PARAMETERIZATION

Question 1

A curve C is defined parametrically

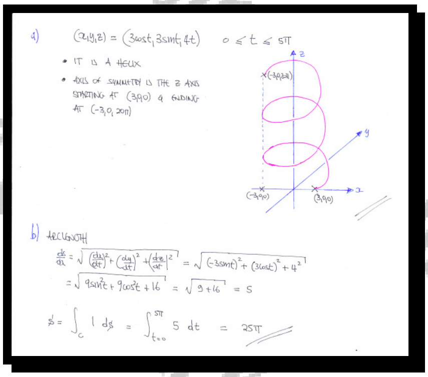
$$(x, y, z) = (3\cos t, 3\sin t, 4t), \quad 0 \le t \le 5\pi .$$

where t is a parameter.

a) Sketch the graph of C.

b) Find the length of C.

$\boxed{25\pi}$



Question 2

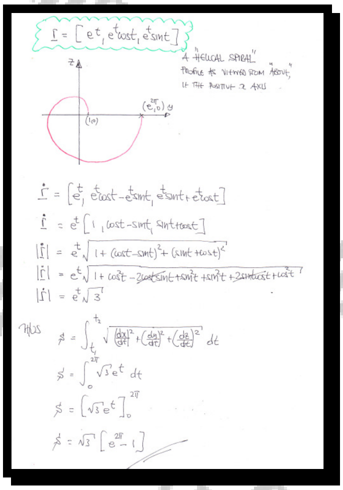
A curve C is defined parametrically

$$(x, y, z) = \left(e^t, e^t \cos t, e^t \sin t\right), \quad 0 \le t \le 2\pi .$$

where t is a parameter.

Describe the graph of C and find its length.

$\boxed{\text{arclength} = \sqrt{3}\left[e^{2\pi} - 1\right]}$


$\underline{r} = \left[e^t, e^t\cos t, e^t\sin t\right]$

A "HELICAL SPIRAL"
PROFILE AS VIEWED FROM "ABOVE", IF THE POSITIVE x AXIS

z, y, (1,0), $(e^{2\pi}, 0)$

$\dot{\underline{r}} = \left[e^t, e^t\cos t - e^t\sin t, e^t\sin t + e^t\cos t\right]$

$\dot{\underline{r}} = e^t\left[1, \cos t - \sin t, \sin t + \cos t\right]$

$|\dot{\underline{r}}| = e^t\sqrt{1 + (\cos t - \sin t)^2 + (\sin t + \cos t)^2}$

$|\dot{\underline{r}}| = e^t\sqrt{1 + \cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t}$

$|\dot{\underline{r}}| = e^t\sqrt{3}$

THUS

$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$s = \int_0^{2\pi} \sqrt{3}e^t \, dt$

$s = \left[\sqrt{3}e^t\right]_0^{2\pi}$

$s = \sqrt{3}\left[e^{2\pi} - 1\right]$

Question 3

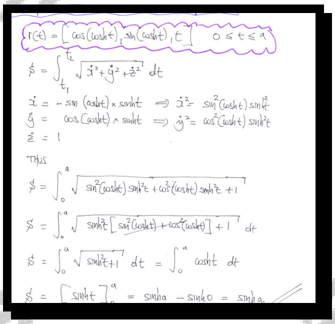
The position vector of a curve C is given by

$$\mathbf{r}(t) = \cos(\cosh t)\,\mathbf{i} + \sin(\cosh t)\,\mathbf{j} + t\,\mathbf{k}\,,$$

where t is a scalar parameter with $0 \le t \le a$, $a \in \mathbb{R}$.

Determine the length of C.

$\boxed{\text{arclength} = \sinh a}$



$\mathbf{r}(t) = [\cos(\cosh t), \sin(\cosh t), t] \quad 0 \le t \le a$

$s = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt$

$\dot{x} = -\sin(\cosh t) \times \sinh t \implies \dot{x}^2 = \sin^2(\cosh t)\sinh^2 t$

$\dot{y} = \cos(\cosh t) \times \sinh t \implies \dot{y}^2 = \cos^2(\cosh t)\sinh^2 t$

$\dot{z} = 1$

Thus

$s = \int_0^a \sqrt{\sin^2(\cosh t)\sinh^2 t + \cos^2(\cosh t)\sinh^2 t + 1}$

$s = \int_0^a \sqrt{\sinh^2 t\,[\sin^2(\cosh t) + \cos^2(\cosh t)] + 1}\, dt$

$s = \int_0^a \sqrt{\sinh^2 t + 1}\, dt = \int_0^a \cosh t\, dt$

$s = [\sinh t]_0^a = \sinh a - \sinh 0 = \sinh a$

Question 4

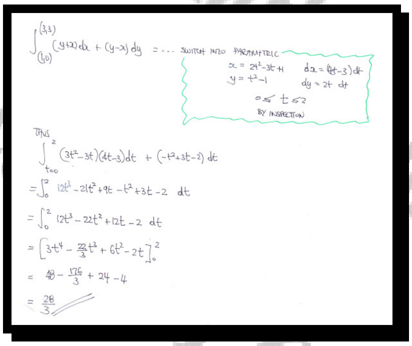
Evaluate the integral

$$\int_{(1,0)}^{(3,3)} (y+x)\,dx + (y-x)\,dy\,,$$

along the curve with parametric equations

$$x = 2t^2 - 3t + 1 \quad \text{and} \quad y = t^2 - 1.$$

$\boxed{\dfrac{28}{3}}$



$$\int_{(1,0)}^{(3,3)} (y+x)\,dx + (y-x)\,dy = \dots$$ SWITCH INTO PARAMETRIC

$x = 2t^2 - 3t + 1$ $\quad dx = (4t-3)\,dt$

$y = t^2 - 1$ $\quad dy = 2t\,dt$

$0 \leqslant t \leqslant 2$ BY INSPECTION

THUS

$$\int_{t=0}^{2} (3t^2 - 3t)(4t-3)\,dt + (-t^2 + 3t - 2)\,dt$$

$$= \int_0^2 12t^3 - 21t^2 + 9t - t^2 + 3t - 2\,dt$$

$$= \int_0^2 12t^3 - 22t^2 + 12t - 2\,dt$$

$$= \left[3t^4 - \frac{22}{3}t^3 + 6t^2 - 2t\right]_0^2$$

$$= 48 - \frac{176}{3} + 24 - 4$$

$$= \frac{28}{3}$$

Question 5

It is given that

$$\mathbf{F}(x, y, z) \equiv \mathbf{j} \wedge \mathbf{r},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

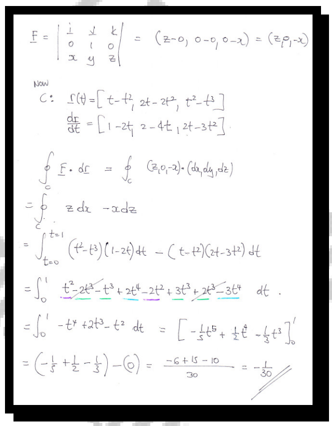
Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr},$$

where C is the closed curve given parametrically by

$$\mathbf{R}(t) = \left(t - t^2\right)\mathbf{i} + \left(2t - 2t^2\right)\mathbf{j} + \left(t^2 - t^3\right)\mathbf{k}, \; 0 \le t \le 1.$$

$$\boxed{-\frac{1}{30}}$$



$$\underline{F} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = (z-0, 0-0, 0-x) = (z, 0, -x)$$

Now

C: $\underline{r}(t) = [t - t^2, 2t - 2t^2, t^2 - t^3]$

$\frac{d\underline{r}}{dt} = [1 - 2t, 2 - 4t, 2t - 3t^2]$.

$$\oint_C \underline{F} \cdot d\underline{r} = \oint_C (z, 0, -x) \cdot (dx, dy, dz)$$

$$= \oint_C z\,dx - x\,dz$$

$$= \int_{t=0}^{t=1} (t^2 - t^3)(1 - 2t)\,dt - (t - t^2)(2t - 3t^2)\,dt$$

$$= \int_0^1 t^2 - 2t^3 - t^3 + 2t^4 - 2t^2 + 3t^3 + 2t^3 - 3t^4 \; dt.$$

$$= \int_0^1 -t^4 + 2t^3 - t^2 \; dt = \left[-\tfrac{1}{5}t^5 + \tfrac{1}{2}t^4 - \tfrac{1}{3}t^3\right]_0^1$$

$$= \left(-\tfrac{1}{5} + \tfrac{1}{2} - \tfrac{1}{3}\right) - (0) = \frac{-6 + 15 - 10}{30} = -\frac{1}{30}$$

Question 6

Evaluate the integral

$$\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx+(y+z)\,dy+\left(1-z^2\right)dz\,,$$

along the straight line segment joining the points with Cartesian coordinates $(1,1,0)$ and $(5,3,4)$.

$\boxed{\frac{32}{3}}$

FIRSTLY $\underline{a}=(1,1,0)$ & $\underline{b}=(5,3,4)$

$\overrightarrow{AB}=\underline{b}-\underline{a}=(5,3,4)-(1,1,0)=(4,2,4)$

PARAMETRIZE THE LINE SEGMENT $x=1+4t$, $y=1+2t$, $z=4t$ $\Rightarrow$ $dx=4\,dt$, $dy=2\,dt$, $dz=4\,dt$

$0 \leqslant t \leqslant 1$

THUS $\int_{(1,1,0)}^{(5,3,4)} (3x-2y)\,dx+(y+z)\,dy+(1-z^2)\,dz$

$= \int_{t=0}^{1} \left[3(1+4t)-2(1+2t)\right](4\,dt)+\left[1+2t+4t\right](2\,dt)+\left[1-16t^2\right](4\,dt)$

$= \int_0^1 4(8t+1)+2(6t+1)+4(1-16t^2)\;dt$

$= \int_0^1 -64t^2+44t+10\;dt$

$= \left[-\frac{64}{3}t^3+22t^2+10t\right]_0^1 = \left(-\frac{64}{3}+22+10\right)-(0) = \frac{32}{3}$

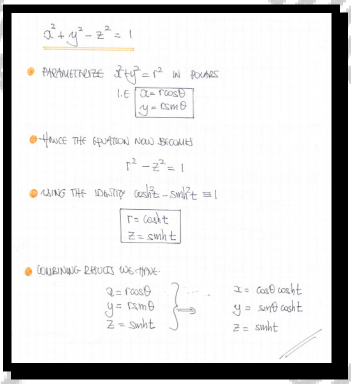
Question 7

A surface S has equation

$$x^2 + y^2 - z^2 = 1.$$

Find a suitable parameterization of S.

$$\boxed{x = \cos\theta \cosh t, \quad y = \sin\theta \cosh t, \quad z = \sinh t}$$



Question 8

The position vector of a curve C is given by

$$\mathbf{r}(t)=\left(\frac{2}{1+t^2}-1\right)\mathbf{i}+\left(\frac{2t}{1+t^2}\right)\mathbf{j},$$

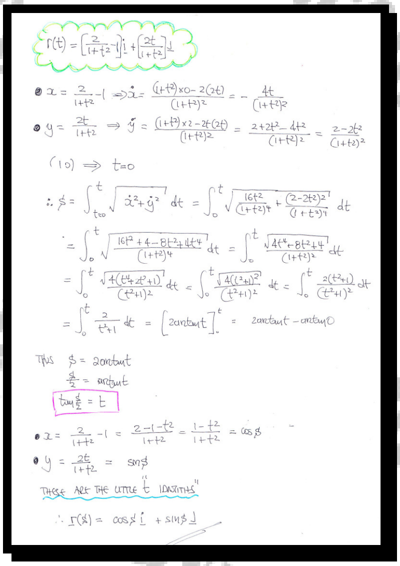
where t is a scalar parameter with $t \in \mathbb{R}$.

Find an expression for the position vector of C, giving the answer in the form

$$\mathbf{r}(s)=f(s)\mathbf{i}+g(s)\mathbf{j},$$

where s is the arc length of a general point on C, measured from the point $(1,0)$.

$$\boxed{\mathbf{r}(s)=(\cos s)\mathbf{i}+(\sin s)\mathbf{j}}$$



$\mathbf{r}(t)=\left[\frac{2}{1+t^2}-1\right]\mathbf{i}+\left[\frac{2t}{1+t^2}\right]\mathbf{j}$

- $x=\frac{2}{1+t^2}-1 \Rightarrow \dot{x}=\frac{(1+t^2)\times 0-2(2t)}{(1+t^2)^2}=-\frac{4t}{(1+t^2)^2}$
- $y=\frac{2t}{1+t^2} \Rightarrow \dot{y}=\frac{(1+t^2)\times 2-2t(2t)}{(1+t^2)^2}=\frac{2+2t^2-4t^2}{(1+t^2)^2}=\frac{2-2t^2}{(1+t^2)^2}$

$(1,0) \Rightarrow t=0$

$\therefore s=\int_{t=0}^{t}\sqrt{\dot{x}^2+\dot{y}^2}\,dt=\int_0^t\sqrt{\frac{16t^2}{(1+t^2)^4}+\frac{(2-2t^2)^2}{(1+t^2)^4}}\,dt$

$=\int_0^t\sqrt{\frac{16t^2+4-8t^2+4t^4}{(1+t^2)^4}}\,dt=\int_0^t\frac{\sqrt{4t^4+8t^2+4}}{(1+t^2)^2}\,dt$

$=\int_0^t\frac{\sqrt{4(t^4+2t^2+1)}}{(t^2+1)^2}\,dt=\int_0^t\frac{\sqrt{4(t^2+1)^2}}{(t^2+1)^2}\,dt=\int_0^t\frac{2(t^2+1)}{(t^2+1)^2}\,dt$

$=\int_0^t\frac{2}{t^2+1}\,dt=\left[2\arctan t\right]_0^t=2\arctan t-\arctan 0$

Thus $s=2\arctan t$

$\frac{s}{2}=\arctan t$

$\tan\frac{s}{2}=t$

- $x=\frac{2}{1+t^2}-1=\frac{2-1-t^2}{1+t^2}=\frac{1-t^2}{1+t^2}=\cos s$
- $y=\frac{2t}{1+t^2}=\sin s$

THESE ARE THE LITTLE "t IDENTITIES"

$\therefore \mathbf{r}(s)=\cos s\,\mathbf{i}+\sin s\,\mathbf{j}$

Question 9

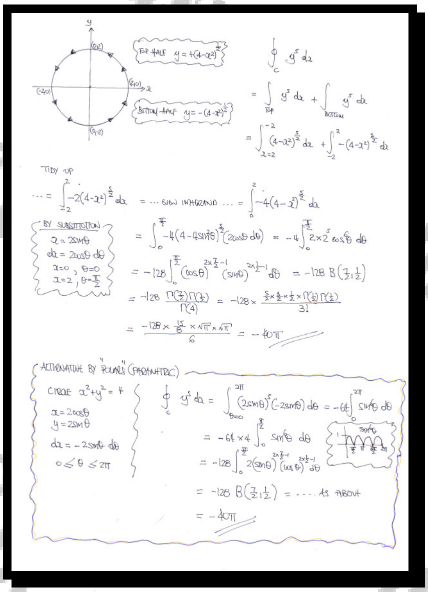
Evaluate the line integral

$$\oint_C y^5 \ dx ,$$

where C is a circle of radius 2, centre at the origin O, traced anticlockwise.

You may not use Green's theorem in this question.

$\boxed{-40\pi}$



TOP HALF $y = +(4-x^2)^{\frac{1}{2}}$

BOTTOM HALF $y = -(4-x^2)^{\frac{1}{2}}$

$\oint_C y^5 \, dx$

$= \int_{\text{TOP}} y^5 \, dx + \int_{\text{BOTTOM}} y^5 \, dx$

$= \int_{x=2}^{-2} (4-x^2)^{\frac{5}{2}} \, dx + \int_{-2}^{2} -(4-x^2)^{\frac{5}{2}} \, dx$

TIDY UP

$\dots = \int_{-2}^{2} -2(4-x^2)^{\frac{5}{2}} \, dx = \dots$ EVEN INTEGRAND $\dots = \int_0^2 -4(4-x^2)^{\frac{5}{2}} \, dx$

BY SUBSTITUTION
$x = 2\sin\theta$
$dx = 2\cos\theta \, d\theta$
$x=0, \ \theta=0$
$x=2, \ \theta=\frac{\pi}{2}$

$= \int_0^{\frac{\pi}{2}} -4(4-4\sin^2\theta)^{\frac{5}{2}}(2\cos\theta \, d\theta) = -4\int_0^{\frac{\pi}{2}} 2 \times 2^5 \cos^6\theta \, d\theta$

$= -128 \int_0^{\frac{\pi}{2}} (\cos\theta)^{2\times\frac{7}{2}-1} (\sin\theta)^{2\times\frac{1}{2}-1} \, d\theta = -128 \, B\left(\frac{7}{2}, \frac{1}{2}\right)$

$= -128 \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{\Gamma(4)} = -128 \times \frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{3!}$

$= \frac{-128 \times \frac{15}{8} \times \sqrt{\pi} \times \sqrt{\pi}}{6} = -40\pi$

ALTERNATIVE BY "POLARS" (PARAMETRIC)

CIRCLE $x^2+y^2=4$
$x = 2\cos\theta$
$y = 2\sin\theta$
$dx = -2\sin\theta \, d\theta$
$0 \le \theta \le 2\pi$

$\oint_C y^5 \, dx = \int_{\theta=0}^{2\pi} (2\sin\theta)^5(-2\sin\theta) \, d\theta = -64\int_0^{2\pi} \sin^6\theta \, d\theta$

$= -64 \times 4 \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta$

$= -128 \int_0^{\frac{\pi}{2}} 2(\sin\theta)^{2\times\frac{7}{2}-1}(\cos\theta)^{2\times\frac{1}{2}-1} \, d\theta$

$= -128 \, B\left(\frac{7}{2}, \frac{1}{2}\right) = \dots$ AS ABOVE

$= -40\pi$

Question 10

A curve C is defined by $\mathbf{r} = \mathbf{r}(t)$, $0 \le t \le 2\pi$ as

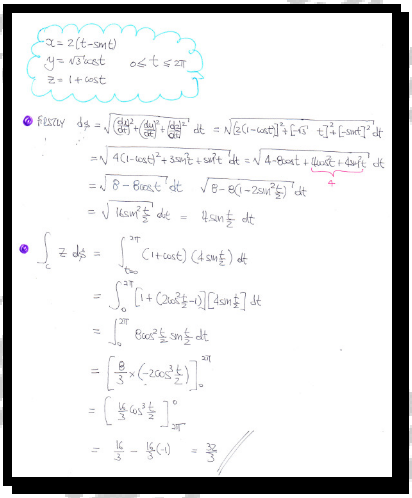
$$\mathbf{r}(t) = (x, y, z) = \left[2(t - \sin t), \sqrt{3}\cos t, 1 + \cos t \right].$$

Evaluate the integral

$$\int_C z \, ds,$$

where s is the arclength along C.

$$\boxed{\frac{32}{3}}$$



$x = 2(t - \sin t)$

$y = \sqrt{3}\cos t \qquad 0 \le t \le 2\pi$

$z = 1 + \cos t$

- FIRSTLY $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{[2(1-\cos t)]^2 + [-\sqrt{3}\ t]^2 + [-\sin t]^2}\, dt$

$= \sqrt{4(1-\cos t)^2 + 3\sin^2 t + \sin^2 t}\, dt = \sqrt{4 - 8\cos t + 4\cos^2 t + 4\sin^2 t}\, dt$ (4)

$= \sqrt{8 - 8\cos t}\, dt \qquad \sqrt{8 - 8(1 - 2\sin^2 \frac{t}{2})}\, dt$

$= \sqrt{16\sin^2 \frac{t}{2}}\, dt = 4\sin\frac{t}{2}\, dt$

- $\int_C z\, ds = \int_{t=0}^{2\pi} (1 + \cos t)(4\sin\frac{t}{2})\, dt$

$= \int_0^{2\pi} \left[1 + (2\cos^2\frac{t}{2} - 1)\right]\left[4\sin\frac{t}{2}\right] dt$

$= \int_0^{2\pi} 8\cos^2\frac{t}{2}\sin\frac{t}{2}\, dt$

$= \left[\frac{8}{3} \times \left(-2\cos^3\frac{t}{2}\right)\right]_0^{2\pi}$

$= \left[\frac{16}{3}\cos^3\frac{t}{2}\right]_{2\pi}^{0}$

$= \frac{16}{3} - \frac{16}{3}(-1) = \frac{32}{3}$

Question 11

$$V(x,y,z) = 60xyz^2 .$$

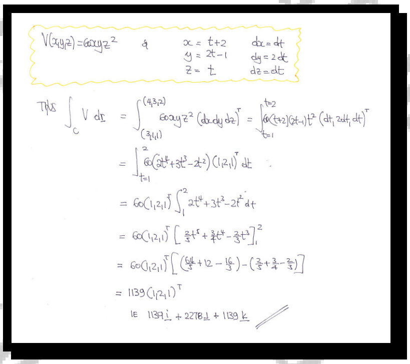
Evaluate the following integral along C, from $(3,1,1)$ to $(4,3,2)$,

$$\int_C V \, \mathbf{dr} , \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} ,$$

where C is the curve with parametric equations

$$x = t + 2 , \quad y = 2t - 1 , \quad z = t .$$

$\boxed{1139\mathbf{i} + 2278\mathbf{j} + 1139\mathbf{k}}$



Question 12

$$\varphi(x, y, z) \equiv 3x + 2y + z .$$

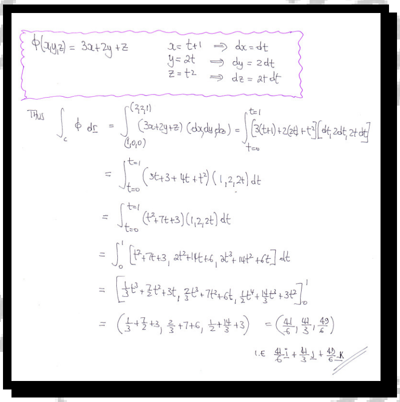
Evaluate the following integral along C, from $(1,0,0)$ to $(2,2,1)$,

$$\int_C \varphi \, \mathbf{dr} , \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} ,$$

where C is the curve with parametric equations

$$x = t + 1, \quad y = 2t , \quad z = t^2 .$$

$$\boxed{\frac{41}{6}\mathbf{i} + \frac{41}{3}\mathbf{j} + \frac{49}{6}\mathbf{k}}$$



$\phi(x,y,z) = 3x + 2y + z$

$x = t+1 \Rightarrow dx = dt$
$y = 2t \Rightarrow dy = 2dt$
$z = t^2 \Rightarrow dz = 2t\,dt$

Thus $\int_c \phi \, d\underline{r} = \int_{(1,0,0)}^{(2,2,1)} (3x+2y+z)(dx,dy,dz) = \int_{t=0}^{t=1} [3(t+1)+2(2t)+t^2][dt, 2dt, 2t\,dt]$

$= \int_{t=0}^{t=1} (3t+3+4t+t^2)(1,2,2t)\,dt$

$= \int_{t=0}^{t=1} (t^2+7t+3)(1,2,2t)\,dt$

$= \int_0^1 [t^2+7t+3, 2t^2+14t+6, 2t^3+14t^2+6t]\,dt$

$= \left[\frac{1}{3}t^3+\frac{7}{2}t^2+3t, \frac{2}{3}t^3+7t^2+6t, \frac{1}{2}t^4+\frac{14}{3}t^3+3t^2\right]_0^1$

$= \left(\frac{1}{3}+\frac{7}{2}+3, \frac{2}{3}+7+6, \frac{1}{2}+\frac{14}{3}+3\right) = \left(\frac{41}{6}, \frac{41}{3}, \frac{49}{6}\right)$

i.e. $\frac{41}{6}\underline{i} + \frac{41}{3}\underline{j} + \frac{49}{6}\underline{k}$

Question 13

$$F(x,y,z) = xyz \, .$$

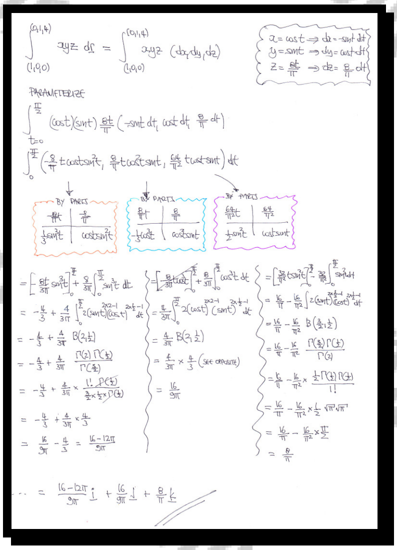
Evaluate the following integral along C, from $(1,0,0)$ to $(0,1,4)$,

$$\int_C F \, \mathbf{dr} \, , \qquad \mathbf{dr} = (dx, dy, dz)^{\mathrm{T}} \, ,$$

where C is the curve with parametric equations

$$x = \cos t \, , \qquad y = \sin t \, , \qquad z = \frac{8t}{\pi} \, .$$

$$\boxed{\frac{16-12\pi}{9\pi}\mathbf{i} + \frac{16}{9\pi}\mathbf{j} + \frac{8}{\pi}\mathbf{k}}$$



Question 14

A curve C has equation

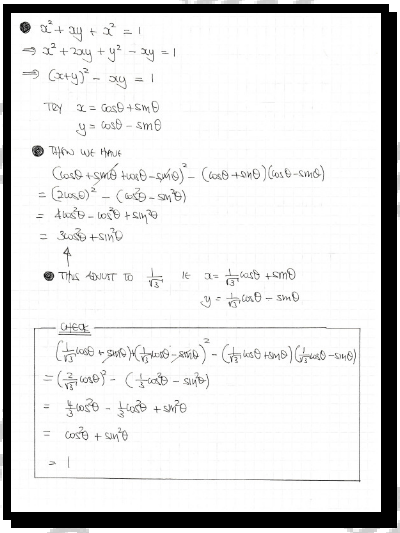
$$x^2 + xy + y^2 = 1. \quad , \quad 0 \le x \le 3.$$

Find a suitable parameterization of C in the form

$$x = A\cos\theta + B\sin\theta \quad \text{and} \quad y = A\cos\theta - B\sin\theta ,$$

where A and B are suitable constants.

$$\boxed{x = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta} , \boxed{x = \frac{1}{\sqrt{3}}\cos\theta - \sin\theta}$$



$x^2 + xy + x^2 = 1$

$\Rightarrow x^2 + 2xy + y^2 - xy = 1$

$\Rightarrow (x+y)^2 - xy = 1$

TRY $x = \cos\theta + \sin\theta$

$y = \cos\theta - \sin\theta$

THEN WE HAVE

$(\cos\theta + \sin\theta + \cos\theta - \sin\theta)^2 - (\cos\theta + \sin\theta)(\cos\theta - \sin\theta)$

$= (2\cos\theta)^2 - (\cos^2\theta - \sin^2\theta)$

$= 4\cos^2\theta - \cos^2\theta + \sin^2\theta$

$= 3\cos^2\theta + \sin^2\theta$

THUS ADJUST TO $\frac{1}{\sqrt{3}}$ IE $x = \frac{1}{\sqrt{3}}\cos\theta + \sin\theta$

$y = \frac{1}{\sqrt{3}}\cos\theta - \sin\theta$

CHECK

$\left(\frac{1}{\sqrt{3}}\cos\theta + \sin\theta + \frac{1}{\sqrt{3}}\cos\theta - \sin\theta\right)^2 - \left(\frac{1}{\sqrt{3}}\cos\theta + \sin\theta\right)\left(\frac{1}{\sqrt{3}}\cos\theta - \sin\theta\right)$

$= \left(\frac{2}{\sqrt{3}}\cos\theta\right)^2 - \left(\frac{1}{3}\cos^2\theta - \sin^2\theta\right)$

$= \frac{4}{3}\cos^2\theta - \frac{1}{3}\cos^2\theta + \sin^2\theta$

$= \cos^2\theta + \sin^2\theta$

$= 1$

Question 15

A surface S is given parametrically by

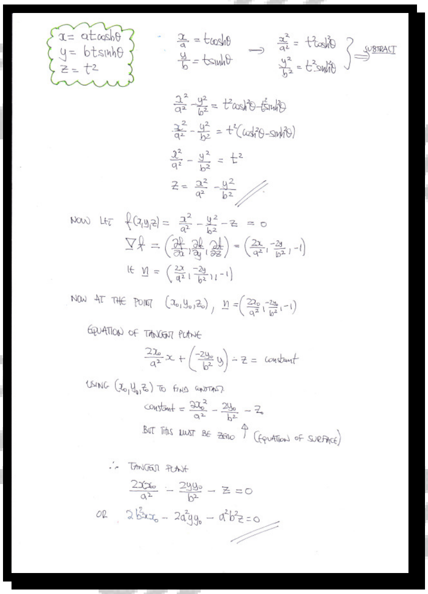
$$x = at\cosh\theta\,,\ x = bt\sinh\theta\,,\ z = t^2\,,$$

where t and θ are real parameters, and a and b are non zero constants .

a) Find a Cartesian equation for S .

b) Determine an equation of the tangent plane on S at the point with Cartesian coordinates (x_0, y_0, z_0) .

$$\boxed{z = \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,,\ \boxed{2b^2 x\,x_0 + 2a^2 y\,y_0 - a^2 b^2 z = 0}$$


$x = at\cosh\theta$, $y = bt\sinh\theta$, $z = t^2$

$\frac{x}{a} = t\cosh\theta$, $\frac{y}{b} = t\sinh\theta$ $\rightarrow$ $\frac{x^2}{a^2} = t^2\cosh^2\theta$, $\frac{y^2}{b^2} = t^2\sinh^2\theta$ $\Big\} \Rightarrow$ SUBTRACT

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2\cosh^2\theta - t^2\sinh^2\theta$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2(\cosh^2\theta - \sinh^2\theta)$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = t^2$

$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

NOW LET $f(x,y,z) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(\frac{2x}{a^2}, \frac{-2y}{b^2}, -1\right)$

i.e. $\underline{n} = \left(\frac{2x}{a^2}, \frac{-2y}{b^2}, -1\right)$

NOW AT THE POINT (x_0, y_0, z_0), $\underline{n} = \left(\frac{2x_0}{a^2}, \frac{-2y_0}{b^2}, -1\right)$

EQUATION OF TANGENT PLANE

$\frac{2x_0}{a^2}x + \left(\frac{-2y_0}{b^2}y\right) - z = \text{constant}$

USING (x_0, y_0, z_0) TO FIND CONSTANT

$\text{constant} = \frac{2x_0^2}{a^2} - \frac{2y_0}{b^2} - z_0$

BUT THIS MUST BE ZERO ↑ (EQUATION OF SURFACE)

∴ TANGENT PLANE

$\frac{2xx_0}{a^2} - \frac{2yy_0}{b^2} - z = 0$

OR $2b^2xx_0 - 2a^2yy_0 - a^2b^2z = 0$

Question 16

In standard notation used for tori, r is the radius of the tube and R is the distance of the centre of the tube from the centre of the torus.

The surface of a torus has parametric equations

$$x(\theta,\varphi) = (R + r\cos\theta)\cos\varphi, \qquad y(\theta,\varphi) = (R + r\cos\theta)\sin\varphi, \qquad z(\theta,\varphi) = r\sin\theta,$$

where $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$.

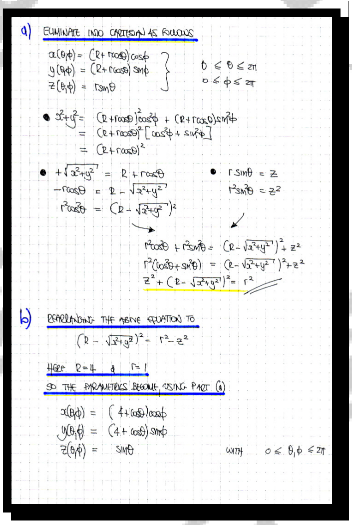
a) Find a general Cartesian equation for the surface of a torus.

A torus T has Cartesian equation

$$\left(4 - \sqrt{x^2 + y^2}\right)^2 = 1 - z^2.$$

b) Use a suitable parameterization of T to find its surface area.

$\boxed{z^2 + \left(R - \sqrt{x^2 + y^2}\right)^2 = r^2}$, $\boxed{\text{area} = (2\pi r)(2\pi R) = 16\pi^2}$


a) ELIMINATE INTO CARTESIAN AS FOLLOWS

$x(\theta,\phi) = (R + r\cos\theta)\cos\phi$, $y(\theta,\phi) = (R + r\cos\theta)\sin\phi$, $z(\theta,\phi) = r\sin\theta$, $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$

- $x^2 + y^2 = (R + r\cos\theta)^2\cos^2\phi + (R + r\cos\theta)^2\sin^2\phi = (R + r\cos\theta)^2[\cos^2\phi + \sin^2\phi] = (R + r\cos\theta)^2$
- $+\sqrt{x^2+y^2} = R + r\cos\theta$, $-r\cos\theta = R - \sqrt{x^2+y^2}$, $r^2\cos^2\theta = (R - \sqrt{x^2+y^2})^2$
- $r\sin\theta = z$, $r^2\sin^2\theta = z^2$

$r^2\cos^2\theta + r^2\sin^2\theta = (R - \sqrt{x^2+y^2})^2 + z^2$

$r^2(\cos^2\theta + \sin^2\theta) = (R - \sqrt{x^2+y^2})^2 + z^2$

$z^2 + (R - \sqrt{x^2+y^2})^2 = r^2$

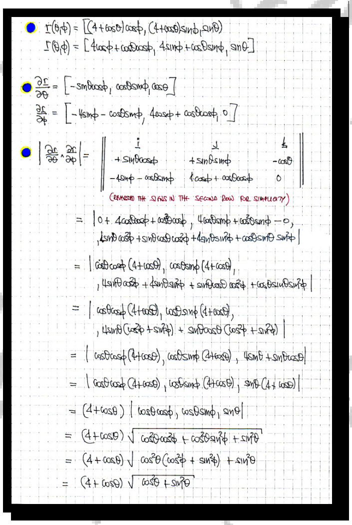
b) REARRANGING THE ABOVE EQUATION TO

$(R - \sqrt{x^2+y^2})^2 = r^2 - z^2$

HERE $R = 4$ & $r = 1$

SO THE PARAMETRICS BECOME, USING PART (a)

$x(\theta,\phi) = (4 + \cos\theta)\cos\phi$, $y(\theta,\phi) = (4 + \cos\theta)\sin\phi$, $z(\theta,\phi) = \sin\theta$ WITH $0 \le \theta,\phi \le 2\pi$


- $\underline{r}(\theta,\phi) = [(4+\cos\theta)\cos\phi, (4+\cos\theta)\sin\phi, \sin\theta]$

$\underline{r}(\theta,\phi) = [4\cos\phi + \cos\theta\cos\phi, 4\sin\phi + \cos\theta\sin\phi, \sin\theta]$

- $\frac{\partial \underline{r}}{\partial \theta} = [-\sin\theta\cos\phi, \cos\theta\sin\phi, \cos\theta]$

$\frac{\partial \underline{r}}{\partial \phi} = [-4\sin\phi - \cos\theta\sin\phi, 4\cos\phi + \cos\theta\cos\phi, 0]$

- $\left|\frac{\partial \underline{r}}{\partial \theta} \wedge \frac{\partial \underline{r}}{\partial \phi}\right| = \left\| \begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ +\sin\theta\cos\phi & +\sin\theta\sin\phi & -\cos\theta \\ -4\sin\phi - \cos\theta\sin\phi & 4\cos\phi + \cos\theta\cos\phi & 0 \end{matrix} \right\|$

(REVERSED THE SIGNS IN THE SECOND ROW FOR SIMPLICITY)

$= |0 + 4\cos\theta\cos\phi + \cos^2\theta\cos\phi, 4\cos\theta\sin\phi + \cos^2\theta\sin\phi - 0, 4\sin\theta\cos^2\phi + \sin\theta\cos\theta\cos^2\phi + 4\sin\theta\sin^2\phi + \cos\theta\sin\theta\sin^2\phi|$

$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), 4\sin\theta\cos^2\phi + 4\sin\theta\sin^2\phi + \sin\theta\cos\theta\cos^2\phi + \cos\theta\sin\theta\sin^2\phi|$

$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), 4\sin\theta(\cos^2\phi + \sin^2\phi) + \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)|$

$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), 4\sin\theta + \sin\theta\cos\theta|$

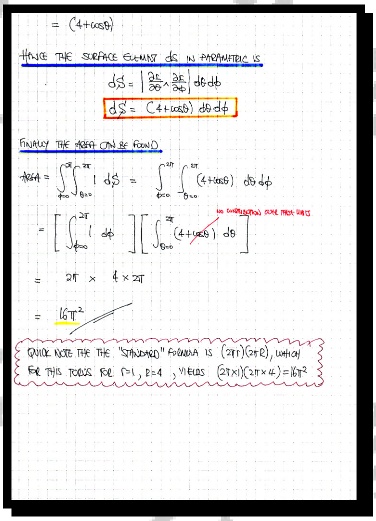
$= |\cos\theta\cos\phi(4+\cos\theta), \cos\theta\sin\phi(4+\cos\theta), \sin\theta(4+\cos\theta)|$

$= (4+\cos\theta)|\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta|$

$= (4+\cos\theta)\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}$

$= (4+\cos\theta)\sqrt{\cos^2\theta(\cos^2\phi + \sin^2\phi) + \sin^2\theta}$

$= (4+\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}$


$= (4+\cos\theta)$

HENCE THE SURFACE ELEMENT dS IN PARAMETRIC IS

$dS = \left|\frac{\partial \underline{r}}{\partial \theta} \wedge \frac{\partial \underline{r}}{\partial \phi}\right| d\theta\, d\phi$

$dS = (4+\cos\theta)\, d\theta\, d\phi$

FINALLY THE AREA CAN BE FOUND

$\text{AREA} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} 1\, dS = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} (4+\cos\theta)\, d\theta\, d\phi$

$= \left[\int_{\phi=0}^{2\pi} 1\, d\phi\right]\left[\int_{\theta=0}^{2\pi} (4+\cos\theta)\, d\theta\right]$ (NO CONTRIBUTION OVER THESE LIMITS)

$= 2\pi \times 4 \times 2\pi$

$= 16\pi^2$

QUICK NOTE THE THE "STANDARD" FORMULA IS $(2\pi r)(2\pi R)$, WHICH FOR THIS TORUS FOR $r=1$, $R=4$, YIELDS $(2\pi\times1)(2\pi\times4) = 16\pi^2$

Question 17

A spiral ramp is modelled by the surface S defined by the vector function

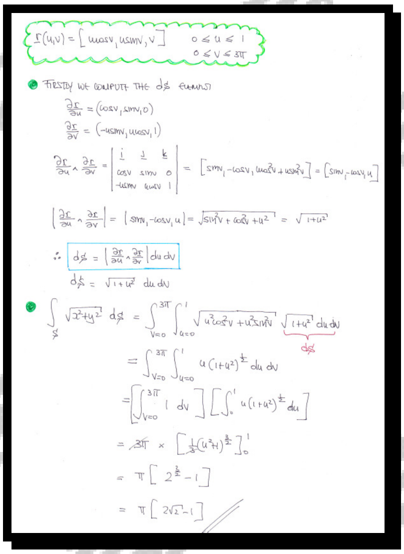
$$\mathbf{r}(u,v) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = (u\cos v)\mathbf{i} + (u\sin v)\mathbf{j} + v\mathbf{k},$$

where $0 \le u \le 1$, $0 \le v \le 3\pi$.

Determine the value of

$$\int_S \sqrt{x^2 + y^2}\ dS$$

$$\boxed{\pi\left[\sqrt{8} - 1\right]}$$



$\underline{r}(u,v) = [u\cos v, u\sin v, v]$ $\quad 0 \le u \le 1$, $0 \le v \le 3\pi$

- FIRSTLY WE COMPUTE THE dS ELEMENT

$$\frac{\partial r}{\partial u} = (\cos v, \sin v, 0)$$

$$\frac{\partial r}{\partial v} = (-u\sin v, u\cos v, 1)$$

$$\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = [\sin v, -\cos v, u\cos^2 v + u\sin^2 v] = [\sin v, -\cos v, u]$$

$$\left|\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right| = |\sin v, -\cos v, u| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\therefore \boxed{dS = \left|\frac{\partial r}{\partial u} \wedge \frac{\partial r}{\partial v}\right| du\,dv}$$

$$dS = \sqrt{1+u^2}\,du\,dv$$

- $$\int_S \sqrt{x^2+y^2}\,dS = \int_{v=0}^{3\pi}\int_{u=0}^{1} \sqrt{u^2\cos^2 v + u^2\sin^2 v}\,\underbrace{\sqrt{1+u^2}\,du\,dv}_{dS}$$

$$= \int_{v=0}^{3\pi}\int_{u=0}^{1} u(1+u^2)^{\frac{1}{2}}\,du\,dv$$

$$= \left[\int_{v=0}^{3\pi} 1\,dv\right]\left[\int_0^1 u(1+u^2)^{\frac{1}{2}}\,du\right]$$

$$= 3\pi \times \left[\tfrac{1}{3}(u^2+1)^{\frac{3}{2}}\right]_0^1$$

$$= \pi\left[2^{\frac{3}{2}} - 1\right]$$

$$= \pi\left[2\sqrt{2} - 1\right]$$

Question 18

The surface S is defined by the vector equation

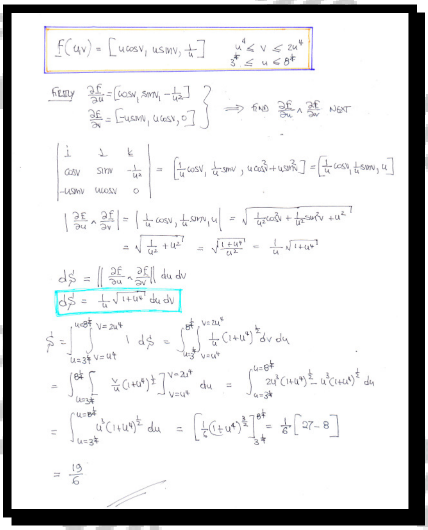
$$\mathbf{F}(u,v) = \left[u\cos v, u\sin v, \frac{1}{u} \right]^T, \; u \neq 0 .$$

Find the area of S lying above the region in the uv plane bounded by the curves

$$v = u^4, \; v = 2u^4,$$

and the straight lines with equations $u = 3^{\frac{1}{4}}$ and $u = 8^{\frac{1}{4}}$.

$\boxed{\frac{19}{6}}$



Question 19

The surface S is defined by the parametric equations

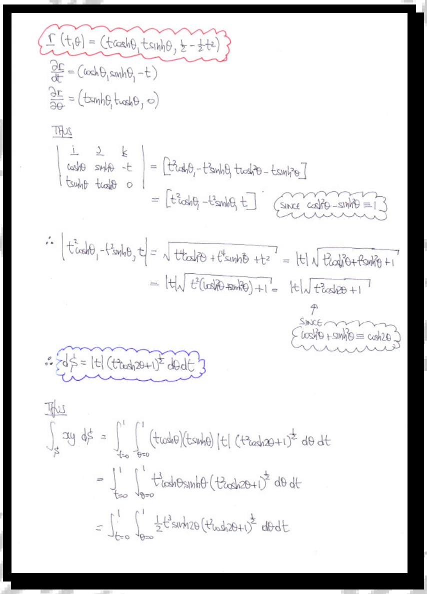
$$x = t\cosh\theta\,,\ \ y = t\sinh\theta\,,\ \ z = \tfrac{1}{2}\left(1 - t^2\right),$$

where t and θ are real parameters such that $0 \le t \le 1$ and $0 \le \theta \le 1$.

Find, in exact form, the value of

$$\int_S xy \ dS\,.$$

$$\boxed{\frac{1}{30}\left[\frac{(\cosh 2 + 1)^{\frac{5}{2}} - 1}{\cosh 2} + 1 - 4\sqrt{2}\right] \approx 0.274397...}$$



$\underline{r}(t,\theta) = \left(t\cosh\theta, t\sinh\theta, \frac{1}{2} - \frac{1}{2}t^2\right)$

$\frac{\partial \underline{r}}{\partial t} = (\cosh\theta, \sinh\theta, -t)$

$\frac{\partial \underline{r}}{\partial \theta} = (t\sinh\theta, t\cosh\theta, 0)$

THUS

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cosh\theta & \sinh\theta & -t \\ t\sinh\theta & t\cosh\theta & 0 \end{vmatrix} = \left[t^2\cosh\theta, -t^2\sinh\theta, t\cosh^2\theta - t\sinh^2\theta\right]$$

$$= \left[t^2\cosh\theta, -t^2\sinh\theta, t\right] \quad \text{SINCE } \cosh^2\theta - \sinh^2\theta \equiv 1$$

$$\therefore \left|t^2\cosh\theta, -t^2\sinh\theta, t\right| = \sqrt{t^4\cosh^2\theta + t^4\sinh^2\theta + t^2} = |t|\sqrt{t^2\cosh^2\theta + t^2\sinh^2\theta + 1}$$

$$= |t|\sqrt{t^2(\cosh^2\theta + \sinh^2\theta) + 1} = |t|\sqrt{t^2\cosh 2\theta + 1}$$

SINCE $\cosh^2\theta + \sinh^2\theta \equiv \cosh 2\theta$

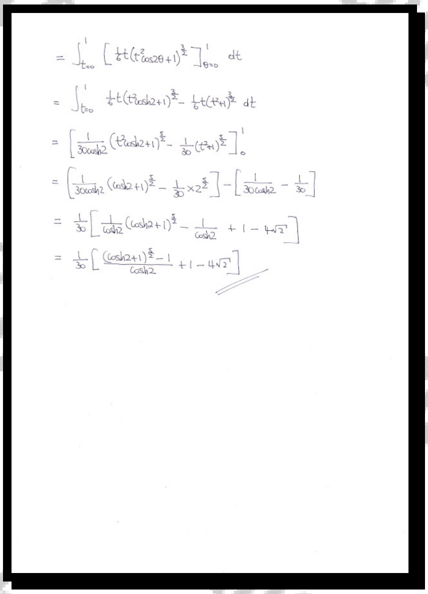
$$\therefore dS = |t|\left(t^2\cosh 2\theta + 1\right)^{\frac{1}{2}} d\theta\, dt$$

THUS

$$\int_S xy\ dS = \int_{t=0}^{1}\int_{\theta=0}^{1} (t\cosh\theta)(t\sinh\theta)\,|t|\left(t^2\cosh 2\theta + 1\right)^{\frac{1}{2}} d\theta\, dt$$

$$= \int_{t=0}^{1}\int_{\theta=0}^{1} t^3\cosh\theta\sinh\theta\left(t^2\cosh 2\theta + 1\right)^{\frac{1}{2}} d\theta\, dt$$

$$= \int_{t=0}^{1}\int_{\theta=0}^{1} \tfrac{1}{2}t^3\sinh 2\theta\left(t^2\cosh 2\theta + 1\right)^{\frac{1}{2}} d\theta\, dt$$



$$= \int_{t=0}^{1}\left[\tfrac{1}{6}t\left(t^2\cos 2\theta + 1\right)^{\frac{3}{2}}\right]_{\theta=0}^{1} dt$$

$$= \int_{t=0}^{1} \tfrac{1}{6}t\left(t^2\cosh 2 + 1\right)^{\frac{3}{2}} - \tfrac{1}{6}t\left(t^2+1\right)^{\frac{3}{2}}\ dt$$

$$= \left[\frac{1}{30\cosh 2}\left(t^2\cosh 2 + 1\right)^{\frac{5}{2}} - \frac{1}{30}\left(t^2+1\right)^{\frac{5}{2}}\right]_0^1$$

$$= \left[\frac{1}{30\cosh 2}(\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{30}\times 2^{\frac{5}{2}}\right] - \left[\frac{1}{30\cosh 2} - \frac{1}{30}\right]$$

$$= \frac{1}{30}\left[\frac{1}{\cosh 2}(\cosh 2 + 1)^{\frac{5}{2}} - \frac{1}{\cosh 2} + 1 - 4\sqrt{2}\right]$$

$$= \frac{1}{30}\left[\frac{(\cosh 2 + 1)^{\frac{5}{2}} - 1}{\cosh 2} + 1 - 4\sqrt{2}\right]$$

Question 20

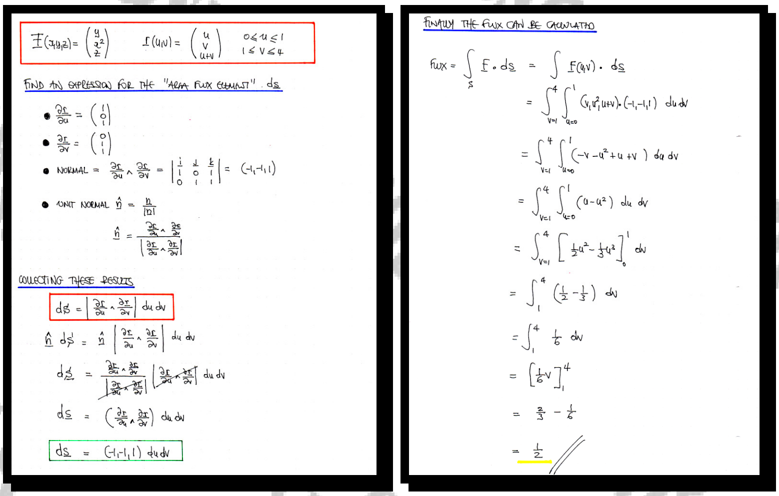
$$\mathbf{F}(x,y,z) \equiv y\,\mathbf{i} + x^2\mathbf{j} + z\,\mathbf{k}\ .$$

Find the magnitude of the flux through the surface with parametric equations

$$\mathbf{r}(u,v) = u\,\mathbf{i} + v\,\mathbf{j} + (u+v)\,\mathbf{k}\ , \quad 0 \le u \le 1,\ 1 \le v \le 4\ .$$

All integrations must be carried out in parametric.

$\boxed{}$, $\boxed{\frac{1}{2}}$



Question 21

Evaluate the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

where S is the surface represented parametrically by

$$\mathbf{r}(u,v) = \begin{bmatrix} u+v \\ u-v \\ u \end{bmatrix}, \quad 0 \le u \le 2, \quad 0 \le v \le 3,$$

and $\mathbf{F}$ is the vector field

$$x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

All integrations must be carried out in parametric.

$\boxed{}$, $\boxed{36}$

PREPARE ALL THE AUXILIARY ITEMS

- $\underline{F}(x,y,z) = (x^2, y^2, z^2)$
- $\underline{r}(u,v) = (u+v, u-v, u)$, $0 \le u \le 2$, $0 \le v \le 3$

(THIS IS IN FACT A PLANE THROUGH O)

- $\frac{\partial \underline{r}}{\partial u} = (1,1,1)$ $\quad \frac{\partial \underline{r}}{\partial v} = (1,-1,0)$
- $\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = (1,1,-2)$

$\therefore$ $\boxed{\underline{n} = \frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} = (1,1,-2)}$

$\boxed{dS = \left\| \frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} \right\| du\, dv}$

HENCE WE NOW HAVE IN PARAMETRIC

$$\int_S \underline{F} \cdot d\underline{S} = \int_S \underline{F} \cdot \hat{\underline{n}}\, dS$$

$$= \int_S \underline{F} \cdot \underbrace{\frac{\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v}}{\left\| \frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} \right\|}}_{\hat{n}} \underbrace{\left\| \frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} \right\| du\, dv}_{dS}$$

$$= \int_S \underline{F} \cdot \left(\frac{\partial \underline{r}}{\partial u} \wedge \frac{\partial \underline{r}}{\partial v} \right) du\, dv$$

SUBSTITUTING FULLY INTO THE REQUIRED SURFACE INTEGRAL

$$\Rightarrow \int_S \underline{F} \cdot d\underline{S} = \int_{v=0}^{3} \int_{u=0}^{2} \left[(u+v)^2, (u-v)^2, u^2 \right] \cdot [1,1,-2]\, du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} (u+v)^2 + (u-v)^2 - 2u^2 \; du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} u^2 + 2uv + v^2 + u^2 - 2uv + v^2 - 2u^2 \; du\, dv$$

$$= \int_{v=0}^{3} \int_{u=0}^{2} 2v^2 \, du\, dv$$

$$= \int_{v=0}^{3} \left[2v^2 u \right]_{u=0}^{2} dv$$

$$= \int_{v=0}^{3} 4v^2 \; dv$$

$$= \left[\tfrac{4}{3} v^3 \right]_0^3$$

$$= \tfrac{4}{3} \times 27$$

$$= 36$$

Question 22

Evaluate the surface integral

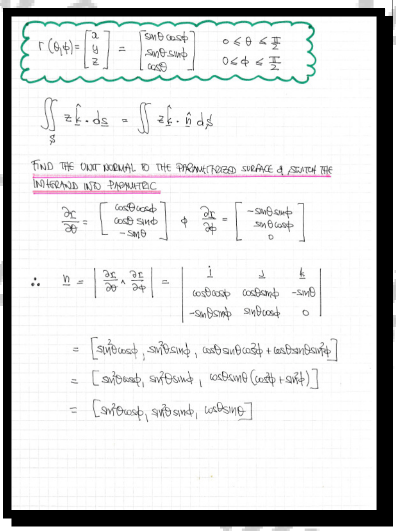
$$\int_S z\mathbf{k}\cdot d\mathbf{S},$$

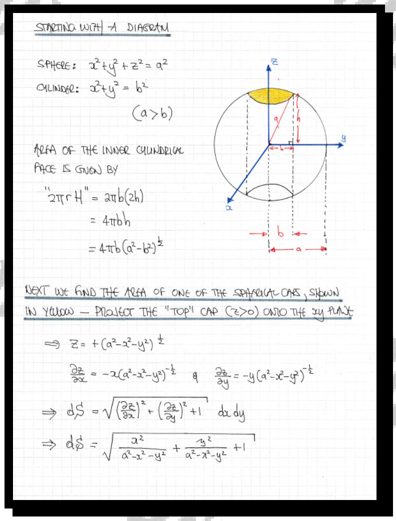
where S is the surface represented parametrically by

$$\mathbf{r}(\theta,\varphi)=\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix},\ 0\le\theta\le\frac{1}{2}\pi,\ 0\le\varphi\le\frac{1}{2}\pi.$$

All integrations must be carried out in parametric.

$\boxed{\frac{1}{6}\pi}$





Question 23

$$\mathbf{F}(x,y,z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k} .$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

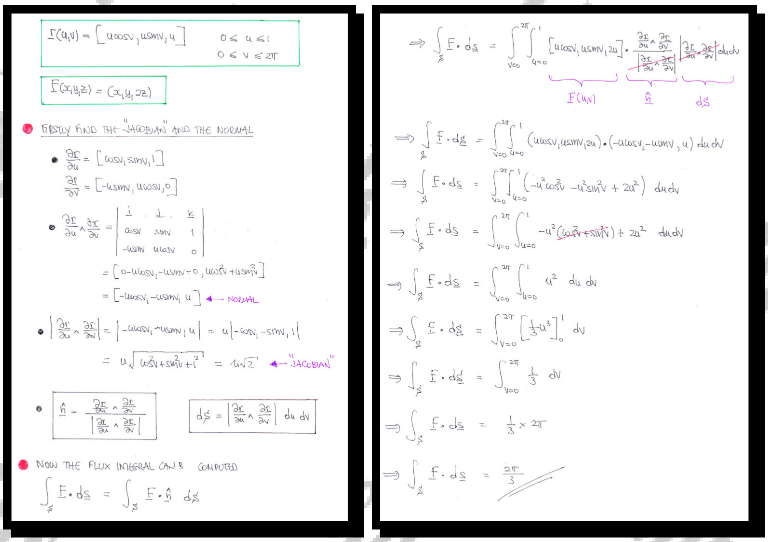
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (u\cos v)\,\mathbf{i} + (u\sin v)\,\mathbf{j} + u\,\mathbf{k} ,$$

such that $0 \le u \le 1$, $0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{\frac{2}{3}\pi}$



Question 24

$$\mathbf{F}(x,y,z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS},$$

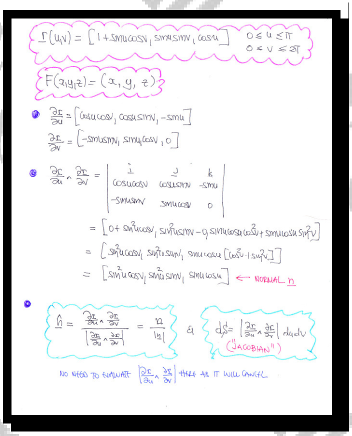
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (1+\sin u \cos v)\mathbf{i} + (\sin u \sin v)\mathbf{j} + (\cos u)\mathbf{k},$$

such that $0 \le u \le \pi$, $0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{4\pi}$


$\mathbf{r}(u,v) = [1+\sin u\cos v, \sin u \sin v, \cos u]$, $0 \le u \le \pi$, $0 \le v \le 2\pi$

$\mathbf{F}(x,y,z) = (x, y, z)$

$\frac{\partial \mathbf{r}}{\partial u} = [\cos u\cos v, \cos u \sin v, -\sin u]$

$\frac{\partial \mathbf{r}}{\partial v} = [-\sin u \sin v, \sin u \cos v, 0]$

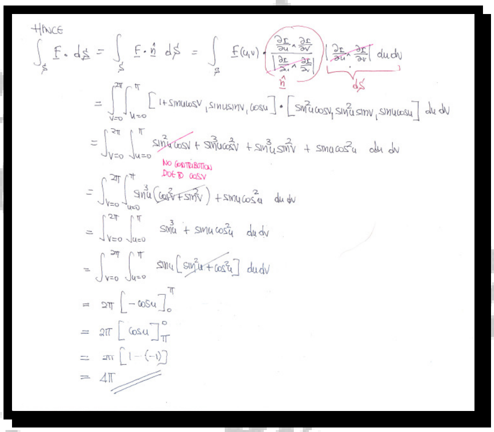
$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin u\sin v & \sin u\cos v & 0 \end{vmatrix}$

$= [0+\sin^2 u\cos v, \sin^2 u \sin v, \ldots]$

$= [\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u]$ ← NORMAL $\mathbf{n}$

$\hat{\mathbf{n}} = \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right|} = \frac{\mathbf{n}}{|\mathbf{n}|}$ & $dS = \left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv$ (JACOBIAN)

NO NEED TO EVALUATE $\left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right|$ HERE AS IT WILL CANCEL


HENCE

$\int_S \mathbf{F}\cdot d\mathbf{S} = \int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_S \mathbf{F}(u,v)\cdot \frac{\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right|}\left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| du\,dv$

$= \int_{v=0}^{2\pi}\int_{u=0}^{\pi} [1+\sin u\cos v, \sin u\sin v, \cos u]\cdot[\sin^2 u\cos v, \sin^2 u\sin v, \sin u\cos u]\,du\,dv$

$= \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin^2 u\cos v + \sin^3 u\cos^2 v + \sin^3 u\sin^2 v + \sin u\cos^2 u\,du\,dv$

NO CONTRIBUTION DUE TO $\cos v$

$= \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin^3 u(\cos^2 v+\sin^2 v) + \sin u\cos^2 u\,du\,dv$

$= \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin^3 u + \sin u\cos^2 u\,du\,dv$

$= \int_{v=0}^{2\pi}\int_{u=0}^{\pi} \sin u[\sin^2 u + \cos^2 u]\,du\,dv$

$= 2\pi\left[-\cos u\right]_0^{\pi}$

$= 2\pi\left[\cos u\right]_{\pi}^{0}$

$= 2\pi\left[1-(-1)\right]$

$= 4\pi$

Question 25

$$\mathbf{F}(x, y, z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} .$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS} ,$$

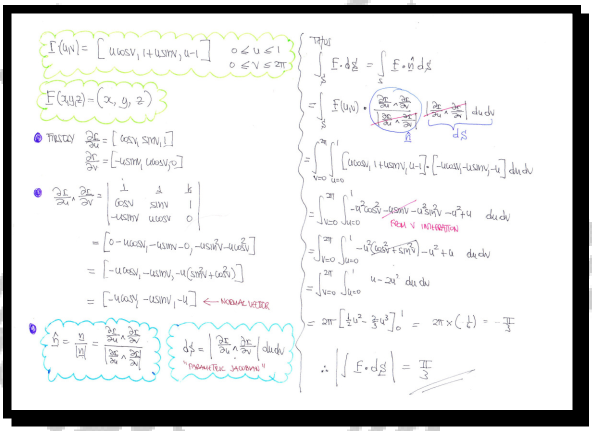
where S is the surface with parametric equations

$$\mathbf{r}(u,v) = (u\cos v)\mathbf{i} + (1 + u\sin v)\mathbf{j} + (u-1)\mathbf{k} ,$$

such that $0 \le u \le 1$, $0 \le v \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{\frac{1}{3}\pi}$



Question 26

$$\mathbf{F}(x, y, z) \equiv x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}\,,$$

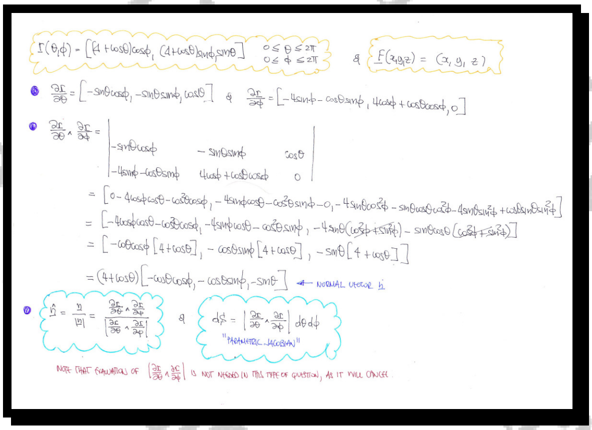
where S is the surface with parametric equations

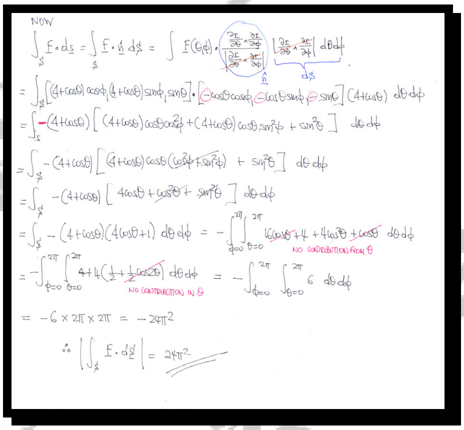
$$\mathbf{r}(\theta, \varphi) = \left[(4 + \cos\theta)\cos\varphi\right]\mathbf{i} + \left[(4 + \cos\theta)\sin\varphi\right]\mathbf{j} + (\sin\theta)\,\mathbf{k}\,,$$

such that $0 \le \theta \le 2\pi$, $0 \le \varphi \le 2\pi$.

All integrations must be carried out in parametric.

$\boxed{24\pi^2}$





Question 27

It is given that the vector field $\mathbf{F}$ satisfies

$$\mathbf{F} = 2y\,\mathbf{i} - 2x\,\mathbf{j} + \mathbf{k}\,.$$

Find the magnitude of the surface integral

$$\int_S \mathbf{F} \cdot \mathbf{dS}\,,$$

where S is the surface with Cartesian equation

$$x^2 + y^2 + z^2 = 1, \quad z \geq 0\,,$$

cut off by the cylinder with cartesian equation

$$x^2 + y^2 = x\,.$$

You **must** find a suitable parameterization for S, and carry out the **integration in parametric**, without using any integral theorems.

$$\boxed{\frac{\pi}{4}}$$