^2+\frac{1}{4}+\frac{1}{4}=\frac{1}{8}$

$\cancel{x^2 = -\frac{3}{8}}$ $\quad \therefore \lambda \neq -1$

SO $x=0$

THUS $x=0$ & $y=z$

$x^2+y^2+z^2=\frac{1}{8}$

$0+y^2+y^2=\frac{1}{8}$

$2y^2=\frac{1}{8}$

$y^2=\frac{1}{16}$

$y = \frac{1}{4}, -\frac{1}{4} \quad z = \frac{1}{4}, -\frac{1}{4}$

$F(x,y,z)_{\text{MAX}} = 0^2+\frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

$F(x,y,z)_{\text{MIN}} = 0^2-\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$

Question 5 (***)

$$f(x,y) = x^2 + y^2 + \sqrt{x^2 + y^2}, \quad x \in \mathbb{R}, \; y \in \mathbb{R}.$$

Find the value of x and the value of y which minimizes value of f, subject to the constraint $x + y = 1$.

$$\boxed{\left(\frac{1}{2}, \frac{1}{2}\right)}$$

MINIMIZE $x^2+y^2+\sqrt{x^2+y^2}$ SUBJECT TO $x+y=1$

LET $f(x,y) = x^2+y^2+(x^2+y^2)^{\frac{1}{2}}$ CONSTRAIN IS $\phi(x,y,z) = x+y-1$

THE LAGRANGIAN L IS $f(x,y)$ $\lambda\phi(x,y)$

$$\boxed{L = x^2+y^2+(x^2+y^2)^{\frac{1}{2}} - \lambda(x+y-1)}$$

$$\left.\begin{aligned} \frac{\partial L}{\partial x} &= 2x + \frac{x}{(x^2+y^2)^{\frac{1}{2}}} - \lambda \\ \frac{\partial L}{\partial y} &= 2y + \frac{y}{(x^2+y^2)^{\frac{1}{2}}} - \lambda \\ \frac{\partial L}{\partial \lambda} &= -(x+y-1) \end{aligned}\right\} \Rightarrow \text{SET TO ZERO} \left\{\begin{aligned} \lambda &= 2x + \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \quad (1) \\ \lambda &= 2y + \frac{y}{(x^2+y^2)^{\frac{1}{2}}} \quad (2) \\ x+y &= 1 \quad (3) \end{aligned}\right.$$

- EQUATE 1 & 2

$$2x + \frac{x}{(x^2+y^2)^{\frac{1}{2}}} = 2y + \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$$

$$x\left[2 + \frac{1}{(x^2+y^2)^{\frac{1}{2}}}\right] = y\left[2 + \frac{1}{(x^2+y^2)^{\frac{1}{2}}}\right]$$

$\therefore \boxed{x=y}$

BUT $x+y=1$

$\therefore x=y=\frac{1}{2}$

NOTE THAT IT IS A MINIMUM AS A RANDOM POINT WHICH SATISFIES THE CONSTRAIN SAY $(0,1)$ GIVES

$$2 > \frac{1}{4} + \frac{1}{4} + \frac{\sqrt{2}}{2}$$

$$2 > \frac{1}{2}(1+\sqrt{2})$$

Question 6 (*+)**

Determine in exact form the shortest distance of the point $(1,2,3)$ from the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

$$\boxed{d_{\min} = \sqrt{14} - 1}$$

- THE DISTANCE OF A POINT (x,y,z) FROM THE POINT $(1,2,3)$ IS $\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$
- THE CONSTRAINT IS THE POINT LIES ON THE SPHERE WITH EQUATION $x^2+y^2+z^2=1$
- FOR SIMPLICITY MINIMIZE

$$f(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2 \quad \text{SUBJECT TO} \quad \phi(x,y,z)=0, \quad x^2+y^2+z^2-1=0$$

- FORM THE LAGRANGIAN

$$L = f(x,y,z) - \lambda\phi(x,y,z)$$
$$L = (x-1)^2+(y-2)^2+(z-3)^2 - \lambda(x^2+y^2+z^2-1)$$

- DIFFERENTIATING & SET TO ZERO

$$\left.\begin{array}{l} \frac{\partial L}{\partial x} = 2(x-1)-2\lambda x \\ \frac{\partial L}{\partial y} = 2(y-2)-2\lambda y \\ \frac{\partial L}{\partial z} = 2(z-3)-2\lambda z \\ \frac{\partial L}{\partial \lambda} = -(x^2+y^2+z^2-1) \end{array}\right\} \Longrightarrow \begin{array}{ll} x-1-\lambda x=0 & \text{— I} \\ y-2-\lambda y=0 & \text{— II} \\ z-3-\lambda z=0 & \text{— III} \\ x^2+y^2+z^2-1=0 & \text{— IV} \end{array}$$

- REARRANGE THE EQUATIONS AS FOLLOWS

$$\begin{array}{ll} (1-\lambda)x=1 & \text{— I} \\ (1-\lambda)y=2 & \text{— II} \\ (1-\lambda)z=3 & \text{— III} \\ x^2+y^2+z^2=1 & \text{— IV} \end{array} \quad \underline{\text{OR}} \quad \begin{array}{l} x=\frac{1}{1-\lambda} \\ y=\frac{2}{1-\lambda} \\ z=\frac{3}{1-\lambda} \end{array}$$

- SUBSTITUTE I, II, III INTO IV

$$\left(\frac{1}{1-\lambda}\right)^2+\left(\frac{2}{1-\lambda}\right)^2+\left(\frac{3}{1-\lambda}\right)^2=1$$
$$\frac{1}{(1-\lambda)^2}+\frac{4}{(1-\lambda)^2}+\frac{9}{(1-\lambda)^2}=1$$
$$\frac{14}{(1-\lambda)^2}=1$$
$$(1-\lambda)^2=14$$
$$1-\lambda = \begin{cases} +\sqrt{14} \\ \cancel{-\sqrt{14}} \end{cases}$$

- THUS EITHER $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ OR $\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$

THIS BY INSPECTION PRODUCES MAX

$$\therefore \sqrt{\left(\frac{1}{\sqrt{14}}-1\right)^2+\left(\frac{2}{\sqrt{14}}-2\right)^2+\left(\frac{3}{\sqrt{14}}-3\right)^2}$$
$$= \sqrt{\frac{(1-\sqrt{14})^2}{14}+\frac{4}{14}(1-\sqrt{14})^2+\frac{9}{14}(1-\sqrt{14})^2}$$
$$= \sqrt{\frac{14}{14}(1-\sqrt{14})^2}$$
$$= \sqrt{14}-1$$

Question 7 (***+)

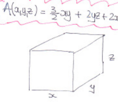
A water tank, in the shape of a cuboid, is to have a capacity of 1 m^3.

Sheet metal is used for the construction of the tank. The sheets are of uniform thickness but the density of the metal used for the lid is half the density of the metal used for the rest of the tank.

Use Lagrange's constrained optimization method to show that the minimum total sheet metal to be used is exactly $3\sqrt[3]{6} \text{ m}^2$.

proof

$A(x,y,z) = \frac{3}{2}xy + 2yz + 2xz$



• CONSTRAINT
$xyz = 1$
$xyz - 1$
$V(x,y,z) = xyz - 1$

(I) $\frac{\partial A}{\partial x} + \lambda \frac{\partial V}{\partial x} = 0 \Rightarrow \frac{3}{2}y + 2z + \lambda(yz) = 0$

(II) $\frac{\partial A}{\partial y} + \lambda \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{3}{2}x + 2z + \lambda(xz) = 0$

SUBTRACT THESE TWO

(III) $\frac{\partial A}{\partial z} + \lambda \frac{\partial V}{\partial z} = 0 \Rightarrow 2y + 2x + \lambda(xy) = 0$

(IV) $V(x,y,z) = 0 \rightarrow xyz = 1$

$\frac{3}{2}y - \frac{3}{2}x + \lambda yz - \lambda xz = 0$

$\frac{3}{2}(y-x) + \lambda z(y-x) = 0$

$(y-x)\left(\frac{3}{2} + \lambda z\right) = 0$

EITHER $\boxed{y = x}$ OR $\boxed{\lambda z = -\frac{3}{2}}$

• IF $\lambda z = -\frac{3}{2}$ EQUATION (I) BECOMES $\frac{3}{2}y + 2z + y\left(-\frac{3}{2}\right) \Rightarrow \cancel{z=0}$ PHYSICALLY AS WELL AS (IV)

• IF $y = x$ THE EQUATIONS REDUCE TO

(I) $\frac{3}{2}x + 2z + \lambda xz = 0$
(II) $\frac{3}{2}x + 2z + \lambda xz = 0$
(III) $2x + 2x + \lambda x^2 = 0$
(IV) $x^2 z = 1$

(A): $\frac{3}{2}x + 2z + \lambda xz = 0$
(B): $4x + \lambda x^2 = 0$
(C): $x^2 z = 1$

Thus $4x + \lambda x^2 = 0$
$x(4 + \lambda x) = 0$
$\boxed{\lambda = -\frac{4}{x}}$ $(x \neq 0)$ SUB INTO (A) & (C)

(A): $\frac{3}{2}x + 2z + \left(-\frac{4}{x}\right)xz = 0$
(C): $x^2 z = 1$
$\Rightarrow \frac{3}{2}x + 2z - 4z = 0$, $x^2 z = 1$ $\Rightarrow 2z = \frac{3}{2}x$, $x^2 z = 1$

$z = \frac{3}{4}x$, $x^2 z = 1$ $\Rightarrow x^2\left(\frac{3}{4}x\right) = 1$
$\Rightarrow x^3 = \frac{4}{3}$

$\therefore x = \left(\frac{4}{3}\right)^{\frac{1}{3}}$

$z = \frac{3}{4}\left(\frac{4}{3}\right)^{\frac{1}{3}} = \frac{3^1 \times 4^{\frac{1}{3}}}{4^1 \times 3^{\frac{1}{3}}} = 3^{\frac{2}{3}} 4^{-\frac{2}{3}} = \left(\frac{3}{4}\right)^{\frac{2}{3}}$

$y = x = \left(\frac{4}{3}\right)^{\frac{1}{3}}$

HENCE

$A = \frac{3}{2}xy + 2yz + 2xz = \frac{3}{2}\left(\frac{4}{3}\right)^{\frac{2}{3}} + 2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}} + 2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$= 2 \times \frac{3}{4}\left(\frac{4}{3}\right)^{\frac{2}{3}} + 2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}} + 2\left(\frac{4}{3}\right)^{\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$= 2 \times \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^{-\frac{2}{3}} + 2\left(\frac{3}{4}\right)^{-\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}} + 2\left(\frac{3}{4}\right)^{-\frac{1}{3}}\left(\frac{3}{4}\right)^{\frac{2}{3}}$

$= 2\left(\frac{3}{4}\right)^{\frac{1}{3}} + 2\left(\frac{3}{4}\right)^{\frac{1}{3}} + 2\left(\frac{3}{4}\right)^{\frac{1}{3}}$

$= 6 \times \left(\frac{3}{4}\right)^{\frac{1}{3}}$

$= 6 \times \frac{3^{\frac{1}{3}}}{4^{\frac{1}{3}}} \times \frac{4^{\frac{2}{3}}}{4^{\frac{2}{3}}}$

$= 6 \times \frac{3^{\frac{1}{3}} \times 16^{\frac{1}{3}}}{4}$

$= \frac{3}{2} \times 48^{\frac{1}{3}} = \frac{3}{2} \times \sqrt[3]{48} = \frac{3}{2}\sqrt[3]{8}\sqrt[3]{6}$

$= \frac{3}{2} \times 2 \times \sqrt[3]{6} = 3\sqrt[3]{6}$

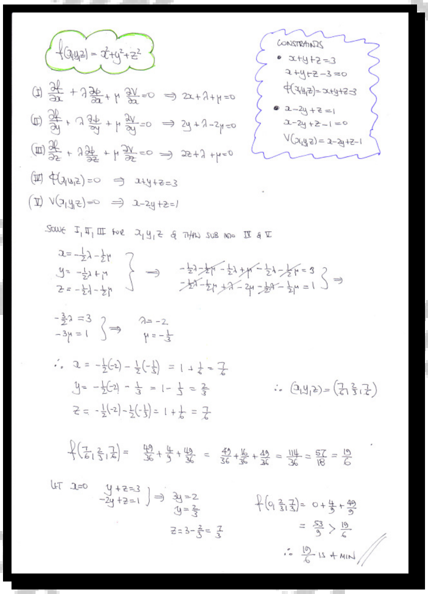
Question 8 (***+)

$$f(x,y,z) = x^2 + y^2 + z^2 .$$

Determine the minimum value of F, subject to the constraints

$$x + y + z = 3 \quad \text{and} \quad x - 2y + z = 1 .$$

$$\boxed{f_{\min} = \frac{19}{6}}$$



$f(x,y,z) = x^2+y^2+z^2$

CONSTRAINTS
- $x+y+z=3$
$x+y+z-3=0$
$\phi(x,y,z)=x+y+z-3$
- $x-2y+z=1$
$x-2y+z-1=0$
$V(x,y,z)=x-2y+z-1$

(I) $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + \mu \frac{\partial V}{\partial x} = 0 \Rightarrow 2x+\lambda+\mu=0$

(II) $\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + \mu \frac{\partial V}{\partial y} = 0 \Rightarrow 2y+\lambda-2\mu=0$

(III) $\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} + \mu \frac{\partial V}{\partial z} = 0 \Rightarrow 2z+\lambda+\mu=0$

(IV) $\phi(x,y,z)=0 \Rightarrow x+y+z=3$

(V) $V(x,y,z)=0 \Rightarrow x-2y+z=1$

SOLVE I, II, III FOR x, y, z & THEN SUB INTO IV & V

$$\left.\begin{array}{l} x=-\frac{1}{2}\lambda-\frac{1}{2}\mu \\ y=-\frac{1}{2}\lambda+\mu \\ z=-\frac{1}{2}\lambda-\frac{1}{2}\mu \end{array}\right\} \Rightarrow \left.\begin{array}{l} -\frac{1}{2}\lambda\cancel{-\frac{1}{2}\mu}-\frac{1}{2}\lambda\cancel{+\mu}-\frac{1}{2}\lambda\cancel{-\frac{1}{2}\mu}=3 \\ \cancel{-\frac{1}{2}\lambda}-\frac{1}{2}\mu\cancel{+\lambda}-2\mu\cancel{-\frac{1}{2}\lambda}-\frac{1}{2}\mu=1 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} -\frac{3}{2}\lambda=3 \\ -3\mu=1 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda=-2 \\ \mu=-\frac{1}{3} \end{array}$$

$\therefore\ x=-\frac{1}{2}(-2)-\frac{1}{2}(-\frac{1}{3})=1+\frac{1}{6}=\frac{7}{6}$

$y=-\frac{1}{2}(-2)-\frac{1}{3}=1-\frac{1}{3}=\frac{2}{3}$

$z=-\frac{1}{2}(-2)-\frac{1}{2}(-\frac{1}{3})=1+\frac{1}{6}=\frac{7}{6}$

$\therefore\ (x,y,z)=(\frac{7}{6},\frac{2}{3},\frac{7}{6})$

$f(\frac{7}{6},\frac{2}{3},\frac{7}{6}) = \frac{49}{36}+\frac{4}{9}+\frac{49}{36} = \frac{49}{36}+\frac{16}{36}+\frac{49}{36} = \frac{114}{36} = \frac{57}{18} = \frac{19}{6}$

LET $x=0$ $\left.\begin{array}{l} y+z=3 \\ -2y+z=1 \end{array}\right\} \Rightarrow 3y=2$, $y=\frac{2}{3}$, $z=3-\frac{2}{3}=\frac{7}{3}$

$f(0,\frac{2}{3},\frac{7}{3}) = 0+\frac{4}{9}+\frac{49}{9} = \frac{53}{9} > \frac{19}{6}$

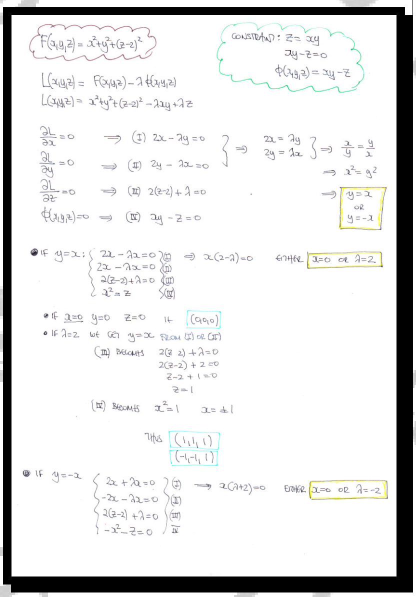
$\therefore\ \frac{19}{6}$ IS A MIN

Question 9 (***+)

$$F(x,y,z) = x^2 + y^2 + (z-2)^2.$$

Determine the minimum value of F, subject to the constraint $z = xy$.

$$\boxed{F_{\min} = 3}$$



$F(x,y,z) = x^2+y^2+(z-2)^2$

CONSTRAINT: $z = xy$
$xy - z = 0$
$\phi(x,y,z) = xy - z$

$L(x,y,z) = F(x,y,z) - \lambda \phi(x,y,z)$
$L(x,y,z) = x^2+y^2+(z-2)^2 - \lambda xy + \lambda z$

$\frac{\partial L}{\partial x} = 0 \implies$ (I) $2x - \lambda y = 0$
$\frac{\partial L}{\partial y} = 0 \implies$ (II) $2y - \lambda x = 0$
$\Bigg\} \implies \begin{array}{l} 2x = \lambda y \\ 2y = \lambda x \end{array} \Bigg\} \implies \frac{x}{y} = \frac{y}{x} \implies x^2 = y^2 \implies \boxed{y = x \text{ or } y = -x}$

$\frac{\partial L}{\partial z} = 0 \implies$ (III) $2(z-2) + \lambda = 0$
$\phi(x,y,z) = 0 \implies$ (IV) $xy - z = 0$

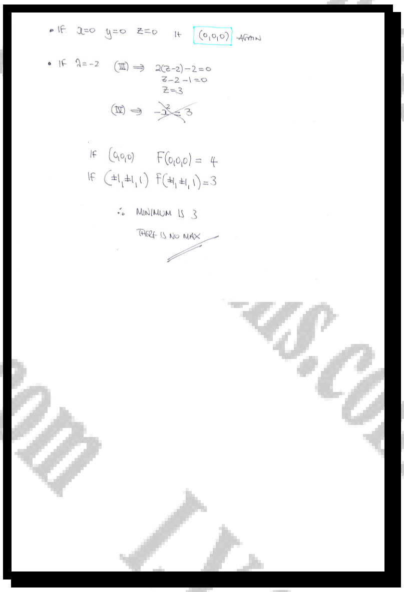
- IF $y = x$: $\left\{\begin{array}{ll} 2x - \lambda x = 0 & \text{(I)} \\ 2x - \lambda x = 0 & \text{(II)} \\ 2(z-2) + \lambda = 0 & \text{(III)} \\ x^2 = z & \text{(IV)} \end{array}\right. \implies x(2-\lambda) = 0$ EITHER $\boxed{x = 0 \text{ or } \lambda = 2}$
  - IF $x = 0$ $y = 0$ $z = 0$ i.e. $\boxed{(0,0,0)}$
  - IF $\lambda = 2$ WE GET $y = x$ FROM (I) OR (II)

    (III) BECOMES $2(z-2) + \lambda = 0$
    $2(z-2) + 2 = 0$
    $z - 2 + 1 = 0$
    $z = 1$

    (IV) BECOMES $x^2 = 1$ $x = \pm 1$

    THUS $\boxed{(1,1,1)}$ $\boxed{(-1,-1,1)}$

- IF $y = -x$ $\left\{\begin{array}{ll} 2x + \lambda x = 0 & \text{(I)} \\ -2x - \lambda x = 0 & \text{(II)} \\ 2(z-2) + \lambda = 0 & \text{(III)} \\ -x^2 - z = 0 & \text{(IV)} \end{array}\right. \longrightarrow x(\lambda+2) = 0$ EITHER $\boxed{x = 0 \text{ or } \lambda = -2}$



- IF $x = 0$ $y = 0$ $z = 0$ i.e. $\boxed{(0,0,0)}$ AGAIN
- IF $\lambda = -2$ (III) $\implies 2(z-2) - 2 = 0$
  $z - 2 - 1 = 0$
  $z = 3$

  (IV) $\implies \cancel{-x^2 = 3}$

IF $(0,0,0)$ $F(0,0,0) = 4$
IF $(\pm 1, \pm 1, 1)$ $F(\pm 1, \pm 1, 1) = 3$

$\therefore$ MINIMUM IS 3

THERE IS NO MAX

Question 10 (***+)

The points P and Q lie on the intersection of the sphere and cylinder with respective Cartesian equations

$$x^2 + y^2 + z^2 = 9 \quad \text{and} \quad x^2 + y^2 = 8$$

The position of P is such so that the distance of P from the point $(5,5,5)$ is least.
The position of Q is such so that the distance of Q from the point $(5,5,5)$ is greatest.

Determine the coordinates of P and the coordinates of Q.

Give the corresponding distance from $(5,5,5)$ in each case.

$\boxed{d_{\text{min}} = \sqrt{34}, \quad P(2,2,1)}$, $\boxed{d_{\text{max}} = \sqrt{134}, \quad P(-2,-2,1)}$

DISTANCE FROM $(5,5,5)$ IS $\sqrt{(x-5)^2+(y-5)^2+(z-5)^2}$

LET $f(x,y,z) = (x-5)^2+(y-5)^2+(z-5)^2$

$\phi(x,y,z) = x^2+y^2-8$
$V(x,y,z) = z^2-1$

SUBJECT TO THE CONSTRAINTS
$x^2+y^2+z^2=9$
$x^2+y^2=8$
IE $x^2+y^2=8$
$z^2=1$

FORM LAGRANGIAN

$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} + \mu\frac{\partial V}{\partial x} \Rightarrow 2(x-5)+2\lambda x+2\mu x=0$

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} + \mu\frac{\partial V}{\partial y} \Rightarrow 2(y-5)+2\lambda y+2\mu y=0$

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} + \mu\frac{\partial V}{\partial z} \Rightarrow 2(z-5)+2\lambda z=0$

TIDY ALL THE AVAILABLE EQUATIONS

$x-5+\lambda x+\mu x=0$ ...... (1)
$y-5+\lambda y+\mu y=0$ ...... (2)
$z-5+\lambda z=0$ ...... (3)
$x^2+y^2=8$ ...... (4)
$z^2=1$ ...... (5)

FROM (3)
$z+\lambda z=5$
$z(\lambda+1)=5$
$z^2(\lambda+1)^2=25$
$(\lambda+1)^2=25$
$\lambda+1 = 5$ or -5
$\lambda = 4$ or -6

$\boxed{\text{IF } \lambda=4}$

- $5x+\mu x=5$
- $5y+\mu y=5$
- $x^2+y^2=8$
- $z^2=1$

$\Rightarrow$

$\left\{\begin{array}{l}(\mu+5)x=5\\(\mu+5)y=5\end{array}\right\} \Rightarrow$

- DIVIDE EQUATIONS

$\frac{x}{y}=1$

$\boxed{y=x}$

$\therefore 2x^2=8$
$x^2=4$
$x = 2$ or -2
$y = 2$ or -2

$z=\pm 1$
WITH ANY PAIR

$\boxed{\text{IF } \lambda=-6}$

- $-5x+\mu x=5$
- $-5y+\mu y=5$
- $x^2+y^2=8$
- $z^2=1$

$\Rightarrow$

$\left\{\begin{array}{l}(\mu-5)x=5\\(\mu-5)y=5\end{array}\right\} \Rightarrow$

- DIVIDE EQUATIONS

$\frac{x}{y}=1$

$y=x$

WHICH LEADS TO THE SAME RESULTS...

THUS

x	2	2	−2	−2
y	2	2	−2	−2
z	1	−1	1	−1
d	$\sqrt{34}$	$\sqrt{54}$	$\sqrt{114}$	$\sqrt{134}$

$\therefore (2,2,1)$ PRODUCES THE MINIMUM DISTANCE OF $\sqrt{34}$
&
$(-2,-2,-1)$ PRODUCES THE MAXIMUM DISTANCE OF $\sqrt{134}$

Question 11 (****)

The function F is defined in cylindrical polar coordinates (r,θ,z) as

$$F(r,\theta,z) = r^2 + \sin^2\theta - z\,,\quad r \ge 0\,,\ 0 \le \theta < 2\pi$$

Determine the minimum value and the maximum value of F, subject to the constraint

$$z = 2r^2 \sin\theta - 1\,.$$

$\boxed{F_{\text{min}} = -1}$, $\boxed{F_{\text{max}} = \frac{1}{2}}$

$\boxed{F(r,\theta,z) = r^2 + \sin^2\theta - z \qquad \text{SUBJECT TO} \qquad z = 2r^2\sin^2\theta - 1,\ 0 \le \theta < 2\pi,\ r > 0}$

- LET THE CONSTRAINT BE $f(r,\theta,z) = z - 2r^2\sin^2\theta + 1$

THEN

$\frac{\partial F}{\partial r} + \lambda \frac{\partial f}{\partial r} = 0 \implies 2r + \lambda(-4r\sin^2\theta) = 0$ (I)

$\frac{\partial F}{\partial \theta} + \lambda \frac{\partial f}{\partial \theta} = 0 \implies 2\sin\theta\cos\theta + \lambda(-4r^2\sin\theta\cos\theta) = 0$ (II)

$\frac{\partial F}{\partial z} + \lambda \frac{\partial f}{\partial z} = 0 \implies -1 + \lambda \times 1 = 0$ (III)

AND

$f(r,\theta,z) = 0 \implies z = 2r^2\sin^2\theta - 1$ (IV)

- TRIVIALLY EQUATION (III) YIELDS $\lambda = 1$ — THUS THE FIRST 2 EQUATIONS SIMPLIFY TO

$2r - 4r\sin^2\theta = 0$ & $2\sin\theta\cos\theta - 4r^2\sin\theta\cos\theta = 0$

$2r(1 - 2\sin^2\theta) = 0$ $2\sin\theta\cos\theta(1 - 2r^2) = 0$

(I) $\boxed{2r\cos 2\theta = 0}$ (II) $\boxed{(1-2r^2)\sin 2\theta = 0}$

- IF $r = 0$ THE FIRST EQUATION IS SATISFIED
  THE SECOND EQUATION GIVES
  $\sin 2\theta = 0$
  $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ (IRRELEVANT IN POLARS IF $r = 0$)
  AND $z = 1$ (FROM THE CONSTRAINT)

- IF $r = +\frac{1}{\sqrt{2}}$ THE SECOND EQUATION IS SATISFIED
  THE FIRST EQUATION GIVES
  $\cos 2\theta = 0$
  $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

AND FROM THE CONSTRAINT $z = 2r^2\sin^2\theta - 1$

$z = 2\left(\frac{1}{\sqrt{2}}\right)^2 \sin^2\left(\text{“ANY OF THESE”}\right) - 1$

$z = 1 \times \frac{1}{2} - 1$

$z = -\frac{1}{2}$

- SUMMARIZING ALL THE RESULTS

r	0	0	0	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
z	1	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

- EVALUATING THESE FROM $F(r,\theta,z) = r^2 + \sin^2\theta - z$ WE OBTAIN IN THE RESPECTIVE ORDER ABOVE

r^2	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\sin^2\theta$	0	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$-z$	-1	-1	-1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$F(r,\theta,z)$	-1	0	-1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\therefore$ F_{MAX} IS $\frac{1}{2}$ AT $r = \frac{1}{2}$, $z = -\frac{1}{2}$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

F_{MIN} IS -1 AT $r = 0$, $z = -1$, $\left[\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right]$

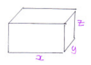
Question 12 (**+)**

A brick is made so that

- the sum of the lengths of all of its edges is 40
- the sum of the area of all of its faces is 24

Use differentiation to find the maximum volume of the brick.

$$\boxed{\frac{1000}{27}}$$



$4x + 4y + 4z = 40 \Rightarrow \boxed{x+y+z=10}$

$2xy + 2yz + 2zx = 24 \Rightarrow \boxed{xy+yz+zx=12}$

- $V = xyz$ • $\phi = x+y+z-10$ • $\psi = xy+yz+zx-12$

(I) $\frac{\partial V}{\partial x} + \lambda \frac{\partial \phi}{\partial x} + \mu \frac{\partial \psi}{\partial x} = 0 \Rightarrow yz + \lambda + \mu(y+z) = 0$ (I)

(II) $\frac{\partial V}{\partial y} + \lambda \frac{\partial \phi}{\partial y} + \mu \frac{\partial \psi}{\partial y} = 0 \Rightarrow xz + \lambda + \mu(x+z) = 0$ (II)

(III) $\frac{\partial V}{\partial z} + \lambda \frac{\partial \phi}{\partial z} + \mu \frac{\partial \psi}{\partial z} = 0 \Rightarrow xy + \lambda + \mu(y+x) = 0$ (III)

(IV) $\phi(x,y,z) = 0 \Rightarrow x+y+z=10$

(V) $\psi(x,y,z) = 0 \Rightarrow xy+yz+zx=12$

- ADDING EQUATIONS I + II + III

$yz + xz + xy + 3\lambda + \mu(2x+2y+2z) = 0$

$12 + 3\lambda + \mu(2 \times 10) = 0$

$\boxed{12 + 3\lambda + 20\mu = 0}$

- $x\text{I} + y\text{II} + z\text{III}$ GIVES $\Rightarrow$

$xyz + x\lambda + \mu(xy + xz) = 0$

$xyz + y\lambda + \mu(xy + zy) = 0$

$xyz + \lambda z + \mu(yz + xz) = 0$

$3xyz + \lambda(x+y+z) + \mu(2xy + 2yz + 2zx) = 0$

$\boxed{3V + 10\lambda + 36\mu = 0}$

- (I) − (II) $(y-x)z + \mu(y-x) = 0$

$\boxed{(y-x)(z+\mu) = 0}$

$y = x$ OR $z = -\mu$

SIMILARLY THE OTHER TWO PAIRINGS

THUS

$x = y$ OR $z = -\mu$

$y = z$ OR $x = -\mu$

$z = x$ OR $y = -\mu$

IN THEORY 8 WAYS THEY COMBINE BUT IN REALITY ONLY 4

- $x = y = z$
- $x = y = -\mu$
- $y = z = -\mu$
- $x = z = -\mu$

IF $x = y = z \Rightarrow x = y = z = \frac{10}{3} \Rightarrow V = \frac{10}{3} \times \frac{10}{3} \times \frac{10}{3} = \boxed{\frac{1000}{27}}$

IF $x = y = -\mu$

$xy + \lambda + \mu(y+x) = 0$

$(-\mu)(-\mu) + \lambda + \mu(-2\mu) = 0$

$\mu^2 - 2\mu^2 + \lambda = 0$

$\boxed{\lambda = \mu^2}$

&

$12 + 3\lambda + 20\mu = 0$

$12 + 3\mu^2 + 20\mu = 0$

$3\mu^2 + 20\mu + 12 = 0$

$(3\mu + 2)(\mu + 6)$

$\mu = -6$ or $-\frac{2}{3}$

$\lambda = 36$ or $\frac{4}{9}$

$\boxed{V = \frac{-36\mu - 10\lambda}{3}}$

IF $\lambda = 36, \mu = -6$ $V = \frac{216 - 360}{3} < 0$

IF $\lambda = \frac{4}{9}, \mu = -\frac{2}{3}$ $V = \frac{-36 \times -\frac{2}{3} - 10 \times \frac{4}{9}}{3} = \frac{24 - \frac{40}{9}}{3} = \frac{216 - 40}{27} = \boxed{\frac{176}{27}}$

BY SYMMETRY THE OTHER TWO PAIRING PRODUCE THE SAME

$\therefore$ MAX VALUE IS $\frac{1000}{27}$

APPLICATIONS TO O.D.E.s

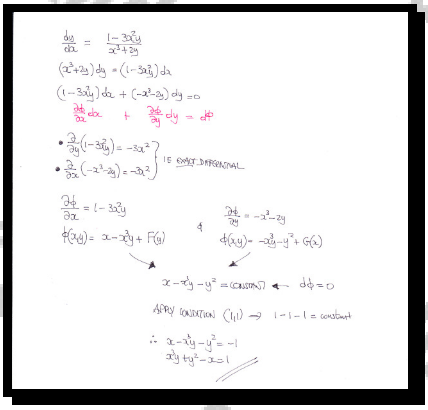
Question 1 (***)

Find the solution of the following differential equation

$$\frac{dy}{dx} = \frac{1-3x^2y}{x^3+2y},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^3y + y^2 - x = 1}$$



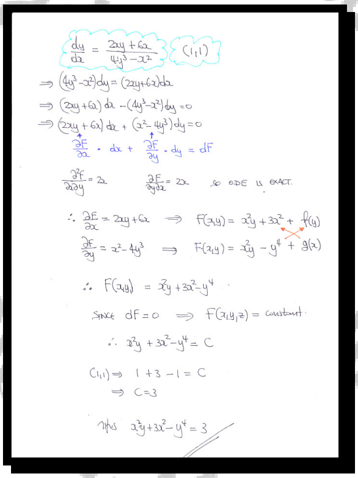
Question 2 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy + 6x}{4y^3 - x^2},$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{x^2 y + 3x^2 - y^4 = 3}$$



$$\frac{dy}{dx} = \frac{2xy+6x}{4y^3-x^2} \qquad (1,1)$$

$\Rightarrow (4y^3 - x^2)dy = (2xy+6x)dx$

$\Rightarrow (2xy+6x)\,dx - (4y^3-x^2)\,dy = 0$

$\Rightarrow (2xy+6x)\,dx + (x^2-4y^3)\,dy = 0$

$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = dF$

$\frac{\partial^2 F}{\partial x \partial y} = 2x \qquad \frac{\partial F}{\partial y \partial x} = 2x$ so O.D.E is EXACT.

$\therefore \frac{\partial F}{\partial x} = 2xy + 6x \Rightarrow F(x,y) = x^2y + 3x^2 + f(y)$

$\frac{\partial F}{\partial y} = x^2 - 4y^3 \Rightarrow F(x,y) = x^2y - y^4 + g(x)$

$\therefore F(x,y) = x^2y + 3x^2 - y^4$

Since $dF = 0 \Rightarrow F(x,y,z) = \text{constant}$

$\therefore x^2y + 3x^2 - y^4 = C$

$(1,1) \Rightarrow 1 + 3 - 1 = C$

$\Rightarrow C = 3$

Thus $x^2y + 3x^2 - y^4 = 3$

Question 3 (***)

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{y\left(y^2-3x^2+1\right)}{x\left(x^2-3y^2-1\right)}.$$

$$\boxed{xy\left(x^2-y^2-1\right)=\text{constant}}$$

$$\frac{dy}{dx}=\frac{y(y^2-3x^2+1)}{x(x^2-3y^2-1)}=\frac{y^3-3x^2y+y}{x^3-3xy^2-x}$$

- $\frac{dy}{dx}=-\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$ WHERE $\phi(x,y)=0$
- DIFFERENTIATE AGAIN TO CHECK

$$\left.\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)&=\underline{3y^2-3x^2+1}\\ \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)&=-(3x^2-3y^2-1)=\underline{3y^2-3x^2+1}\end{aligned}\right\}\text{ so "EXACT"}$$

- $\frac{\partial\phi}{\partial x}=y^3-3x^2y+y$

  $\phi(x,y)=y^3x-x^3y+xy+F(y)$

- $\frac{\partial\phi}{\partial y}=-(x^3-3xy^2-x)$

  $\frac{\partial\phi}{\partial y}=-x^3+3xy^2+x$

  $\phi(x,y)=-x^3y+y^3x+xy+G(x)$

COMPARING

$\phi(x,y)=xy+y^3x-x^3y+C$

OR

$xy\left[y^2-x^2+1\right]=k$

Question 4 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{4e^{2x} - y\left(2e^{2x}+1\right)}{e^{2x}+x},$$

subject to the boundary condition $y = 2$ at $x = 0$.

$$\boxed{y = \frac{2e^{2x}}{e^{2x}+x}}$$

$$\frac{dy}{dx} = \frac{4e^{2x} - y(2e^{2x}+1)}{e^{2x}+x} \quad \text{SUBJECT TO } (0,2)$$

$$(e^{2x}+x)\,dy = \left[4e^{2x} - y(2e^{2x}+1)\right]dx$$

$$0 = \left[4e^{2x} - y(2e^{2x}+1)\right]dx - (e^{2x}+x)\,dy$$

$$(4e^{2x} - 2ye^{2x} - y)\,dx + (-e^{2x} - x)\,dy = 0$$

$$\frac{\partial F}{\partial x}\cdot dx + \frac{\partial F}{\partial y}\cdot dy = dF$$

$$\frac{\partial^2 F}{\partial x \partial y} = -2e^{2x} - 1 \qquad \frac{\partial^2 F}{\partial y \partial x} = -2e^{2x} - 1 \qquad \therefore \text{ EXACT DIFFERENTIAL}$$

- $\frac{\partial F}{\partial x} = 4e^{2x} - 2ye^{2x} - y \implies F(x,y) = 2e^{2x} - ye^{2x} - xy + f(y)$ constant
- $\frac{\partial F}{\partial y} = -e^{2x} - x \implies F(x,y) = -ye^{2x} - xy + g(x)$

$$\therefore F(x,y) = 2e^{2x} - ye^{2x} - xy$$

SINCE $dF = 0$

$$F(x,y) = \text{constant}$$

$$2e^{2x} - ye^{2x} - xy = C$$

APPLY $(0,2) \implies 2 - 2 - 0 = C$

$$C = 0$$

$$\therefore 2e^{2x} - ye^{2x} - xy = 0$$

$$2e^{2x} = ye^{2x} + xy$$

$$2e^{2x} = y(e^{2x}+x)$$

$$y = \frac{2e^{2x}}{e^{2x}+x}$$

Question 5 (***+)

Find a general solution of the following differential equation

$$\frac{dy}{dx}=\frac{\cos x\cos y+\sin^2 x}{\sin x\sin y+\cos^2 y}.$$

$$\boxed{\sin x\cos y-\frac{1}{4}\left(\sin 2x+\sin 2y\right)+\frac{1}{2}\left(x-y\right)=\text{constant}}$$

$$\frac{dy}{dx}=\frac{\cos x\cos y+\sin^2 x}{\sin x\sin y+\cos^2 y}$$

$$\Rightarrow (\sin x\sin y+\cos^2 y)\,dy=(\cos x\cos y+\sin^2 x)\,dx$$

$$\Rightarrow \underbrace{(\cos x\cos y+\sin^2 x)}_{M(x,y)}\,dx\ \underbrace{-(\sin x\sin y+\cos^2 y)}_{N(x,y)}\,dy=0$$

- $\frac{\partial M}{\partial y}=-\cos x\sin y$
- $\frac{\partial N}{\partial x}=-\cos x\sin y$

i.e. $\frac{\partial M}{\partial y}=\frac{\partial M}{\partial x}$

$$\Longrightarrow dF=\frac{\partial F}{\partial x}\,dx+\frac{\partial F}{\partial y}\,dy=0$$

$$\Longrightarrow dF=(\cos x\cos y+\sin^2 x)\,dx+(-\sin x\sin y-\cos^2 y)\,dy=0$$

Thus

$$\frac{\partial}{\partial x}\left[F(x,y)\right]=\cos x\cos y+\sin^2 x$$

$$\frac{\partial}{\partial x}\left[F(x,y)\right]=\cos x\cos y+\frac{1}{2}-\frac{1}{2}\cos 2x$$

$$F(x,y)=\sin x\cos y+\frac{1}{2}x-\frac{1}{4}\sin 2x+f(y)$$

$$\frac{\partial}{\partial y}\left[F(x,y)\right]=-\sin x\sin y-\cos^2 y$$

$$\frac{\partial}{\partial y}\left[F(x,y)\right]=-\sin x\sin y-\frac{1}{2}-\frac{1}{2}\cos 2y$$

$$F(x,y)=\sin x\cos y-\frac{1}{2}y-\frac{1}{4}\sin 2y+g(x)$$

COMPARING AND NOTING THAT $dF=0$ IMPLIES $F(x,y)=\text{constant}$

$$F(x,y)=\sin x\cos y-\frac{1}{4}\sin 2x-\frac{1}{4}\sin 2y+\frac{1}{2}x-\frac{1}{2}y+\text{constant}$$

$$\therefore\ \sin x\cos y-\frac{1}{4}(\sin 2x+\sin 2y)+\frac{1}{2}(x-y)=\text{constant}$$