DIFFERENTIAL EQUATIONS 1st order

SEPARATION OF VARIABLES

Question 1 (**)

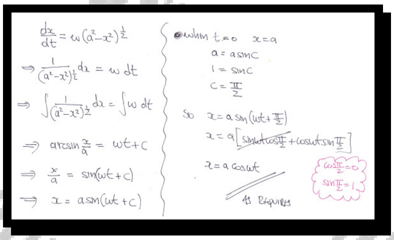
Show that if $y = a$ at $t = 0$, the solution of the differential equation

$$\frac{dy}{dt} = \omega\left(a^2 - y^2\right)^{\frac{1}{2}},$$

where a and ω are positive constants, can be written as

$$y = a\cos\omega t.$$

$\boxed{\text{proof}}$



$\frac{dx}{dt} = \omega(a^2 - x^2)^{\frac{1}{2}}$

$\Rightarrow \frac{1}{(a^2-x^2)^{\frac{1}{2}}}dx = \omega\, dt$

$\Rightarrow \int \frac{1}{(a^2-x^2)^{\frac{1}{2}}}dx = \int \omega\, dt$

$\Rightarrow \arcsin\frac{x}{a} = \omega t + C$

$\Rightarrow \frac{x}{a} = \sin(\omega t + C)$

$\Rightarrow x = a\sin(\omega t + C)$

• when $t = 0$ $x = a$

$a = a\sin C$

$1 = \sin C$

$C = \frac{\pi}{2}$

So $x = a\sin(\omega t + \frac{\pi}{2})$

$x = a\left[\cancel{\sin\omega t\cos\frac{\pi}{2}} + \cos\omega t\sin\frac{\pi}{2}\right]$

$x = a\cos\omega t$

As required

$\cos\frac{\pi}{2} = 0$

$\sin\frac{\pi}{2} = 1$

Question 2 **(**+)**

Show that a general solution of the differential equation

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

is given by

$$y = \frac{Ae^x - 3}{2Ae^x - 1},$$

where A is an arbitrary constant.

SYN, proof

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

START BY SEPARATING VARIABLES

$$\Rightarrow 5\,dy = (2y^2 - 7y + 3)\,dx$$

$$\Rightarrow \frac{5}{2y^2 - 7y + 3}\,dy = 1\,dx$$

$$\Rightarrow \frac{5}{(2y-1)(y-3)}\,dy = 1\,dy$$

PARTIAL FRACTIONS ON THE L.H.S OF THE O.D.E

$$\Rightarrow \frac{5}{(2y-1)(y-3)} \equiv \frac{P}{2y-1} + \frac{Q}{y-3}$$

$$\Rightarrow \boxed{5 \equiv P(y-3) + Q(2y-1)}$$

- IF $y=3 \Rightarrow 5 = 5Q$
  $\Rightarrow Q = 1$
- IF $y=0 \Rightarrow 5 = -3P - Q$
  $\Rightarrow 5 = -3P - 1$
  $\Rightarrow 3P = -6$
  $\Rightarrow P = -2$

RETURNING TO THE O.D.E

$$\Rightarrow \int \frac{1}{y-3} - \frac{2}{2y-1}\,dy = \int 1\,dx$$

$$\Rightarrow \ln|y-3| - \ln|2y-1| = x + C$$

$$\Rightarrow \ln\left|\frac{y-3}{2y-1}\right| = x + C$$

$$\Rightarrow \frac{y-3}{2y-1} = e^{x+C}$$

$$\Rightarrow \frac{y-3}{2y-1} = Ae^x, \quad \text{WHERE } A = e^C$$

$$\Rightarrow y - 3 = 2Aye^x - Ae^x$$

$$\Rightarrow Ae^x - 3 = 2Aye^x - y$$

$$\Rightarrow Ae^x - 3 = y(2Ae^x - 1)$$

$$\Rightarrow y = \frac{Ae^x - 3}{2Ae^x - 1}$$

AS REQUIRED

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Question 3 (+)**

Show that a general solution of the differential equation

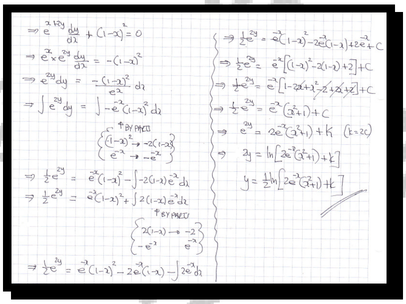
$$e^{x+2y}\frac{dy}{dx}+(1-x)^2=0$$

is given by

$$y=\frac{1}{2}\ln\left[2e^{-x}\left(x^2+1\right)+K\right],$$

where K is an arbitrary constant.

$\boxed{\text{proof}}$



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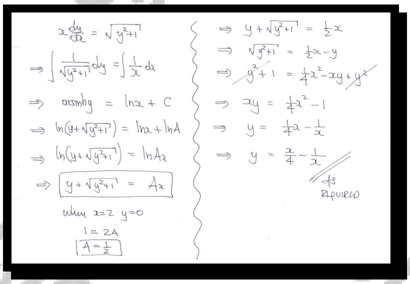
Question 4 (**+)

$$x\frac{dy}{dx}=\sqrt{y^2+1}\ ,\ x>0,\ \text{with } y=0 \text{ at } x=2.$$

Show that the solution of the above differential equation is

$$y=\frac{x}{4}-\frac{1}{x}.$$

proof



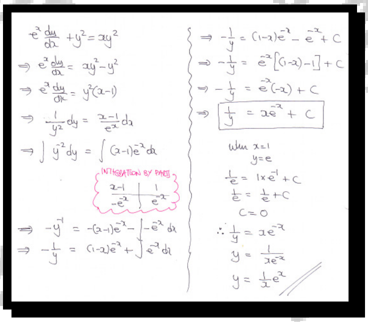
Question 5 (***)

$$e^x\frac{dy}{dx}+y^2=xy^2,\ x>0,\ y>0$$

Show that the solution of the above differential equation subject to $y=e$ at $x=1$, is

$$y=\frac{1}{x}e^x.$$

proof



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Question 6 (***)

A curve $y = f(x)$ satisfies the differential equation

$$y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}, \quad y > 1,\, x > -1$$

a) Solve the differential equation to show that

$$\ln(y-5) + \frac{1}{2}x^2 + 4x - 2\ln(x+1) = C\,.$$

When $x = 0$, $y = 2$.

b) Show further that

$$y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2}\,.$$

$\boxed{\text{proof}}$

(a)

$$y = 1 - \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)}$$

$$\Rightarrow \frac{dy}{dx}\frac{x+1}{(x-1)(x+2)} = 1 - y$$

$$\Rightarrow \frac{1}{1-y}\,dy = \frac{(x-1)(x+2)}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x^2+x-2}{x+1}\,dx$$

BY SUBSTITUTION $u = x+1$ OR BY ALGEBRAIC MANIPULATION

$$\Rightarrow \int \frac{1}{1-y}\,dy = \int \frac{x(x+1)-2}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{1-y}\,dy = \int x - \frac{2}{x+1}\,dx$$

$$\Rightarrow -\ln|1-y| = \tfrac{1}{2}x^2 - 2\ln|x+1| + C$$

BUT $y > 1$ & $x > -1$

$$\Rightarrow -\ln(y-1) = \tfrac{1}{2}x^2 - 2\ln(x+1) + C$$

$$\Rightarrow \ln(y-1) = -\tfrac{1}{2}x^2 + 2\ln(x+1) + C$$

$$\Rightarrow \ln(y-1) + \tfrac{1}{2}x^2 - 2\ln(x+1) = C$$

AS REQUIRED

(b)

when $x=0$ $y=2$

$\ln 1 + 0 - 2\ln 1 = C$

$C = 0$

$$\therefore \ln(y-1) + \tfrac{1}{2}x^2 - 2\ln(x+1) = 0$$

$$\Rightarrow \ln(y-1) = 2\ln(x+1) - \tfrac{1}{2}x^2$$

$$\Rightarrow \ln(y-1) = \ln(x+1)^2 - \tfrac{1}{2}x^2$$

$$\Rightarrow y - 1 = e^{\ln(x+1)^2 - \frac{1}{2}x^2}$$

$$\Rightarrow y - 1 = e^{\ln(x+1)^2} \times e^{-\frac{1}{2}x^2}$$

$$\Rightarrow y - 1 = (x+1)^2 \times e^{-\frac{1}{2}x^2}$$

$$\Rightarrow y = 1 + (x+1)^2 e^{-\frac{1}{2}x^2}$$

AS REQUIRED

Question 7 (***)

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\ x>0.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

$\boxed{\text{proof}}$

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{(x^2+2)(4x^2+3)}$$

INTEGRATING FACTOR

$$\Rightarrow e^{\int \frac{1}{x}dx}=e^{\ln x}=x$$

$$\Rightarrow \frac{d}{dx}(yx)=\frac{5x}{(x^2+2)(4x^2+3)}$$

PARTIAL FRACTIONS

$$\frac{5x}{(x^2+2)(4x^2+3)}\equiv\frac{Ax+B}{x^2+2}+\frac{Cx+D}{4x^2+3}$$

$$\boxed{5x\equiv(Ax+B)(4x^2+3)+(Cx+D)(x^2+2)}$$

$$5x\equiv 4Ax^3+3Ax+4Bx^2+3B+Cx^3+2Cx+Dx^2+2D$$

$$5x\equiv(4A+C)x^3+(4B+D)x^2+(3A+2C)x+(3B+2D)$$

- $\left.\begin{array}{l}4A+C=0\\3A+2C=5\end{array}\right\}\Rightarrow\left.\begin{array}{l}8A+2C=0\\3A+2C=5\end{array}\right\}\Rightarrow\begin{array}{l}\boxed{A=-1}\\\boxed{C=4}\end{array}$
- $\left.\begin{array}{l}4B+D=0\\3B+2D=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}8B+2D=0\\3B+2D=0\end{array}\right\}\Rightarrow\begin{array}{l}\boxed{B=0}\\\boxed{D=0}\end{array}$

$$\Rightarrow yx=\int\frac{4x}{4x^2+3}-\frac{x}{x^2+2}\,dx$$

$$\Rightarrow yx=\frac{1}{2}\ln(4x^2+3)-\frac{1}{2}\ln(x^2+2)+\frac{1}{2}\ln A$$

$$\Rightarrow yx=\frac{1}{2}\ln\left(\frac{A(4x^2+3)}{x^2+2}\right)$$

WHEN $x=1$ $y=\frac{1}{2}\ln\frac{7}{6}$

$$\frac{1}{2}\ln\frac{7}{6}=\frac{1}{2}\ln\left(\frac{7A}{3}\right)$$

$$\frac{7}{6}=\frac{7A}{3}$$

$$A=\frac{1}{2}$$

$$\therefore\ yx=\frac{1}{2}\ln\left(\frac{4x^2+3}{2(x^2+2)}\right)\qquad \text{i.e.}\qquad y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right)$$

Question 8 (***)

$$\frac{dy}{dx} = 1 - \sqrt{y}, \; y \geq 0, \; y \neq 1.$$

Find the solution of the above differential equation subject to the condition $y = 0$ at $x = 0$, giving the answer in the form $x = f(y)$.

$$\boxed{x = 2\ln\left|\frac{1}{1-\sqrt{y}}\right| - 2\sqrt{y}}$$

$\frac{dy}{dx} = 1 - \sqrt{y}$ SUBJECT TO $y=0$ AT $x=0$

- $\frac{dy}{dx} = 1 - \sqrt{y}$

$\Rightarrow \frac{1}{1-\sqrt{y}}\,dy = 1\,dx$

$\Rightarrow \int \frac{1}{1-y^{\frac{1}{2}}}\,dy = \int 1\,dx$

SUBSTITUTION
$u = 1 - y^{\frac{1}{2}}$
$y^{\frac{1}{2}} = 1 - u$
$y = (1-u)^2$
$\frac{dy}{du} = -2(1-u)$
$dy = -2(1-u)\,du$

$\Rightarrow \int \frac{1}{u}(-2(1-u))\,du = \int 1\,dx$

$\Rightarrow \int \frac{2u-2}{u}\,du = \int 1\,dx$

$\Rightarrow \int 2 - \frac{2}{u}\,du = \int 1\,dx$

$\Rightarrow 2u - 2\ln|u| = x + C$

$\Rightarrow 2(1-y^{\frac{1}{2}}) - 2\ln|1-y^{\frac{1}{2}}| = x + C$

$\Rightarrow 2 - 2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x + C$

$\Rightarrow -2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x + C$

APPLY $x=0$ $y=0$

$0 - \cancel{2\ln 1} = 0 + C$

$\therefore C = 0$

$\Rightarrow -2y^{\frac{1}{2}} - 2\ln|1-y^{\frac{1}{2}}| = x$

$\Rightarrow x = -2\ln|1-y^{\frac{1}{2}}| - 2y^{\frac{1}{2}}$

$\Rightarrow x = 2\ln\left|\frac{1}{1-\sqrt{y}}\right| - 2\sqrt{y}$

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Question 9 **(***)**

Solve the differential equation

$$\frac{dy}{dx} = 2 - \frac{2}{y^2},$$

subject to the condition $y = 2$ at $x = 1$, giving the answer in the form $x = f(y)$.

$$\boxed{x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|}$$

$$\frac{dy}{dx} = 2 - \frac{2}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 - 2}{y^2} = \frac{2(y^2-1)}{y^2}$$

$$\Rightarrow \frac{y^2}{y^2-1}\,dy = 2\,dx$$

IMPROPER

$*$ DIVIDE OR MANIPULATE

OR

$$\frac{y^2}{(y+1)(y-1)} \equiv A + \frac{B}{y+1} + \frac{C}{y-1}$$

$$\boxed{y^2 \equiv A(y+1)(y-1) + B(y-1) + C(y+1)}$$

If $y = 1$ $\quad 1 = 2C \Rightarrow C = \frac{1}{2}$

If $y = -1$ $\quad 1 = 2B \Rightarrow B = -\frac{1}{2}$

If $y = 2$ $\quad 4 = 3A + B + 3C$

$4 = 3A - \frac{1}{2} + \frac{3}{2}$

$3 = 3A$

$A = 1$

$*$

$$\frac{y^2}{y^2-1} = \frac{(y^2-1)+1}{y^2-1} = 1 + \frac{1}{y^2-1}$$

$$= 1 + \frac{1}{(y-1)(y+1)}$$

$$= 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}$$

$$\Rightarrow \int 1 + \frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\,dy = \int 2\,dx$$

$$\Rightarrow y + \frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = 2x + C$$

$$\Rightarrow y + \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = 2x + C$$

$$\Rightarrow \boxed{\frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| = x + C}$$

- APPLY CONDITION

$x = 1 \quad y = 2$

$1 + \frac{1}{4}\ln\frac{1}{3} = 1 + C$

$C = \frac{1}{4}\ln\frac{1}{3}$

$\therefore \quad x + \frac{1}{4}\ln\frac{1}{3} = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right|$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| - \frac{1}{4}\ln\frac{1}{3}$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right| + \frac{1}{4}\ln 3$

$x = \frac{1}{2}y + \frac{1}{4}\ln\left|\frac{3y-3}{y+1}\right|$

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Question 10 **(***+)**

The function $y = f(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2\left(x+\frac{1}{6}\pi\right)},$$

subject to the condition $y = 1$ at $x = 0$.

Find the exact value of y when $x = \frac{\pi}{12}$.

$\boxed{\text{MP2-S}}$, $\boxed{y = \frac{1}{e^{\frac{1}{6}\pi} - 1}}$

SOLVE THE O.D.E. BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{2xy(y+1)}{\sin^2(x+\frac{\pi}{6})}$$

$$\Rightarrow \frac{1}{y(y+1)}\,dy = \frac{2x}{\sin^2(x+\frac{\pi}{6})}\,dx$$

$$\Rightarrow \int \frac{1}{y(y+1)}\,dy = \int 2x\,\mathrm{cosec}^2(x+\tfrac{\pi}{6})\,dx$$

THE L.H.S REQUIRES PARTIAL FRACTIONS (BY INSPECTION) AND THE R.H.S INTEGRATION BY PARTS

$2x$	2
$-\cot(x+\frac{\pi}{6})$	$\mathrm{cosec}^2(x+\frac{\pi}{6})$

$$\Rightarrow \int \frac{1}{y} - \frac{1}{y+1}\,dy = -2x\cot(x+\tfrac{\pi}{6}) - \int -2\cot(x+\tfrac{\pi}{6})\,dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = -2x\cot(x+\tfrac{\pi}{6}) + \int 2\cot(x+\tfrac{\pi}{6})\,dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = -2x\cot(x+\tfrac{\pi}{6}) + 2\ln|\sin(x+\tfrac{\pi}{6})| + C$$

$\int \cot x\,dx = \ln|\sin x| + C$

APPLY CONDITION $x=0$, $y=1$

$$\cancel{\ln 1} - \ln 2 = 0 + 2\ln(\sin\tfrac{\pi}{6}) + C$$

$$-\ln 2 = 2\ln\tfrac{1}{2} + C$$

$$-\ln 2 = -2\ln 2 + C$$

$$C = \ln 2$$

THUS WE NOW HAVE

$$\ln|y| - \ln|y+1| = \ln 2 - 2x\cot(x+\tfrac{\pi}{6}) + 2\ln|\sin(x+\tfrac{\pi}{6})|$$

WHEN $x = \frac{\pi}{12}$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - 2\times\left(\tfrac{\pi}{12}\right)\cot\tfrac{\pi}{4} + 2\ln(\sin\tfrac{\pi}{4})$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \tfrac{\pi}{6} + 2\ln\left(\tfrac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \ln 2 - \tfrac{\pi}{6} + 2\ln 2^{-\frac{1}{2}}$$

$$\Rightarrow \ln\left|\frac{y}{y+1}\right| = \cancel{\ln 2} - \tfrac{\pi}{6} - \cancel{\ln 2}$$

$$\Rightarrow \frac{y}{y+1} = e^{-\frac{\pi}{6}}$$

$$\Rightarrow \frac{y+1}{y} = e^{\frac{\pi}{6}}$$

$$\Rightarrow 1 + \frac{1}{y} = e^{\frac{\pi}{6}}$$

$$\Rightarrow \frac{1}{y} = e^{\frac{\pi}{6}} - 1$$

$$\Rightarrow y = \frac{1}{e^{\frac{\pi}{6}} - 1}$$

Question 11 (****+)

A curve passes through the point with coordinates $\left[1\ ,\ \log_2\left(\log_2 \mathrm{e}\right)\right]$ and its gradient function satisfies

$$\frac{dy}{dx} = 2^y, \quad x \in \mathbb{R}, \quad x < 2.$$

Find the equation of the curve in the form $y = f(x)$

$\square$, $\boxed{y = -\log_2\left[(2-x)\ln 2\right]}$

REWRITE IN TERMS OF THE EXPONENTIAL FUNCTION & SEPARATE VARIABLES

$\Rightarrow \frac{dy}{dx} = 2^y$

$\Rightarrow \frac{dy}{dx} = e^{\ln(2^y)}$

$\Rightarrow \frac{dy}{dx} = e^{y\ln 2}$

$\Rightarrow dy = e^{y\ln 2}\,dx$

$\Rightarrow \frac{1}{e^{y\ln 2}}\,dy = 1\,dx$

$\Rightarrow \int e^{-y\ln 2}\,dy = \int 1\,dx$

$\Rightarrow \frac{1}{-\ln 2}e^{-y\ln 2} = x + C$

$\Rightarrow e^{-y\ln 2} = (-\ln 2)(x+C)$

$\Rightarrow \frac{1}{e^{y\ln 2}} = (A-x)\ln 2$

$\Rightarrow e^{y\ln 2} = \frac{1}{(A-x)\ln 2}$

NOW IT IS BETTER TO MANIPULATE FURTHER BEFORE APPLYING THE BOUNDARY CONDITION $[1, \log_2(\log_2 e)]$

$\Rightarrow 2^y = \frac{1}{(A-x)\ln 2}$

$\Rightarrow 2^{\log_2(\log_2 e)} = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \log_2 e = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \frac{\log_e e}{\log_e 2} = \frac{1}{(A-1)\ln 2}$

$\Rightarrow \frac{1}{\cancel{\ln 2}} = \frac{1}{(A-1)\cancel{\ln 2}}$

$\Rightarrow A-1 = 1$

$\Rightarrow A = 2$

$\Rightarrow 2^y = \frac{1}{(2-x)\ln 2}$

$\Rightarrow \log_2 2^y = \log_2\left[\frac{1}{(2-x)\ln 2}\right]$

$\Rightarrow y = -\log_2\left[(2-x)\ln 2\right]$

Question 12 (****)

$$\frac{dy}{dx} = \sqrt{\frac{y^4 - y^2}{x^4 - x^2}}, \quad x > 0, \; y > 0.$$

Find the solution of the above differential equation subject to the boundary condition $y = \frac{2}{\sqrt{3}}$ at $x = 2$.

Give the answer in the form $y = \frac{2x}{f(x)}$, where $f(x)$ is a function to be found.

$\boxed{}$, $\boxed{f(x) = \sqrt{3} + \sqrt{x^2 - 1}}$

SOLVE THE O.D.E BY SEPARATION OF VARIABLES

$$\frac{dy}{dx} = \sqrt{\frac{y^4-y^2}{x^4-x^2}} = \frac{|y|\sqrt{y^2-1}}{|x|\sqrt{x^2-1}} = \frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}} \quad \text{AS } x,y>0$$

$$\Rightarrow \int \frac{1}{y\sqrt{y^2-1}}\,dy = \int \frac{1}{x\sqrt{x^2-1}}\,dx$$

IDENTICAL SUBSTITUTION OR DIRECTLY RECOGNISING THE DERIVATIVE OF "arcsec"

LET $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta = \int \frac{\cancel{\sec\theta\tan\theta}}{\cancel{\sec\theta\tan\theta}}\,d\theta$$

$$= \int 1\,d\theta = \theta + C = \operatorname{arcsec}x + C$$

RETURNING TO THE O.D.E.

$$\Rightarrow \operatorname{arcsec}y = \operatorname{arcsec}x + C$$

APPLY CONDITION $\left(2, \frac{2}{\sqrt{3}}\right)$

$$\operatorname{arcsec}\frac{2}{\sqrt{3}} = \operatorname{arcsec}2 + C$$

$$\frac{\pi}{6} = \frac{\pi}{3} + C$$

$$C = -\frac{\pi}{6}$$

$$\Rightarrow \operatorname{arcsec}y = \operatorname{arcsec}x - \frac{\pi}{6}$$

$$\Rightarrow \cos(\operatorname{arcsec}y) = \cos\left(\operatorname{arcsec}x - \frac{\pi}{6}\right)$$

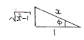
$$\Rightarrow \cos(\operatorname{arcsec}y) = \cos(\operatorname{arcsec}x)\cos\frac{\pi}{6} + \sin(\operatorname{arcsec}x)\sin\frac{\pi}{6}$$

NEXT REMOVE THE "arcsec"

$\operatorname{arcsec}x = \phi$

$\sec\phi = x$

$\cos\phi = \frac{1}{x}$ & $\sin\phi = \frac{\sqrt{x^2-1}}{x}$



$\therefore \cos(\operatorname{arcsec}x) = \frac{1}{x}$ $\qquad \sin(\operatorname{arcsec}x) = \frac{\sqrt{x^2-1}}{x}$

$\cos(\operatorname{arcsec}y) = \frac{1}{y}$

RETURNING TO THE O.D.E

$$\frac{1}{y} = \frac{1}{x}\times\frac{\sqrt{3}}{2} + \frac{\sqrt{x^2-1}}{x}\times\frac{1}{2}$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{\sqrt{x^2-1}}{2x}$$

$$\frac{1}{y} = \frac{\sqrt{3}+\sqrt{x^2-1}}{2x}$$

$$y = \frac{2x}{\sqrt{3}+\sqrt{x^2-1}} \quad //$$

1^{ST} ORDER BY STANDARD INTEGRATING FACTORS

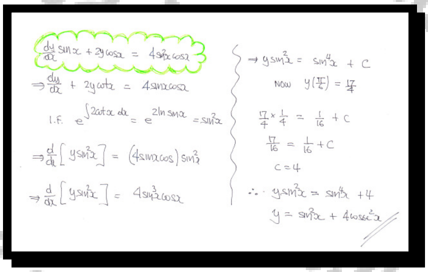
Question 1 (**)

Solve the differential equation

$$\frac{dy}{dx}\sin x + 2y\cos x = 4\sin^2 x\cos x, \qquad y\left(\tfrac{1}{6}\pi\right) = \frac{17}{4}.$$

Give the answer in the form $y = f(x)$.

$$\boxed{y = \sin^2 x + 4\operatorname{cosec}^2 x}$$

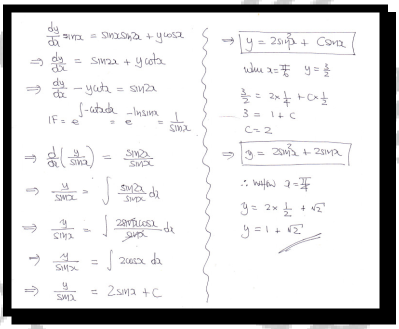


Question 2 (**)

$$\frac{dy}{dx}\sin x = \sin x \sin 2x + y\cos x.$$

Given that $y = \dfrac{3}{2}$ at $x = \dfrac{\pi}{6}$, find the exact value of y at $x = \dfrac{\pi}{4}$.

$$\boxed{1 + \sqrt{2}}$$



Question 3 **(**)**

$$x\frac{dy}{dx}+2y=9x\left(x^3+1\right)^{\frac{1}{2}} \text{ , with } y=\frac{27}{2} \text{ at } x=2.$$

Show that the solution of the above differential equation is

$$y=\frac{2}{x^2}\left(x^3+1\right)^{\frac{3}{2}}.$$

$\boxed{\text{proof}}$

$$x\frac{dy}{dx}+2y=9x(x^3+1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y=9(x^3+1)^{\frac{1}{2}}$$

$$\boxed{\text{I.F.}=e^{\int \frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2}$$

$$\Rightarrow \frac{d}{dx}(yx^2)=9x^2(x^3+1)^{\frac{1}{2}}$$

$$\Rightarrow yx^2=\int 9x^2(x^3+1)^{\frac{1}{2}}\,dx$$

$$\Rightarrow yx^2=2(x^3+1)^{\frac{3}{2}}+C$$

$$\Rightarrow \boxed{y=\frac{2(x^3+1)^{\frac{3}{2}}}{x^2}+\frac{C}{x^2}}$$

- when $x=2$ $y=\frac{27}{2}$

$$\frac{27}{2}=\frac{27}{2}+\frac{C}{4}$$

$$C=0$$

Hence

$$y=\frac{2(x^3+1)^{\frac{3}{2}}}{x^2}$$

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Question 4 **(**)**

20 grams of salt are dissolved into a beaker containing 1 litre of a certain chemical.

The mass of salt, M grams, which remains undissolved t seconds later, is modelled by the differential equation

$$\frac{dM}{dt} + \frac{2M}{20-t} + 1 = 0, \; t \geq 0.$$

Show clearly that

$$M = \frac{1}{10}(10-t)(20-t).$$

proof

$$\frac{dM}{dt} + \frac{2M}{20-t} + 1 = 0$$

$$\frac{dM}{dt} + \frac{2}{20-t}M = -1$$

$$\text{I.F.} = e^{\int \frac{2}{20-t}\,dt} = e^{-2\ln|20-t|} = e^{\ln\left(\frac{1}{(20-t)^2}\right)} = \frac{1}{(20-t)^2}$$

$$\frac{d}{dt}\left(\frac{M}{(20-t)^2}\right) = -\frac{1}{(20-t)^2}$$

$$\frac{M}{(20-t)^2} = \int -(20-t)^{-2}\,dt$$

$$\frac{M}{(20-t)^2} = -(20-t)^{-1} + A$$

$$\frac{M}{(20-t)^2} = A - \frac{1}{20-t}$$

$$M = A(20-t)^2 - (20-t)$$

- APPLY CONDITION

$t=0 \quad M=20$

$$20 = A \times 20^2 - 20$$

$$20 = 400A - 20$$

$$40 = 400A$$

$$A = \frac{1}{10}$$

- THUS

$$M = \frac{1}{10}(20-t)^2 - (20-t)$$

$$M = \frac{1}{10}(20-t)\left[(20-t) - 10\right]$$

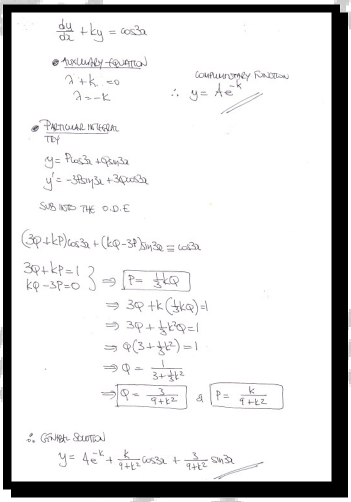
$$M = \frac{1}{10}(20-t)(10-t)$$

Question 5 (**+)

$$\frac{dy}{dx} + ky = \cos 3x \,, \quad k \text{ is a non zero constant.}$$

By finding a complimentary function and a particular integral, or otherwise, find the general of the above differential equation.

$$\boxed{y = A\mathrm{e}^{-x} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x}$$



$\frac{dy}{dx} + ky = \cos 3x$

• AUXILIARY EQUATION

$\lambda + k = 0$

$\lambda = -k$

COMPLIMENTARY FUNCTION

$\therefore \; y = Ae^{-k}$

• PARTICULAR INTEGRAL

TRY

$y = P\cos 3x + Q\sin 3x$

$y' = -3P\sin 3x + 3Q\cos 3x$

SUB INTO THE O.D.E

$(3Q + kP)\cos 3x + (kQ - 3P)\sin 3x \equiv \cos 3x$

$\left.\begin{array}{l} 3Q + kP = 1 \\ kQ - 3P = 0 \end{array}\right\} \Rightarrow \boxed{P = \frac{1}{3}kQ}$

$\Rightarrow 3Q + k\left(\frac{1}{3}kQ\right) = 1$

$\Rightarrow 3Q + \frac{1}{3}k^2Q = 1$

$\Rightarrow Q\left(3 + \frac{1}{3}k^2\right) = 1$

$\Rightarrow Q = \frac{1}{3 + \frac{1}{3}k^2}$

$\Rightarrow \boxed{Q = \frac{3}{9+k^2}}$ & $\boxed{P = \frac{k}{9+k^2}}$

$\therefore$ GENERAL SOLUTION

$y = Ae^{-k} + \frac{k}{9+k^2}\cos 3x + \frac{3}{9+k^2}\sin 3x$

Question 6 (+)**

Given that $z = f(x)$ and $y = g(x)$ satisfy the following differential equations

$$\frac{dz}{dx} + 2z = \mathrm{e}^{-2x} \quad \text{and} \quad \frac{dy}{dx} + 2y = z,$$

a) Find z in the form $z = f(x)$

b) Express y in the form $y = g(x)$, given further that at $x = 0$, $y = 1$, $\dfrac{dy}{dx} = 0$

$\boxed{z = (x + C)\mathrm{e}^{-2x}}$, $\boxed{y = \left(\frac{1}{2}x^2 + 2x + 1\right)\mathrm{e}^{-2x}}$

(a) $\frac{dz}{dx} + 2z = e^{-2x}$

- I.F. $e^{\int 2\,dx} = e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = e^{-2x}e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = 1$

$\Rightarrow ze^{2x} = \int 1\,dx$

$\Rightarrow ze^{2x} = x + C$

$\Rightarrow z = xe^{-2x} + Ce^{-2x}$

(b) $\frac{dy}{dx} + 2y = z$

$\Rightarrow \frac{dy}{dx} + 2y = xe^{-2x} + Ce^{-2x}$

- IF $= e^{2x}$ AS BEFORE

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = \left(xe^{-2x} + Ce^{-2x}\right)e^{2x}$

$\Rightarrow \frac{d}{dx}\left(ye^{2x}\right) = x + C$

$\Rightarrow ye^{2x} = \int x + C\,dx$

$\Rightarrow ye^{2x} = \frac{1}{2}x^2 + Cx + D$

$\Rightarrow \boxed{y = \left(\frac{1}{2}x^2 + Cx + D\right)e^{-2x}}$

- $x = 0 \quad y = 1$

$\boxed{1 = D}$

$\Rightarrow y = \left(\frac{1}{2}x^2 + Cx + 1\right)e^{-2x}$

- WHEN $x = 0$, $y = 1$, $\frac{dy}{dx} = 0$

FROM THE 2ND O.D.E

$0 + 2 = z$

$\therefore \boxed{z = 2}$

HENCE FROM 1ST ODE:

$\boxed{2 = C}$

$\therefore y = \left(\frac{1}{2}x^2 + 2x + 1\right)e^{-2x}$

Question 7 (***)

A curve C, with equation $y = f(x)$, passes through the points with coordinates $(1,1)$ and $(2,k)$, where k is a constant.

Given further that the equation of C satisfies the differential equation

$$x^2 \frac{dy}{dx} + xy(x+3) = 1,$$

determine the exact value of k.

$$\boxed{k = \frac{e+1}{8e}}$$

REWRITE THE O.D.E IN "STANDARD" FORM

$$\Rightarrow x^2 \frac{dy}{dx} + xy(x+3) = 1$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{x^2+3x}{x^2}\right) = \frac{1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{x+3}{x}\right) = \frac{1}{x^2}$$

OBTAIN THE INTEGRATING FACTOR

$$\text{I.F.} = e^{\int \frac{x+3}{x}\,dx} = e^{\int 1+\frac{3}{x}\,dx} = e^{x+3\ln x} = e^x \times e^{3\ln x}$$

$$= e^x \times e^{\ln x^3} = x^3 e^x$$

MULTIPLY THROUGH MAKES L.H.S EXACT

$$\Rightarrow \frac{d}{dx}\left[y x^3 e^x\right] = \frac{1}{x^2} x^3 e^x$$

$$\Rightarrow y x^3 e^x = \int x e^x \, dx$$

INTEGRATE THE R.H.S BY PARTS

$$\Rightarrow y x^3 e^x = x e^x - \int e^x \, dx$$

x	1
e^x	e^x

$$\Rightarrow y x^3 e^x = x e^x - e^x + A$$

$$\Rightarrow y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{A}{x^3} e^{-x}$$

APPLY THE BOUNDARY CONDITION $(1,1)$

$$\Rightarrow 1 = 1 - 1 + \frac{A}{1} e^{-1}$$

$$\Rightarrow 1 = A e^{-1}$$

$$\Rightarrow A = \frac{1}{e^{-1}} = e^1 = e$$

FINALLY LET $x=2$

$$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{e}{x^3} e^{-x}$$

$$k = \frac{1}{2^2} - \frac{1}{2^3} + \frac{e \times e^{-2}}{2^3}$$

$$k = \frac{1}{4} - \frac{1}{8} + \frac{e^{-1}}{8}$$

$$k = \frac{e^{-1}}{8} + \frac{1}{8}$$

$$k = \frac{1}{8}\left(\frac{1}{e} + 1\right)$$

$$k = \frac{1}{8}\left(\frac{e+1}{e}\right)$$

$$k = \frac{e+1}{8e}$$

Question 8 (***)

A curve C, with equation $y = f(x)$, meets the y axis the point $(0,1)$.

It is further given that the equation of C satisfies the differential equation

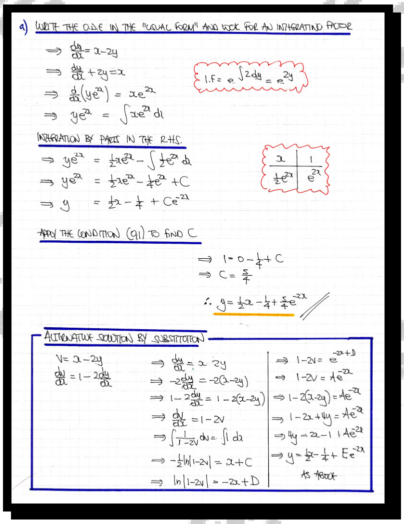
$$\frac{dy}{dx} = x - 2y.$$

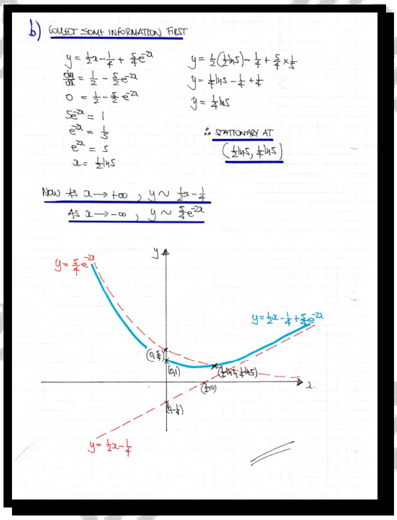
a) Determine an equation of C.

b) Sketch the graph of C.

The graph must include in exact simplified form the coordinates of the stationary point of the curve and the equation of its asymptote.

$\boxed{}$, $\boxed{y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}}$





Question 9 **(***)**

$$\left(1-x^2\right)\frac{dy}{dx}+y=\left(1-x^2\right)\left(1-x\right)^{\frac{1}{2}}\ ,\ -1<x<1\,.$$

Given that $y=\dfrac{\sqrt{2}}{2}$ at $x=\dfrac{1}{2}$, show that the solution of the above differential equation can be written as

$$y=\frac{2}{3}\sqrt{\left(1-x^2\right)\left(1+x\right)}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

$$\boxed{(1-x^2)\frac{dy}{dx}+y=(1-x^2)(1-x)^{\frac{1}{2}}}$$

REWRITE THE O.D.E IN "STANDARD" FORM AND LOOK FOR AN INTEGRATING FACTOR

$$\Rightarrow \frac{dy}{dx}+\frac{1}{1-x^2}\frac{dy}{dx}=(1-x)^{\frac{1}{2}}$$

- $\text{I.F.}=e^{\int\frac{1}{1-x^2}dx}=e^{\int\frac{1}{(1+x)(1-x)}dx}=\dots$ PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$=e^{\int\frac{\frac{1}{2}}{1+x}+\frac{\frac{1}{2}}{1-x}dx}=e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}=e^{\ln\sqrt{\frac{1+x}{1-x}}}=\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

$$\Rightarrow \frac{d}{dx}\left[y\left(\frac{\sqrt{1+x}}{\sqrt{1-x}}\right)\right]=\cancel{(1-x)^{\frac{1}{2}}}\left(\frac{\sqrt{1+x}}{\cancel{\sqrt{1-x}}}\right)$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=\int(1+x)^{\frac{1}{2}}\,dx$$

$$\Rightarrow \frac{y(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}=\frac{2}{3}(1+x)^{\frac{3}{2}}+A$$

$$\Rightarrow \boxed{y=\frac{2}{3}(1+x)^1(1-x)^{\frac{1}{2}}+A\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}}$$

APPLY $x=\frac{1}{2}$, $y=\frac{\sqrt{2}}{2}$

$$\Rightarrow \frac{\sqrt{2}}{2}=\frac{2}{3}\times\frac{3}{2}\times\frac{\sqrt{2}}{2}+A\,\frac{\sqrt{2}/2}{3/2}$$

$$\Rightarrow \frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}+A\frac{\sqrt{2}}{3}$$

$$\Rightarrow A=0$$

$$\Rightarrow y=\frac{2}{3}(1+x)(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{(1+x)(1-x)}$$

$$\Rightarrow y=\frac{2}{3}(1+x)^{\frac{1}{2}}\sqrt{1-x^2}$$

$$\Rightarrow y=\frac{2}{3}\sqrt{(1+x)(1-x^2)}$$

// AS REQUIRED

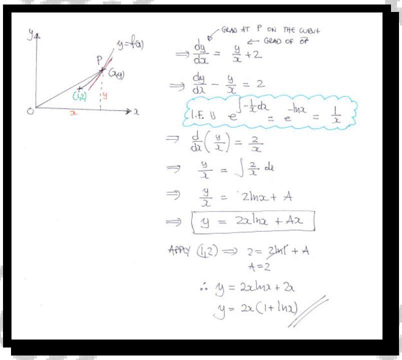
Question 10 (***)

The general point P lies on the curve with equation $y = f(x)$.

The gradient of the curve at P is 2 more than the gradient of the straight line segment OP.

Given further that the curve passes through $Q(1,2)$, express y in terms of x.

$$\boxed{y = 2x(1+\ln x)}$$



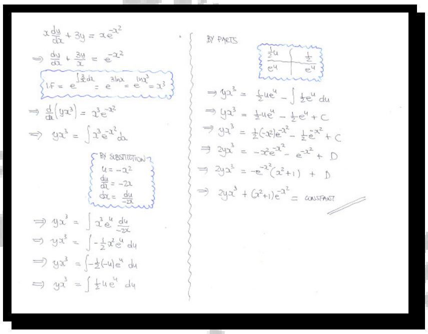
Question 11 **(***)**

$$x\frac{dy}{dx}+3y=x\,e^{-x^2}\ ,\ x>0\,.$$

Show clearly that the general solution of the above differential equation can be written in the form

$$2yx^3+\left(x^2+1\right)e^{-x^2}=\text{constant}\,.$$

$\boxed{\text{proof}}$



$x\frac{dy}{dx}+3y=xe^{-x^2}$

$\Rightarrow \frac{dy}{dx}+\frac{3y}{x}=e^{-x^2}$

$\text{I.F.}=e^{\int\frac{3}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$

$\Rightarrow \frac{d}{dx}\left(yx^3\right)=x^3e^{-x^2}$

$\Rightarrow yx^3=\int x^3e^{-x^2}\,dx$

BY SUBSTITUTION

$u=-x^2$

$\frac{du}{dx}=-2x$

$dx=\frac{du}{-2x}$

$\Rightarrow yx^3=\int x^3e^{u}\frac{du}{-2x}$

$\Rightarrow yx^3=\int -\frac{1}{2}x^2e^{u}\,du$

$\Rightarrow yx^3=\int -\frac{1}{2}(-u)e^{u}\,du$

$\Rightarrow yx^3=\int \frac{1}{2}ue^{u}\,du$

BY PARTS

$\frac{1}{2}u$	$\frac{1}{2}$
e^{u}	e^{u}

$\Rightarrow yx^3=\frac{1}{2}ue^{u}-\int\frac{1}{2}e^{u}\,du$

$\Rightarrow yx^3=\frac{1}{2}ue^{u}-\frac{1}{2}e^{u}+C$

$\Rightarrow yx^3=\frac{1}{2}(-x^2)e^{-x^2}-\frac{1}{2}e^{-x^2}+C$

$\Rightarrow 2yx^3=-x^2e^{-x^2}-e^{-x^2}+D$

$\Rightarrow 2yx^3=-e^{-x^2}(x^2+1)+D$

$\Rightarrow 2yx^3+(x^2+1)e^{-x^2}=\text{CONSTANT}$

Question 11 (***+)

The curve with equation $y = f(x)$ passes through the origin, and satisfies the relationship

$$\frac{d}{dx}\left[y\left(x^2+1\right)\right] = x^5 + 2x^3 + x + 3xy \, .$$

Determine a simplified expression for the equation of the curve.

$\boxed{}$, $\boxed{y = \frac{1}{3}\left(x^2+1\right)^2 - \frac{1}{3}\left(x^2+1\right)^{\frac{1}{2}}}$

PROCEED AS FOLLOWS

$$\Rightarrow \frac{d}{dx}\left[y(x^2+1)\right] = x^5+2x^3+x+3xy$$

$$\Rightarrow \frac{dy}{dx}(x^2+1) + 2xy = x^5+2x^3+x+3xy$$

$$\Rightarrow \frac{dy}{dx}(x^2+1) - xy = x^5+2x^3+x$$

$$\Rightarrow \frac{dy}{dx} - \frac{xy}{x^2+1} = \frac{x^5+2x^3+x}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = \frac{x(x^4+2x+1)}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = \frac{x(x^2+1)^2}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{x^2+1}\right)y = x(x^2+1)$$

LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -\frac{x}{x^2+1}\,dx} = e^{\int -\frac{1}{2}\frac{2x}{x^2+1}\,dx} = e^{-\frac{1}{2}\ln(x^2+1)} = e^{\ln(x^2+1)^{-\frac{1}{2}}}$$

$$= (x^2+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2+1}}$$

WE NOW HAVE

$$\Rightarrow \frac{d}{dx}\left[y \times \frac{1}{\sqrt{x^2+1}}\right] = x(x^2+1) \times \frac{1}{\sqrt{x^2+1}}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{y}{\sqrt{x^2+1}}\right] = x(x^2+1)^{\frac{1}{2}}$$

$$\Rightarrow \frac{y}{(x^2+1)^{\frac{1}{2}}} = \int x(x^2+1)^{\frac{1}{2}}\,dx$$

$$\Rightarrow \frac{y}{(x^2+1)^{\frac{1}{2}}} = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + A$$

$$\Rightarrow y = \frac{1}{3}(x^2+1)^2 + A(x^2+1)^{\frac{1}{2}}$$

FINALLY APPLYING THE CONDITION (0,0)

$$\Rightarrow 0 = \frac{1}{3} + A$$

$$\Rightarrow A = -\frac{1}{3}$$

$$\Rightarrow y = \frac{1}{3}(x^2+1)^2 - \frac{1}{3}(x^2+1)^{\frac{1}{2}}$$

Question 12 **(***+)**

$$\frac{dy}{dx}+\frac{y}{x}=\frac{5}{\left(x^2+2\right)\left(4x^2+3\right)},\ x>0.$$

Given that $y=\frac{1}{2}\ln\frac{7}{6}$ at $x=1$, show that the solution of the above differential equation can be written as

$$y=\frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right).$$

$\boxed{}$, $\boxed{\text{proof}}$

WRITE THE O.D.E IN THE USUAL ORDER

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=\frac{5}{(x^2+2)(4x^2+3)}$$

INTEGRATING FACTOR CAN BE FOUND

$$e^{\int \frac{1}{x}\,dx}=e^{\ln x}=x$$

HENCE WE OBTAIN

$$\Rightarrow \frac{d}{dx}(yx)=\frac{5x}{(x^2+2)(4x^2+3)}$$

$$\Rightarrow yx=\int \frac{5x}{(x^2+2)(4x^2+3)}\,dx$$

PARTIAL FRACTIONS ARE NEEDED

$$\frac{5x}{(x^2+2)(4x^2+3)}\equiv\frac{Ax+B}{x^2+2}+\frac{Cx+D}{4x^2+3}$$

$$\boxed{5x\equiv(Ax+B)(4x^2+3)+(x^2+2)(Cx+D)}$$

$$5x\equiv 4Ax^3+4Bx^2+3Ax+3B$$
$$Cx^3+Dx^2+2Cx+2D$$

$$5x\equiv(4A+C)x^3+(4B+D)x^2+(3A+2C)x+(3B+2D)$$

$$\left.\begin{array}{l}4A+C=0\\3A+2C=5\end{array}\right\}\Rightarrow\left.\begin{array}{l}8A+2C=0\\3A+2C=5\end{array}\right\}\Rightarrow\begin{array}{l}A=-1\\C=4\end{array}$$

$$\left.\begin{array}{l}4B+D=0\\3B+2D=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}8B+2D=0\\3B+2D=0\end{array}\right\}\Rightarrow\begin{array}{l}B=0\\D=0\end{array}$$

CARRYING OUT THE REQUIRED INTEGRATION

$$\Rightarrow yx=\int \frac{4x}{4x^2+3}-\frac{x}{x^2+2}\,dx$$

$$\Rightarrow 2yx=\int \frac{8x}{4x^2+3}-\frac{2x}{x^2+2}\,dx$$

$$\Rightarrow 2yx=\ln(4x^2+3)-\ln(x^2+2)+\ln A$$

$$\Rightarrow 2yx=\ln\left[\frac{A(4x^2+3)}{x^2+2}\right]$$

APPLY CONDITION $x=1$, $y=\frac{1}{2}\ln\frac{7}{6}$

$$\Rightarrow 2\times\frac{1}{2}\ln\frac{7}{6}\times 1=\ln\left(\frac{7A}{3}\right)$$

$$\Rightarrow \ln\frac{7}{6}=\ln\frac{7A}{3}$$

$$\Rightarrow \frac{7}{6}=\frac{7A}{3}$$

$$\Rightarrow A=\frac{1}{2}$$

FINALLY WE HAVE

$$\Rightarrow 2yx=\ln\left[\frac{4x^2+3}{2(x^2+2)}\right]$$

$$\Rightarrow y=\frac{1}{2x}\ln\left[\frac{4x^2+3}{2x^2+4}\right]$$

AS REQUIRED

Question 13 (***+)

$$\left(2x-4y^{2}\right)\frac{dy}{dx}+y=0.$$

By reversing the role of x and y in the above differential equation, or otherwise, find its general solution.

$\boxed{}$, $\boxed{xy^{2}=y^{4}+C}$

USING THE SUGGESTION GIVEN

$\Rightarrow \left(2x-4y^{2}\right)\frac{dy}{dx}+y=0$

LET $x \mapsto Y$ & $y \mapsto X$

$\Rightarrow \left(2Y-4X^{2}\right)\frac{dX}{dY}+X=0$

$\Rightarrow \frac{dX}{dY}=-\frac{X}{2Y-4X^{2}}$

$\Rightarrow \frac{dY}{dX}=\frac{4X^{2}-2Y}{X}$

$\Rightarrow \frac{dY}{dX}=4X-\frac{2Y}{X}$

$\Rightarrow \frac{dY}{dX}+\frac{2}{X}Y=4X$

INTEGRATING FACTOR

$e^{\int \frac{2}{X}dX}=e^{2\ln X}=e^{\ln X^{2}}=X^{2}$

MULTIPLYING THROUGH BY THE INTEGRATING FACTOR TO MAKE THE LEFT SIDE EXACT

$\Rightarrow \frac{d}{dX}\left(YX^{2}\right)=4X^{3}$

$\Rightarrow YX^{2}=\int 4X^{3}\,dX$

$\Rightarrow YX^{2}=X^{4}+C$

$\Rightarrow xy^{2}=y^{4}+C$

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Question 14 (****)

It is given that a curve with equation $y = f(x)$ passes through the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and satisfies the differential equation

$$\left(\frac{dy}{dx} - \sqrt{\tan x}\right)\sin 2x = y .$$

Find an equation for the curve in the form $y = f(x)$.

$\boxed{\quad}$, $\boxed{y = x\sqrt{\tan x}}$

MULTIPLY OUT FIRST & REARRANGE

$$\Rightarrow \left[\frac{dy}{dx} - \sqrt{\tan x}\right]\sin 2x = y$$

$$\Rightarrow \frac{dy}{dx}\sin 2x - \sin 2x\sqrt{\tan x} = y$$

$$\Rightarrow \frac{dy}{dx}\sin 2x - y = \sin 2x\sqrt{\tan x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{\sin 2x} = \sqrt{\tan x}$$

LOOK FOR AN INTEGRATING FACTOR

$$e^{\int -\frac{1}{\sin 2x}\,dx} = e^{\int -\operatorname{cosec} 2x\,dx} = e^{\frac{1}{2}\ln|\operatorname{cosec} 2x + \cot 2x|}$$

$$= e^{\ln(\operatorname{cosec} 2x + \cot 2x)^{\frac{1}{2}}} = (\operatorname{cosec} 2x + \cot 2x)^{\frac{1}{2}}$$

$$= \left(\frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x}\right)^{\frac{1}{2}} = \left(\frac{1 + \cos 2x}{\sin 2x}\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{1 + (2\cos^2 x - 1)}{2\sin x\cos x}} = \sqrt{\frac{2\cos^2 x}{2\sin x\cos x}}$$

$$= \sqrt{\frac{\cos x}{\sin x}} = \sqrt{\cot x}$$

RETURNING TO THE O.D.E

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{\cot x}\right) = \sqrt{\tan x}\sqrt{\cot x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{\sqrt{\tan x}}\right) = 1$$

$$\Rightarrow \frac{y}{\sqrt{\tan x}} = \int 1\,dx$$

$$\Rightarrow \frac{y}{\sqrt{\tan x}} = x + C$$

$$\Rightarrow y = x\sqrt{\tan x} + C\sqrt{\tan x}$$

APPLY BOUNDARY CONDITION $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4}\sqrt{\tan\frac{\pi}{4}} + C\sqrt{\tan\frac{\pi}{4}}$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4}\times 1 + C\times 1$$

$$\Rightarrow \frac{\pi}{4} = \frac{\pi}{4} + C$$

$$\Rightarrow C = 0$$

$$\therefore y = x\sqrt{\tan x}$$

Question 15 (****)

Find a simplified general solution for the following differential equation.

$$x\frac{dy}{dx} = 2x^2 + 2xy + y\,.$$

$$\boxed{y = Ax\mathrm{e}^{-x} - x}$$

$$\Rightarrow x\frac{dy}{dx} = 2x^2 + 2xy + y$$

$$\Rightarrow x\frac{dy}{dx} - 2xy - y = 2x^2$$

$$\Rightarrow \frac{dy}{dx} - 2y - \frac{y}{x} = 2x$$

$$\Rightarrow \frac{dy}{dx} + y\left(-2 - \frac{1}{x}\right) = 2x$$

- INTEGRATING FACTOR

$$e^{\int -2-\frac{1}{x}\,dx} = e^{-2x-\ln x} = e^{-2x} \times e^{-\ln x} = e^{-2x} \times e^{\ln\frac{1}{x}} = \frac{1}{x}e^{-2x}$$

- HENCE WE OBTAIN

$$\Rightarrow \frac{d}{dx}\left[y\frac{1}{x}e^{-2x}\right] = 2x\left(\frac{1}{x}e^{-2x}\right)$$

$$\Rightarrow \frac{d}{dx}\left[\frac{y}{x}e^{-2x}\right] = 2e^{-2x}$$

$$\Rightarrow \frac{y}{x}e^{-2x} = \int 2e^{-2x}\,dx$$

$$\Rightarrow \frac{y}{x}e^{-2x} = -e^{-2x} + A$$

$$\Rightarrow ye^{-2x} = -xe^{-2x} + Ax$$

$$\Rightarrow y = -x + Axe^{2x}$$

Question 16 (****)

The curve with equation $y = f(x)$ has the line $y = 1$ as an asymptote and satisfies the differential equation

$$x^3 \frac{dy}{dx} - x = xy + 1, \quad x \neq 0.$$

Solve the above differential equation, giving the solution in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = e^{-\frac{1}{x}} - \frac{1}{x}}$

REWRITE THE O.D.E

$$x^3\frac{dy}{dx} - x = xy + 1$$
$$x^3\frac{dy}{dx} - xy = x + 1$$
$$\frac{dy}{dx} - \frac{y}{x^2} = \frac{x+1}{x^3}$$

LOOK FOR AN INTEGRATING FACTOR

$$\text{I.F} = e^{\int -\frac{1}{x^2}dx} = e^{\frac{1}{x}}$$
$$\Rightarrow \frac{d}{dx}\left(e^{\frac{1}{x}}y\right) = \left(\frac{x+1}{x^3}\right)e^{\frac{1}{x}}$$
$$\Rightarrow e^{\frac{1}{x}}y = \int \left(\frac{1}{x^2} + \frac{1}{x^3}\right)e^{\frac{1}{x}}\,dx$$

PROCEED WITH A SUBSTITUTION

$$\boxed{u = \frac{1}{x} \Leftrightarrow x = \frac{1}{u}, \quad dx = -\frac{1}{u^2}du}$$

$$\Rightarrow ye^{\frac{1}{x}} = \int (u^2 + u^3)e^u\left(-\frac{1}{u^2}du\right)$$
$$\Rightarrow ye^{\frac{1}{x}} = -\int (1+u)e^u\,du$$
$$\Rightarrow ye^{\frac{1}{x}} = \int -e^u - ue^u\,du$$
$$\Rightarrow ye^{\frac{1}{x}} = \int -e^u\,du - \int ue^u\,du$$

NOW INTEGRATION BY PARTS / INSPECTION

$$\boxed{\begin{aligned} \frac{d}{du}(ue^u) &= e^u + ue^u \\ ue^u &= e^u + \int ue^u\,du \\ \int ue^u\,du &= ue^u - e^u + C \end{aligned}}$$

$$\Rightarrow ye^{\frac{1}{x}} = -e^u - \left[ue^u - e^u\right] + C$$
$$\Rightarrow ye^{\frac{1}{x}} = \cancel{-e^u} - ue^u \cancel{+e^u} + C$$
$$\Rightarrow ye^{\frac{1}{x}} = -ue^u + C$$
$$\Rightarrow ye^{\frac{1}{x}} = -\frac{1}{x}e^{\frac{1}{x}} + C$$
$$\Rightarrow y = -\frac{1}{x} + Ce^{-\frac{1}{x}}$$

NOW $y=1$ IS AN ASYMPTOTE

$$\Rightarrow \text{AS } x \to \pm\infty,\ y \to 1$$
$$\Rightarrow 1 = 0 + Ce^0$$
$$\Rightarrow C = 1$$

$$\Rightarrow y = e^{-\frac{1}{x}} - \frac{1}{x}$$

Question 17 (****)

It is given that a curve with equation $x = f(y)$ passes through the point $\left(0, \frac{1}{2}\right)$ and satisfies the differential equation

$$(2y + 3x)\frac{dy}{dx} = y.$$

Find an equation for the curve in the form $x = f(y)$.

$\boxed{}$, $\boxed{x = 4y^3 - y}$

METHOD A

REARRANGE & TREAT y AS THE INDEPENDENT VARIABLE

$$\Rightarrow (2y+3x)\frac{dy}{dx} = y$$
$$\Rightarrow 2y + 3x = y\frac{dx}{dy}$$
$$\Rightarrow y\frac{dx}{dy} - 3x = 2y$$
$$\Rightarrow \frac{dx}{dy} - \frac{3}{y}x = 2$$

INTEGRATING FACTOR CAN NOW BE FOUND

$$e^{\int -\frac{3}{y}\,dy} = e^{-3\ln y} = e^{\ln \frac{1}{y^3}} = \frac{1}{y^3}$$

HENCE WE NOW HAVE

$$\Rightarrow \frac{d}{dy}\left(x \cdot \frac{1}{y^3}\right) = 2 \cdot \frac{1}{y^3}$$
$$\Rightarrow \frac{x}{y^3} = \int \frac{2}{y^3}\,dy$$
$$\Rightarrow \frac{x}{y^3} = -\frac{1}{y^2} + A$$
$$\Rightarrow x = Ay^3 - y$$

APPLY CONDITION $(0, \frac{1}{2})$

$$\Rightarrow 0 = A\left(\tfrac{1}{8}\right) - \tfrac{1}{2}$$
$$\Rightarrow 0 = A - 4$$
$$\Rightarrow A = 4$$

$$\therefore x = 4y^3 - y$$

METHOD B

PROCEED BY A SUBSTITUTION

$$\boxed{\begin{aligned} y &= x\,V(x) \\ \frac{dy}{dx} &= V(x) + x\frac{dV}{dx} \end{aligned}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2y+3x}$$
$$\Rightarrow V + x\frac{dV}{dx} = \frac{\cancel{x}V}{2\cancel{x}V + 3\cancel{x}}$$
$$\Rightarrow x\frac{dV}{dx} = \frac{V}{2V+3} - V$$
$$\Rightarrow x\frac{dV}{dx} = \frac{V - 2V^2 - 3V}{2V+3} = \frac{-2V^2 - 2V}{2V+3} = \frac{-2V(V+1)}{2V+3}$$

SEPARATING VARIABLES

$$\Rightarrow \frac{2V+3}{V(V+1)}\,dV = -\frac{2}{x}\,dx$$
$$\Rightarrow \int \frac{3}{V} - \frac{1}{V+1}\,dV = \int -\frac{2}{x}\,dx$$

(PARTIAL FRACTIONS BY INSPECTION)

$$\Rightarrow 3\ln|V| - \ln|V+1| = -2\ln|x| + \ln A$$
$$\Rightarrow \ln\left|\frac{V^3}{V+1}\right| = \ln\left|\frac{A}{x^2}\right|$$
$$\Rightarrow \frac{V^3}{V+1} = \frac{A}{x^2}$$
$$\Rightarrow \frac{\frac{y^3}{x^3}}{\frac{y}{x}+1} = \frac{A}{x^2}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION IN THE L.H.S. BY x^3

$$\Rightarrow \frac{y^3}{yx^2 + x^3} = \frac{A}{x^2}$$

MULTIPLY BOTH SIDES BY x^2

$$\Rightarrow \frac{y^3}{y + x} = A$$

APPLY THE CONDITION $(0, \frac{1}{2})$

$$\Rightarrow \frac{\frac{1}{8}}{\frac{1}{2}+0} = A$$
$$\Rightarrow A = \frac{1}{4}$$

$$\therefore \frac{y^3}{y+x} = \frac{1}{4}$$
$$4y^3 = y + x$$
$$x = 4y^3 - y$$

AS BEFORE

Question 18 (****)

It is given that a curve passes through the point $(-2,0)$ and satisfies the ordinary differential equation

$$\frac{dy}{dx} = \frac{1}{x+y^2}.$$

Show that an equation of C is

$$(y+1)^2 + x + 1 = 0.$$

proof

$$\boxed{\frac{dy}{dx} = \frac{1}{x+y^2}, \text{ SUBJECT TO } y=0 \text{ AT } x=-2}$$

RECIPROCATE THE EQUATION AND REARRANGE

$$\Rightarrow \frac{dx}{dy} = x + y^2$$

$$\Rightarrow \frac{dx}{dy} - x = y^2$$

INTEGRATING FACTOR IS $e^{\int -1\,dy} = e^{-y}$

THUS AFTER MULTIPLYING THROUGH WE OBTAIN

$$\Rightarrow \frac{d}{dy}\left(xe^{-y}\right) = y^2e^{-y}$$

$$\Rightarrow xe^{-y} = \int y^2e^{-y}\,dy \quad \leftarrow \text{BY PARTS TWICE}$$

$$\Rightarrow xe^{-y} = -y^2e^{-y} + \int 2ye^{-y}\,dy$$

$$\Rightarrow xe^{-y} = -y^2e^{-y} - 2ye^{-y} + \int 2e^{-y}\,dy$$

$$\Rightarrow xe^{-y} = -y^2e^{-y} - 2ye^{-y} - 2e^{-y} + A$$

$$\Rightarrow x = Ae^{y} - y^2 - 2y - 2$$

y^2	$2y$
$-e^{-y}$	e^{-y}

$2y$	2
$-e^{-y}$	e^{-y}

APPLY CONDITION $x=-2,\ y=0$

$$-2 = A - 2$$
$$A = 0$$

THUS $x = -y^2 - 2y - 2$

$$y^2 + 2y + 1 + x + 1 = 0$$
$$(y+1)^2 + x + 1 = 0$$

Question 19 (****)

The variables x and y satisfy

$$\left(2y-x\right)\frac{dy}{dx}=y\,,\quad y>0\,,\ x>0\,.$$

If $y=1$ at $x=2$, show that $x=y+\frac{1}{y}$.

$\boxed{\text{V}}$, $\boxed{}$, $\boxed{\text{proof}}$

METHOD A – EXACT EQUATIONS & INTEGRATING FACTORS

$\Rightarrow (2y-x)\frac{dy}{dx}=y$

$\Rightarrow \frac{dy}{dx}=\frac{y}{2y-x}$

$\Rightarrow \frac{dx}{dy}=\frac{2y-x}{y}$

$\Rightarrow y\frac{dx}{dy}=2y-x$

$\Rightarrow y\frac{dx}{dy}+x=2y$

NOW NOTE THAT O.D.E. IS EXACT

$\frac{d}{dy}(xy)=\frac{dx}{dy}\times y+x\times 1$

$\frac{d}{dy}(xy)=y\frac{dx}{dy}+x$

$\therefore \frac{d}{dy}(xy)=2y$

OR BY INTEGRATING FACTOR IN y

$y\frac{dx}{dy}+x=2y$

$\frac{dx}{dy}+\frac{x}{y}=2$

$e^{\int\frac{1}{y}dy}=e^{\ln y}=y$

$\frac{d}{dy}(xy)=2y$

INTEGRATING W.R.T y

$\Rightarrow xy=\int 2y\,dy$

$\Rightarrow xy=y^2+A$

APPLY CONDITION $(2,1)$

$\Rightarrow 2=1+A$

$\Rightarrow A=1$

THUS WE HAVE

$xy=y^2+1$ OR $x=y+\frac{1}{y}$

METHOD B – BY SUBSTITUTION AS THE O.D.E IS HOMOGENEOUS

$\Rightarrow (2y-x)\frac{dy}{dx}=y$

$\Rightarrow \frac{dy}{dx}=\frac{y}{2y-x}$

LET $y=xV(x)$ OR $V(x)=\frac{y(x)}{x}$

$\frac{dy}{dx}=1\times V+x\frac{dV}{dx}$

$\frac{dy}{dx}=V+x\frac{dV}{dx}$

$\Rightarrow V+x\frac{dV}{dx}=\frac{Vx}{2Vx-x}$

$\Rightarrow V+x\frac{dV}{dx}=\frac{V}{2V-1}$

$\Rightarrow x\frac{dV}{dx}=\frac{V}{2V-1}-V$

$\Rightarrow x\frac{dV}{dx}=\frac{V-2V^2+V}{2V-1}$

SEPARATE VARIABLES & MANIPULATE

$\Rightarrow \frac{2V-1}{-2V^2+2V}\,dV=\frac{1}{x}\,dx$

$\Rightarrow \frac{2V-1}{2V^2-2V}\,dV=-\frac{1}{x}\,dx$

$\Rightarrow \frac{2V-1}{V^2-V}\,dV=-\frac{2}{x}\,dx$

$\Rightarrow \int\frac{2V-1}{V^2-V}\,dV=\int-\frac{2}{x}\,dx$

$\Rightarrow \ln|V^2-V|=-2\ln|x|+A$

$\Rightarrow \ln|V^2-V|=\ln\left(\frac{1}{x^2}\right)+\ln B$

$\Rightarrow \ln|V^2-V|=\ln\left(\frac{B}{x^2}\right)$

$\Rightarrow V^2-V=\frac{B}{x^2}$

$\Rightarrow \frac{y^2}{x^2}-\frac{y}{x}=\frac{B}{x^2}$

FINALLY APPLY CONDITION $(2,1)$

$\frac{1}{4}-\frac{1}{2}=\frac{B}{4}$

$-\frac{1}{4}=\frac{B}{4}$

$B=-1$

$\therefore \frac{y^2}{x^2}-\frac{y}{x}=-\frac{1}{x^2}$

$y^2-xy=-1$

$y^2+1=xy$

$xy=y^2+1$

$x=y+\frac{1}{y}$ // AS BEFORE

Question 20

The variables x and y satisfy

$$\frac{dy}{dx} = \frac{y(y+1)}{y-x-xy-1}, \quad y>0.$$

If $y=1$ at $x=1-\ln 4$, show that $y+\ln(y+1)=0$ at $x=3$.

V, , proof

START MANIPULATING AS FOLLOWS

$$\frac{dy}{dx} = \frac{y(y+1)}{y-1-xy-x} = \frac{y(y+1)}{(y-1)-x(y+1)}$$

"INVERTING" TO SWAP DEPENDENT & INDEPENDENT VARIABLES

$$\Rightarrow \frac{dx}{dy} = \frac{(y-1)-x(y+1)}{y(y+1)}$$

SPLITTING R.H.S

$$\Rightarrow \frac{dx}{dy} = \frac{y-1}{y(y+1)} - \frac{x}{y} \quad (y>0)$$

$$\Rightarrow y\frac{dx}{dy} = \frac{y-1}{y+1} - x$$

$$\Rightarrow y\frac{dx}{dy} + x = \frac{y-1}{y+1}$$

NOW THE L.H.S IS EXACT IN y (OR INTEGRATING FACTOR)

- $\frac{d}{dy}(xy) = \frac{dx}{dy}\cdot y + x\cdot 1$
  $= y\frac{dx}{dy} + x$
  $\therefore \frac{d}{dy}(xy) = \frac{y-1}{y+1}$

- $y\frac{dx}{dy} + x = \frac{y-1}{y+1}$
  $\frac{dx}{dy} + \frac{x}{y} = \frac{y-1}{y(y+1)}$
  $e^{\int \frac{1}{y}dy} = e^{\ln y} = y$
  $\therefore \frac{d}{dy}(xy) = \frac{y(y-1)}{y(y+1)} \quad (y>0)$
  $\frac{d}{dy}(xy) = \frac{y-1}{y+1}$ AS OPPOSITE

INTEGRATING W.R.T y

$$\Rightarrow xy = \int \frac{y-1}{y+1}\,dy$$

$$\Rightarrow xy = \int \frac{(y+1)-2}{y+1}\,dy$$

$$\Rightarrow xy = \int 1 - \frac{2}{y+1}\,dy$$

$$\Rightarrow xy = y - 2\ln(y+1) + A \qquad y>0$$

NO MODULI NEEDED

APPLY BOUNDARY CONDITION GIVEN

$x = 1-\ln 4,\ y=1$

$$\Rightarrow (1-\ln 4)\times 1 = 1 - 2\ln 2 + A$$
$$\Rightarrow 1-\ln 4 = 1 - \ln 4 + A$$
$$\Rightarrow A = 0$$

$$\therefore xy = y - 2\ln(y+1)$$

WHEN $x=3$

$$\Rightarrow 3y = y - 2\ln(y+1)$$
$$\Rightarrow 2y = -2\ln(y+1)$$
$$\Rightarrow y = -\ln(y+1)$$
$$\Rightarrow y + \ln(y+1) = 0$$

Question 21 (*****)

Use suitable manipulations to solve this **exact** differential equation.

$$4x\frac{dy}{dx}+\sin 2y = 4\cos^2 y\,, \quad y\left(\tfrac{1}{4}\right)=0\,.$$

Given the answer in the form $y = f(x)$.

V, , $\boxed{y=\arctan\left[2-\dfrac{1}{\sqrt{x}}\right]}$

SWITCH INTO SINES & COSINES AND TIDY

$\Rightarrow 4x\frac{dy}{dx}+\sin 2y = 4\cos^2 y$ ← CHANGING THIS TO $4\left(\frac{1}{2}+\frac{1}{2}\cos 2y\right)$ $2+2\cos 2y$ WAS ALSO TRIED

$\Rightarrow 4x\frac{dy}{dx}+2\sin y\cos y = 4\cos^2 y$

$\Rightarrow 2x\frac{dy}{dx}+\sin y\cos y = 2\cos^2 y$

$\Rightarrow 2x\sec^2 y\frac{dy}{dx}+\sin y\cos y\sec^2 y = 2\cos^2 y\sec^2 y$

$\Rightarrow 2x\sec^2 y\frac{dy}{dx}+\tan y = 2$

THIS IS "AN EXACT" EQUATION "ALMOST", IF $\frac{d}{dx}(\tan y \times ?)$

AFTER A BIT OF "TRYING" DIVIDE THROUGH BY $x^{1/2}$ (OR MULTIPLY BY $x^{-\frac{1}{2}}$)

$\Rightarrow 2x^{\frac{1}{2}}\sec^2 y\frac{dy}{dx}+x^{-\frac{1}{2}}\tan y = 2x^{-\frac{1}{2}}$

$\Rightarrow \frac{d}{dx}\left[2x^{\frac{1}{2}}\tan y\right] = 2x^{-\frac{1}{2}}$

$\Rightarrow 2x^{\frac{1}{2}}\tan y = \int 2x^{-\frac{1}{2}}\,dx$

$\Rightarrow 2x^{\frac{1}{2}}\tan y = 4x^{\frac{1}{2}}+C$

$\Rightarrow \tan y = 2+Ax^{-\frac{1}{2}}$

$\quad\quad 2+\frac{A}{x^{\frac{1}{2}}}$

APPLY CONDITION GIVEN

$\tan 0 = 2+\frac{A}{\sqrt{1/4}}$

$0 = 2+\frac{A}{1/2}$

$A=-1$

$\Rightarrow \tan y = 2-\frac{1}{\sqrt{x}}$

$\Rightarrow y=\arctan\left(2-\frac{1}{\sqrt{x}}\right)$

1^{ST} ORDER HOMOGENEOUS

Question 1 (+)**

Solve the differential equation

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2, \quad x > 0,$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y = \frac{x}{1+\ln x}}$$

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$v = \frac{y}{x}$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x \times \frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = v - v^2$

$\Rightarrow x\frac{dv}{dx} = -v^2$

$\Rightarrow -\frac{1}{v^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int -v^{-2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow v^{-1} = \ln x + C$

$\Rightarrow \frac{x}{y} = \ln x + C$

$\Rightarrow \frac{x}{\ln x + C} = y$

$\Rightarrow$ APPLY CONDITION $(1,1)$

$\frac{1}{\ln 1 + C} = 1$

$C = 1$

$\therefore y = \frac{x}{1+\ln x}$

Question 2 (**+)

$$\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}, \quad x > 0.$$

a) Use a suitable substitution to show that the above differential equation can be transformed to

$$x\frac{dv}{dx} = (v+2)^2.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y = f(x)$.

c) Use the boundary condition $y = -1$ at $x = 1$, to show that a specific solution of the original differential equation is

$$y = \frac{x}{1 - \ln x} - 2x.$$

$$\boxed{y = \frac{x}{A - \ln x} - 2x}$$

(a) $\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}$

$y = vx$

$\frac{dy}{dx} = \frac{dv}{dx}x + v \times 1$

$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

$v = \frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{4x^2+5xy+y^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5x(vx)+(vx)^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{4x^2+5vx^2+v^2x^2}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = 4+5v+v^2$

$\Rightarrow x\frac{dv}{dx} = v^2+4v+4$

$\Rightarrow x\frac{dv}{dx} = (v+2)^2$ // AS REQUIRED

(b) $\frac{1}{(v+2)^2}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{(v+2)^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{v+2} = \ln x + C$

$\Rightarrow \frac{1}{v+2} = A - \ln x$

$\Rightarrow v+2 = \frac{1}{A-\ln x}$

$\Rightarrow v = \frac{1}{A-\ln x} - 2$

$\Rightarrow \frac{y}{x} = \frac{1}{A-\ln x} - 2$

$\Rightarrow y = \frac{x}{A-\ln x} - 2x$ //

(c) $x=1 \quad y=-1$

$-1 = \frac{1}{A-\ln 1} - 2$

$1 = \frac{1}{A}$

$\boxed{A = 1}$

$\therefore y = \frac{x}{1-\ln x} - 2x$ //

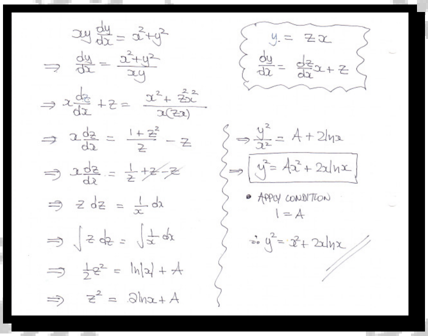
Question 3 (**+)

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx} = x^2 + y^2, \quad x > 0,$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{y = x^2(1 + 2\ln x)}$$



$$xy\frac{dy}{dx} = x^2 + y^2$$
$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{xy}$$
$$\Rightarrow x\frac{dz}{dx} + z = \frac{x^2 + z^2x^2}{x(zx)}$$
$$\Rightarrow x\frac{dz}{dx} = \frac{1+z^2}{z} - z$$
$$\Rightarrow x\frac{dz}{dx} = \frac{1}{z} + \cancel{z} - \cancel{z}$$
$$\Rightarrow z\,dz = \frac{1}{x}\,dx$$
$$\Rightarrow \int z\,dz = \int \frac{1}{x}\,dx$$
$$\Rightarrow \frac{1}{2}z^2 = \ln|x| + A$$
$$\Rightarrow z^2 = 2\ln x + A$$

$$y = zx$$
$$\frac{dy}{dx} = \frac{dz}{dx}x + z$$

$$\Rightarrow \frac{y^2}{x^2} = A + 2\ln x$$
$$\Rightarrow \boxed{y^2 = Ax^2 + 2x\ln x}$$

- APPLY CONDITION

$1 = A$

$$\therefore y^2 = x^2 + 2x\ln x$$

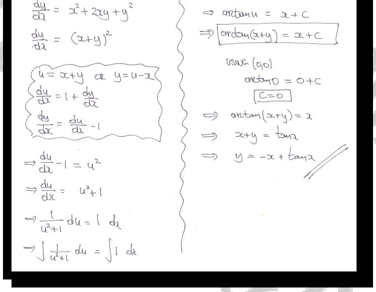
Question 4 (**+)

By using a suitable substitution, or otherwise, solve the differential equation

$$\frac{dy}{dx} = x^2 + 2xy + y^2,$$

subject to the condition $y(0) = 0$.

$$\boxed{y = -x + \tan x}$$



$$\frac{dy}{dx} = x^2 + 2xy + y^2$$
$$\frac{dy}{dx} = (x+y)^2$$

$$u = x + y \text{ or } y = u - x$$
$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\Rightarrow \frac{du}{dx} - 1 = u^2$$
$$\Rightarrow \frac{du}{dx} = u^2 + 1$$
$$\Rightarrow \frac{1}{u^2+1}\,du = 1\,dx$$
$$\Rightarrow \int \frac{1}{u^2+1}\,du = \int 1\,dx$$
$$\Rightarrow \arctan u = x + C$$
$$\Rightarrow \boxed{\arctan(x+y) = x + C}$$

USING $(0,0)$

$$\arctan 0 = 0 + C$$
$$\boxed{C = 0}$$

$$\Rightarrow \arctan(x+y) = x$$
$$\Rightarrow x + y = \tan x$$
$$\Rightarrow y = -x + \tan x$$

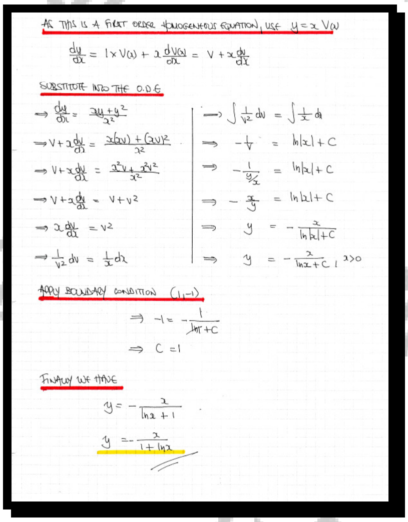
Question 5 (**+)

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2}, \; x > 0,$$

subject to the condition $y = -1$ at $x = 1$.

$$\boxed{y = -\frac{x}{1 + \ln x}}$$



AS THIS IS A FIRST ORDER HOMOGENEOUS EQUATION, USE $y = xV(x)$

$$\frac{dy}{dx} = 1 \times V(x) + x\frac{dV(x)}{dx} = V + x\frac{dv}{dx}$$

SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{xy + y^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x(xv) + (xv)^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = \frac{x^2v + x^2v^2}{x^2}$$

$$\Rightarrow V + x\frac{dv}{dx} = V + v^2$$

$$\Rightarrow x\frac{dv}{dx} = v^2$$

$$\Rightarrow \frac{1}{v^2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{V} = \ln|x| + C$$

$$\Rightarrow -\frac{1}{y/x} = \ln|x| + C$$

$$\Rightarrow -\frac{x}{y} = \ln|x| + C$$

$$\Rightarrow y = -\frac{x}{\ln|x| + C}$$

$$\Rightarrow y = -\frac{x}{\ln x + C}, \; x > 0$$

APPLY BOUNDARY CONDITION $(1, -1)$

$$\Rightarrow -1 = -\frac{1}{\cancel{\ln 1} + C}$$

$$\Rightarrow C = 1$$

FINALLY WE HAVE

$$y = -\frac{x}{\ln x + 1}$$

$$y = -\frac{x}{1 + \ln x}$$

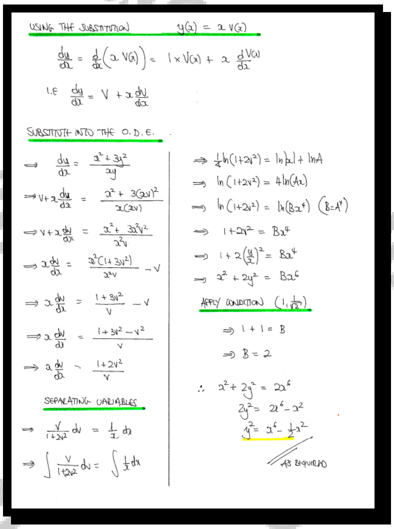
Question 6 (***)

$$\frac{dy}{dx} = \frac{x^2+3y^2}{xy}, \quad x>0, \quad y>0.$$

Given the boundary condition $y(1) = \frac{1}{\sqrt{2}}$, show that

$$y^2 = x^6 - \frac{1}{2}x^2.$$

$\boxed{}$, $\boxed{\text{proof}}$


USING THE SUBSTITUTION $y(x) = xV(x)$

$$\frac{dy}{dx} = \frac{d}{dx}\big(xV(x)\big) = 1\times V(x) + x\frac{dV(x)}{dx}$$

I.E $\frac{dy}{dx} = V + x\frac{dV}{dx}$

SUBSTITUTE INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{xy}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2+3(xV)^2}{x(xV)}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2+3x^2V^2}{x^2V}$

$\Rightarrow x\frac{dV}{dx} = \frac{x^2(1+3V^2)}{x^2V} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2}{V} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{1+3V^2-V^2}{V}$

$\Rightarrow x\frac{dV}{dx} = \frac{1+2V^2}{V}$

SEPARATING VARIABLES

$\Rightarrow \frac{V}{1+2V^2}\,dV = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{V}{1+2V^2}\,dV = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{1}{4}\ln(1+2V^2) = \ln|x| + \ln A$

$\Rightarrow \ln(1+2V^2) = 4\ln(Ax)$

$\Rightarrow \ln(1+2V^2) = \ln(Bx^4) \quad (B=A^4)$

$\Rightarrow 1+2V^2 = Bx^4$

$\Rightarrow 1+2\left(\frac{y}{x}\right)^2 = Bx^4$

$\Rightarrow x^2+2y^2 = Bx^6$

APPLY CONDITION $\left(1, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow 1+1 = B$

$\Rightarrow B = 2$

$\therefore x^2+2y^2 = 2x^6$

$2y^2 = 2x^6 - x^2$

$y^2 = x^6 - \frac{1}{2}x^2$

AS REQUIRED

Question 7 (***)

By using a suitable substitution, solve the differential equation

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2},$$

subject to the condition $y = 1$ at $x = 1$.

$$\boxed{y^3 = x^3\left(3\ln x + 1\right)}$$

$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$

$v = \frac{y}{x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3+x^3v^3}{x(x^2v^2)}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{1}{v^2} + \cancel{v} - \cancel{v}$

$\Rightarrow v^2\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int v^2\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{1}{3}v^3 = \ln|x| + A$

$\Rightarrow v^3 = 3\ln|x| + B$

$\Rightarrow \frac{y^3}{x^3} = 3\ln|x| + B$

$\Rightarrow y^3 = 3x^3\ln|x| + Bx^3$

$1 = B$ SINCE $x=1$, $y=1$

$\Rightarrow y^3 = 3x^3\ln x + x^3$

Question 8 (***)

By using a suitable substitution, solve the differential equation

$$2x^2\frac{dy}{dx} = x^2 + y^2, \quad x > 0,$$

subject to the condition $y(1) = 0$.

$$\boxed{y = x - \frac{2x}{2 + \ln x}}$$

$2x^2\frac{dy}{dx} = x^2 + y^2$

$y = xz(x)$

$\frac{dy}{dx} = 1 \times z + x\frac{dz}{dx}$

$z = \frac{y}{x}$

$\Rightarrow 2\cancel{x^2}\left[z + x\frac{dz}{dx}\right] = \cancel{x^2} + \cancel{x^2}z^2$

$\Rightarrow 2z + 2x\frac{dz}{dx} = 1 + z^2$

$\Rightarrow 2x\frac{dz}{dx} = 1 - 2z + z^2$

$\Rightarrow 2x\frac{dz}{dx} = (z-1)^2$

$\Rightarrow \frac{1}{(z-1)^2}\,dz = \frac{1}{2x}\,dx$

$\Rightarrow \int \frac{1}{(z-1)^2}\,dz = \int \frac{1}{2x}\,dx$

$\Rightarrow -\frac{1}{z-1} = \frac{1}{2}\ln x + C$

$\Rightarrow \frac{1}{z-1} = C - \frac{1}{2}\ln x$

$\Rightarrow z - 1 = \frac{1}{C - \frac{1}{2}\ln x}$

$\Rightarrow z = \frac{1}{C - \frac{1}{2}\ln x} + 1$

$\Rightarrow \frac{y}{x} = \frac{2}{A - \ln x} + 1$

$\Rightarrow \boxed{y = \frac{2x}{A - \ln x} + x}$

APPLY $(1,0) \Rightarrow 0 = \frac{2}{A} + 1$

$\Rightarrow A = -$

$\therefore y = x - \frac{2x}{2 + \ln x}$

Question 9 (***)

By using a suitable substitution, solve the differential equation

$$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right), \ x \neq 0,$$

subject to the condition $y(4) = \pi$.

The final answer may not involve natural logarithms.

$$\boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x\left(1+\sqrt{2}\right)}$$

$x\frac{dy}{dx} - y = x\cos\left(\frac{y}{x}\right)$

$\Rightarrow x\left(v + x\frac{dv}{dx}\right) - xv = x\cos\left(\frac{xv}{x}\right)$

$\Rightarrow \cancel{xv} + x^2\frac{dv}{dx} - \cancel{xv} = x\cos v$

$\Rightarrow x\frac{dv}{dx} = \cos v$

$\Rightarrow \frac{1}{\cos v}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \sec v\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \ln|\sec v + \tan v| = \ln|x| + \ln A$

$\Rightarrow \ln|\sec v + \tan v| = \ln|Ax|$

$\Rightarrow \sec v + \tan v = Ax$

$\Rightarrow \boxed{\sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = Ax}$

$y = xv$

$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$

$v = \frac{y}{x}$

APPLY CONDITION $x=4 \quad y=\pi$

$\sec\frac{\pi}{4} + \tan\frac{\pi}{4} = 4A$

$\sqrt{2} + 1 = 4A$

$A = \frac{1}{4}(1+\sqrt{2})$

$\therefore \sec\left(\frac{y}{x}\right) + \tan\left(\frac{y}{x}\right) = \frac{1}{4}x(1+\sqrt{2})$

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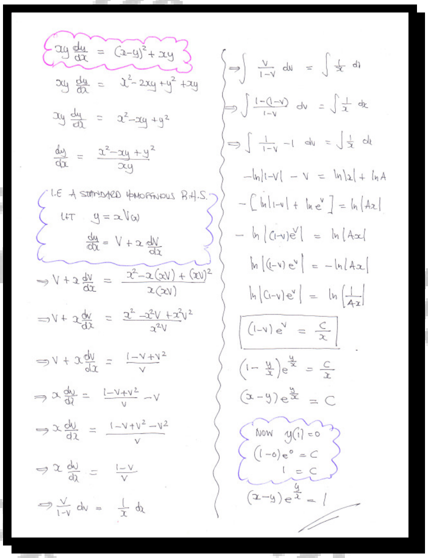
Question 10 (***)

$$xy\frac{dy}{dx} = (x-y)^2 + xy\,, \quad y(1) = 0\,.$$

Show that the solution of the above differential equation is

$$(x-y)\mathrm{e}^{\frac{y}{x}} = 1\,.$$

$\boxed{\text{proof}}$



$$xy\frac{dy}{dx} = (x-y)^2 + xy$$

$$xy\frac{dy}{dx} = x^2 - 2xy + y^2 + xy$$

$$xy\frac{dy}{dx} = x^2 - xy + y^2$$

$$\frac{dy}{dx} = \frac{x^2 - xy + y^2}{xy}$$

I.E A STANDARD HOMOGENEOUS R.H.S.

LET $y = xV(x)$

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 - x(xV) + (xV)^2}{x(xV)}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^2 - x^2V + x^2V^2}{x^2V}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{1 - V + V^2}{V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V + V^2}{V} - V$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V + V^2 - V^2}{V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1 - V}{V}$$

$$\Rightarrow \frac{V}{1-V}\,dV = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{V}{1-V}\,dV = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1-(1-V)}{1-V}\,dV = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{1-V} - 1\,dV = \int \frac{1}{x}\,dx$$

$$-\ln|1-V| - V = \ln|x| + \ln A$$

$$-\left[\ln|1-V| + \ln e^{V}\right] = \ln|Ax|$$

$$-\ln\left|(1-V)e^{V}\right| = \ln|Ax|$$

$$\ln\left|(1-V)e^{V}\right| = -\ln|Ax|$$

$$\ln\left|(1-V)e^{V}\right| = \ln\left|\frac{1}{Ax}\right|$$

$$\boxed{(1-V)e^{V} = \frac{C}{x}}$$

$$\left(1 - \frac{y}{x}\right)e^{\frac{y}{x}} = \frac{C}{x}$$

$$(x-y)e^{\frac{y}{x}} = C$$

NOW $y(1) = 0$

$$(1-0)e^{0} = C$$

$$1 = C$$

$$(x-y)e^{\frac{y}{x}} = 1$$

Question 11 (***)

Use the substitution $y = xv$, where $v = v(x)$, to solve the following differential equation

$$2\frac{dy}{dx} = 1 + \frac{y^2}{x^2}, \quad y(\mathrm{e}) = -\mathrm{e}.$$

$\boxed{}$, $\boxed{y = x - \dfrac{2x}{\ln x}}$

IF $y = xv(x) \Rightarrow v(x) = \frac{y(x)}{x}$

$$\frac{dy}{dx} = 1 \times v + x\frac{dv}{dx}$$

TRANSFORM THE O.D.E.

$$\Rightarrow 2\frac{dy}{dx} = 1 + \frac{y^2}{x^2}$$
$$\Rightarrow 2\left[v + x\frac{dv}{dx}\right] = 1 + v^2$$
$$\Rightarrow 2v + 2x\frac{dv}{dx} = 1 + v^2$$
$$\Rightarrow 2x\frac{dv}{dx} = v^2 - 2v + 1$$
$$\Rightarrow 2x\frac{dv}{dx} = (v-1)^2$$
$$\Rightarrow \frac{2}{(v-1)^2}\,dv = \frac{1}{x}\,dx$$

INTEGRATING BOTH SIDES SUBJECT

$x = e \quad y = -e \quad v = -1$

$$\Rightarrow \int_{v=-1}^{v} \frac{2}{(v-1)^2}\,dv = \int_{x=e}^{x} \frac{1}{x}\,dx$$
$$\Rightarrow \left[-\frac{2}{v-1}\right]_{-1}^{v} = \left[\ln x\right]_{e}^{x}$$
$$\Rightarrow \left[\frac{2}{v-1}\right]_{v}^{-1} = \left[\ln x\right]_{e}^{x}$$
$$\Rightarrow \frac{2}{-2} - \frac{2}{v-1} = \ln x - \ln e$$
$$\Rightarrow -1 - \frac{2}{v-1} = \ln x - 1$$
$$\Rightarrow -\frac{2}{v-1} = \ln x$$
$$\Rightarrow -\frac{2}{\ln x} = v - 1$$
$$\Rightarrow v = 1 - \frac{2}{\ln x}$$
$$\Rightarrow \frac{y}{x} = 1 - \frac{2}{\ln x}$$
$$\Rightarrow y = x - \frac{2x}{\ln x}$$

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Question 12 (***)

Solve the following differential equation

$$\frac{dy}{dx} = \frac{3x+2y}{3y-2x}, \quad y(1) = 3.$$

Give the final answer in the form $F(x, y) = 12$

$\boxed{}$, $\boxed{3y^2 - 4xy - 3x^2 = 12}$

USING THE SUBSTITUTION GIVEN

$\Rightarrow y = xV(x)$

$\Rightarrow \frac{dy}{dx} = 1 \times V(x) + x\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = V + x\frac{dv}{dx}$

SUBSTITUTING INTO THE O.D.E.

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3x+2xv}{3xv-2x}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{3+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v}{3v-2} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{3+2v-3v^2+2v}{3v-2}$

$\Rightarrow x\frac{dv}{dx} = \frac{3+4v-3v^2}{3v-2}$

SEPARATING VARIABLES

$\Rightarrow \int \frac{3v-2}{3+4v-3v^2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \int \frac{3v-2}{3v^2-4v-3}\,dv = \int -\frac{1}{x}\,dx$

$\Rightarrow \int \frac{6v-4}{3v^2-4v-3}\,dv = \int -\frac{2}{x}\,dx$

$\Rightarrow \ln|3v^2-4v-3| = -2\ln|x| + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left(\frac{1}{x^2}\right) + \ln A$

$\Rightarrow \ln|3v^2-4v-3| = \ln\left|\frac{A}{x^2}\right|$

$\Rightarrow 3v^2 - 4v - 3 = \frac{A}{x^2}$

REVERSING THE TRANSFORMATION

$\Rightarrow 3\left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) - 3 = \frac{A}{x^2}$

$\Rightarrow \frac{3y^2}{x^2} - \frac{4y}{x} - 3 = \frac{A}{x^2}$

$\Rightarrow 3y^2 - 4xy - 3x^2 = A$

APPLY CONDITION $(1,3)$

$3\times3^2 - 4\times1\times3 - 3\times1^2 = A$

$27 - 12 - 3 = A$

$A = 12$

$\therefore 3y^2 - 4xy - 3x^2 = 12$

ALTERNATIVE SUBSTITUTION

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow 3\frac{dy}{dx} = \frac{9x+6y}{3y-2x}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{9x+(2v+4x)}{v}$

$\Rightarrow \frac{dv}{dx} + 2 = \frac{2v+13x}{v}$

$\Rightarrow \frac{dv}{dx} + \cancel{2} = \cancel{2} + \frac{13x}{v}$

$\Rightarrow \int v\,dv = \int 13x\,dx$

$\Rightarrow \frac{1}{2}v^2 = \frac{13}{2}x^2 + C$

$\Rightarrow v^2 = 13x^2 + C$

$\Rightarrow (3y-2x)^2 = 13x^2 + C$

LET $V(x) = 3y - 2x$

$\frac{dv}{dx} = 3\frac{dy}{dx} - 2$

$3\frac{dy}{dx} = \frac{dv}{dx} + 2$

ALSO WE HAVE

$3y = v + 2x$

$6y = 2v + 4x$

APPLY $(1,3)$ $\Rightarrow (9-2)^2 = 13\times1^2 + C$

$\Rightarrow 49 = 13 + C$

$\Rightarrow C = 36$

$\Rightarrow (3y-2x)^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy + 4x^2 = 13x^2 + 36$

$\Rightarrow 9y^2 - 12xy - 9x^2 = 36$

$\Rightarrow 3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

ALTERNATIVE BY MULTIVARIABLE CALCULUS

$\Rightarrow \frac{dy}{dx} = \frac{3x+2y}{3y-2x}$

$\Rightarrow (3y-2x)\,dy = (3x+2y)\,dx$

$\Rightarrow (3x+2y)\,dx + (2x-3y)\,dy = 0$

$\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = dG$

EVIDENTLY THIS IS EXACT AS $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} = 2$

HENCE WE HAVE BY DIRECT INTEGRATION

- $dG = 0 \Rightarrow G(x,y) = \text{constant}$
- $\frac{\partial G}{\partial x} = 3x + 2y \Rightarrow G(x,y) = \frac{3}{2}x^2 + 2xy + f(y)$
- $\frac{\partial G}{\partial y} = 2x - 3y \Rightarrow G(x,y) = 2xy - \frac{3}{2}y^2 + g(x)$

$\therefore f(y) = -\frac{3}{2}y^2$ & $g(x) = \frac{3}{2}x^2$

THUS WE OBTAIN

$G(x,y) = \text{constant}$

$\frac{3}{2}x^2 + 2xy - \frac{3}{2}y^2 = \text{constant}$

$3x^2 + 4xy - 3y^2 = \text{constant}$

$(1,3)$ NOW YIELDS

$3 + 12 - 27 = \text{constant}$

$\text{constant} = -12$

$3x^2 + 4xy - 3y^2 = -12$

$3y^2 - 4xy - 3x^2 = 12$

AS BEFORE

Question 13 (***+)

Find a general solution for the following differential equation

$$(2x+y)\frac{dy}{dx}+x=0.$$

The final answer must not contain natural logarithms.

$$\boxed{y+x=Ae^{\frac{x}{x+y}}}$$

$$(2x+y)\frac{dy}{dx}+x=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{2x+y}$$

THE R.H.S. IS HOMOGENEOUS ; USE

$$\boxed{\begin{aligned} &y=xV(x)\\ &\frac{dy}{dx}=1\times V(x)+x\frac{dV(x)}{dx}\\ &\frac{dy}{dx}=V+x\frac{dv}{dx}\end{aligned}}$$

THUS

$$\Rightarrow V+x\frac{dv}{dx}=\frac{-x}{2x+xV}$$

$$\Rightarrow V+x\frac{dv}{dx}=\frac{-1}{2+V}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-1}{V+2}-V$$

$$\Rightarrow x\frac{dv}{dx}=\frac{-1-V^2-2V}{V+2}$$

$$\Rightarrow -x\frac{dv}{dx}=\frac{V^2+2V+1}{V+2}$$

$$\Rightarrow -x\frac{dv}{dx}=\frac{(V+1)^2}{V+2}$$

$$\Rightarrow \frac{V+2}{(V+1)^2}\,dv=-\frac{1}{x}\,dx$$

$$\Rightarrow \frac{(V+1)+1}{(V+1)^2}\,dv=-\frac{1}{x}\,dx$$

$$\Rightarrow \left[\frac{1}{V+1}+\frac{1}{(V+1)^2}\right]dv=-\frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{V+1}+\frac{1}{(V+1)^2}\,dv=\int -\frac{1}{x}\,dx$$

$$\Rightarrow \ln|V+1|-\frac{1}{V+1}=-\ln|x|+A$$

$$\Rightarrow \ln\left(\frac{y}{x}+1\right)-\frac{1}{\frac{y}{x}+1}=-\ln|x|+A$$

$$\Rightarrow \ln\left(\frac{y+x}{x}\right)-\frac{x}{x+y}=-\ln|x|+A$$

$$\Rightarrow \ln\left(\frac{y+x}{x}\right)+\ln x=\frac{x}{x+y}+A$$

$$\Rightarrow \ln\left(\frac{y+x}{x}\times x\right)=\frac{x}{x+y}+A$$

$$\Rightarrow \ln(y+x)=\frac{x}{x+y}+A$$

$$\Rightarrow y+x=e^{\frac{x}{x+y}+A}$$

$$\Rightarrow y+x=e^{\frac{x}{x+y}}\times e^{A}$$

$$\Rightarrow y+x=Be^{\frac{x}{x+y}}$$

Question 14 (***+)

Solve the following differential equation.

$$\left(xy+4x^2\right)\frac{dy}{dx}=2y^2+9xy+6x^2\,,\quad y\left(\tfrac{4}{3}\right)=0\,.$$

$$\boxed{(y+2x)^2=x^2(y+3x)}$$

$$\left(xy+4x^2\right)\frac{dy}{dx}=2y^2+9yx+6x^2,\quad y\left(\tfrac{4}{3}\right)=0$$

$$\Longrightarrow \frac{dy}{dx}=\frac{2y^2+9xy+6x^2}{xy+4x^2}$$

- THE R.H.S IS HOMOGENEOUS IN xy, SO THE STANDARD SUBSTITUTION $y=xV(x)$ MAY BE USED.

$$\frac{dy}{dx}=V(x)+x\frac{dV(x)}{dx}$$

- HENCE THE O.D.E BECOMES

$$\Longrightarrow V+x\frac{dV}{dx}=\frac{2(xV)^2+9x(xV)+6x^2}{x(xV)+4x^2}$$

$$\Longrightarrow V+x\frac{dV}{dx}=\frac{2x^2V^2+9x^2V+6x^2}{x^2V+4x^2}$$

$$\Longrightarrow V+x\frac{dV}{dx}=\frac{2V^2+9V+6}{V+4}$$

$$\Longrightarrow x\frac{dV}{dx}=\frac{2V^2+9V+6}{V+4}-V$$

$$\Longrightarrow x\frac{dV}{dx}=\frac{2V^2+9V+6-V^2-4V}{V+4}$$

$$\Longrightarrow x\frac{dV}{dx}=\frac{V^2+5V+6}{V+4}$$

$$\Longrightarrow \frac{V+4}{V^2+5V+6}\,dV=\frac{1}{x}\,dx$$

$$\Longrightarrow \int\frac{V+4}{(V+2)(V+3)}\,dV=\int\frac{1}{x}\,dx$$

- PARTIAL FRACTIONS BY INSPECTION (COVER UP)

$$\Longrightarrow \int\frac{2}{V+2}+\frac{-1}{V+3}\,dV=\int\frac{1}{x}\,dx$$

$$\Longrightarrow 2\ln|V+2|-\ln|V+3|=\ln x+\ln A$$

$$\Longrightarrow \ln\left|\frac{(V+2)^2}{V+3}\right|=\ln|Ax|$$

$$\Longrightarrow \frac{(V+2)^2}{V+3}=Ax$$

$$\Longrightarrow \frac{\left(\frac{y}{x}+2\right)^2}{\frac{y}{x}+3}=Ax$$

$$\Longrightarrow \frac{\frac{(y+2x)^2}{x^2}}{\frac{y+3x}{x}}=Ax$$

$$\Longrightarrow \frac{(y+2x)^2}{x^2}=A(y+3x)$$

$$\Longrightarrow (y+2x)^2=Ax^2(y+3x)$$

- APPLY CONDITION $y\left(\frac{4}{3}\right)=0 \Longrightarrow \left(\frac{8}{3}\right)^2=A\left(\frac{4}{3}\right)^2\left[3\times\frac{4}{3}\right]$

$$\frac{64}{9}=A\times\frac{16}{9}\times 4$$

$$A=1$$

$$\Longrightarrow (y+2x)^2=x^2(y+3x)$$

Question 15 (****)

Solve the differential equation

$$\frac{d}{dx}\left(xy^2\right) = \frac{x^4 + x^2y^2 + y^4}{x^2}, \quad y(\mathrm{e}) = \sqrt{2}\,\mathrm{e}.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{x^2(1+\ln x)}{\ln x}}$$

TIDY UP THE O.D.E.

$$\Rightarrow \frac{d}{dx}\left(xy^2\right) = \frac{x^4 + x^2y^2 + y^4}{x^2}$$

$$\Rightarrow y^2 + 2xy\frac{dy}{dx} = \frac{x^4 + x^2y^2 + y^4}{x^2}$$

$$\Rightarrow 2xy\frac{dy}{dx} = \frac{x^4 + x^2y^2 + y^4}{x^2} - y^2$$

$$\Rightarrow 2xy\frac{dy}{dx} = \frac{x^4 + \cancel{x^2y^2} + y^4 - \cancel{x^2y^2}}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^4 + y^4}{2x^3y}$$

THIS IS A "HOMOGENEOUS" EQUATION — USE A TRANSFORMATION

$$y = xV(x)$$
$$\frac{dy}{dx} = 1 \times V(x) + x\frac{dV}{dx}$$
$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

TRANSFORMING THE O.D.E.

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^4 + x^4V^4}{2x^3(xV)}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x^4(1+V^4)}{2x^4V}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{1+V^4}{2V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+V^4}{2V} - V$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+V^4-2V^2}{2V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{(V^2-1)^2}{2V}$$

SEPARATING VARIABLES & INTEGRATING

$$\Rightarrow \frac{2V}{(V^2-1)^2}\,dV = \frac{1}{x}\,dx$$

$$\Rightarrow \int 2V(V^2-1)^{-2}\,dV = \int \frac{1}{x}\,dx$$

$$\Rightarrow -(V^2-1)^{-1} = \ln x + C$$

$$\Rightarrow \frac{1}{V^2-1} = \ln x + C$$

$$\Rightarrow \frac{1}{\frac{y^2}{x^2}-1} = \ln x + C$$

$$\Rightarrow \frac{x^2}{y^2-x^2} = \ln x + C$$

APPLY CONDITION $(\mathrm{e}, \sqrt{2}\mathrm{e})$

$$\Rightarrow \frac{\mathrm{e}^2}{2\mathrm{e}^2-\mathrm{e}^2} = \ln \mathrm{e} + C$$

$$\Rightarrow \frac{\mathrm{e}^2}{\mathrm{e}^2} = 1 + C$$

$$\Rightarrow C = 0$$

$$\therefore \frac{x^2}{y^2-x^2} = \ln x$$

FINALLY REARRANGING

$$\Rightarrow x^2 = y^2\ln x - x^2\ln x$$

$$\Rightarrow x^2 + x^2\ln x = y^2\ln x$$

$$\Rightarrow y^2\ln x = x^2(1+\ln x)$$

$$\Rightarrow y^2 = \frac{x^2(1+\ln x)}{\ln x}$$

Question 16 (****)

Solve the differential equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}, \quad y(1) = 1.$$

$\boxed{}$, $\boxed{y^2 + 2xy - x^2 = 2}$

AS THIS IS A STANDARD HOMOGENEOUS O.D.E WE USE THE SUBSTITUTION $y = vx$, WHERE $v = v(x)$

$$\Rightarrow y = vx$$
$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \times x + v \times 1$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

HENCE WE CAN TRANSFORM THE O.D.E

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow \frac{dy}{dx} = \frac{x - vx}{x + vx}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-v-v(1+v)}{1+v}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$
$$\Rightarrow \frac{v+1}{-v^2-2v+1}\,dv = \frac{1}{x}\,dx$$

$\times(-2)$

$$\Rightarrow \int \frac{-2v-2}{-v^2-2v+1}\,dv = \int \frac{-2}{x}\,dx$$
$$\Rightarrow \ln|-v^2-2v+1| = -2\ln|x| + \ln A$$
$$\Rightarrow \ln|1-2v-v^2| = \ln\left|\frac{A}{x^2}\right|$$
$$\Rightarrow 1-2v-v^2 = \frac{A}{x^2}$$

REVERSING THE TRANSFORMATIONS WE OBTAIN

$$\Rightarrow 1 - 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 = \frac{A}{x^2}$$
$$\Rightarrow 1 - \frac{2y}{x} - \frac{y^2}{x^2} = \frac{A}{x^2}$$
$$\Rightarrow x^2 - 2xy - y^2 = A$$

APPLYING THE CONDITION $(1,1)$ YIELDS $A = -2$

$$\Rightarrow x^2 - 2xy - y^2 = -2$$
$$\Rightarrow y^2 + 2xy - x^2 = 2$$

ALTERNATIVE USING PARTIAL DIFFERENTIATION

$$\Rightarrow \frac{dy}{dx} = \frac{x-y}{x+y}$$
$$\Rightarrow (x-y)\,dx = (x+y)\,dy$$
$$\Rightarrow (x-y)\,dx + (-x-y)\,dy = 0$$
$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = dF$$

CHECK FOR "EXACTNESS"

- $\frac{\partial F}{\partial x} = x - y \Rightarrow \frac{\partial^2 F}{\partial y \partial x} = -1$
- $\frac{\partial F}{\partial y} = -x - y \Rightarrow \frac{\partial^2 F}{\partial x \partial y} = -1$

$\therefore$ EXACT

- $\frac{\partial F}{\partial x} = x - y$, $\quad F(x,y) = \frac{1}{2}x^2 - xy + f(y)$
- $\frac{\partial F}{\partial y} = -x - y$, $\quad F(x,y) = -xy - \frac{1}{2}y^2 + g(x)$

COMPARING EXPRESSIONS FOR $F(x,y)$ GIVES

$$f(y) = -\frac{1}{2}y^2 \quad \& \quad g(x) = \frac{1}{2}x^2$$

FINALLY WE HAVE

$$F(x,y) = \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$$

& SINCE $dF = 0$

$$F(x,y) = \text{CONSTANT}$$
$$\Rightarrow \frac{1}{2}x^2 - xy - \frac{1}{2}y^2 = \text{CONSTANT}$$
$$\Rightarrow y^2 + 2xy - x^2 = \text{CONSTANT}$$

& USING $(1,1)$ FINDS THE CONSTANT AS 2

$$\therefore y^2 + 2yx - x^2 = 2$$

AS BEFORE

Question 17 (****)

It is given that a curve with equation $f(x,y)=0$ passes through the point $(0,1)$ and satisfies the differential equation

$$\frac{dy}{dx}=\frac{xy}{x^2+y^2}.$$

By solving the differential equation, show that an equation for the curve is

$$y=\exp\left[\frac{x^2}{2y^2}\right].$$

$\boxed{}$, $\boxed{\text{proof}}$

- THIS IS A FIRST ORDER O.D.E WITH A HOMOGENEOUS R.H.S WHICH IS SOLVED BY THE STANDARD SUBSTITUTION

$$y = xv(x)$$
$$\frac{dy}{dx} = v(x) + x\frac{dv}{dx}$$

- TRANSFORMING THE O.D.E. AND SOLVE BY SEPARATING VARIABLES

$$\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{x(xv)}{x^2+x^2v^2}$$
$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^2}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$$
$$\Rightarrow x\frac{dv}{dx} = \frac{v-(v+v^3)}{1+v^2}$$
$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3}{1+v^2}$$
$$\Rightarrow x\frac{dv}{dx} = -\frac{v^3}{v^2+1}$$
$$\Rightarrow \frac{v^2+1}{v^3}\,dv = -\frac{1}{x}\,dx$$
$$\Rightarrow \int \frac{1}{v}+\frac{1}{v^3}\,dv = \int -\frac{1}{x}\,dx$$
$$\Rightarrow \ln v - \frac{1}{2v^2} = -\ln x + A$$
$$\Rightarrow \ln v + \ln x = A + \frac{1}{2v^2}$$
$$\Rightarrow \ln(xv) = A + \frac{1}{2v^2}$$
$$\Rightarrow \ln y = A + \frac{1}{2\left(\frac{y}{x}\right)^2}$$
$$\Rightarrow \ln y = A + \frac{x^2}{2y^2}$$
$$\Rightarrow y = e^{A+\frac{x^2}{2y^2}}$$
$$\Rightarrow y = e^A \times e^{\frac{x^2}{2y^2}}$$
$$\Rightarrow y = Be^{\frac{x^2}{2y^2}}$$

when $x=0$ $y=1$

$\therefore B=1$

$$\therefore y = e^{\frac{x^2}{2y^2}}$$

1^{ST} ORDER BERNOULLI TYPE

Question 1 (**+)

$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2, \quad y > 0.$$

a) Show that the substitution $z = \frac{1}{y^2}$ transforms the above differential equation into the new differential equation

$$\frac{dz}{dx} + 2z = -4x.$$

b) Hence find the general solution of the original differential equation, giving the answer in the form $y^2 = f(x)$.

$\boxed{}$, $\boxed{y^2 = \frac{1}{Ae^{-2x} - 2x + 1}}$

a) USING THE SUBSTITUTION GIVEN

$$z = \frac{1}{y^2} \Rightarrow \frac{d}{dx}(z) = \frac{d}{dx}\left(\frac{1}{y^2}\right)$$
$$\Rightarrow \frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = -\frac{y^3}{2}\frac{dz}{dx}$$

SUBSTITUTE INTO THE O.D.E.

$$\frac{1}{y}\frac{dy}{dx} = 1 + 2xy^2$$
$$\frac{dy}{dx} = y + 2xy^3$$
$$-\frac{y^3}{2}\frac{dz}{dx} = y + 2xy^3 \quad \times\left(-\frac{2}{y^3}\right)$$
$$\frac{dz}{dx} = -\frac{2}{y^2} - 4x$$
$$\frac{dz}{dx} = -2z - 4x$$
$$\frac{dz}{dx} + 2z = -4x$$

AS REQUIRED

b) LOOKING FOR AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int 2\,dx} = e^{2x}$$
$$\Rightarrow \frac{d}{dx}\left(ze^{2x}\right) = -4xe^{2x}$$
$$\Rightarrow ze^{2x} = \int -4xe^{2x}\,dx$$

INTEGRATION BY PARTS ON THE R.H.S.

$-4x$	-4
$\frac{1}{2}e^{2x}$	e^{2x}

$$\int -4xe^{2x}\,dx = -2xe^{2x} - \int -2e^{2x}\,dx$$
$$= -2xe^{2x} + \int 2e^{2x}\,dx$$
$$= e^{2x} - 2xe^{2x} + C$$

RETURNING TO THE O.D.E.

$$ze^{2x} = e^{2x} - 2xe^{2x} + C$$
$$z = 1 - 2x + Ce^{-2x}$$
$$\frac{1}{y^2} = 1 - 2x + Ce^{-2x}$$
$$y^2 = \frac{1}{1 - 2x + Ce^{-2x}}$$

Question 2 (***)

a) Use the suitable substitution to solve the differential equation

$$x^2\frac{dy}{dx}+xy=y^2, \qquad y\left(\tfrac{1}{2}\right)=2.$$

Give the answer in the form $y=f(x)$.

b) Verify the answer of part **(a)** by solving the above differential equation with an alternative method.

$$\boxed{y=\frac{2x}{1-2x^2}}$$

a) START BY RE-WRITING THE O.D.E

$$\Rightarrow x^2\frac{dy}{dx}+xy=y^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2-xy}{x^2}$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{y}{x}\right)^2-\left(\frac{y}{x}\right)$$

THIS IS A STANDARD FIRST ORDER HOMOGENEOUS O.D.E AS IT IS OF THE FORM $y=f\left(\frac{y}{x}\right)$

$$v=\frac{y}{x} \Longrightarrow y=xv$$

$$\Longrightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

HENCE WE MAY TRANSFORM THE O.D.E TO

$$\Rightarrow v+x\frac{dv}{dx}=v^2-v$$

$$\Rightarrow x\frac{dv}{dx}=v^2-2v$$

$$\Rightarrow \frac{1}{v^2-2v}\,dv=\frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{1}{v(v-2)}\,dv=\int\frac{1}{x}\,dx$$

PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow \int\frac{\frac{1}{2}}{v-2}-\frac{\frac{1}{2}}{v}\,dv=\int\frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{1}{v-2}-\frac{1}{v}\,dv=\int\frac{2}{x}\,dx$$

$$\Rightarrow \ln|v-2|-\ln|v|=2\ln|x|+\ln A$$

$$\Rightarrow \ln\left|\frac{v-2}{v}\right|=\ln(Ax^2)$$

$$\Rightarrow \frac{v-2}{v}=Ax^2$$

$$\Rightarrow \frac{\frac{y}{x}-2}{\frac{y}{x}}=Ax^2$$

$$\Rightarrow \frac{y-2x}{y}=Ax^2$$

$$\Rightarrow 1-\frac{2x}{y}=Ax^2$$

$$\Rightarrow 1+Ax^2=\frac{2x}{y}$$

$$\Rightarrow y=\frac{2x}{1+Ax^2}$$

FINALLY APPLY CONDITIONS $y\left(\frac{1}{2}\right)=2$

$$2=\frac{1}{1+A\times\frac{1}{4}}$$

$$1+\tfrac{1}{4}A=\tfrac{1}{2}$$

$$\tfrac{1}{4}A=-\tfrac{1}{2}$$

$$A=-2$$

HENCE WE OBTAIN

$$y=\frac{2x}{1-2x^2}$$

b) LOOKING AT THE ODE ONCE DIVIDED THROUGH BY x^2

$$\frac{dy}{dx}+\frac{y}{x}=\frac{y^2}{x^2}$$

THIS IS A BERNOULLI EQUATION ALSO, AS IT IS OF THE FORM

$$\frac{dy}{dx}+yf(x)=y^n g(x) \qquad n\neq -1$$

WHICH IS SOLVED BY THE SUBSTITUTION

$$z=\frac{1}{y^{n-1}}$$

IN THIS EXAMPLE WE HAVE

$$z=\frac{1}{y}$$

$$\frac{dz}{dx}=-\frac{1}{y^2}\frac{dy}{dx}$$

MULTIPLY THE O.D.E BY $-\frac{1}{y^2}$ TO OBTAIN

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx}-\frac{1}{xy}=-\frac{1}{x^2}$$

$$\Rightarrow \frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x^2}$$

THIS IS A STANDARD FIRST ORDER WITH INTEGRATING FACTOR

$$e^{\int -\frac{1}{x}\,dx}=e^{-\ln x}=e^{\ln\left(\frac{1}{x}\right)}=\frac{1}{x}$$

MULTIPLYING BY THE INTEGRATING ~~FACTOR~~ MAKES THE ODE EXACT

$$\Rightarrow \frac{1}{x}\frac{dz}{dx}-\frac{z}{x^2}=-\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{z}{x}\right)=-\frac{1}{x^3}$$

$$\Rightarrow \frac{z}{x}=\int-\frac{1}{x^3}\,dx$$

$$\Rightarrow \frac{z}{x}=\frac{1}{2x^2}+C$$

$$\Rightarrow z=\frac{1}{2x}+Cx$$

$$\Rightarrow \frac{1}{y}=\frac{1+Cx^2}{2x}$$

$$\Rightarrow y=\frac{2x}{1+Cx^2}$$

APPLYING CONDITION $y\left(\frac{1}{2}\right)=2$, GIVES $C=-2$ AS IN PART (a), YIELDING THE SAME SOLUTION

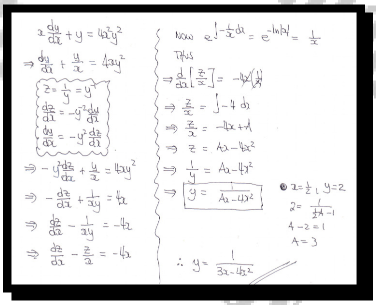
$$y=\frac{2x}{1-2x^2}$$

Question 3 **(***)**

Solve the differential equation

$$x\frac{dy}{dx} + y = 4x^2y^2, \quad y\left(\tfrac{1}{2}\right) = 2.$$

$$\boxed{y = \frac{1}{3x - 4x^2}}$$



$x\frac{dy}{dx} + y = 4x^2y^2$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = 4xy^2$

$z = \frac{1}{y} = y^{-1}$

$\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$

$\frac{dy}{dx} = -y^2\frac{dz}{dx}$

$\Rightarrow -y^2\frac{dz}{dx} + \frac{y}{x} = 4xy^2$

$\Rightarrow -\frac{dz}{dx} + \frac{1}{xy} = 4x$

$\Rightarrow \frac{dz}{dx} - \frac{1}{xy} = -4x$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -4x$

Now $e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

Thus

$\Rightarrow \frac{d}{dx}\left[\frac{z}{x}\right] = -4\cancel{x}\left(\frac{1}{\cancel{x}}\right)$

$\Rightarrow \frac{z}{x} = \int -4\, dx$

$\Rightarrow \frac{z}{x} = -4x + A$

$\Rightarrow z = Ax - 4x^2$

$\Rightarrow \frac{1}{y} = Ax - 4x^2$

$\Rightarrow \boxed{y = \frac{1}{Ax - 4x^2}}$

$x = \frac{1}{2},\ y = 2$

$2 = \frac{1}{\frac{1}{2}A - 1}$

$A - 2 = 1$

$A = 3$

$\therefore y = \frac{1}{3x - 4x^2}$

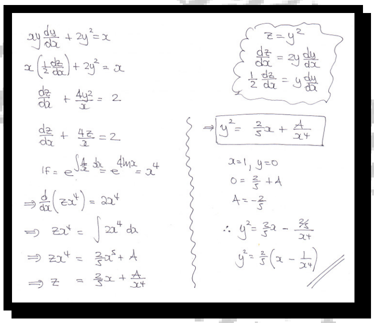
Question 4 **(***)**

By using a suitable substitution, solve the differential equation

$$xy\frac{dy}{dx} + 2y^2 = x, \quad y(1) = 0.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)}$$



$xy\frac{dy}{dx} + 2y^2 = x$

$z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{1}{2}\frac{dz}{dx} = y\frac{dy}{dx}$

$x\left(\frac{1}{2}\frac{dz}{dx}\right) + 2y^2 = x$

$\frac{dz}{dx} + \frac{4y^2}{x} = 2$

$\frac{dz}{dx} + \frac{4z}{x} = 2$

$\text{IF} = e^{\int \frac{4}{x}dx} = e^{4\ln x} = x^4$

$\Rightarrow \frac{d}{dx}\left(zx^4\right) = 2x^4$

$\Rightarrow zx^4 = \int 2x^4\, dx$

$\Rightarrow zx^4 = \frac{2}{5}x^5 + A$

$\Rightarrow z = \frac{2}{5}x + \frac{A}{x^4}$

$\Rightarrow \boxed{y^2 = \frac{2}{5}x + \frac{A}{x^4}}$

$x = 1,\ y = 0$

$0 = \frac{2}{5} + A$

$A = -\frac{2}{5}$

$\therefore y^2 = \frac{2}{5}x - \frac{2/5}{x^4}$

$y^2 = \frac{2}{5}\left(x - \frac{1}{x^4}\right)$

Question 5 (***)

Solve the differential equation

$$\frac{dy}{dx} + y = 4xy^3, \quad y(0) = \frac{1}{\sqrt{2}}.$$

Give the answer in the form $y^2 = f(x)$.

$$\boxed{y^2 = \frac{1}{4x+2}}$$

$$\frac{dy}{dx} + y = 4xy^3$$

$$\boxed{\begin{aligned} &\bullet\, t = \frac{1}{y^2} = y^{-2} \\ &\frac{dt}{dx} = -2y^{-3}\frac{dy}{dx} \\ &\frac{dt}{dx} = -\frac{2}{y^3}\frac{dy}{dx} \\ &\frac{dy}{dx} = -\frac{y^3}{2}\frac{dt}{dx} \end{aligned}}$$

$\Rightarrow -\frac{y^3}{2}\frac{dt}{dx} + y = 4xy^3$

$\Rightarrow -\frac{1}{2}\frac{dt}{dx} + \frac{1}{y^2} = 4x$

$\Rightarrow \frac{dt}{dx} - \frac{2}{y^2} = -8x$

$\Rightarrow \frac{dt}{dx} - 2t = -8x$

$\text{IF} = e^{\int -2\,dx} = e^{-2x}$

HENCE

$\Rightarrow \frac{d}{dx}\left(te^{-2x}\right) = -8xe^{-2x}$

$\Rightarrow te^{-2x} = \int -8xe^{-2x}\,dx$

BY PARTS

$-8x \to -8$

$-\frac{1}{2}e^{-2x} \leftarrow e^{-2x}$

$\Rightarrow te^{-2x} = 4xe^{-2x} - \int 4e^{-2x}\,dx$

$\Rightarrow te^{-2x} = 4xe^{-2x} + 2e^{-2x} + A$

$\Rightarrow t = 4x + 2 + Ae^{2x}$

$\Rightarrow \frac{1}{y^2} = 4x + 2 + Ae^{2x}$

$\Rightarrow \boxed{y^2 = \frac{1}{4x+2+Ae^{2x}}}$

APPLY CONDITION $x=0 \quad y = \frac{1}{\sqrt{2}}$

$\frac{1}{2} = \frac{1}{0+2+A}$

$A = 0$

HENCE $y^2 = \frac{1}{4x+2}$

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Question 6 **(***+)**

$$\frac{dy}{dx} + \frac{2y}{x} = y^4, \quad x > 0, \quad y > 0.$$

Given that $y(1) = 1$, show that

$$y^3 = \frac{5}{3x + 2x^6}.$$

proof

$$\boxed{\frac{dy}{dx} + \frac{2y}{x} = y^4}$$

$u = \frac{1}{y^3}$ OR $u = y^{-3}$

$$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$\frac{du}{dx} = -\frac{3}{y^4}\frac{dy}{dx}$$

$$\boxed{-\frac{y^4}{3}\frac{du}{dx} = \frac{dy}{dx}}$$

THUS TRANSFORMING THE O.D.E

$$\Rightarrow -\frac{y^4}{3}\frac{du}{dx} + \frac{2y}{x} = y^4 \quad \left(\text{MULTIPLY BY } \frac{-3}{y^4}\right)$$

$$\Rightarrow \frac{du}{dx} - \frac{6}{xy^3} = -3$$

$$\Rightarrow \frac{du}{dx} - \frac{6}{x}\left(\frac{1}{y^3}\right) = -3$$

$$\Rightarrow \boxed{\frac{du}{dx} - \frac{6u}{x} = -3}$$

$$\text{I.F.} = e^{\int -\frac{6}{x}\,dx} = e^{-6\ln x} = e^{\ln x^{-6}} = \frac{1}{x^6}$$

THUS

$$\frac{d}{dx}\left(u \times \frac{1}{x^6}\right) = -3 \times \frac{1}{x^6}$$

$$\Rightarrow \frac{u}{x^6} = \int -\frac{3}{x^6}\,dx$$

$$\Rightarrow \frac{u}{x^6} = \int -3x^{-6}\,dx$$

$$\Rightarrow \frac{u}{x^6} = \frac{3}{5}x^{-5} + A$$

$$\Rightarrow u = \frac{3}{5}x + Ax^6$$

$$\Rightarrow \frac{1}{y^3} = \frac{3}{5}x + Ax^6$$

$$\Rightarrow y^3 = \frac{1}{\frac{3}{5}x + Ax^6}$$

$$\Rightarrow y^3 = \frac{5}{3x + Bx^6}$$

NOW $x=1$ $y=1$ $\Rightarrow 1 = \frac{5}{3+B}$

$$\Rightarrow B + 3 = 5$$

$$\Rightarrow \boxed{B = 2}$$

$$\therefore y^3 = \frac{5}{3x + 2x^6}$$

AS REQUIRED

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Question 7 (***+)

Solve the differential equation

$$\frac{dy}{dx}+\frac{xy}{1+x^2}=y^3,\ \ y(0)=1.$$

Give the answer in the form $y^2=f(x)$.

$$\boxed{y^2=\frac{1}{\left(1+x^2\right)\left(1-2\arctan x\right)}}$$

$$\frac{dy}{dx}+\frac{xy}{1+x^2}=y^3$$

$$t=\frac{1}{y^2}$$

$$\frac{dt}{dx}=-\frac{2}{y^3}\frac{dy}{dx}$$

$$\frac{dy}{dx}=-\frac{1}{2}y^3\frac{dt}{dx}$$

$$\Rightarrow -\frac{1}{2}y^3\frac{dt}{dx}+\frac{xy}{1+x^2}=y^3$$

$$\Rightarrow \frac{dt}{dx}-\frac{2x}{y^2(1+x^2)}=-2$$

$$\Rightarrow \frac{dt}{dx}-\frac{2xt}{1+x^2}=-2$$

$$\text{I.F.}=e^{\int\frac{-2x}{1+x^2}\,dx}=e^{-\ln(1+x^2)}=\frac{1}{1+x^2}$$

$$\Rightarrow \frac{d}{dx}\left(t\times\frac{1}{1+x^2}\right)=\frac{-2}{1+x^2}$$

$$\Rightarrow \frac{t}{1+x^2}=\int\frac{-2}{1+x^2}\,dx$$

$$\Rightarrow \frac{t}{1+x^2}=A-2\arctan x$$

$$\Rightarrow t=A(1+x^2)-2(1+x^2)\arctan x$$

$$\Rightarrow \frac{1}{y^2}=(1+x^2)(A-2\arctan x)$$

$$\Rightarrow \boxed{y^2=\frac{1}{(1+x^2)(A-2\arctan x)}}$$

when $x=0$ $y=1$ $\quad 1=\dfrac{1}{1\times(A-0)}$

$$A=1$$

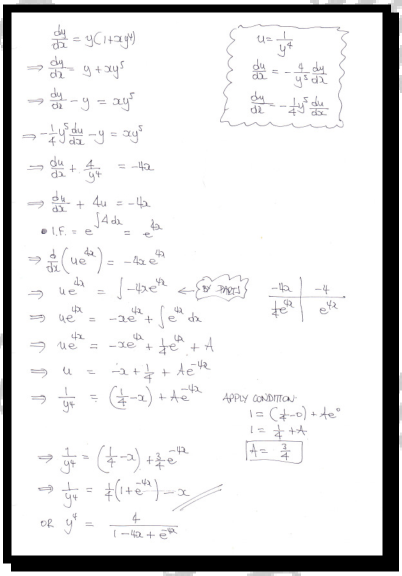
$$\therefore y^2=\frac{1}{(1+x^2)(1-2\arctan x)}$$

Question 8 (***+)

Solve the differential equation

$$\frac{dy}{dx} = y\left(1 + xy^4\right), \quad y(0) = 1.$$

$$\boxed{\frac{1}{y^4} = \frac{1}{4}\left(1 + 3e^{-4x}\right) - x}$$



$$\frac{dy}{dx} = y(1+xy^4)$$

$$u = \frac{1}{y^4}$$

$$\frac{du}{dx} = -\frac{4}{y^5}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{4}y^5\frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = y + xy^5$$

$$\Rightarrow \frac{dy}{dx} - y = xy^5$$

$$\Rightarrow -\frac{1}{4}y^5\frac{du}{dx} - y = xy^5$$

$$\Rightarrow \frac{du}{dx} + \frac{4}{y^4} = -4x$$

$$\Rightarrow \frac{du}{dx} + 4u = -4x$$

- I.F. $= e^{\int 4\,dx} = e^{4x}$

$$\Rightarrow \frac{d}{dx}\left(ue^{4x}\right) = -4xe^{4x}$$

$$\Rightarrow ue^{4x} = \int -4xe^{4x} \leftarrow \text{BY PARTS}$$

$-4x$	-4
$\frac{1}{4}e^{4x}$	e^{4x}

$$\Rightarrow ue^{4x} = -xe^{4x} + \int e^{4x}\,dx$$

$$\Rightarrow ue^{4x} = -xe^{4x} + \frac{1}{4}e^{4x} + A$$

$$\Rightarrow u = -x + \frac{1}{4} + Ae^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + Ae^{-4x}$$

APPLY CONDITION:

$$1 = \left(\frac{1}{4} - 0\right) + Ae^0$$

$$1 = \frac{1}{4} + A$$

$$\boxed{A = \frac{3}{4}}$$

$$\Rightarrow \frac{1}{y^4} = \left(\frac{1}{4} - x\right) + \frac{3}{4}e^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \frac{1}{4}\left(1 + e^{-4x}\right) - x$$

OR $$y^4 = \frac{4}{1 - 4x + e^{-4x}}$$

Question 9 **(****)**

A curve C passes through the point $(1,1)$ and satisfies the differential equation

$$\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}, \quad x > 0, \quad y > 0,$$

subject to the condition $y = 1$ at $x = 1$.

a) Find an equation of C by using the substitution $z = y^4$.

b) Find an equation of C by using the substitution $v = \dfrac{x}{y}$.

Give the answer in the form $y^4 = f(x)$.

$$\boxed{y^4 = x^4(1 + \ln x)}$$

(a) $\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$z = \frac{1}{y^{-4}} = y^4$

$\frac{dz}{dx} = 4y^3\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{4y^3}\frac{dz}{dx}$

$\Rightarrow \frac{1}{4y^3}\frac{dz}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$\Rightarrow \frac{dz}{dx} - \frac{4y^4}{x} = x^3$

$\Rightarrow \frac{dz}{dx} - \frac{4z}{x} = x^3$

$\text{I.F.} = e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = \frac{1}{x^4}$

$\Rightarrow \frac{d}{dx}\left(\frac{z}{x^4}\right) = \frac{x^3}{x^4}$

$\Rightarrow \frac{z}{x^4} = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{z}{x^4} = \ln x + A$

$\Rightarrow z = x^4(\ln x + A)$

$\Rightarrow \boxed{y^4 = x^4(\ln x + A)}$

• $x=1 \quad y=1$

$1 = 1(\ln 1 + A) \Rightarrow A = 1$

$\therefore y^4 = x^4(1 + \ln x)$

(b) $\frac{dy}{dx} - \frac{y}{x} = \frac{x^3}{4y^3}$

$v = \frac{x}{y} \Rightarrow y = \frac{x}{v}$

$\Rightarrow y = xv^{-1}$

$\Rightarrow \frac{dy}{dx} = v^{-1} + x(-v^{-2})\frac{dv}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx}$

$\Rightarrow \left(\frac{1}{v} - \frac{x}{v^2}\frac{dv}{dx}\right) - \frac{1}{v} = \frac{1}{4}v^3$

$\Rightarrow -\frac{x}{v^2}\frac{dv}{dx} = \frac{1}{4}v^3$

$\Rightarrow -\frac{4}{v^5}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int -4v^{-5}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow v^{-4} = \ln x + B$

$\Rightarrow \frac{1}{v^4} = \ln x + B$

$\Rightarrow \frac{y^4}{x^4} = \ln x + B$

$\Rightarrow y^4 = x^4(\ln x + B)$

AS BEFORE IF APPLY SAME CONDITIONS

1^{ST} ORDER BY PARTIAL DIFFERENTIATION TECHNIQUES

Question 1 (**+)

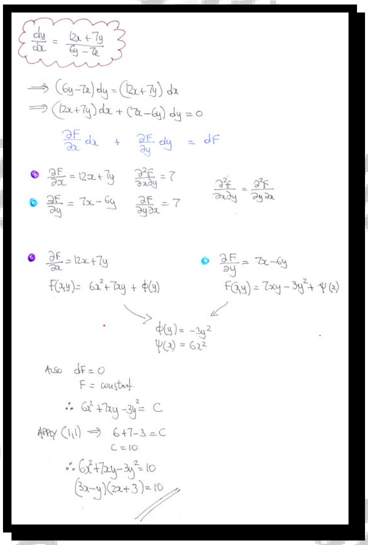
$$\frac{dy}{dx}=\frac{12x+7y}{6y-7x}, \ \ y(1)=1.$$

Use a method involving partial differentiation to show that the solution of the above differential equation can be written as

$$(ax+by)(cx+dy)=10,$$

where a, b, c and d are integers to be found.

$$\boxed{(3x-y)(2x+3y)=k}$$



$$\frac{dy}{dx}=\frac{12x+7y}{6y-7x}$$

$\Longrightarrow (6y-7x)\,dy=(12x+7y)\,dx$

$\Longrightarrow (12x+7y)\,dx+(7x-6y)\,dy=0$

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=dF$$

- $\frac{\partial F}{\partial x}=12x+7y$ $\quad \frac{\partial^2 F}{\partial x \partial y}=7$ $\quad \frac{\partial^2 F}{\partial x\partial y}=\frac{\partial^2 F}{\partial y \partial x}$
- $\frac{\partial F}{\partial y}=7x-6y$ $\quad \frac{\partial F}{\partial y \partial x}=7$

- $\frac{\partial F}{\partial x}=12x+7y$
  $F(x,y)=6x^2+7xy+\phi(y)$
- $\frac{\partial F}{\partial y}=7x-6y$
  $F(x,y)=7xy-3y^2+\psi(x)$

$\phi(y)=-3y^2$
$\psi(x)=6x^2$

Also $dF=0$
$F=$ constant.

$\therefore 6x^2+7xy-3y^2=C$

Apply $(1,1) \Longrightarrow 6+7-3=C$
$C=10$

$\therefore 6x^2+7xy-3y^2=10$
$(3x-y)(2x+3)=10$

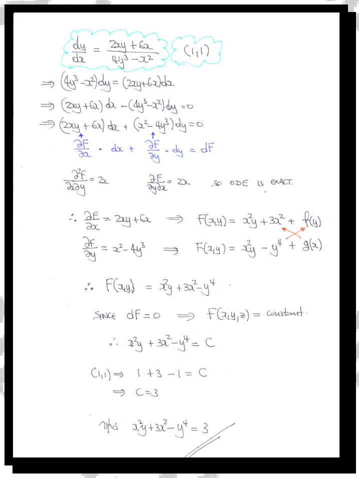
Question 2 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy + 6x}{4y^3 - x^2},$$

subject to the boundary condition $y = 1$ at $x = 1$.

$$\boxed{x^2 y + 3x^2 - y^4 = 3}$$



$$\frac{dy}{dx} = \frac{2xy+6x}{4y^3-x^2} \quad (1,1)$$

$\Rightarrow (4y^3-x^2)\,dy = (2xy+6x)\,dx$

$\Rightarrow (2xy+6x)\,dx - (4y^3-x^2)\,dy = 0$

$\Rightarrow (2xy+6x)\,dx + (x^2-4y^3)\,dy = 0$

$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = dF$

$\frac{\partial^2 F}{\partial x \partial y} = 2x \qquad \frac{\partial F}{\partial y \partial x} = 2x$ so O.D.E is EXACT.

$\therefore \frac{\partial F}{\partial x} = 2xy + 6x \Rightarrow F(x,y) = x^2 y + 3x^2 + f(y)$

$\frac{\partial F}{\partial y} = x^2 - 4y^3 \Rightarrow F(x,y) = x^2 y - y^4 + g(x)$

$\therefore F(x,y) = x^2 y + 3x^2 - y^4$

Since $dF = 0 \Rightarrow F(x,y,z) = \text{constant}$

$\therefore x^2 y + 3x^2 - y^4 = C$

$(1,1) \Rightarrow 1 + 3 - 1 = C$

$\Rightarrow C = 3$

Thus $x^2 y + 3x^2 - y^4 = 3$

Question 3 (***)

Find a general solution of the following differential equation

$$\frac{dy}{dx} = \frac{y\left(y^2 - 3x^2 + 1\right)}{x\left(x^2 - 3y^2 - 1\right)}.$$

$$\boxed{xy\left(x^2 - y^2 - 1\right) = \text{constant}}$$

$$\frac{dy}{dx} = \frac{y(y^2-3x^2+1)}{x(x^2-3y^2-1)} = \frac{y^3-3x^2y+y}{x^3-3xy^2-x}$$

$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$ WHERE $\phi(x,y)=0$

DIFFERENTIATE AGAIN TO CHECK

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = 3y^2-3x^2+1$$
$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = -(3x^2-3y^2-1) = 3y^2-3x^2+1$$

SO "EXACT"

$$\frac{\partial\phi}{\partial x} = y^3-3x^2y+y$$
$$\phi(x,y) = y^3x - x^3y + xy + F(y)$$

$$\frac{\partial\phi}{\partial y} = -(x^3-3xy^2-x)$$
$$\frac{\partial\phi}{\partial y} = -x^3+3xy^2+x$$
$$\phi(x,y) = -x^3y + y^3x + xy + G(x)$$

COMPARING

$$\phi(x,y) = xy + y^3x - x^3y + C$$

OR

$$xy\left[y^2-x^2+1\right] = k$$

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Question 4 (***)

Find the solution of the following differential equation

$$\frac{dy}{dx} = \frac{1-3x^2y}{x^3+2y},$$

subject to the boundary condition $y=1$ at $x=1$.

$$\boxed{x^3y + y^2 - x = 1}$$

$$\frac{dy}{dx} = \frac{1-3x^2y}{x^3+2y}$$

$$(x^3+2y)\,dy = (1-3x^2y)\,dx$$

$$(1-3x^2y)\,dx + (-x^3-2y)\,dy = 0$$

$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = d\phi$$

- $\frac{\partial}{\partial y}(1-3x^2y) = -3x^2$
- $\frac{\partial}{\partial x}(-x^3-2y) = -3x^2$

$\Big\}$ IE EXACT DIFFERENTIAL

$$\frac{\partial\phi}{\partial x} = 1-3x^2y \qquad \frac{\partial\phi}{\partial y} = -x^3-2y$$

$$\phi(x,y) = x - x^3y + F(y) \qquad \phi(x,y) = -x^3y - y^2 + G(x)$$

$$x - x^3y - y^2 = \text{CONSTANT} \leftarrow d\phi = 0$$

APPLY CONDITION $(1,1) \Rightarrow 1-1-1 = \text{constant}$

$$\therefore\ x - x^3y - y^2 = -1$$

$$x^3y + y^2 - x = 1$$

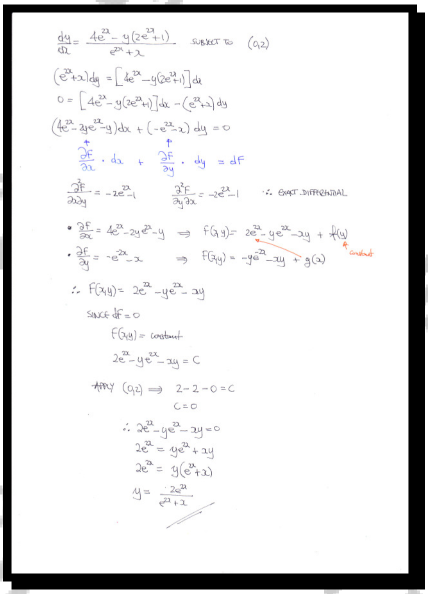
Question 5 (***)

Solve the differential equation

$$\frac{dy}{dx} = \frac{4e^{2x} - y\left(2e^{2x} + 1\right)}{e^{2x} + x},$$

subject to the boundary condition $y = 2$ at $x = 0$.

$$\boxed{y = \frac{2e^{2x}}{e^{2x} + x}}$$



$\frac{dy}{dx} = \frac{4e^{2x} - y(2e^{2x}+1)}{e^{2x}+x}$ SUBJECT TO $(0,2)$

$(e^{2x}+x)dy = \left[4e^{2x} - y(2e^{2x}+1)\right]dx$

$0 = \left[4e^{2x} - y(2e^{2x}+1)\right]dx - (e^{2x}+x)dy$

$(4e^{2x} - 2ye^{2x} - y)dx + (-e^{2x} - x)dy = 0$

$\frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = dF$

$\frac{\partial^2 F}{\partial x \partial y} = -2e^{2x} - 1$ $\quad \frac{\partial^2 F}{\partial y \partial x} = -2e^{2x} - 1$ $\quad \therefore$ EXACT DIFFERENTIAL

- $\frac{\partial F}{\partial x} = 4e^{2x} - 2ye^{2x} - y \Rightarrow F(x,y) = 2e^{2x} - ye^{2x} - xy + f(y)$ constant
- $\frac{\partial F}{\partial y} = -e^{2x} - x \Rightarrow F(x,y) = -ye^{2x} - xy + g(x)$

$\therefore F(x,y) = 2e^{2x} - ye^{2x} - xy$

SINCE $dF = 0$

$F(x,y) = \text{constant}$

$2e^{2x} - ye^{2x} - xy = C$

APPLY $(0,2) \Rightarrow 2 - 2 - 0 = C$

$C = 0$

$\therefore 2e^{2x} - ye^{2x} - xy = 0$

$2e^{2x} = ye^{2x} + xy$

$2e^{2x} = y(e^{2x}+x)$

$y = \frac{2e^{2x}}{e^{2x}+x}$

Question 6 (*+)**

Find a general solution of the following differential equation

$$\frac{dy}{dx} = \frac{\cos x \cos y + \sin^2 x}{\sin x \sin y + \cos^2 y}.$$

$$\boxed{\sin x \cos y - \frac{1}{4}(\sin 2x + \sin 2y) + \frac{1}{2}(x - y) = \text{constant}}$$

$$\frac{dy}{dx} = \frac{\cos x \cos y + \sin^2 x}{\sin x \sin y + \cos^2 y}$$

$$\Rightarrow (\sin x \sin y + \cos^2 y)\,dy = (\cos x \cos y + \sin^2 x)\,dx$$

$$\Rightarrow \underbrace{(\cos x \cos y + \sin^2 x)}_{M(x,y)}\,dx \underbrace{-(\sin x \sin y + \cos^2 y)}_{N(x,y)}\,dy = 0$$

- $\frac{\partial M}{\partial y} = -\cos x \sin y$
- $\frac{\partial N}{\partial x} = -\cos x \sin y$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}$

$$\Longrightarrow dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

$$\Longrightarrow dF = (\cos x \cos y + \sin^2 x)\,dx + (-\sin x \sin y - \cos^2 y)\,dy = 0$$

Thus

$$\frac{\partial}{\partial x}[F(x,y)] = \cos x \cos y + \sin^2 x$$

$$\frac{\partial}{\partial x}[F(x,y)] = \cos x \cos y + \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$F(x,y) = \sin x \cos y + \frac{1}{2}x - \frac{1}{4}\sin 2x + f(y)$$

$$\frac{\partial}{\partial y}[F(x,y)] = -\sin x \sin y - \cos^2 y$$

$$\frac{\partial}{\partial y}[F(x,y)] = -\sin x \sin y - \frac{1}{2} - \frac{1}{2}\cos 2y$$

$$F(x,y) = \sin x \cos y - \frac{1}{2}y - \frac{1}{4}\sin 2y + g(x)$$

COMPARING AND NOTING THAT $dF = 0$ IMPLIES $F(x,y) = \text{constant}$.

$$F(x,y) = \sin x \cos y - \frac{1}{4}\sin 2x - \frac{1}{4}\sin 2y + \frac{1}{2}x - \frac{1}{2}y + \text{constant}$$

$$\therefore \sin x \cos y - \frac{1}{4}(\sin 2x + \sin 2y) + \frac{1}{2}(x - y) = \text{constant}$$

Question 7 (***)

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx} = \frac{(x+y)^2}{1-2xy-x^2}, \quad y(1) = 1.$$

$$\boxed{x^3 - 3y + 3xy(x+y) = 4}$$

$$\boxed{\frac{dy}{dx} = \frac{(x+y)^2}{1-2xy-x^2}, \text{ SUBJECT TO } y=1 \text{ AT } x=1}$$

- AS THE EQUATION IS NOT HOMOGENOUS AND THERE IS NO OBVIOUS SUBSTITUTION, REWRITE THE O.D.E TO CHECK FOR EXACTNESS

$$\Rightarrow (1-2xy-x^2)dy = (x+y)^2dx$$

$$\Rightarrow \underbrace{(x^2+2xy+y^2)}_{M(x,y)}dx + \underbrace{(x^2+2xy-1)}_{N(x,y)}dy = 0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y} &= 2x+2y \\ \frac{\partial N}{\partial x} &= 2x+2y\end{aligned}\right\} \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y} \quad \text{THE EQUATION IS } \underline{\underline{\text{EXACT}}}$$

- REGROUP THE TERMS AND INTEGRATE BY INSPECTION

$$\Rightarrow x^2dx - 1dy + \left[2xy\,dx + x^2dy\right] + \left[y^2dx + 2xy\,dy\right] = 0$$

$$\Rightarrow \tfrac{1}{3}x^3 - y + x^2y + xy^2 = C$$

$$\Rightarrow x^3 - 3y + 3x^2y + 3xy^2 = C$$

- APPLY CONDITION $x=1$ $y=1$ TO OBTAIN $C=4$

$$\therefore \quad x^3 - 3y + 3xy(x+y) = 4$$

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Question 8 (***+)

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}+\frac{x+y}{x\ln x}=0.$$

$$\boxed{x+y\ln x=C}$$

$$\boxed{\frac{dy}{dx}+\frac{x+y}{x\ln x}=0}$$

- THE ODE IS NOT SEPARABLE, NOT HOMOGENEOUS, NO OBVIOUS SUBSTITUTIONS, SO WE CHECK FOR EXACTNESS, BY REWRITING IN DIFFERENTIAL FORM

$$\Rightarrow \underbrace{(x+y)}_{M(x,y)}dx+\underbrace{(x\ln x)}_{N(x,y)}dy=0$$

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=1\\[4pt] \frac{\partial N}{\partial x}=\ln x+1\end{array}\right\}$$ AS $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ THE O.D.E IS NOT EXACT IN ITS CURRENT FORM

- LOOK FOR POSSIBLE INTEGRATING FACTORS TO MAKE IT EXACT

IF $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$, THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR

IF $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)$, THEN $e^{\int -g(y)\,dy}$ IS AN INTEGRATING FACTOR

- HERE WE OBTAIN

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=-\ln x$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-\ln x}{x\ln x}=-\frac{1}{x}$$

- THUS WE CAN FIND AN INTEGRATING FACTOR

$$e^{\int -\frac{1}{x}\,dx}=e^{-\ln x}=\frac{1}{x}$$

- MULTIPLY THE O.D.E BY $\frac{1}{x}$, AND INTEGRATE BY INSPECTION

$$\Rightarrow \left(1+\frac{y}{x}\right)dx+(\ln x)\,dy=0$$

$$\Rightarrow 1\,dx+\left(\frac{y}{x}dx+\ln x\,dy\right)=0$$

$$\Rightarrow x+y\ln x=C$$

ALTERNATIVE WITHOUT INSPECTION

$$\left(1+\frac{y}{x}\right)dx+(\ln x)\,dy=0$$

$$\frac{\partial \phi}{\partial x}\cdot dx+\frac{\partial \phi}{\partial y}\cdot dy=d\phi$$

- $\frac{\partial \phi}{\partial x}=1+\frac{y}{x}$

$\phi(x,y)=x+y\ln x+f(y)$

- $\frac{\partial \phi}{\partial y}=\ln x$

$\phi(x,y)=y\ln x+g(x)$

COMPARING

$g(x)=x$

$f(y)=$ CONSTANT

SO $d\phi=0$

$$\phi(x,y)=C$$

$$x+y\ln x=C$$

Question 9 (***+)

$$\frac{dy}{dx} = \frac{2xy}{3x^2 - y^2}, \quad y(0) = 1.$$

a) Find an integrating factor for the above differential equation and hence show

$$y^3 = y^2 - x^2.$$

b) Verify the answer of part **(a)** by a solving the differential equation by a suitable substitution.

proof

a) $\boxed{\frac{dy}{dx} = \frac{2xy}{3x^2-y^2} \quad \text{SUBJECT TO } y=1 \text{ AT } x=0}$

REWRITE THE O.D.E IN THE "STANDARD FORM"

$\Rightarrow (3x^2-y^2)\,dy = 2xy\,dx$

$\Rightarrow \underbrace{2xy}_{M}\,dx + \underbrace{(y^2-3x^2)}_{N}\,dy = 0$

CHECK FOR EXACTNESS

$\left.\begin{array}{l} \frac{\partial M}{\partial y} = 2x \\ \frac{\partial N}{\partial x} = -6x \end{array}\right\} \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ SO THE O.D.E IS NOT EXACT IN ITS CURRENT FORM

LOOK FOR POSSIBLE INTEGRATING FACTORS

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, THEN $e^{\int -g(y)\,dy}$ IS AN INTEGRATING FACTOR

HERE $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{2x+6x}{2xy} = \frac{8x}{2xy} = \frac{4}{y}$

THUS THERE IS AN INTEGRATING FACTOR

$e^{\int -\frac{4}{y}\,dy} = e^{-4\ln y} = \frac{1}{y^4}$

MULTIPLY THE O.D.E BY THE INTEGRATING FACTOR

$\Rightarrow \frac{2x}{y^3}\,dx + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)dy = 0$

$\Rightarrow \underbrace{\left[\frac{2x}{y^3}\,dx - \frac{3x^2}{y^4}\,dy\right]}_{\text{EXACT}} + \frac{1}{y^2}\,dy = 0$

INTEGRATING (BY INSPECTION)

$\Rightarrow \frac{x^2}{y^3} - \frac{1}{y} = C$

$\Rightarrow x^2 - y^2 = Cy^3$

APPLY CONDITION $(0,1)$ TO OBTAIN $C=-1$

$\Rightarrow x^2 - y^2 = -y^3$

$\Rightarrow y^2 - x^2 = y^3$

b) $\boxed{\frac{dy}{dx} = \frac{2xy}{3x^2-y^2}, \quad \text{SUBJECT TO } y=1 \text{ AT } x=0}$

AS THE R.H.S IS HOMOGENEOUS WE MAY USE A SUBSTITUTION

$y = xV(x)$

$\frac{dy}{dx} = V + x\frac{dV}{dx}$

O.D.E NOW BECOMES

$\Rightarrow V + x\frac{dV}{dx} = \frac{2x(xV)}{3x^2 - x^2V^2} = \frac{2x^2V}{3x^2 - x^2V^2}$

$\Rightarrow V + x\frac{dV}{dx} = \frac{2V}{3-V^2}$

$\Rightarrow x\frac{dV}{dx} = \frac{2V}{3-V^2} - V$

$\Rightarrow x\frac{dV}{dx} = \frac{2V - 3V + V^3}{3-V^2}$

$\Rightarrow x\frac{dV}{dx} = \frac{V^3 - V}{3-V^2}$

$\Rightarrow \frac{3-V^2}{V^3-V}\,dV = \frac{1}{x}\,dx$

$\Rightarrow \frac{3-V^2}{V(V-1)(V+1)}\,dV = \frac{1}{x}\,dx$

BY PARTIAL FRACTIONS (COVER UP)

$\frac{3-V^2}{V(V-1)(V+1)} = \frac{\frac{3}{-1}}{V} + \frac{\frac{2}{2}}{V-1} + \frac{\frac{2}{2}}{V+1}$

$= \frac{1}{V-1} + \frac{1}{V+1} - \frac{3}{V}$

$\Rightarrow \int \frac{1}{V-1} + \frac{1}{V+1} - \frac{3}{V}\,dV = \int \frac{1}{x}\,dx$

$\Rightarrow \ln|V-1| + \ln|V+1| - 3\ln|V| = \ln|x| + \ln A$

$\Rightarrow \ln\left|\frac{V^2-1}{V^3}\right| = \ln|Ax|$

$\Rightarrow \frac{V^2-1}{V^3} = Ax$

$\Rightarrow \frac{\frac{y^2}{x^2} - 1}{\frac{y^3}{x^3}} = Ax$

$\Rightarrow \frac{xy^2 - x^3}{y^3} = Ax$

$\Rightarrow \frac{y^2 - x^2}{y^3} = A$

$\Rightarrow y^2 - x^2 = Ay^3$ APPLY $(0,1)$

$\Rightarrow y^2 - x^2 = y^3$

AS BEFORE

Question 10 (***+)

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}\left(x^2+2y^2+2\right)-xy=0.$$

$$\boxed{x^2=-2+4y^2\ln y+Cy^2}$$

$$\boxed{\frac{dy}{dx}(x^2+2y^2+2)-xy=0}$$

REWRITE THE O.D.E IN THE "USUAL" FORM

$$\Rightarrow dy\,(x^2+2y^2+2)-(xy)\,dx=0$$

$$\Rightarrow \underbrace{(xy)}_{M}\,dx+\underbrace{(-x^2-2y^2-2)}_{N}\,dy=0$$

CHECK FOR EXACTNESS

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=x\\ \frac{\partial N}{\partial x}=-2x\end{array}\right\}\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\ \text{NOT EXACT}$$

LOOK FOR INTEGRATING FACTORS

$$\boxed{\begin{array}{l}\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x),\ \text{THEN } e^{\int f(x)\,dx} \text{ IS AN INTEGRATING FACTOR OF THE O.D.E}\\[2ex] \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=f(y),\ \text{THEN } e^{\int -f(y)\,dy} \text{ IS AN INTEGRATING FACTOR OF THE O.D.E}\end{array}}$$

EVIDENTLY HERE $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\dfrac{x+2x}{xy}=\dfrac{3}{y}$

THUS THE INTEGRATING FACTOR HERE WILL BE

$$e^{\int -\frac{3}{y}\,dy}=e^{-3\ln y}=\frac{1}{y^3}$$

MULTIPLY THE O.D.E THROUGH BY $\frac{1}{y^3}$

$$\Rightarrow \frac{x}{y^2}\,dx+\left(-\frac{x^2}{y^3}-\frac{2}{y}-\frac{2}{y^3}\right)dy=0$$

$$\Rightarrow \underbrace{\left[\frac{x}{y^2}\,dx-\frac{x^2}{y^3}\,dy\right]}_{\text{EXACT}}-\frac{2}{y}\,dy-\frac{2}{y^3}\,dy=0$$

BY DIRECT INTEGRATION (INSPECTION)

$$\frac{x^2}{2y^2}-2\ln y+\frac{1}{y^2}=C$$

$$x^2-4y^2\ln y+2=Cy^2$$

$$x^2=-2+4y^2\ln y+Cy^2$$

Question 11 (***+)

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{2y-2y^2}{2xy-x}.$$

$$\boxed{x^2y(1-y)=C}$$

$$\boxed{\frac{dy}{dx}=\frac{2y-2y^2}{2xy-x}}$$

REWRITE THE EQUATION AS FOLLOWS

$$\Rightarrow (2xy-x)\,dy=(2y-2y^2)\,dx$$

$$\Rightarrow \underbrace{(2y-2y^2)}_{M}\,dx+\underbrace{(x-2xy)}_{N}\,dy=0$$

CHECK FOR EXACTNESS

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=2-4y\\ \frac{\partial N}{\partial x}=1-2y\end{array}\right\}\ \text{AS } \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \text{ THE O.D.E IS NOT EXACT IN ITS CURRENT FORM}$$

LOOK FOR POSSIBLE INTEGRATING FACTORS

$$\boxed{\begin{array}{l}\text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x),\ \text{THEN } e^{\int f(x)\,dx} \text{ IS AN INTEGRATING FACTOR}\\[2ex] \text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y),\ \text{THEN } e^{\int -g(y)\,dy} \text{ IS AN INTEGRATING FACTOR}\end{array}}$$

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=1-2y \quad \& \quad N(x,y)=x(1-2y)$$

$$\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{\cancel{1-2y}}{x\cancel{(1-2y)}}=\frac{1}{x} \text{ IF AN INTEGRATING FACTOR CAN BE FOUND}$$

MULTIPLY THE O.D.E BY THE INTEGRATING FACTOR

$$\text{I.F.}=e^{\int\frac{1}{x}\,dx}=e^{\ln x}=x$$

$$\Rightarrow (2xy-2xy^2)\,dx+(x^2-2x^2y)\,dy=0$$

$$\Rightarrow \underbrace{\left[2xy\,dx+x^2\,dy\right]}_{\text{EXACT}}+\underbrace{\left[-2xy^2\,dx-2x^2y\,dy\right]}_{\text{EXACT}}=0$$

INTEGRATING BY INSPECTION

$$\Rightarrow x^2y-x^2y^2=C$$

$$\Rightarrow x^2y(1-y)=C$$

Question 12 (***+)

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx} = \frac{6xy}{4y+9x^2}.$$

$$\boxed{3x^2y^3 + y^4 = C}$$

$$\boxed{\frac{dy}{dx} + \frac{6xy}{4y+9x^2} = 0}$$

- THE ODE IS NOT SEPARABLE, IS NOT HOMOGENEOUS, NO OBVIOUS SUBSTITUTIONS — WRITE THE ODE. IN DIFFERENTIAL FORM TO CHECK FOR EXACTNESS

$$\Rightarrow (4y+9x^2)\,dy + 6xy\,dx = 0$$

$$\Rightarrow \underbrace{(6xy)}_{M(x,y)}\,dx + \underbrace{(4y+9x^2)}_{N(x,y)}\,dy = 0$$

- CHECKING FOR EXACTNESS

$$\left.\begin{array}{l} \frac{\partial M}{\partial y} = 6x \\ \frac{\partial N}{\partial x} = 18x \end{array}\right\} \text{ AS } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ THE ODE. IS NOT EXACT IN ITS CURRENT FORM

- LOOK FOR POSSIBLE INTEGRATING FACTORS

$$\boxed{\begin{array}{l} \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x), \text{ THEN } e^{\int f(x)\,dx} \text{ IS AN INTEGRATING FACTOR} \\ \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y), \text{ THEN } e^{\int -g(y)\,dy} \text{ IS AN INTEGRATING FACTOR} \end{array}}$$

- HERE WE HAVE

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6x - 18x = -12x \quad \& \quad M(x,y) = 6xy$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-12x}{6xy} = -\frac{2}{y}$$

- THUS WE CAN FIND AN INTEGRATING FACTOR

$$\text{I.F.} = e^{\int +\frac{2}{y}\,dy} = e^{2\ln y} = y^2$$

- MULTIPLY THE O.D.E THROUGH BY y^2, AND INTEGRATE BY INSPECTION

$$\Rightarrow (6xy^3)\,dx + (4y^3 + 9x^2y^2)\,dy = 0$$

$$\Rightarrow \underbrace{\left[6xy^3\,dx + 9x^2y^2\,dy\right]}_{\text{EXACT}} + 4y^3\,dy = 0$$

$$\Rightarrow 3x^2y^3 + y^4 = C$$

ALTERNATIVE WITHOUT INSPECTION

$$(6xy^3)\,dx + (4y^3+9x^2y^2)\,dy = 0$$

$$\frac{\partial \psi}{\partial x}\cdot dx + \frac{\partial \psi}{\partial y}\cdot dy = d\psi$$

$$\frac{\partial \psi}{\partial x} = 6xy^3 \qquad \frac{\partial \psi}{\partial y} = 4y^3 + 9x^2y^2$$

$$\psi(x,y) = 3x^2y^3 + f(y) \qquad \psi(x,y) = y^4 + 3x^2y^2 + g(x)$$

COMPARING

$$f(y) = y^4$$

$$g(x) = \text{CONSTANT.}$$

$$d\psi = 0$$

$$\psi(x,y) = \text{CONSTANT}$$

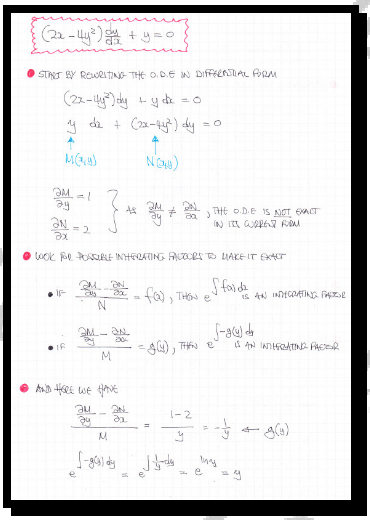
$$3x^2y^3 + y^4 = C$$

Question 13 (***+)

$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0.$$

By finding a suitable integrating factor for the above differential equation determine its general solution.

$$\boxed{xy^2-y^4=C}$$



$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0$$

START BY REWRITING THE O.D.E IN DIFFERENTIAL FORM

$$\left(2x-4y^2\right)dy+y\,dx=0$$

$$\underbrace{y}_{M(x,y)}\,dx+\underbrace{\left(2x-4y^2\right)}_{N(x,y)}\,dy=0$$

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=1\\ \frac{\partial N}{\partial x}&=2\end{aligned}\right\}$$ AS $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, THE O.D.E IS NOT EXACT IN ITS CURRENT FORM

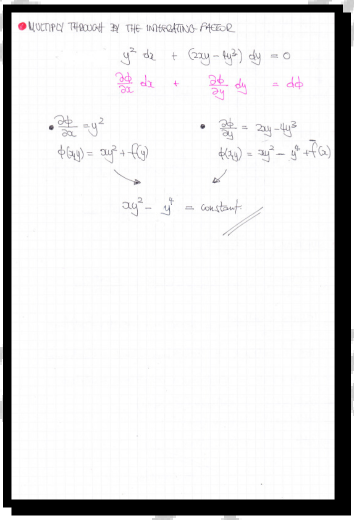
LOOK FOR POSSIBLE INTEGRATING FACTORS TO MAKE IT EXACT

- IF $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$, THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR
- IF $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y)$, THEN $e^{\int -g(y)\,dy}$ IS AN INTEGRATING FACTOR

AND HERE WE HAVE

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=\frac{1-2}{y}=-\frac{1}{y}\leftarrow g(y)$$

$$e^{\int -g(y)\,dy}=e^{\int \frac{1}{y}dy}=e^{\ln y}=y$$



MULTIPLY THROUGH BY THE INTEGRATING FACTOR

$$y^2\,dx+\left(2xy-4y^3\right)dy=0$$

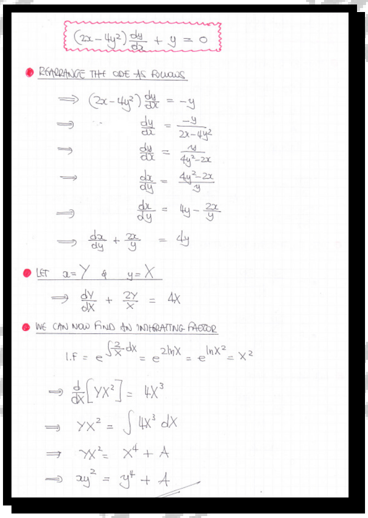
$$\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy=d\phi$$

- $\frac{\partial\phi}{\partial x}=y^2$

  $\phi(x,y)=xy^2+f(y)$
- $\frac{\partial\phi}{\partial y}=2xy-4y^3$

  $\phi(x,y)=xy^2-y^4+\bar{f}(x)$

$$xy^2-y^4=\text{constant}$$



$$\left(2x-4y^2\right)\frac{dy}{dx}+y=0$$

REARRANGE THE ODE AS FOLLOWS

$$\Rightarrow \left(2x-4y^2\right)\frac{dy}{dx}=-y$$

$$\Rightarrow \frac{dy}{dx}=\frac{-y}{2x-4y^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{4y^2-2x}$$

$$\Rightarrow \frac{dx}{dy}=\frac{4y^2-2x}{y}$$

$$\Rightarrow \frac{dx}{dy}=4y-\frac{2x}{y}$$

$$\Rightarrow \frac{dx}{dy}+\frac{2x}{y}=4y$$

LET $x=Y$ & $y=X$

$$\Rightarrow \frac{dY}{dX}+\frac{2Y}{X}=4X$$

WE CAN NOW FIND AN INTEGRATING FACTOR

$$\text{I.F}=e^{\int\frac{2}{X}dX}=e^{2\ln X}=e^{\ln X^2}=X^2$$

$$\Rightarrow \frac{d}{dX}\left[YX^2\right]=4X^3$$

$$\Rightarrow YX^2=\int 4X^3\,dX$$

$$\Rightarrow YX^2=X^4+A$$

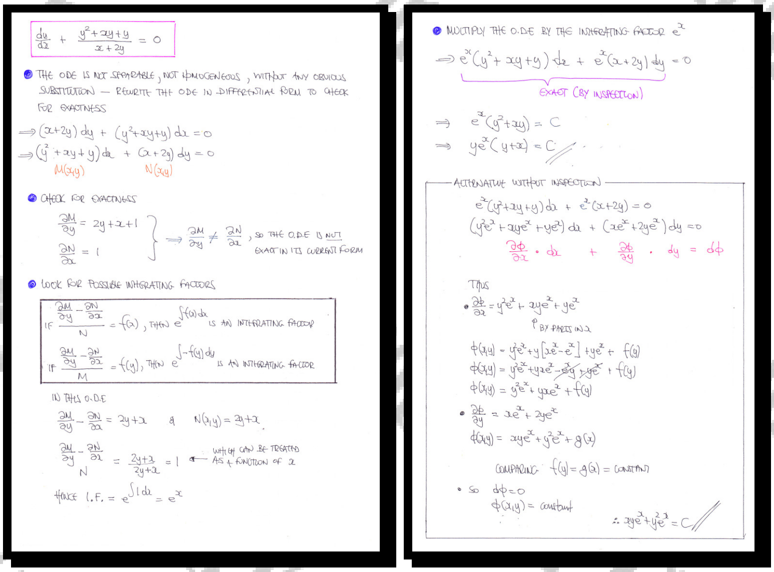
$$\Rightarrow xy^2=y^4+A$$

Question 14 (***+)

Find a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}=\frac{y^2+xy+y}{x+2y}.$$

$$\boxed{ye^x\left(y+x\right)=C}$$



$$\boxed{\frac{dy}{dx}+\frac{y^2+xy+y}{x+2y}=0}$$

THE ODE IS NOT SEPARABLE, NOT HOMOGENEOUS, WITHOUT ANY OBVIOUS SUBSTITUTION — REWRITE THE ODE IN DIFFERENTIAL FORM TO CHECK FOR EXACTNESS

$$\Longrightarrow (x+2y)\,dy+(y^2+xy+y)\,dx=0$$

$$\Longrightarrow \underbrace{(y^2+xy+y)}_{M(x,y)}\,dx+\underbrace{(x+2y)}_{N(x,y)}\,dy=0$$

CHECK FOR EXACTNESS

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=2y+x+1\\ \frac{\partial N}{\partial x}=1\end{array}\right\}\Longrightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$, SO THE O.D.E IS NOT EXACT IN ITS CURRENT FORM

LOOK FOR POSSIBLE INTEGRATING FACTORS

$$\boxed{\begin{array}{l}\text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)\text{, THEN } e^{\int f(x)\,dx}\text{ IS AN INTEGRATING FACTOR}\\ \text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=f(y)\text{, THEN } e^{\int -f(y)\,dy}\text{ IS AN INTEGRATING FACTOR}\end{array}}$$

IN THIS O.D.E

$$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=2y+x \quad \& \quad N(x,y)=2y+x.$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y+x}{2y+x}=1$$ ← WHICH CAN BE TREATED AS A FUNCTION OF x

HENCE $\text{I.F.}=e^{\int 1\,dx}=e^x$

MULTIPLY THE O.D.E BY THE INTEGRATING FACTOR e^x

$$\Longrightarrow \underbrace{e^x(y^2+xy+y)\,dx+e^x(x+2y)\,dy}_{\text{EXACT (BY INSPECTION)}}=0$$

$$\Longrightarrow e^x(y^2+xy)=C$$

$$\Longrightarrow ye^x(y+x)=C$$

ALTERNATIVE WITHOUT INSPECTION

$$e^x(y^2+xy+y)\,dx+e^x(x+2y)=0$$

$$(y^2e^x+xye^x+ye^x)\,dx+(xe^x+2ye^x)\,dy=0$$

$$\frac{\partial\phi}{\partial x}\cdot dx+\frac{\partial\phi}{\partial y}\cdot dy=d\phi$$

THUS

- $\frac{\partial\phi}{\partial x}=y^2e^x+xye^x+ye^x$ ← BY PARTS IN x

$$\phi(x,y)=y^2e^x+y\left[xe^x-e^x\right]+ye^x+f(y)$$

$$\phi(x,y)=y^2e^x+yxe^x\cancel{-e^xy}\cancel{+ye^x}+f(y)$$

$$\phi(x,y)=y^2e^x+yxe^x+f(y)$$

- $\frac{\partial\phi}{\partial y}=xe^x+2ye^x$

$$\phi(x,y)=xye^x+y^2e^x+g(x)$$

COMPARING $f(y)=g(x)=\text{CONSTANT}$

- SO $d\phi=0$

$$\phi(x,y)=\text{constant}$$

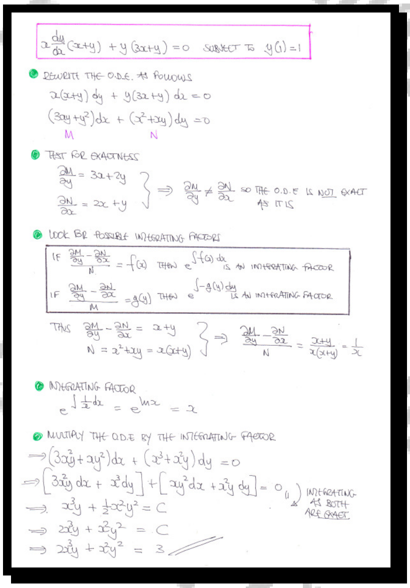
$$\therefore xye^x+y^2e^x=C$$

Question 15 (***+)

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$x\frac{dy}{dx}(x+y)+y(3x+y)=0, \quad y(1)=1.$$

$$\boxed{2yx^3+x^2y^2=3}$$



$$\boxed{x\frac{dy}{dx}(x+y)+y(3x+y)=0 \quad \text{SUBJECT TO } y(1)=1}$$

- REWRITE THE O.D.E. AS FOLLOWS

$$x(x+y)\,dy+y(3x+y)\,dx=0$$
$$\underbrace{(3xy+y^2)}_{M}dx+\underbrace{(x^2+xy)}_{N}dy=0$$

- TEST FOR EXACTNESS

$$\left.\begin{array}{l}\frac{\partial M}{\partial y}=3x+2y\\ \frac{\partial N}{\partial x}=2x+y\end{array}\right\}\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \text{ SO THE O.D.E IS NOT EXACT AS IT IS}$$

- LOOK FOR POSSIBLE INTEGRATING FACTORS

$$\boxed{\begin{array}{l}\text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x) \text{ THEN } e^{\int f(x)\,dx} \text{ IS AN INTEGRATING FACTOR}\\ \text{IF } \dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=g(y) \text{ THEN } e^{\int -g(y)\,dy} \text{ IS AN INTEGRATING FACTOR}\end{array}}$$

$$\text{THUS } \left.\begin{array}{l}\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=x+y\\ N=x^2+xy=x(x+y)\end{array}\right\}\Rightarrow \frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{x+y}{x(x+y)}=\frac{1}{x}$$

- INTEGRATING FACTOR

$$e^{\int\frac{1}{x}dx}=e^{\ln x}=x$$

- MULTIPLY THE O.D.E BY THE INTEGRATING FACTOR

$$\Rightarrow (3x^2y+xy^2)dx+(x^3+x^2y)dy=0$$
$$\Rightarrow \left[3x^2y\,dx+x^3dy\right]+\left[xy^2dx+x^2y\,dy\right]=0$$

INTEGRATING AS BOTH ARE EXACT

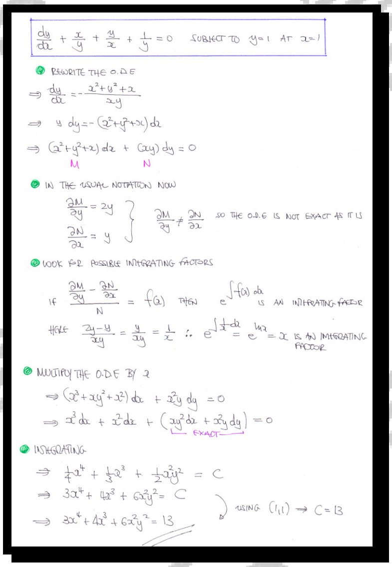
$$\Rightarrow x^3y+\tfrac{1}{2}x^2y^2=C$$
$$\Rightarrow 2x^3y+x^2y^2=C$$
$$\Rightarrow 2x^3y+x^2y^2=3$$

Question 16 (***+)

Determine the solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx}+\frac{x}{y}+\frac{y}{x}+\frac{1}{y}=0, \quad y(1)=1.$$

$$\boxed{3x^4+4x^3+6x^2y^2=13}$$



$$\boxed{\frac{dy}{dx}+\frac{x}{y}+\frac{y}{x}+\frac{1}{y}=0 \quad \text{SUBJECT TO } y=1 \text{ AT } x=1}$$

REWRITE THE O.D.E

$$\Rightarrow \frac{dy}{dx}=-\frac{x^2+y^2+x}{xy}$$

$$\Rightarrow xy\,dy=-(x^2+y^2+x)\,dx$$

$$\Rightarrow \underbrace{(x^2+y^2+x)}_{M}\,dx+\underbrace{(xy)}_{N}\,dy=0$$

IN THE USUAL NOTATION NOW

$$\left.\begin{aligned}\frac{\partial M}{\partial y}&=2y\\ \frac{\partial N}{\partial x}&=y\end{aligned}\right\}\ \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$$ SO THE O.D.E IS NOT EXACT AS IT IS

LOOK FOR POSSIBLE INTEGRATING FACTORS

IF $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$ THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR

HERE $\dfrac{2y-y}{xy}=\dfrac{y}{xy}=\dfrac{1}{x}$ $\therefore$ $e^{\int \frac{1}{x}dx}=e^{\ln x}=x$ IS AN INTEGRATING FACTOR

MULTIPLY THE O.D.E BY x

$$\Rightarrow (x^3+xy^2+x^2)\,dx+x^2y\,dy=0$$

$$\Rightarrow x^3\,dx+x^2\,dx+\underbrace{\left(xy^2\,dx+x^2y\,dy\right)}_{\text{EXACT}}=0$$

INTEGRATING

$$\Rightarrow \tfrac{1}{4}x^4+\tfrac{1}{3}x^3+\tfrac{1}{2}x^2y^2=C$$

$$\Rightarrow 3x^4+4x^3+6x^2y^2=C$$

USING $(1,1)\Rightarrow C=13$

$$\Rightarrow 3x^4+4x^3+6x^2y^2=13$$

Question 17 (***+)

Determine the solution of the following differential equation.

$$\frac{dy}{dx} = \frac{x^2}{y} + \frac{2y}{x}, \quad y(1) = 2.$$

$$\boxed{y^2 = 6x^4 - 2x^3}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2}{y} + \frac{2y}{x} \quad \text{SUBJECT TO} \quad y(1) = 2}$$

- REWRITE THE O.D.E AS FOLLOWS

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 + 2y^2}{xy}$$

$$\Rightarrow xy\,dy = (x^3 + 2y^2)\,dx$$

$$\Rightarrow \underbrace{(x^3 + 2y^2)}_{M}\,dx + \underbrace{(-xy)}_{N}\,dy$$

- NOW $\frac{\partial M}{\partial y} = 4y$ & $\frac{\partial N}{\partial x} = -y$
  - AS $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ THE O.D.E IS NOT EXACT
  - LOOK FOR POSSIBLE INTEGRATING FACTORS
  - $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{4y + y}{-xy} = \dfrac{5y}{-xy} = -\dfrac{5}{x}$ ← A FUNCTION OF x ONLY

$\therefore$ INTEGRATING FACTOR

$$e^{\int -\frac{5}{x}\,dx} = e^{-5\ln x} = \frac{1}{x^5}$$

- MULTIPLY THE O.D.E BY THE INTEGRATING FACTOR

$$\Rightarrow (x^{-2} + 2y^2x^{-5})\,dx + (-x^{-4}y)\,dy = 0$$

$$\Rightarrow \frac{1}{x^2}dx + \underbrace{\left[\frac{2y^2}{x^5}dx - \frac{y}{x^4}dy\right]}_{\text{EXACT}} = 0$$

INTEGRATING

$$\Rightarrow -\frac{1}{x} - \frac{y^2}{2x^4} = A$$

- APPLY CONDITION $y(1) = 2 \Rightarrow -1 - 2 = A$
  $A = -3$

$$\Rightarrow -\frac{1}{x} - \frac{y^2}{2x^4} = -3$$

$$\Rightarrow \frac{1}{x} + \frac{y^2}{2x^4} = 3$$

$$\Rightarrow \frac{y^2}{2x^4} = 3 - \frac{1}{x}$$

$$\Rightarrow y^2 = 6x^4 - 2x^3$$

ALTERNATIVE BY SUBSTITUTION

$$\boxed{\frac{dy}{dx} = \frac{x^2}{y} + \frac{2y}{x}, \quad y(1) = 2}$$

LET $V = \frac{y}{x^2}$

$y = x^2 V$

$\frac{dy}{dx} = 2xV + x^2\frac{dV}{dx}$

$$2xV + x^2\frac{dV}{dx} = \frac{1}{V} + \frac{2(x^2V)}{x}$$

$$\cancel{2xV} + x^2\frac{dV}{dx} = \frac{1}{V} + \cancel{2xV}$$

$$V\,dV = \frac{1}{x^2}\,dx$$

INTEGRATING

$$\frac{1}{2}V^2 = -\frac{1}{x} + C$$

$$V^2 = C - \frac{2}{x}$$

$$\frac{y^2}{x^4} = C - \frac{2}{x}$$

$$y^2 = Cx^4 - 2x^3$$

APPLY CONDITION $(1,2)$

$4 = C - 2$

$C = 6$

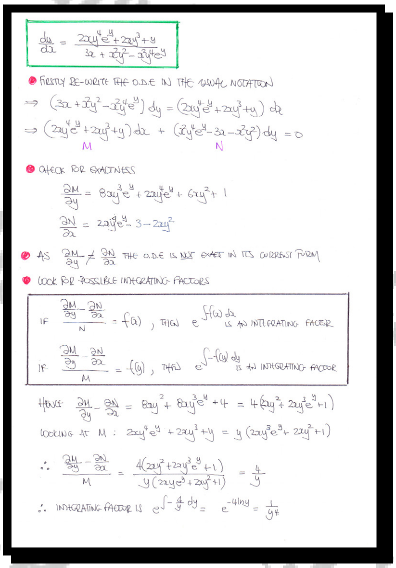
$$y^2 = 6x^4 - 2x^3$$ AS BEFORE

Question 18 (****)

Determine a general solution of the following differential equation by looking for a suitable integrating factor.

$$\frac{dy}{dx} = \frac{2xy^4 e^y + 2xy^3 + y}{3x + x^2 y^2 - x^2 y^4 e^y}.$$

$$\boxed{x^2 e^y + \frac{x^2}{y} + \frac{x}{y^3} = C}$$



$$\boxed{\frac{dy}{dx} = \frac{2xy^4e^y + 2xy^3 + y}{3x + x^2y^2 - x^2y^4e^y}}$$

- FIRSTLY RE-WRITE THE O.D.E IN THE USUAL NOTATION

$\Rightarrow (3x + x^2y^2 - x^2y^4e^y)\,dy = (2xy^4e^y + 2xy^3 + y)\,dx$

$\Rightarrow \underbrace{(2xy^4e^y + 2xy^3 + y)}_{M}\,dx + \underbrace{(x^2y^4e^y - 3x - x^2y^2)}_{N}\,dy = 0$

- CHECK FOR EXACTNESS

$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 3 - 2xy^2$

- AS $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ THE O.D.E IS NOT EXACT IN ITS CURRENT FORM
- LOOK FOR POSSIBLE INTEGRATING FACTORS

IF $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, THEN $e^{\int f(x)\,dx}$ IS AN INTEGRATING FACTOR

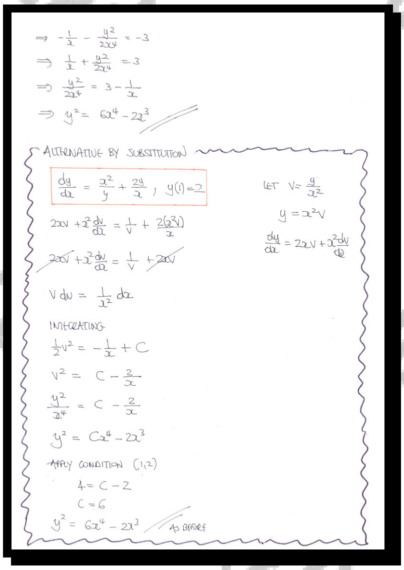
IF $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(y)$, THEN $e^{\int -f(y)\,dy}$ IS AN INTEGRATING FACTOR

HENCE $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 8xy^2 + 8xy^3e^y + 4 = 4(2xy^2 + 2xy^3e^y + 1)$

LOOKING AT M: $2xy^4e^y + 2xy^3 + y = y(2xy^3e^y + 2xy^2 + 1)$

$\therefore \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \dfrac{4(2xy^2 + 2xy^3e^y + 1)}{y(2xye^y + 2xy^2 + 1)} = \dfrac{4}{y}$

$\therefore$ INTEGRATING FACTOR IS $e^{\int -\frac{4}{y}\,dy} = e^{-4\ln y} = \frac{1}{y^4}$



$\Rightarrow -\frac{1}{x} - \frac{y^2}{2x^4} = -3$

$\Rightarrow \frac{1}{x} + \frac{y^2}{2x^4} = 3$

$\Rightarrow \frac{y^2}{2x^4} = 3 - \frac{1}{x}$

$\Rightarrow y^2 = 6x^4 - 2x^3$

ALTERNATIVE BY SUBSTITUTION

$\boxed{\frac{dy}{dx} = \frac{x^2}{y} + \frac{2y}{x}, \quad y(1) = 2}$

LET $V = \frac{y}{x^2}$

$y = x^2V$

$\frac{dy}{dx} = 2xV + x^2\frac{dV}{dx}$

$2xV + x^2\frac{dV}{dx} = \frac{1}{V} + \frac{2(x^2V)}{x}$

$\cancel{2xV} + x^2\frac{dV}{dx} = \frac{1}{V} + \cancel{2xV}$

$V\,dV = \frac{1}{x^2}\,dx$

INTEGRATING

$\frac{1}{2}V^2 = -\frac{1}{x} + C$

$V^2 = C - \frac{2}{x}$

$\frac{y^2}{x^4} = C - \frac{2}{x}$

$y^2 = Cx^4 - 2x^3$

APPLY CONDITION $(1,2)$

$4 = C - 2$

$C = 6$

$y^2 = 6x^4 - 2x^3$ AS BEFORE

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1^{ST} ORDER BY VARIOUS TECHNIQUES

Question 1 (**)

By using a suitable substitution find a general solution of the differential equation

$$\frac{dy}{dx} = x + y,$$

giving the answer in the form $y = f(x)$.

$$\boxed{y = Ae^{x} - x - 1}$$

$\frac{dy}{dx} = x + y$

LET $V = x + y$

$\frac{dV}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dV}{dx} - 1$

THUS $\frac{dV}{dx} - 1 = V$

$\Rightarrow \frac{dV}{dx} = V + 1$

$\Rightarrow \frac{1}{V+1}\,dV = 1\,dx$

$\Rightarrow \int \frac{1}{V+1}\,dV = \int 1\,dx$

$\Rightarrow \ln|V+1| = x + C$

$\Rightarrow V + 1 = e^{x+C}$

$\Rightarrow V + 1 = Ae^{x}$ $\quad (A = e^{C})$

$\Rightarrow x + y + 1 = Ae^{x}$

$\Rightarrow y = Ae^{x} - x - 1$ //

Question 2 (**)

$$\frac{dy}{dx} = x + 2y, \text{ with } y = -\frac{1}{4} \text{ at } x = 0.$$

By using a suitable substitution, show that the solution of the differential equation is

$$y = -\frac{1}{4}(2x+1).$$

$$\boxed{\text{proof}}$$

$V = x + 2y$

$\frac{dV}{dx} = 1 + 2\frac{dy}{dx}$

$2\frac{dy}{dx} = \frac{dV}{dx} - 1$

$\frac{dy}{dx} = x + 2y$

$\Rightarrow \frac{1}{2}\left(\frac{dV}{dx} - 1\right) = V$

$\Rightarrow \frac{dV}{dx} - 1 = 2V$

$\Rightarrow \frac{dV}{dx} = 2V + 1$

$\Rightarrow \int \frac{1}{2V+1}\,dV = \int 1\,dx$

$\Rightarrow \frac{1}{2}\ln|2V+1| = x + C$

$\Rightarrow \ln|2V+1| = 2x + C$

$\Rightarrow 2V + 1 = Ae^{2x}$

$\Rightarrow V = \frac{1}{2}(-1 + Ae^{2x})$

$\Rightarrow \boxed{x + 2y = \frac{1}{2}(-1 + Ae^{2x})}$

APPLY CONDITION $(0, -\frac{1}{4})$

$-\frac{1}{2} = \frac{1}{2}(-1 + Ae^{0})$

$-\frac{1}{2} = -\frac{1}{2} + \frac{1}{2}A$

$\boxed{A = 0}$

$\therefore x + 2y = -\frac{1}{2}$

$\Rightarrow 2y = -\frac{1}{2} - x$

$\Rightarrow 2y = -\frac{1}{2}(1 + 2x)$

$\Rightarrow y = -\frac{1}{4}(2x+1)$ // AS REQUIRED

ALTERNATIVE

$\frac{dy}{dx} - 2y = x$

$\text{IF} = e^{\int -2\,dx} = e^{-2x}$

$\Rightarrow \frac{d}{dx}\left(e^{-2x}y\right) = xe^{-2x}$

$\Rightarrow ye^{-2x} = \int xe^{-2x}\,dx$

BY PARTS $\begin{array}{c|c} x & 1 \\ -\frac{1}{2}e^{-2x} & e^{-2x} \end{array}$

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$

$ye^{-2x} = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + A$

$\boxed{y = -\frac{1}{2}x - \frac{1}{4} + Ae^{2x}}$

APPLY $(0, -\frac{1}{4}) \Rightarrow A = 0$

$\therefore y = -\frac{1}{2}x - \frac{1}{4}$

$y = -\frac{1}{4}(2x+1)$ // AS BEFORE

Question 3 (**)

Use the substitution $t = \sqrt{y}$ to solve the following differential equation.

$$\frac{dy}{dx} = y + \sqrt{y}, \quad y > 0, \quad y(0) = 4.$$

Given the answer in the form $y = f(x)$.

$\boxed{y = 9\mathrm{e}^{x} - 6\mathrm{e}^{\frac{1}{2}x} + 1}$

USING THE SUBSTITUTION GIVEN WE CAN TRANSFORM THE O.D.E.

$t = \sqrt{y}$

$t^2 = y$

$\frac{dy}{dx} = 2t\frac{dt}{dx}$

$\Rightarrow \frac{dy}{dx} = y + \sqrt{y}$

$\Rightarrow 2t\frac{dt}{dx} = t^2 + t$

$\Rightarrow 2\frac{dt}{dx} = t + 1 \quad (t \neq 0)$

$\Rightarrow \frac{2}{t+1}\,dt = 1\,dx$

INTEGRATING BOTH SIDES

$\Rightarrow 2\ln|t+1| = x + C$

$\Rightarrow \ln|t+1| = \frac{1}{2}x + D$

$\Rightarrow |t+1| = Ae^{\frac{1}{2}x}$

$\Rightarrow |\sqrt{y}+1| = Ae^{\frac{1}{2}x}$

$\Rightarrow \sqrt{y}+1 = Ae^{\frac{1}{2}x}$

APPLY CONDITION $x=0$ $y=4$ YIELDS $A=3$

$\Rightarrow \sqrt{y}+1 = 3e^{\frac{1}{2}x}$

$\Rightarrow \sqrt{y} = 3e^{\frac{1}{2}x} - 1$

$\Rightarrow y = \left(3e^{\frac{1}{2}x} - 1\right)^2$

$\Rightarrow y = 9e^{x} - 6e^{\frac{1}{2}x} + 1$

Question 4 **(***)**

Solve the differential equation

$$\frac{dy}{dx} = (9x + 4y + 1)^2, \quad y(0) = -\frac{1}{4}.$$

Give the answer in the form $y = f(x)$.

$$\boxed{y = -\frac{1}{4} - \frac{9}{4}x + \frac{3}{8}\tan 6x}$$

$$\frac{dy}{dx} = (9x+4y+1)^2, \quad y(0) = -\frac{1}{4}$$

- LET $u = 9x + 4y + 1$
- $\frac{du}{dx} = 9 + 4\frac{dy}{dx}$

$\Rightarrow 4\frac{dy}{dx} = 4(9x+4y+1)^2$

$\Rightarrow 4\frac{dy}{dx} + 9 = 4(9x+4y+1)^2 + 9$

$\Rightarrow \frac{du}{dx} = 4u^2 + 9$

$\Rightarrow \frac{1}{4u^2+9}\,du = 1\,dx$

$\Rightarrow \frac{4}{4u^2+9}\,du = 4\,dx$

$\Rightarrow \frac{1}{u^2+\frac{9}{4}}\,du = 4\,dx$

$\Rightarrow \int \frac{1}{u^2+\left(\frac{3}{2}\right)^2}\,du = \int 4\,dx$

$\Rightarrow \frac{1}{\frac{3}{2}}\arctan\left(\frac{u}{3/2}\right) = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left(\frac{2}{3}u\right) = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left[\frac{2}{3}(9x+4y+1)\right] = 4x + A$

$\Rightarrow \frac{2}{3}\arctan\left[6x + \frac{8}{3}y + \frac{2}{3}\right] = 4x + A$

$\Rightarrow \arctan\left(6x + \frac{8}{3}y + \frac{2}{3}\right) = 6x + A$

$\Rightarrow 6x + \frac{8}{3}y + \frac{2}{3} = \tan(6x + A)$

$\Rightarrow \boxed{18x + 8y + 2 = 3\tan(6x + A)}$

when $x = 0$ $y = -\frac{1}{4}$

$0 - 2 + 2 = 3\tan A$

$\tan A = 0$

$A = 0$

$18x + 8y + 2 = 3\tan 6x$

$8y = -2 - 18x + 3\tan 6x$

$y = -\frac{1}{4} - \frac{9}{4}x + \frac{3}{8}\tan 6x$

Question 5 **(***)**

Use a suitable substitution to solve the differential equation

$$\frac{dy}{dx} = \frac{x + y}{4 - 3(x + y)}, \quad y(0) = 1.$$

$$\boxed{2\ln|x + y - 2| = 3 - x - 3y}$$

$$\frac{dy}{dx} = \frac{x+y}{4-3(x+y)}$$

$z = x + y \Leftrightarrow y = z - x$

$\frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\Rightarrow \frac{dz}{dx} - 1 = \frac{z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{z}{4-3z} + 1$

$\Rightarrow \frac{dz}{dx} = \frac{z+4-3z}{4-3z}$

$\Rightarrow \frac{dz}{dx} = \frac{4-2z}{4-3z}$

$\Rightarrow \int \frac{4-3z}{4-2z}\,dz = \int 1\,dx$

$\Rightarrow \int \frac{8-6z}{4-2z} = \int 2\,dx$

$\Rightarrow \int \frac{3(4-2z)-4}{4-2z}\,dz \int 2\,dx$

$\Rightarrow \int 3 - \frac{4}{4-2z}\,dz = \int 2\,dx$

$\Rightarrow 3z + 2\ln|2-z| = 2x + C$

$x = 0,\ y = 1 \Rightarrow z = 1$

$3 + \cancel{2\ln 1} = 0 + C$

$\boxed{C = 3}$

$\Rightarrow 3(x+y) + 2\ln|2-x-y| = 2x + 3$

$\Rightarrow -x - 3y + 3 = 2\ln|2-x-y|$

$\left(\text{OR } 3 - x - 3y = 2\ln|x+y-2|\right)$

Question 6 **(***)**

Use the substitution $y = e^z$ to solve the differential equation

$$x\frac{dy}{dx} + y\ln y = 2xy, \quad y(1) = e^2.$$

$$\boxed{y = e^{x+\frac{1}{x}}}$$

$x\frac{dy}{dx} + y\ln y = 2xy$

$y = e^z$

$\frac{dy}{dx} = e^z\frac{dz}{dx}$

$\Rightarrow x\left(\cancel{e^z}\frac{dz}{dx}\right) + \cancel{e^z}\left(\ln e^z\right) = 2x\cancel{e^z}$

$\Rightarrow x\frac{dz}{dx} + z = 2x$

BY INTEGRATING FACTOR OR NOTICING LHS IS EXACT

$\Rightarrow \frac{d}{dx}(xz) = 2x$

$\Rightarrow xz = \int 2x\, dx.$

$\Rightarrow xz = x^2 + A$

$\Rightarrow \boxed{x\ln y = x^2 + A}$

$(1, e^2) \Rightarrow 2 = 1 + A$

$\Rightarrow \boxed{A = 1}$

$\Rightarrow x\ln y = x^2 + 1$

$\Rightarrow \ln y = x + \frac{1}{x}$

$\Rightarrow y = e^{x+\frac{1}{x}}$

Question 7 **(***)**

Use the substitution $z = \sin y$ to solve the differential equation

$$x\frac{dy}{dx}\cos y - \sin y = x^2\ln x, \quad y(1) = 0$$

subject to the condition $y = 0$ at $x = 1$.

$$\boxed{\sin y = x^2\ln x - x^2 + x}$$

- $x\frac{dy}{dx}\cos y - \sin y = x^2\ln x$

$z = \sin y$

$\frac{dz}{dx} = \cos y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{\cos y}\frac{dz}{dx}$

$\frac{x}{\cos y}\frac{dz}{dx}\cos y - z = x^2\ln x$

$x\frac{dz}{dx} - z = x^2\ln x$

$\boxed{\frac{dz}{dx} - \frac{z}{x} = x\ln x}$

$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$\therefore \frac{d}{dx}\left(z \times \frac{1}{x}\right) = \ln x$

$\Rightarrow \frac{z}{x} = \int \ln x\, dx$

$\Rightarrow \frac{z}{x} = x\ln x - x + C$

STANDARD RESULT OR PARTS

$\Rightarrow \boxed{z = x^2\ln x - x^2 + Cx}$

$\Rightarrow \sin y = x^2\ln x - x^2 + Cx$

USING $(1,0)$

$0 = 0 - 1 + C$

$\boxed{C = 1}$

$\therefore \sin y = x^2\ln x - x^2 + x$

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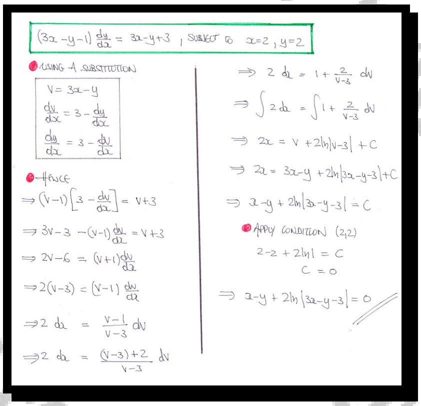
Question 8 (***+)

Use a suitable substitution to find the solution of the following differential equation.

$$(3x - y - 1)\frac{dy}{dx} = 3x - y + 3, \quad y(2) = 2.$$

$$\boxed{x - y + 2\ln|3x - y - 3| = 0}$$



$\boxed{(3x-y-1)\frac{dy}{dx} = 3x-y+3 \text{ , SUBJECT TO } x=2 \text{ , } y=2}$

USING A SUBSTITUTION

$$\boxed{\begin{gathered} v = 3x-y \\ \frac{dv}{dx} = 3 - \frac{dy}{dx} \\ \frac{dy}{dx} = 3 - \frac{dv}{dx} \end{gathered}}$$

HENCE

$\Rightarrow (v-1)\left[3 - \frac{dv}{dx}\right] = v+3$

$\Rightarrow 3v-3 - (v-1)\frac{dv}{dx} = v+3$

$\Rightarrow 2v-6 = (v+1)\frac{dv}{dx}$

$\Rightarrow 2(v-3) = (v-1)\frac{dv}{dx}$

$\Rightarrow 2\,dx = \frac{v-1}{v-3}\,dv$

$\Rightarrow 2\,dx = \frac{(v-3)+2}{v-3}\,dv$

$\Rightarrow 2\,dx = 1 + \frac{2}{v-3}\,dv$

$\Rightarrow \int 2\,dx = \int 1 + \frac{2}{v-3}\,dv$

$\Rightarrow 2x = v + 2\ln|v-3| + C$

$\Rightarrow 2x = 3x-y + 2\ln|3x-y-3| + C$

$\Rightarrow x-y + 2\ln|3x-y-3| = C$

APPLY CONDITION (2,2)

$2-2 + 2\ln 1 = C$

$C = 0$

$\Rightarrow x-y + 2\ln|3x-y-3| = 0$

Question 9 (***+)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+\sqrt{y+1}=y+1, \quad y>-1, \quad y(0)=3.$$

Given the answer in the form $y=f(x)$.

$\boxed{}$, $\boxed{y=e^{x}\pm 2e^{\frac{1}{2}x}}$

USE THE SUBSTITUTION $v=\sqrt{y+1}$

$v^2=y+1$

$2v\frac{dv}{dx}=\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}+\sqrt{y+1}=y+1$

$\Rightarrow 2v\frac{dv}{dx}+v=v^2$

$v\neq 0$

$\Rightarrow 2\frac{dv}{dx}+1=v$

$\Rightarrow 2\frac{dv}{dx}=v-1$

$\Rightarrow \frac{2}{v-1}\,dv=1\,dx$

INTEGRATE BOTH SIDES

$\Rightarrow 2\ln|v-1|=x+C$

$\Rightarrow \ln|v-1|=\frac{1}{2}x+D$

$\Rightarrow |v-1|=e^{\frac{1}{2}x+D}$

$\Rightarrow |v-1|=Ae^{\frac{1}{2}x}$

$\Rightarrow |+\sqrt{y+1}-1|=Ae^{\frac{1}{2}x}$ (ORIGINAL SUBSTITUTION)

APPLY CONDITION

$y(0)=3 \Rightarrow |2-1|=A$

$\Rightarrow A=1$

$\Rightarrow |\sqrt{y+1}-1|=e^{\frac{1}{2}x}$

GETTING THE RESULTS AND TIDY UP

$|\sqrt{y+1}-1|=e^{\frac{1}{2}x}$

$\sqrt{y+1}-1=e^{\frac{1}{2}x}$

$\sqrt{y+1}=e^{\frac{1}{2}x}+1$

$(\sqrt{y+1})^2=e^{x}+2e^{\frac{1}{2}x}+1$

$y+1=e^{x}+2e^{\frac{1}{2}x}+1$

$y=e^{x}+2e^{\frac{1}{2}x}$

$-\sqrt{y+1}+1=e^{\frac{1}{2}x}$

$\sqrt{y+1}-1=-e^{\frac{1}{2}x}$

$\sqrt{y+1}=1-e^{\frac{1}{2}x}$

$y+1=1-2e^{\frac{1}{2}x}+e^{x}$

$y=e^{x}-2e^{\frac{1}{2}x}$

$\therefore y=e^{x}\pm 2e^{\frac{1}{2}x}$

Question 10 (***+)

a) By using the substitution $z = x^2 + y^2$, solve the following differential equation

$$2xy\frac{dy}{dx} + y^2 = 2x - 3x^2,$$

subject to the condition $y = 1$ at $x = 1$.

b) Verify the answer to part **(a)** by using the substitution $z = y^2$ to solve the same differential equation and subject to the same condition.

$\boxed{}$, $\boxed{y^2 = x - x^2 + \frac{1}{x}}$

a) USING THE SUBSTITUTION GIVEN

$\Rightarrow z = x^2 + y^2$

$\Rightarrow \frac{dz}{dx} = 2x + 2y\frac{dy}{dx}$

$\Rightarrow 2y\frac{dy}{dx} = \frac{dz}{dx} - 2x$

$\Rightarrow 2xy\frac{dy}{dx} = x\frac{dz}{dx} - 2x^2$

SUBSTITUTE INTO THE O.D.E

$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2 \qquad [x=1,\ y=1]$

$\Rightarrow \left[x\frac{dz}{dx} - 2x^2\right] + y^2 = 2x - 3x^2 \qquad [x=1,\ z=2]$

$\Rightarrow x\frac{dz}{dx} - 2x^2 + (z - x^2) = 2x - 3x^2$

$\Rightarrow x\frac{dz}{dx} \cancel{-3x^2} + z = 2x \cancel{-3x^2}$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2$

INTEGRATING FACTOR NEXT (IN FACT THE ODE WAS EXACT)

$e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

THUS WE FINALLY HAVE

$\Rightarrow \frac{d}{dx}(zx) = 2x$

$\Rightarrow \left[zx\right]_{(1,2)}^{(x,z)} = \left[x^2\right]_1^x$

$\Rightarrow zx - 2 = x^2 - 1$

$\Rightarrow z = x^2 + 1$

$\Rightarrow z = \frac{x^2+1}{x}$

$\Rightarrow x^2 + y^2 = x + \frac{1}{x}$

$\Rightarrow y^2 = x + \frac{1}{x} - x^2$

b) REWRITE THE O.D.E AS

$\Rightarrow 2xy\frac{dy}{dx} + y^2 = 2x - 3x^2$

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \frac{2x - 3x^2}{2xy}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1} \leftarrow n = -1$

THIS IS A BERNOULLI TYPE, SO WE USE THE SUBSTITUTION

$z = \frac{1}{y^{n-1}}$ HERE $z = \frac{1}{y^{-2}}$

- $z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

RETURNING TO THE O.D.E

$\Rightarrow \frac{dy}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$

$\Rightarrow \frac{1}{2y}\frac{dz}{dx} + \frac{y}{2x} = \left(1 - \frac{3}{2}x\right)y^{-1}$

$\Rightarrow \frac{dz}{dx} + \frac{y^2}{x} = 2\left(1 - \frac{3}{2}x\right)$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x} = 2 - 3x$

MULTIPLY THROUGH BY x — OR INTEGRATING FACTOR

$\Rightarrow x\frac{dz}{dx} + z = 2x - 3x^2 \qquad [x=1,\ y=1,\ z=1]$

$\Rightarrow \frac{d}{dx}(xz) = 2x - 3x^2$

$\Rightarrow \left[xz\right]_{(1,1)}^{(x,z)} = \int_1^x 2x - 3x^2\ dx$

$\Rightarrow xz - 1 = \left[x^2 - x^3\right]_1^x$

$\Rightarrow xz - 1 = (x^2 - x^3) - \cancel{(1-1)}$

$\Rightarrow xz = x^2 + 1 - x^3$

$\Rightarrow z = x + \frac{1}{x} - x^2$

$\Rightarrow y^2 = x + \frac{1}{x} - x^2$

AS BEFORE

Question 11 (***+)

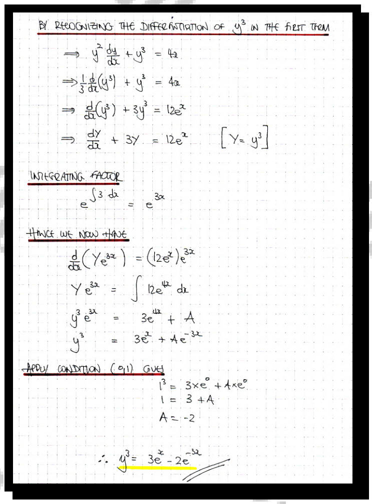
A curve with equation $y = f(x)$ passes through the point with coordinates $(0,1)$ and satisfies the differential equation

$$y^2 \frac{dy}{dx} + y^3 = 4e^x .$$

By finding a suitable integrating factor, solve the differential equation to show that

$$y^3 = 3e^x - 2e^{-3x} .$$

$\boxed{}$, $\boxed{\text{proof}}$


BY RECOGNIZING THE DIFFERENTIATION OF y^3 IN THE FIRST TERM

$$\Rightarrow y^2 \frac{dy}{dx} + y^3 = 4e^x$$
$$\Rightarrow \frac{1}{3}\frac{d}{dx}(y^3) + y^3 = 4e^x$$
$$\Rightarrow \frac{d}{dx}(y^3) + 3y^3 = 12e^x$$
$$\Rightarrow \frac{dY}{dx} + 3Y = 12e^x \qquad [Y = y^3]$$

INTEGRATING FACTOR

$$e^{\int 3\,dx} = e^{3x}$$

HENCE WE NOW HAVE

$$\frac{d}{dx}(Ye^{3x}) = (12e^x)e^{3x}$$
$$Ye^{3x} = \int 12e^{4x}\,dx$$
$$y^3e^{3x} = 3e^{4x} + A$$
$$y^3 = 3e^x + Ae^{-3x}$$

APPLY CONDITION $(0,1)$ GIVES

$$1^3 = 3 \times e^0 + A \times e^0$$
$$1 = 3 + A$$
$$A = -2$$

$$\therefore y^3 = 3e^x - 2e^{-3x}$$

Question 12 (***+)

A curve with equation $y = f(x)$ passes through the origin and satisfies the differential equation

$$2y\left(1+x^2\right)\frac{dy}{dx} + xy^2 = \left(1+x^2\right)^{\frac{3}{2}}.$$

By finding a suitable integrating factor, or otherwise, show that

$$y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}.$$

$\boxed{\text{SP}}$, $\boxed{\text{proof}}$

$$2y(1+x^2)\frac{dy}{dx} + xy^2 = (1+x^2)^{\frac{3}{2}}$$

$$\Rightarrow 2y\frac{dy}{dx} + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx}(y^2) + \frac{x}{1+x^2}y^2 = (1+x^2)^{\frac{1}{2}}$$

$$\text{I.F} = e^{\int \frac{x}{1+x^2}\,dx} = e^{\frac{1}{2}\ln(1+x^2)} = (1+x^2)^{\frac{1}{2}}$$

$$\Rightarrow \frac{d}{dx}\left(y^2(1+x^2)^{\frac{1}{2}}\right) = 1+x^2$$

$$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = \int 1+x^2\,dx$$

$$\Rightarrow y^2(1+x^2)^{\frac{1}{2}} = x + \frac{1}{3}x^3 + C$$

$$\Rightarrow y^2 = \frac{x+\frac{1}{3}x^3+C}{(1+x^2)^{1/2}}$$

$$\Rightarrow y^2 = \frac{3x+x^3+A}{3(1+x^2)^{1/2}}$$

$$\text{Now } (0,0) \Rightarrow A=0$$

$$\Rightarrow y^2 = \frac{x^3+3x}{3\sqrt{x^2+1}}$$

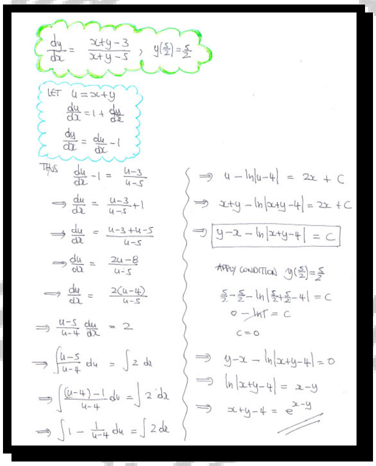
Question 13 (****)

Solve the differential equation

$$\frac{dy}{dx} = \frac{x+y-3}{x+y-5},$$

subject to the condition $y = \frac{5}{2}$ at $x = \frac{5}{2}$.

$$\boxed{x + y - 4 = e^{x-y}}$$



$\frac{dy}{dx} = \frac{x+y-3}{x+y-5}$, $y\left(\frac{5}{2}\right) = \frac{5}{2}$

LET $u = x+y$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

THUS $\frac{du}{dx} - 1 = \frac{u-3}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{u-3}{u-5} + 1$

$\Rightarrow \frac{du}{dx} = \frac{u-3+u-5}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{2u-8}{u-5}$

$\Rightarrow \frac{du}{dx} = \frac{2(u-4)}{u-5}$

$\Rightarrow \frac{u-5}{u-4}\frac{du}{dx} = 2$

$\Rightarrow \int \frac{u-5}{u-4}\, du = \int 2\, dx$

$\Rightarrow \int \frac{(u-4)-1}{u-4}\, du = \int 2\, dx$

$\Rightarrow \int 1 - \frac{1}{u-4}\, du = \int 2\, dx$

$\Rightarrow u - \ln|u-4| = 2x + C$

$\Rightarrow x+y - \ln|x+y-4| = 2x + C$

$\Rightarrow \boxed{y - x - \ln|x+y-4| = C}$

APPLY CONDITION $y\left(\frac{5}{2}\right) = \frac{5}{2}$

$\frac{5}{2} - \frac{5}{2} - \ln\left|\frac{5}{2} + \frac{5}{2} - 4\right| = C$

$0 - \cancel{\ln 1} = C$

$C = 0$

$\Rightarrow y - x - \ln|x+y-4| = 0$

$\Rightarrow \ln|x+y-4| = x - y$

$\Rightarrow x + y - 4 = e^{x-y}$

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Question 14 (****)

Find a general solution of the following differential equation

$$y\frac{dy}{dx}+x=2y.$$

$$\boxed{(x-y)e^{\frac{y}{x-y}}=C}$$

- $y\frac{dy}{dx}+x=2y$

$\Rightarrow \frac{dy}{dx}+\frac{x}{y}=2$

LET $v=\frac{x}{y} \Rightarrow yv=x$

DIFF WITH RESPECT TO x

$\Rightarrow \frac{dy}{dx}v+y\frac{dv}{dx}=1$

- MULTIPLY THE O.D.E BY v BEFORE SUBSTITUTING IN

$\Rightarrow v\frac{dy}{dx}+\frac{x}{y}v=2v$

$\Rightarrow \left(1-y\frac{dv}{dx}\right)+v^2=2v$

$\Rightarrow 1-\frac{x}{v}\frac{dv}{dx}+v^2=2v$

$\Rightarrow v^2-2v+1=\frac{x}{v}\frac{dv}{dx}$

$\Rightarrow (v-1)^2=\frac{x}{v}\frac{dv}{dx}$

$\Rightarrow v(v-1)^2=x\frac{dv}{dx}$

$\Rightarrow \frac{1}{x}\,dx=\frac{1}{v(v-1)^2}\,dv$

- EXPRESS INTO PARTIAL FRACTIONS

$\Rightarrow \frac{1}{v(v-1)^2}\equiv\frac{A}{v}+\frac{B}{(v-1)^2}+\frac{C}{v-1}$

$\Rightarrow 1\equiv A(v-1)^2+Bv+Cv(v-1)$

$\Rightarrow 1\equiv Av^2-2Av+A+Bv+Cv^2-Cv$

$\Rightarrow 1\equiv(A+C)v^2+(B-C-2A)v+A$

COMPARING COEFFICIENTS

- $\boxed{A=1}$
- $A+C=0$, $\boxed{C=-1}$
- $B-C-2A=0$, $B+1-2=0$, $\boxed{B=1}$

- RETURNING TO THE O.D.E.

$\Rightarrow \int\frac{1}{x}\,dx=\int\frac{1}{v}+\frac{1}{(v-1)^2}-\frac{1}{v-1}\,dv$

$\Rightarrow \ln|x|=\ln|v|-\frac{1}{v-1}-\ln|v-1|+C$

$\Rightarrow \ln|x|=\ln\left|\frac{v}{v-1}\right|-\frac{1}{v-1}+C$

$\Rightarrow \ln|x|=\ln\left|\frac{\frac{x}{y}}{\frac{x}{y}-1}\right|-\frac{1}{\frac{x}{y}-1}+C$

$\Rightarrow \ln|x|=\ln\left|\frac{x}{x-y}\right|-\frac{y}{x-y}+C$

$\Rightarrow \frac{y}{x-y}+C=\ln\left|\frac{x}{x-y}\right|-\ln x$

$\Rightarrow \frac{y}{x-y}+C=\ln\left|\frac{x}{x-y}\right|+\ln\left|\frac{1}{x}\right|$

$\Rightarrow \frac{y}{x-y}+C=\ln\left|\frac{1}{x-y}\right|$

$\Rightarrow \frac{1}{x-y}=e^{C+\frac{y}{x-y}}$

$\Rightarrow \frac{1}{x-y}=Ae^{\frac{y}{x-y}}$

$\Rightarrow (x-y)e^{\frac{y}{x-y}}=C$

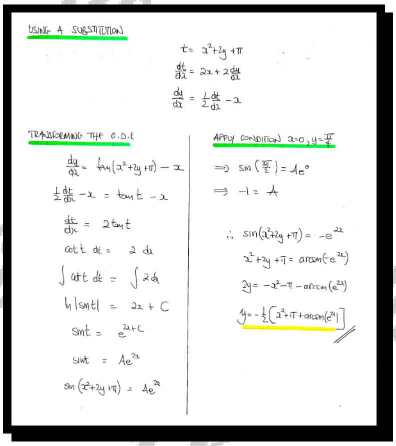
Question 15 (****)

$$\frac{dy}{dx} = \tan\left(x^2 + 2y + \pi\right) - x, \quad y(0) = \frac{1}{4}\pi \ .$$

Solve the above differential equation to show that

$$y = -\frac{1}{2}\left[x^2 + \pi + \arcsin\left(e^{2x}\right)\right].$$

$\boxed{}$, $\boxed{\text{proof}}$



USING A SUBSTITUTION

$$t = x^2 + 2y + \pi$$
$$\frac{dt}{dx} = 2x + 2\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx} - x$$

TRANSFORMING THE O.D.E

$$\frac{dy}{dx} = \tan(x^2+2y+\pi) - x$$
$$\frac{1}{2}\frac{dt}{dx} - x = \tan t - x$$
$$\frac{dt}{dx} = 2\tan t$$
$$\cot t \ dt = 2 \ dx$$
$$\int \cot t \ dt = \int 2 \ dx$$
$$\ln|\sin t| = 2x + C$$
$$\sin t = e^{2x+C}$$
$$\sin t = Ae^{2x}$$
$$\sin(x^2+2y+\pi) = Ae^{2x}$$

APPLY CONDITION $x=0$, $y=\frac{\pi}{4}$

$$\Rightarrow \sin\left(\frac{3\pi}{2}\right) = Ae^0$$
$$\Rightarrow -1 = A$$

$$\therefore \sin(x^2+2y+\pi) = -e^{2x}$$
$$x^2+2y+\pi = \arcsin(-e^{2x})$$
$$2y = -x^2 - \pi - \arcsin(e^{2x})$$
$$y = -\frac{1}{2}\left[x^2 + \pi + \arcsin(e^{2x})\right]$$

Question 16 (****)

$$\frac{dy}{dx}\left(x+y^2\right)=y.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$ by considering $\frac{dx}{dy}$, followed by a suitable substitution..

b) Verify the validity of the answer obtained in part (**a**).

$$\boxed{y^2=x}$$

a) $\boxed{\frac{dy}{dx}(x+y^2)=y}$

$\Rightarrow \frac{dy}{dx}=\frac{y}{x+y^2}$

$\Rightarrow \frac{dx}{dy}=\frac{x+y^2}{y}$

$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+y$

LET $z=\frac{x}{y}$

$x=zy$

DIFF W.R.T y

$\frac{dx}{dy}=\frac{dz}{dy}y+z$

$\Rightarrow \cancel{z}+y\frac{dz}{dy}=\cancel{z}+y$

$\Rightarrow y\frac{dz}{dy}=y$

$\Rightarrow \frac{dz}{dy}=1$

$\Rightarrow dz=dy$

$\Rightarrow z=y+C$

$\Rightarrow \boxed{\frac{x}{y}=y+C}$

APPLY CONDITIONS $(1,1)$ $\therefore \boxed{C=0}$

$\Rightarrow \frac{x}{y}=y$

$\Rightarrow y^2=x$

b) $y^2=x$

Diff w.r.t. x

$2y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{2y}$

MULTIPLY BY $(x+y^2)$

$(x+y^2)\frac{dy}{dx}=\frac{1}{2y}(x+y^2)$

$(x+y^2)\frac{dy}{dx}=\frac{x+y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=\frac{y^2+y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=\frac{2y^2}{2y}$

$(x+y^2)\frac{dy}{dx}=y$

Question 17 (****)

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}, \quad y(0) = 0.$$

Show that the solution of the above differential equation is

$$y - x - 2\ln(x+y+1) = 0.$$

proof

$$\frac{dy}{dx} = \frac{x+y+3}{x+y-1}$$

$$z = x+y \Rightarrow y = z-x$$

$$\frac{dz}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

$$\Rightarrow \frac{dz}{dx} - 1 = \frac{z+3}{z-1}$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+3}{z-1} + 1$$

$$\Rightarrow \frac{dz}{dx} = \frac{z+3+z-1}{z-1}$$

$$\Rightarrow \frac{dz}{dx} = \frac{2z+2}{z-1}$$

$$\Rightarrow \frac{z-1}{2(z+1)}\,dz = 1\,dx$$

$$\Rightarrow \int \frac{z-1}{z+1}\,dz = \int 2\,dx$$

$$\Rightarrow \int \frac{(z+1)-2}{(z+1)}\,dz = \int 2\,dx$$

$$\Rightarrow \int 1 - \frac{2}{z+1}\,dx = \int 2\,dx$$

$$\Rightarrow z - 2\ln|z+1| = 2x + C$$

$$\Rightarrow \boxed{x+y-2\ln|x+y+1| = 2x + C}$$

USING $(0,0) \Rightarrow 0+0-\cancel{2\ln 1} = 0 + C$

$$\Rightarrow \boxed{C=0}$$

$$\Rightarrow x+y-2\ln|x+y+1| = 2x$$

$$\Rightarrow y-x-2\ln|x+y+1| = 0$$

AS REQUIRED

Question 18 (****)

Given that $v = yx^{-2}$ find a general solution for the following differential equation.

$$\frac{dy}{dx} - \frac{2y}{x} = \log_v \mathrm{e}, \quad u > 0, \quad u \neq 1.$$

Given the answer in the form $f(x, y) = \text{constant}$.

$\boxed{}$, $\boxed{\frac{1}{x} - \frac{y}{x^2}\left[1 - \ln\left(yx^{-2}\right)\right] = \text{constant}}$

USING THE DEFINITION OF V AS A SUBSTITUTION

$$V = yx^{-2} = \frac{y}{x^2} \implies \frac{dv}{dx} = \frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}(y)$$
$$\frac{1}{x^2}\frac{dy}{dx} = \frac{dv}{dx} + \frac{2y}{x^3}$$

RETURNING TO THE O.D.E.

$$\implies \frac{dy}{dx} - \frac{2y}{x} = \log_v e$$
$$\implies \frac{1}{x^2}\frac{dy}{dx} - \frac{1}{x^2}\left(\frac{2y}{x}\right) = \frac{1}{x^2}\log_v e$$
$$\implies \frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x^2}\log_v e$$
$$\implies \left[\frac{dv}{dx} + \frac{2y}{x^3}\right] - \frac{2y}{x^3} = \frac{1}{x^2}\left[\frac{\log_e e}{\log_e v}\right]$$
$$\implies \frac{dv}{dx} + \cancel{\frac{2y}{x^3}} - \cancel{\frac{2y}{x^3}} = \frac{1}{x^2} \times \frac{1}{\ln v}$$
$$\implies \frac{dv}{dx} = \frac{1}{x^2} \times \frac{1}{\ln v}$$

SEPARATE VARIABLES

$$\implies \ln v \, dv = \frac{1}{x^2}\,dx$$
$$\implies \int \ln v \, dv = \int \frac{1}{x^2}\,dx$$

QUOTING THE INTEGRAL OF $\ln v$, OTHERWISE INTEGRATION BY PARTS

$$\implies v\ln v - v = -\frac{1}{x} + C$$
$$\implies \frac{y}{x^2}\ln\frac{y}{x^2} - \frac{y}{x^2} = -\frac{1}{x} + C$$

$$\therefore \frac{1}{x} \quad \frac{y}{x^2}\left[1 - \ln\left(\frac{y}{x^2}\right)\right] = C$$

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Question 19 (****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx}+8xy=y^2+16x^2\,,\quad y(0)=-6\,.$$

Given the answer in the form $y=f(x)$.

$$\boxed{}\,,\quad \boxed{y=\frac{4x\left(2e^{4x}-1\right)-2\left(2e^{4x}+1\right)}{2e^{4x}-1}}$$

REWRITE THE O.D.E

$$\Rightarrow \frac{dy}{dx}+8xy=y^2+16x^2$$

$$\Rightarrow \frac{dy}{dx}=y^2-8xy+16x^2$$

$$\Rightarrow \frac{dy}{dx}=(y-4x)^2$$

NOW A SUBSTITUTION

$$\Rightarrow v=y-4x$$

$$\Rightarrow \frac{dv}{dx}=\frac{dy}{dx}-4$$

$$\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}+4$$

TRANSFORM THE O.D.E

$$\Rightarrow \frac{dv}{dx}+4=v^2$$

$$\Rightarrow \frac{dv}{dx}=v^2-4$$

$$\Rightarrow \frac{dv}{dx}=(v-2)(v+2)$$

SEPARATE VARIABLES

$$\Rightarrow \frac{1}{(v-2)(v+2)}\,dv=1\,dx$$

$$\Rightarrow \int\frac{1}{(v-2)(v+2)}\,dv=\int 1\,dx$$

PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow \int \frac{\frac{1}{4}}{v-2}-\frac{\frac{1}{4}}{v+2}\,dv=\int 1\,dx$$

$$\Rightarrow \int \frac{1}{v-2}-\frac{1}{v+2}\,dv=\int 4\,dx$$

$$\Rightarrow \ln|v-2|-\ln|v+2|=4x+C$$

$$\Rightarrow \ln\left|\frac{v-2}{v+2}\right|=4x+C$$

$$\Rightarrow \frac{v-2}{v+2}=Ae^{4x}$$

$$\Rightarrow \frac{y-4x-2}{y-4x+2}=Ae^{4x}$$

APPLY BOUNDARY CONDITION

$$(0,-6)\Rightarrow \frac{-6-2}{-6+2}=A$$

$$\Rightarrow A=\frac{-8}{-4}=2$$

FINALLY MAKE y THE SUBJECT

$$\Rightarrow y-4x-2=2e^{4x}(y-4x+2)$$

$$\Rightarrow y-4x-2=2ye^{4x}-8xe^{4x}+4e^{4x}$$

$$\Rightarrow 8xe^{4x}-4x-4e^{4x}-2=2ye^{4x}-y$$

$$\Rightarrow 4x(2e^{4x}-1)-2(2e^{4x}+1)=y(2e^{4x}-1)$$

$$\Rightarrow y=\frac{4x(2e^{4x}-1)-2(2e^{4x}+1)}{2e^{4x}-1}$$

Question 20 (****)

Sketch the curve which passes through the point with coordinates $(1,2)$ and satisfies

$$\frac{1}{2}\frac{dy}{dx}+\frac{x}{3y^2}=\frac{\sqrt{x^2+y^3}}{y^2}.$$

$\boxed{}$, $\boxed{\text{graph}}$

ESSENTIALLY WE NEED TO SOLVE THE O.D.E. – TRY A SUBSTITUTION

$\Rightarrow v = x^2+y^3$

$\Rightarrow \frac{dv}{dx} = 2x + 3y^2\frac{dy}{dx}$

$\Rightarrow 3y^2\frac{dy}{dx} = \frac{dv}{dx} - 2$

TRANSFORM THE O.D.E.

$\Rightarrow \frac{1}{2}\frac{dy}{dx} + \frac{x}{3y^2} = \frac{\sqrt{x^2+y^3}}{y^2}$ $\times 3y^2$

$\Rightarrow \frac{3}{2}y^2\frac{dy}{dx} + x = 3\sqrt{x^2+y^3}$ $\times 2$

$\Rightarrow 3y^2\frac{dy}{dx} + 2x = 6\sqrt{x^2+y^3}$

$\Rightarrow \left[\frac{dv}{dx} - 2x\right] + 2x = 6v^{\frac{1}{2}}$

$\Rightarrow \frac{dv}{dx} = 6v^{\frac{1}{2}}$

$\Rightarrow \frac{1}{v^{\frac{1}{2}}}dv = 6\,dx$

$\Rightarrow \int v^{-\frac{1}{2}}\,dv = \int 6\,dx$

INTEGRATE SUBJECT TO THE CONDITION $x=1, y=2 \Rightarrow v=9$

$\Rightarrow \left[2v^{\frac{1}{2}}\right]_{v=9}^{v} = \left[6x\right]_{x=1}^{x}$

$\Rightarrow 2v^{\frac{1}{2}} - 2\times 3 = 6x - 6$

$\Rightarrow 2v^{\frac{1}{2}} = 6x$

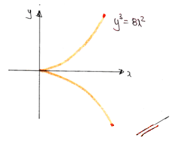
$\Rightarrow v^{\frac{1}{2}} = 3x$

$\Rightarrow v = 9x^2$

$\Rightarrow x^2+y^3 = 9x^2$

$\Rightarrow y^3 = 8x^2$

IT A SEMICUBICAL PARABOLA



Question 21 (****)

Use a suitable substitution to solve the following differential equation.

$$\frac{dy}{dx} = (x-y+2)^2, \quad y(0) = 4.$$

Given the answer in the form $y = f(x)$.

$$\boxed{}, \quad \boxed{y = \frac{(x+1)e^{2x} \pm 3(x+3)}{e^{2x} \pm 3} = \frac{\pm(x+1)e^{2x} - 3(x+3)}{\pm e^{2x} - 3}}$$

START WITH THE OBVIOUS SUBSTITUTION

$$t = x - y + 2$$
$$\frac{dt}{dx} = 1 - \frac{dy}{dx}$$
$$\frac{dy}{dx} = 1 - \frac{dt}{dx}$$

TRANSFORM THE O.D.E

$$\Rightarrow \frac{dy}{dx} = (x-y+2)^2$$
$$\Rightarrow 1 - \frac{dt}{dx} = t^2$$
$$\Rightarrow 1 - t^2 = \frac{dt}{dx}$$

SEPARATE VARIABLES

$$\Rightarrow 1\,dx = \frac{1}{1-t^2}\,dt$$
$$\Rightarrow 1\,dx = \frac{1}{(1+t)(1-t)}\,dt$$

PARTIAL FRACTIONS BY INSPECTION

$$\Rightarrow 1\,dx = \left[\frac{\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{1-t}\right]dt$$
$$\Rightarrow 2\,dx = \left(\frac{1}{1+t} + \frac{1}{1-t}\right)dt$$
$$\Rightarrow \int 2\,dx = \int \frac{1}{1+t} + \frac{1}{1-t}\,dt$$
$$\Rightarrow 2x + C = \ln|1+t| - \ln|1-t|$$
$$\Rightarrow \ln\left|\frac{1+t}{1-t}\right| = 2x + C$$
$$\Rightarrow \left|\frac{1+t}{1-t}\right| = Ae^{2x}$$
$$\Rightarrow \left|\frac{1+x-y+2}{1-x+y-2}\right| = Ae^{2x}$$
$$\Rightarrow \left|\frac{x-y+3}{-x+y-1}\right| = Ae^{2x}$$
$$\Rightarrow \left|\frac{x-y+3}{x-y+1}\right| = Ae^{2x}.$$

APPLY CONDITION $y(0) = 4$

$$\left|\frac{0-4+3}{0-4+1}\right| = A$$
$$A = \left|\frac{-1}{-3}\right|$$
$$A = \frac{1}{3}$$

HENCE IN THE ABSENCE OF ANY OTHER CONSTRAINTS, THERE ARE 2 CASES

$$\frac{x-y+3}{x-y+1} = \pm\frac{1}{3}e^{2x}$$

CONSIDER EACH CASE SEPARATELY & LET $\pm\frac{1}{3}e^{2x} = E$

$$\Rightarrow \frac{x-y+3}{x-y+1} = E$$
$$\Rightarrow x - y + 3 = Ex - Ey + E$$
$$\Rightarrow Ey - y = Ex - x + E - 3$$
$$\Rightarrow y[E-1] = Ex + E - (x+3)$$
$$\Rightarrow y = \frac{E(x+1) - (x+3)}{E-1}$$
$$\Rightarrow y = \frac{\pm\frac{1}{3}e^{2x}(x+1) - (x+3)}{\pm\frac{1}{3}e^{2x} - 1}$$
$$\Rightarrow y = \frac{\pm(x+1)e^{2x} - 3(x+3)}{\pm e^{2x} - 3}$$

IN OTHER WORDS WE HAVE

$$y = \frac{(x+1)e^{2x} - 3(x+3)}{e^{2x} - 3} \quad \text{OR} \quad y = \frac{-(x+1)e^{2x} - 3(x+3)}{-e^{2x} - 3}$$
$$y = \frac{(x+1)e^{2x} + 3(x+3)}{e^{2x} + 3}$$

Question 22 (****+)

$$\frac{d^2y}{dx^2} + \frac{xy+4}{x^2} = y^2\,, \ x \neq 0\,.$$

Find a general solution for the above differential equation.

$$\boxed{y = \frac{2+2Ax^4}{x - Ax^5}}$$

$$\frac{dy}{dx} + \frac{xy+4}{x^2} = y^2\,,\ x \neq 0$$

- TIDY THE O.D.E LOOKING FOR CLUES

$$\frac{dy}{dx} = y^2 - \frac{xy+4}{x^2}$$

$$\frac{dy}{dx} = \frac{y^2x^2 - xy - 4}{x^2}$$

- WE TRY THE SUBSTITUTION $v = xy$ — DIFFERENTIATE W.R.T

$$\frac{dv}{dx} = 1 \times y + x\frac{dy}{dx}$$
$$\frac{dv}{dx} = \frac{v}{x} + x\frac{dy}{dx}$$
$$\frac{dv}{dx} - \frac{v}{x} = x\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}$$

- SUBSTITUTE INTO THE O.D.E

$$\Rightarrow \left[\frac{1}{x}\frac{dv}{dx} - \frac{v}{x^2}\right] = \frac{v^2 - v - 4}{x^2}$$

$$\Rightarrow x\frac{dv}{dx} - v = v^2 - v - 4$$

$$\Rightarrow x\frac{dv}{dx} = v^2 - 4$$

$$\Rightarrow \frac{1}{v^2-4}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{(v-2)(v+2)}\,dv = \frac{1}{x}\,dx$$

- PARTIAL FRACTIONS BY INSPECTIONS (COVER UP)

$$\Rightarrow \frac{\frac{1}{4}}{v-2} - \frac{\frac{1}{4}}{v+2}\,dv = \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{v-2} - \frac{1}{v+2}\,dv = \frac{4}{x}\,dx$$

- INTEGRATING BOTH SIDES YIELDS

$$\Rightarrow \ln|v-2| - \ln|v+2| = 4\ln x + \ln A$$

$$\Rightarrow \ln\left|\frac{v-2}{v+2}\right| = \ln Ax^4$$

$$\Rightarrow \frac{v-2}{v+2} = Ax^4$$

$$\Rightarrow \frac{xy-2}{xy+2} = Ax^4$$

$$\Rightarrow xy - 2 = Ax^5y + 2Ax^4$$

$$\Rightarrow xy - Ax^5y = 2 + 2Ax^4$$

$$\Rightarrow y(x - Ax^5) = 2 + 2Ax^4$$

$$\Rightarrow y = \frac{2+2Ax^4}{x-Ax^5}$$

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Question 23 (****+)

$$\frac{dy}{dx} = \frac{3x - y + 1}{x + y + 1}, \quad y(1) = 2.$$

Solve the differential equation to show that

$$(y - x)(y + 3x + 2) = 7.$$

proof

$$\frac{dy}{dx} = \frac{3x-y+1}{x+y+1}, \quad y(1)=2$$

- FIRST TRY TO MAKE THE R.H.S HOMOGENEOUS BY "TRANSLATING" THE ORIGIN

$$\left.\begin{array}{l} 3x-y+1=0 \\ x+y+1=0 \end{array}\right\} \Rightarrow 4x+2=0 \Rightarrow \boxed{x=-\tfrac{1}{2}} \Rightarrow \boxed{y=-\tfrac{1}{2}}$$

- THUS, PLACE THE ORIGIN AT $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

$$\boxed{\begin{array}{l} x = X-\frac{1}{2} \\ y = Y-\frac{1}{2} \end{array}} \qquad \begin{array}{l} dx = dX \\ dy = dY \end{array} \Rightarrow \boxed{\frac{dy}{dx} = \frac{dY}{dX}}$$

$$\therefore \frac{dY}{dX} = \frac{3\left(X-\frac{1}{2}\right)-\left(Y-\frac{1}{2}\right)+1}{\left(X-\frac{1}{2}\right)+\left(Y-\frac{1}{2}\right)+1} = \frac{3X-\frac{3}{2}-Y+\frac{1}{2}+1}{X-\frac{1}{2}+Y-\frac{1}{2}+1}$$

$$\boxed{\frac{dY}{dX} = \frac{3X-Y}{X+Y}}$$

- BY SUBSTITUTION

$$Y = XV(X)$$

$$\frac{dY}{dX} = 1 \times V + X\frac{dV}{dX}$$

HENCE $V + X\dfrac{dV}{dX} = \dfrac{3X-XV}{X+XV}$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{3-V}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{3-V}{1+V} - V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{3-V-V-V^2}{1+V}$$

$$\Rightarrow X\frac{dV}{dX} = -\frac{V^2+2V-3}{V+1}$$

$$\Rightarrow \frac{V+1}{V^2+2V-3}\,dV = -\frac{1}{X}\,dX$$

- BY PARTIAL FRACTIONS OR NOTING THAT $\int \frac{2f'(V)}{f(V)}\,dV$

$$\Rightarrow \int \frac{2V+2}{V^2+2V-3}\,dV = \int -\frac{2}{X}\,dX$$

$$\Rightarrow \ln|V^2+2V-3| = \ln A - 2\ln X$$

$$\Rightarrow \ln|V^2+2V-3| = \ln\left|\frac{A}{X^2}\right|$$

$$\Rightarrow V^2+2V-3 = \frac{A}{X^2}$$

$$\Rightarrow (V+3)(V-1) = \frac{A}{X^2}$$

$$\Rightarrow \left(\frac{Y}{X}+3\right)\left(\frac{Y}{X}-1\right) = \frac{A}{X^2}$$

$$\Rightarrow \frac{Y+3X}{X} \cdot \frac{Y-X}{X} = \frac{A}{X^2}$$

$$\Rightarrow (Y+3X)(Y-X) = A$$

$$\Rightarrow \left[\left(y+\tfrac{1}{2}\right)+3\left(x+\tfrac{1}{2}\right)\right]\left[\left(y+\tfrac{1}{2}\right)-\left(x+\tfrac{1}{2}\right)\right] = A$$

$$\Rightarrow (y+3x+2)(y-x) = A$$

- APPLY CONDITION $x=1$ $y=2$

$$(2+3+2)(2-1) = A$$

$$A = 7$$

$$\therefore (y+3x+2)(y-x) = 7$$

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Question 24 (****+)

$$\frac{dy}{dx} = \frac{2x+5y+3}{4x+y-3}, \quad y(1)=1.$$

Solve the differential equation to show that

$$(y-2x+3)^2 = 2(x+y).$$

$\boxed{\text{proof}}$

$$\frac{dy}{dx} = \frac{2x+5y+3}{4x+y-3}, \quad y(1)=1$$

- FIRSTLY WE SHALL TRY TO MAKE THE RHS HOMOGENEOUS

$$\left.\begin{array}{l} \bullet\ 2x+5y+3=0 \quad (\times 2) \\ \bullet\ 4x+y-3=0 \quad (\times 1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4x+10y+6=0 \\ 4x+y-3=0 \end{array}\right\} \Rightarrow 9y+9=0$$

$$\Rightarrow \boxed{y=-1}$$

$$\Rightarrow \boxed{x=1}$$

- THUS "TRANSFER" THE ORIGIN AT $(1,-1)$

$$\left.\begin{array}{l} x = X+1 \\ y = Y-1 \end{array}\right\} \Rightarrow \begin{array}{l} dx = dX \\ dy = dY \end{array} \Rightarrow \frac{dy}{dx} = \frac{dY}{dX}$$

- $$\frac{dY}{dX} = \frac{2(X+1)+5(Y-1)+3}{4(X+1)+(Y-1)-3} = \frac{2X+2+5Y-5+3}{4X+4+Y-1-3}$$

$$\frac{dY}{dX} = \frac{2X+5Y}{4X+Y}$$

- NOW THIS IS A HOMOGENEOUS EQUATION
USE THE SUBSTITUTION $Y = XV(X)$

$$\frac{dY}{dX} = 1 \times V + X\frac{dV}{dX}$$

- $$V + X\frac{dV}{dX} = \frac{2X+5XV}{4X+XV}$$

$$\Rightarrow V + X\frac{dV}{dX} = \frac{2+5V}{4+V}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{5V+2}{V+4} - V$$

$$\Rightarrow X\frac{dV}{dX} = \frac{5V+2-V^2-4V}{V+4}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{-V^2+V+2}{V+4}$$

$$\Rightarrow X\frac{dV}{dX} = -\frac{V^2-V-2}{V+4}$$

$$\Rightarrow \frac{V+4}{V^2-V-2}\,dV = -\frac{1}{X}\,dX$$

$$\Rightarrow \int \frac{V+4}{(V-2)(V+1)}\,dV = \int -\frac{1}{X}\,dX$$

- BY PARTIAL FRACTIONS

$$\Rightarrow \int \frac{2}{V-2} - \frac{1}{V+1}\,dV = \int -\frac{1}{X}\,dX$$

$$\Rightarrow 2\ln|V-2| - \ln|V+1| = -\ln|X| + \ln A$$

$$\Rightarrow \ln\left|\frac{(V-2)^2}{V+1}\right| = \ln\left|\frac{A}{X}\right|$$

$$\Rightarrow \frac{(V-2)^2}{V+1} = \frac{A}{X}$$

$$\Rightarrow X(V-2)^2 = A(V+1)$$

$$\Rightarrow X\left(\frac{Y}{X}-2\right)^2 = A\left(\frac{Y}{X}+1\right)$$

$$\Rightarrow X\left(\frac{Y-2X}{X}\right)^2 = A\left(\frac{Y+X}{X}\right)$$

$$\Rightarrow X\frac{(Y-2X)^2}{X^2} = \frac{A(Y+X)}{X}$$

$$\Rightarrow \frac{(Y-2X)^2}{X} = \frac{A(Y+X)}{X}$$

$$\Rightarrow (Y-2X)^2 = A(Y+X)$$

- BUT $X = x-1$
$Y = y+1$

$$[(y+1)-2(x-1)]^2 = A((y+1)+(x-1))$$

$$[y-2x+3]^2 = A(x+y)$$

- APPLY CONDITIONS

$x=1,\ y=1$

$$(1-2+3)^2 = A(1+1)$$

$$4 = 2A$$

$$A = 2$$

$$\therefore (y-2x+3)^2 = 2(x+y)$$

Question 25 (****+)

Solve the following differential equation

$$\frac{dy}{dx}=\frac{2x+y-1}{x+2y+1},$$

to show that

$$(x-y)(x+y-2)(x-y-2)^2=\text{constant}.$$

proof

$$\frac{dy}{dx}=\frac{2x+y-1}{x+2y+1}$$

- ATTEMPT TO TRANSLATE THE ORIGIN

$$\left.\begin{array}{l}2x+y-1=0\\x+2y+1=0\end{array}\right\}\Rightarrow\left.\begin{array}{r}-4x-2y+2=0\\x+2y+1=0\end{array}\right\}\Rightarrow -3x+3=0$$

$$\underline{\underline{x=1}}$$

$$\text{\& }\ \underline{\underline{y=-1}}$$

- USE THE SUBSTITUTIONS

$$\boxed{\begin{array}{ll}x=X+1\,, & dx=dX\\ y=Y-1 & dy=dY\end{array}}\quad\Rightarrow\frac{dY}{dX}=\frac{2(X+1)+(Y-1)-1}{X+1+2(Y-1)+1}$$

$$\Rightarrow\frac{dY}{dX}=\frac{2X+Y}{X+2Y}$$

- THIS HAS NOW REDUCED TO A HOMOGENEOUS EQUATION

LET $\boxed{\begin{array}{c}Y=XV\\ \frac{dY}{dX}=V+X\frac{dV}{dX}\end{array}}$

$$\Rightarrow V+X\frac{dV}{dX}=\frac{2X+XV}{X+2XY}$$

$$\Rightarrow V+X\frac{dV}{dX}=\frac{2+V}{1+2V}$$

$$\Rightarrow X\frac{dV}{dX}=\frac{V+2}{2V+1}-V$$

$$\Rightarrow X\frac{dV}{dX}=\frac{\cancel{V}+2-2V^2-\cancel{V}}{2V+1}$$

$$\Rightarrow X\frac{dV}{dX}=\frac{2(1-V^2)}{2V+1}$$

$$\Rightarrow\frac{2V+1}{1-V^2}\,dV=\frac{2}{X}\,dX$$

$$\Rightarrow\int\frac{2V+1}{(1-V)(1+V)}\,dV=\int\frac{2}{X}\,dX$$

- BY PARTIAL FRACTIONS

$$\Rightarrow\int\frac{\frac{3}{2}}{1-V}-\frac{\frac{1}{2}}{1+V}\,dV=\int\frac{2}{X}\,dX$$

$$\Rightarrow\int\frac{3}{1-V}-\frac{1}{1+V}\,dV=\int\frac{4}{X}\,dX$$

$$\Rightarrow -3\ln|1-V|-\ln|1+V|=4\ln X+\ln A$$

$$\Rightarrow 3\ln|1-V|+\ln|1+V|=\ln A-4\ln X$$

$$\Rightarrow\ln\left[(1-V)^3(1+V)\right]=\ln\left(\frac{A}{X^4}\right)$$

$$\Rightarrow(1-V)^3(1+V)=\frac{A}{X^4}$$

$$\Rightarrow(1-V)^2(1-V^2)=\frac{A}{X^4}$$

- REVERSING THE TRANSFORMATIONS

$$\Rightarrow\left(1-\frac{Y}{X}\right)^2\left(1-\frac{Y^2}{X^2}\right)=\frac{A}{X^4}$$

$$\Rightarrow\frac{1}{X^4}(X-Y)^2(X^2-Y^2)=\frac{A}{X^4}$$

$$\Rightarrow\left[(x-1)-(y+1)\right]^2\left[(x-1)^2-(y-1)^2\right]=A$$

$$\Rightarrow(x-y-2)^2(x-1+y-1)(x-1-y+1)=A$$

$$\Rightarrow(x-y-2)^2(x+y-2)(x-y)=A$$

Question 26 (****+)

Solve the following differential equation

$$\frac{dy}{dx}=\frac{2x+3y-7}{3x+2y-8}, \quad y(1)=1$$

Give the answer in the form $(y-x-1)^5 = f(x,y)$, where $f(x,y)$ is a function to be found.

$$\boxed{(y-x-1)^5 = y+x-3}$$

$$\boxed{\frac{dy}{dx}=\frac{2x+3y-7}{3x+2y-8} \quad \text{SUBJECT TO } y=1 \text{ AT } x=1}$$

FIND THE INTERSECTION OF THE TWO LINES

$$\left.\begin{aligned}2x+3y-7&=0\\3x+2y-8&=0\end{aligned}\right\}\begin{aligned}&\times 3\\&\times(-2)\end{aligned} \Rightarrow \left.\begin{aligned}6x+9y-21&=0\\-6x-4y+16&=0\end{aligned}\right\} \xrightarrow{\text{ADD}}$$

$$5y-5=0$$

$$\boxed{y=1} \ \& \ \boxed{x=2}$$

SHIFT THE ORIGIN AT $(2,1)$

$$\text{IE} \ \boxed{\begin{aligned}x&=X+2 & & dx=dX\\ y&=Y+1 & \& \ \ & dy=dY\end{aligned}}$$

$$\Rightarrow \frac{dY}{dX}=\frac{2(X+2)+3(Y+1)-7}{3(X+2)+2(Y+1)-8}$$

$$\Rightarrow \frac{dY}{dX}=\frac{2X+3Y}{3X+2Y}$$

THE O.D.E IS NOW HOMOGENEOUS

$$\text{LET} \ \boxed{\begin{aligned}&Y=XV \quad \text{WHERE } V=f(X)\\ &\frac{dY}{dX}=V+X\frac{dV}{dX}\end{aligned}}$$

$$\Rightarrow V+X\frac{dV}{dX}=\frac{2X+3XV}{3X+2XV}$$

$$\Rightarrow V+X\frac{dV}{dX}=\frac{2+3V}{3+2V}$$

$$\Rightarrow X\frac{dV}{dX}=\frac{2+3V}{3+2V}-V$$

$$\Rightarrow X\frac{dV}{dX}=\frac{2+\cancel{3V}-\cancel{3V}-2V^2}{3+2V}$$

$$\Rightarrow X\frac{dV}{dX}=\frac{2-2V^2}{3+2V}$$

$$\Rightarrow \frac{3+2V}{1-V^2}\,dV=\frac{2}{X}\,dX$$

$$\Rightarrow \frac{2V+3}{(1-V)(1+V)}\,dV=\frac{2}{X}\,dX$$

PARTIAL FRACTIONS ON THE LHS BY INSPECTION (COVER UP)

$$\Rightarrow \int \frac{5/2}{1-V}+\frac{1/2}{1+V}\,dV=\int\frac{2}{X}\,dX$$

$$\Rightarrow \int \frac{5}{1-V}+\frac{1}{1+V}\,dV=\int\frac{4}{X}\,dX$$

$$\Rightarrow \ln|1+V|-5\ln|1-V|=4\ln|X|+\ln A$$

$$\Rightarrow \ln\left|\frac{1+V}{(1-V)^5}\right|=\ln|AX^4|$$

$$\Rightarrow \frac{1+V}{(1-V)^5}=AX^4$$

$$\Rightarrow \frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^5}=AX^4$$

$$\Rightarrow \frac{1+\frac{Y}{X}}{\left(\frac{X-Y}{X}\right)^5}=AX^4$$

$$\Rightarrow \frac{1+\frac{Y}{X}}{\frac{(X-Y)^5}{X^5}}=AX^4$$

$$\Rightarrow 1+\frac{Y}{X}=AX^4\cdot\frac{(X-Y)^5}{X^5}$$

$$\Rightarrow 1+\frac{Y}{X}=\frac{A(X-Y)^5}{X}$$

$$\Rightarrow X+Y=A(X-Y)^5$$

$$\Rightarrow (x-2)+(y-1)=A\left[(x-2)-(y-1)\right]^5$$

$$\Rightarrow (x+y-3)=A(x-y+1)^5$$

APPLY CONDITION $x=1$, $y=1$

$$-1=A$$

$$\Rightarrow x+y-3=-(x-y+1)^5$$

$$\Rightarrow x+y-3=(y-x-1)^5$$

Question 27 (****+)

$$\frac{dy}{dx}\left(x+y^2\right)=y\,.$$

a) Solve the above differential equation, subject to $y=1$ at $x=1$.

b) Verify the validity of the answer obtained in part **(a)**.

$$\boxed{y^2=x}$$

a) $\boxed{\frac{dy}{dx}(x+y^2)=y}$

- LET $p=\frac{dy}{dx}$ AND DIFFERENTIATE WITH RESPECT TO y

$$\frac{dp}{dy}=\frac{d^2y}{dx^2}\times\frac{dx}{dy}=\frac{d^2y}{dx^2}\times\frac{1}{p}\Rightarrow\boxed{\frac{d^2y}{dx^2}=p\frac{dp}{dy}\ \text{ OR }\ \frac{dp}{dx}=p\frac{dp}{dy}}$$

- NOW DIFFERENTIATE THE O.D.E WRT x

$$\Rightarrow \frac{d^2y}{dx^2}(x+y^2)+\frac{dy}{dx}\left(1+2y\frac{dy}{dx}\right)=\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}(x+y^2)+\cancel{\frac{dy}{dx}}+2y\left(\frac{dy}{dx}\right)^2=\cancel{\frac{dy}{dx}}$$

- WRITE COMPACTLY USING SOME OF THE ABOVE DERIVED RESULTS

$$\Rightarrow p\frac{dp}{dy}(x+y^2)+2yp^2=0$$

- RETURNING TO THE ORIGINAL O.D.E $\boxed{\frac{dy}{dx}(x+y^2)=y \quad p(x+y^2)=y}$

$$\Rightarrow y\frac{dp}{dy}+2yp^2=0$$

$$\Rightarrow \frac{dp}{dy}+2p^2=0$$

$$\Rightarrow \frac{1}{p^2}\,dp=-2\,dy$$

$$\Rightarrow -\frac{1}{p}=-2y+C$$

$$\Rightarrow \frac{1}{p}=2y+C$$

$$\Rightarrow \boxed{p=\frac{1}{2y+C}}\ \text{(I)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2y+C}$$

$$\Rightarrow (2y+C)\,dy=1\,dx$$

$$\Rightarrow \boxed{y^2+Cy=x+k}\ \text{(II)}$$

- CONDITIONS $x=1$ $y=1$ & FROM THE ORIGINAL O.D.E $\frac{dy}{dx}=\frac{1}{2}$ IE $p=\frac{1}{2}$

USING (I) $\frac{1}{2}=\frac{1}{2+C}\Rightarrow\boxed{C=0}$

USING (II) $1=1+k\Rightarrow\boxed{k=0}$

$$\therefore\ y^2=x$$

b) CHECK

$$y^2=x$$
$$2y\frac{dy}{dx}=1$$
$$\frac{dy}{dx}=\frac{1}{2y}$$
$$(x+y^2)\frac{dy}{dx}=\frac{1}{2y}(x+y^2)$$
$$(x+y^2)\frac{dy}{dx}=\frac{1}{2y}(y^2+y^2)$$
$$(x+y^2)\frac{dy}{dx}=y$$

AS REQUIRED

ALTERNATIVE

a) $\boxed{\frac{dy}{dx}(x+y^2)=y}$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x+y^2}$$

$$\Rightarrow \frac{dx}{dy}=\frac{x+y^2}{y}$$

$$\Rightarrow \frac{dx}{dy}=\frac{x}{y}+y$$

LET $z=\frac{x}{y}$

$x=zy$

DIFF W.R.T y

$\frac{dx}{dy}=\frac{dz}{dy}y+z$

$$\Rightarrow \cancel{z}+y\frac{dz}{dy}=\cancel{z}+y$$

$$\Rightarrow y\frac{dz}{dy}=y$$

$$\Rightarrow \frac{dz}{dy}=1$$

$$\Rightarrow dz=dy$$

$$\Rightarrow z=y+C$$

$$\Rightarrow \boxed{\frac{x}{y}=y+C}$$

APPLY CONDITIONS $(1,1)$ $\therefore\ \boxed{C=0}$

$$\Rightarrow \frac{x}{y}=y$$

$$\Rightarrow y^2=x$$

b) $y^2=x$

DIFF W.R.T. x

$$2y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{2y}$$

MULTIPLY BY $(x+y^2)$

$$(x+y^2)\frac{dy}{dx}=\frac{1}{2y}(x+y^2)$$

$$(x+y^2)\frac{dy}{dx}=\frac{x+y^2}{2y}$$

$$(x+y^2)\frac{dy}{dx}=\frac{y^2+y^2}{2y}$$

$$(x+y^2)\frac{dy}{dx}=\frac{2y^2}{2y}$$

$$(x+y^2)\frac{dy}{dx}=y$$

Question 28 (****+)

Find a general solution for the following differential equation

$$\frac{dy}{dx}=\frac{1+y^2}{-x+\arctan y}.$$

$$\boxed{x=-1+\arctan y+A\,\mathrm{e}^{-\arctan y}}$$

$$\frac{dy}{dx}=\frac{1+y^2}{\arctan y - x}$$

- O.D.E DOES NOT SEPARATE & NO OBVIOUS TYPE SO ATTEMPT DIFFERENTIATION

$$\Rightarrow \frac{dy}{dx}(\arctan y - x)=1+y^2$$

- DIFF W.R.T x

$$\Rightarrow \frac{d^2y}{dx^2}(\arctan y - x)+\frac{dy}{dx}\left(\frac{1}{1+y^2}\frac{dy}{dx}-1\right)=2y\frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}(\arctan y - x)+\left(\frac{dy}{dx}\right)^2\frac{1}{1+y^2}-\frac{dy}{dx}=2y\frac{dy}{dx}$$

NOW LET $\boxed{p=\frac{dy}{dx}}$ $\quad \frac{dp}{dx}=\frac{d^2y}{dx^2}$

HOWEVER TO ELIMINATE x COMPLETELY DIFF WITH RESPECT TO y

$$\frac{dy}{dx}=p \Rightarrow \frac{d^2y}{dx^2}\frac{dx}{dy}=\frac{dp}{dy}$$

$$\Rightarrow \frac{d^2y}{dx^2}\times\frac{1}{p}=\frac{dp}{dy}$$

$$\Rightarrow \boxed{\frac{d^2y}{dx^2}=p\frac{dp}{dy}} \quad \text{OR} \quad \frac{dp}{dx}=p\frac{dp}{dy}$$

- RETURNING TO THE ODE

$$\Rightarrow p\frac{dp}{dy}(\arctan y - x)+p^2\frac{1}{1+y^2}-p=2yp$$

- FROM THE ORIGINAL ODE: $p(\arctan y - x)=1+y^2$

$$\Rightarrow (1+y^2)\frac{dp}{dy}+\frac{p^2}{1+y^2}-p=2yp$$

$$\Rightarrow \frac{dp}{dy}+\frac{p^2}{(1+y^2)^2}-\frac{p}{1+y^2}=\frac{2yp}{1+y^2}$$

$$\Rightarrow \frac{dp}{dy}-\frac{2yp+p}{1+y^2}=-\frac{p^2}{(1+y^2)^2}$$

$$\Rightarrow \frac{dp}{dy}-\frac{p(2y+1)}{1+y^2}=-\frac{p^2}{(1+y^2)^2}$$

- THIS IS A FIRST ORDER, BERNOULLI TYPE

LET $z=\frac{1}{p}$ $\quad \frac{dz}{dy}=-\frac{1}{p^2}\frac{dp}{dy}$

$$\frac{dp}{dy}=-p^2\frac{dz}{dy}$$

$$\Rightarrow -p^2\frac{dz}{dy}-p\frac{(2y+1)}{1+y^2}=-\frac{p^2}{(1+y^2)^2}$$

$$\Rightarrow \frac{dz}{dy}+\frac{1}{p}\,\frac{2y+1}{1+y^2}=\frac{1}{(1+y^2)^2}$$

$$\Rightarrow \frac{dz}{dy}+z\,\frac{2y+1}{1+y^2}=\frac{1}{(1+y^2)^2}$$

- NOW INTEGRATING FACTOR

$$e^{\int\frac{2y+1}{1+y^2}dy}=e^{\int\frac{2y}{1+y^2}+\frac{1}{1+y^2}dy}=e^{\ln(1+y^2)}e^{\arctan y}$$

$$=(1+y^2)e^{\arctan y}$$

$$\Rightarrow \frac{d}{dy}\left[z(1+y^2)e^{\arctan y}\right]=\frac{1}{(1+y^2)^2}(1+y^2)e^{\arctan y}$$

$$\Rightarrow \frac{d}{dy}\left[z(1+y^2)e^{\arctan y}\right]=\frac{e^{\arctan y}}{1+y^2}$$

$$\Rightarrow z(1+y^2)e^{\arctan y}=\int\frac{e^{\arctan y}}{1+y^2}dy$$

$$\Rightarrow z(1+y^2)e^{\arctan y}=e^{\arctan y}+C$$

$$\Rightarrow z=\frac{1}{1+y^2}+\frac{C}{1+y^2}e^{-\arctan y}$$

$$\Rightarrow \frac{1}{p}=\frac{1}{1+y^2}\left(Ce^{-\arctan y}+1\right)$$

$$\Rightarrow \frac{dx}{dy}=\frac{1}{1+y^2}+\frac{Ce^{-\arctan y}}{1+y^2}$$

$$\Rightarrow \int 1\,dx=\int\frac{1}{1+y^2}+\frac{Ce^{-\arctan y}}{1+y^2}dy$$

$$\Rightarrow x=\arctan y-Ce^{-\arctan y}+k$$

- NOW THE GENERAL SOLUTION HAS TWO CONSTANTS DUE TO "OUR" DIFFERENTIATION — THUS WE SHOULD BE ABLE TO EVALUATE IT FROM THE ORIGINAL O.D.E

$$\frac{dx}{dy}=\frac{1}{1+y^2}+\frac{Ce^{-\arctan y}}{1+y^2}$$

$$\frac{dx}{dy}=\frac{1}{1+y^2}+\frac{\arctan y - x+k}{1+y^2}$$

$$\frac{dx}{dy}=\frac{(\arctan y - x)+(k+1)}{1+y^2}$$

$$\frac{dy}{dx}=\frac{1+y^2}{\arctan y - x} \quad \text{IF } k=-1$$

- THUS RELABELLING THE CONSTANT WE OBTAIN

$$x=\arctan y+Ae^{-\arctan y}-1$$

CHECK

$$\frac{dx}{dy}=\frac{1}{1+y^2}-\frac{Ae^{-\arctan y}}{1+y^2}$$

$$\frac{dx}{dy}=\frac{1-Ae^{-\arctan y}}{1+y^2}$$

$$\frac{dy}{dx}=\frac{1+y^2}{1-Ae^{-\arctan y}}$$

BUT $x=\arctan y+Ae^{-\arctan y}-1$

$$1-Ae^{-\arctan y}=\arctan y - x$$

$$\therefore \frac{dy}{dx}=\frac{1+y^2}{\arctan y - x}$$

$\therefore$ INDEED THE GENERAL SOLUTION

ALTERNATIVE METHOD BY SUBSTITUTION

$$\boxed{\frac{dy}{dx}=\frac{1+y^2}{\arctan y - x}}$$

LET $y=\tan\theta$

$$\frac{dy}{dx}=\sec^2\theta\frac{d\theta}{dx}$$

$$\Rightarrow \sec^2\theta\frac{d\theta}{dx}=\frac{1+\tan^2\theta}{\arctan(\tan\theta)-x}$$

$$\Rightarrow \sec^2\theta\frac{d\theta}{dx}=\frac{\sec^2\theta}{\theta-x}$$

$$\Rightarrow \frac{d\theta}{dx}=\frac{1}{\theta-x}$$

$$\Rightarrow \frac{dx}{d\theta}=\theta-x$$

$$\Rightarrow \frac{dx}{d\theta}+x=\theta$$

$$\boxed{\text{I.F } e^{\int 1\,d\theta}=e^{\theta}}$$

$$\Rightarrow \frac{d}{d\theta}\left(xe^{\theta}\right)=\theta e^{\theta}$$

$$\Rightarrow xe^{\theta}=\int\theta e^{\theta}\,d\theta$$

BY PARTS

θ	1
e^{θ}	e^{θ}

$$\Rightarrow xe^{\theta}=\theta e^{\theta}-\int e^{\theta}\,d\theta$$

$$\Rightarrow xe^{\theta}=\theta e^{\theta}-e^{\theta}+A$$

REVERSING & TIDYING

$$\Rightarrow x=\theta-1+Ae^{-\theta}$$

$$\Rightarrow x=\arctan y-1+Ae^{-\arctan y}$$

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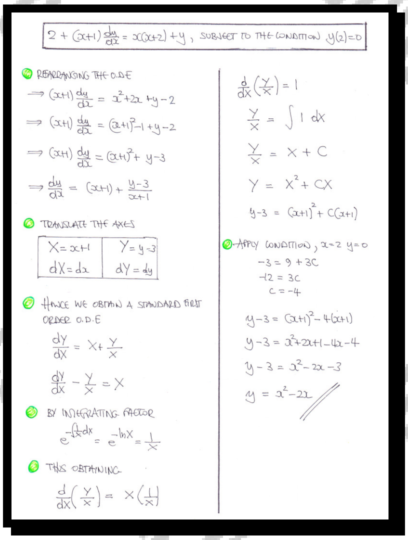
Question 29 **(****+)**

$$2+(x+1)\frac{dy}{dx}=x(x+2)+y.$$

Solve the above differential equation, subject to $y(2)=0$.

$$\boxed{y=x^2-2x}$$



$\boxed{2+(x+1)\frac{dy}{dx}=x(x+2)+y \text{, SUBJECT TO THE CONDITION } y(2)=0}$

REARRANGING THE O.D.E

$\Rightarrow (x+1)\frac{dy}{dx}=x^2+2x+y-2$

$\Rightarrow (x+1)\frac{dy}{dx}=(x+1)^2-1+y-2$

$\Rightarrow (x+1)\frac{dy}{dx}=(x+1)^2+y-3$

$\Rightarrow \frac{dy}{dx}=(x+1)+\frac{y-3}{x+1}$

TRANSLATE THE AXES

$X=x+1$	$Y=y-3$
$dX=dx$	$dY=dy$

HENCE WE OBTAIN A STANDARD FIRST ORDER O.D.E

$\frac{dY}{dX}=X+\frac{Y}{X}$

$\frac{dY}{dX}-\frac{Y}{X}=X$

BY INTEGRATING FACTOR

$e^{-\int\frac{1}{X}dX}=e^{-\ln X}=\frac{1}{X}$

THUS OBTAINING

$\frac{d}{dX}\left(\frac{Y}{X}\right)=X\left(\frac{1}{X}\right)$

$\frac{d}{dX}\left(\frac{Y}{X}\right)=1$

$\frac{Y}{X}=\int 1\,dX$

$\frac{Y}{X}=X+C$

$Y=X^2+CX$

$y-3=(x+1)^2+C(x+1)$

APPLY CONDITION, $x=2$ $y=0$

$-3=9+3C$

$-12=3C$

$C=-4$

$y-3=(x+1)^2-4(x+1)$

$y-3=x^2+2x+1-4x-4$

$y-3=x^2-2x-3$

$y=x^2-2x$

Created by T. Madas

Question 30 (****+)

Use the substitution $v = \frac{y-x}{y+x}$, $y+x \neq 0$, to solve the following differential equation

$$x\frac{dy}{dx} - y = \frac{(1-x)\left(x^2-y^2\right)}{x^3+x^2+x+1}, \quad y(0)=1.$$

Give the answer in the form $y = f(x)$.

$\boxed{}$, $\boxed{y = x^2+x+1}$

$$\boxed{x\frac{dy}{dx} - y = \frac{(1-x)(x^2-y^2)}{x^3+x^2+x+1} \qquad y(0)=1}$$

START WITH THE GIVEN SUBSTITUTION

$$v = \frac{y-x}{y+x} \implies yv + xv = y - x$$
$$\implies xv + x = y - yv$$
$$\implies x(v+1) = y(1-v)$$
$$\implies \boxed{y = \frac{x(v+1)}{1-v}}$$

NEXT DIFFERENTIATE THE ORIGINAL SUBSTITUTION W.R.T x

$$\implies v = \frac{y-x}{y+x} = \frac{y+x-2x}{y+x} = 1 - \frac{2x}{y+x}$$
$$\implies \frac{dv}{dx} = -\frac{(y+x)\times 2 - 2x\left(\frac{dy}{dx}+1\right)}{(y+x)^2}$$
$$\implies \frac{dv}{dx} = -\frac{2y + \cancel{2x} - 2x\frac{dy}{dx} - \cancel{2x}}{(y+x)^2}$$
$$\implies \boxed{\frac{dv}{dx} = \frac{2\left(x\frac{dy}{dx} - y\right)}{(y+x)^2}} \iff \boxed{x\frac{dy}{dx} - y = \frac{(y+x)^2}{2}\frac{dv}{dx}}$$

SUBSTITUTE INTO THE O.D.E.

$$\implies \frac{(y+x)^2}{2}\frac{dv}{dx} = \frac{(1-x)(x^2-y^2)}{x^3+x^2+x+1}$$
$$\implies \frac{dv}{dx} = \frac{2(1-x)(x^2-y^2)}{(x^3+x^2+x+1)(y+x)^2}$$
$$\implies \frac{dv}{dx} = \frac{2(1-x)(x-y)\cancel{(x+y)}}{(x^3+x^2+x+1)(x+y)^{\cancel{2}}} \qquad (x+y \neq 0)$$
$$\implies \frac{dv}{dx} = \frac{2(1-x)}{x^3+x^2+x+1} \times \frac{x-y}{x+y}$$

$$\implies \frac{dv}{dx} = \frac{2(1-x)}{x^2(x+1)+(x+1)} \times (-v)$$
$$\implies \frac{dv}{dx} = \frac{2v(x-1)}{(x+1)(x^2+1)}$$
$$\implies \int \frac{1}{v}\,dv = \int \frac{2(x-1)}{(x+1)(x^2+1)}\,dx$$

PROCEED BY OBTAINING THE PARTIAL FRACTIONS

$$\frac{2(x-1)}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$
$$\boxed{2(x-1) \equiv A(x^2+1) + (Bx+C)(x+1)}$$

- IF $x=-1$: $-4 = 2A$, $A=-2$
- IF $x=0$: $-2 = A+C$, $-2 = -2+C$, $C=0$
- IF $x=2$: $2 = 5A+6B$, $2 = -10+6B$, $12 = 6B$, $B=2$

RETURNING TO THE O.D.E

$$\implies \int \frac{1}{v}\,dv = \int \frac{2x}{x^2+1} - \frac{2}{x+1}\,dx$$
$$\implies \ln|v| = \ln|x^2+1| - 2\ln|x+1| + \ln A$$
$$\implies \ln|v| = \ln\left|\frac{A(x^2+1)}{(x+1)^2}\right|$$
$$\implies v = \frac{A(x^2+1)}{(x+1)^2}$$
$$\implies \boxed{\frac{y-x}{y+x} = \frac{A(x^2+1)}{(x+1)^2}}$$

APPLY BOUNDARY CONDITION $y(0)=1$

$$\implies \frac{1-0}{1+0} = \frac{A\times 1}{(0+1)^2}$$
$$\implies A = 1$$
$$\implies \boxed{\frac{y-x}{y+x} = \frac{x^2+1}{(x+1)^2}}$$

REARRANGE THE SOLUTION IN THE FORM $y=f(x)$

$$\implies \frac{y-x}{y+x} = F(x) \qquad \left[F(x) = \frac{x^2+1}{(x+1)^2}\right]$$
$$\implies y - x = yF(x) + xF(x)$$
$$\implies y - F(x)y = xF(x) + x$$
$$\implies y(1-F(x)) = x(F(x)+1)$$
$$\implies y = \frac{x(1+F(x))}{1-F(x)}$$
$$\implies y = \frac{x\left[1 + \frac{x^2+1}{(x+1)^2}\right]}{\left[1 - \frac{x^2+1}{(x+1)^2}\right]}$$

MULTIPLY "TOP & BOTTOM" OF THE FRACTION IN THE R.H.S BY $(x+1)^2$

$$\implies y = \frac{x\left[(x+1)^2 + (x^2+1)\right]}{(x+1)^2 - (x^2+1)}$$
$$\implies y = \frac{x(x^2+2x+1+x^2+1)}{\cancel{x^2}+2x\cancel{+1}\cancel{-x^2}\cancel{-1}}$$
$$\implies y = \frac{\cancel{x}(2x^2+2x+2)}{2\cancel{x}}$$
$$y = x^2+x+1$$

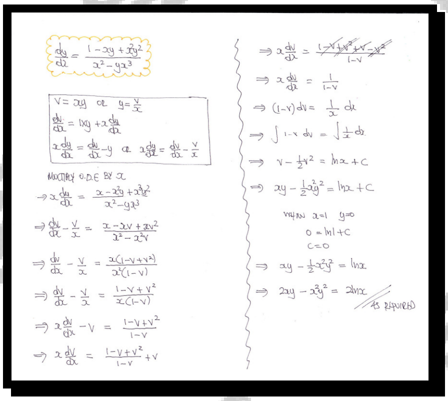
Question 31 (*****)

Solve the differential equation

$$\frac{dy}{dx} = \frac{1 - xy + x^2y^2}{x^2 - yx^3}, \quad x > 0,$$

subject to the condition $y(1) = 0$.

$$\boxed{2xy - x^2y^2 = 2\ln x}$$



$$\frac{dy}{dx} = \frac{1 - xy + x^2y^2}{x^2 - yx^3}$$

$V = xy$ or $y = \frac{V}{x}$

$\frac{dV}{dx} = 1 \times y + x\frac{dy}{dx}$

$x\frac{dy}{dx} = \frac{dV}{dx} - y$ or $x\frac{dy}{dx} = \frac{dV}{dx} - \frac{V}{x}$

MULTIPLY O.D.E BY x

$\Rightarrow x\frac{dy}{dx} = \frac{x - x^2y + x^3y^2}{x^2 - yx^3}$

$\Rightarrow \frac{dV}{dx} - \frac{V}{x} = \frac{x - xV + xV^2}{x^2 - x^2V}$

$\Rightarrow \frac{dV}{dx} - \frac{V}{x} = \frac{x(1 - V + V^2)}{x^2(1 - V)}$

$\Rightarrow \frac{dV}{dx} - \frac{V}{x} = \frac{1 - V + V^2}{x(1 - V)}$

$\Rightarrow x\frac{dV}{dx} - V = \frac{1 - V + V^2}{1 - V}$

$\Rightarrow x\frac{dV}{dx} = \frac{1 - V + V^2}{1 - V} + V$

$\Rightarrow x\frac{dV}{dx} = \frac{1 \cancel{- V} \cancel{+ V^2} \cancel{+ V} \cancel{- V^2}}{1 - V}$

$\Rightarrow x\frac{dV}{dx} = \frac{1}{1 - V}$

$\Rightarrow (1 - V)\,dV = \frac{1}{x}\,dx$

$\Rightarrow \int 1 - V\,dV = \int \frac{1}{x}\,dx$

$\Rightarrow V - \frac{1}{2}V^2 = \ln x + C$

$\Rightarrow xy - \frac{1}{2}x^2y^2 = \ln x + C$

WHEN $x = 1$ $y = 0$

$0 = \ln 1 + C$

$C = 0$

$\Rightarrow xy - \frac{1}{2}x^2y^2 = \ln x$

$\Rightarrow 2xy - x^2y^2 = 2\ln x$

AS REQUIRED

Question 32 (*****)

Find a simplified general solution for the following differential equation.

$$\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1.$$

$$\boxed{\left(A+y\sqrt{x^2-1}\right)\left(y+Bx+C\right)=0}$$

$$\boxed{\left(x^2-1\right)\left(\frac{dy}{dx}\right)^2-2xy\left(\frac{dy}{dx}\right)+y^2=1}$$

- DIFFERENTIATE THE DIFFERENTIAL EQUATION W.R.T x

$$\Rightarrow 2x\left(\frac{dy}{dx}\right)^2-(x^2-1)\times 2\left(\frac{dy}{dx}\right)^1\frac{d^2y}{dx^2}-2y\frac{dy}{dx}-2x\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)-2xy\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)=0$$

$$\Rightarrow \cancel{2x\left(\frac{dy}{dx}\right)^2}-2(x^2-1)\frac{dy}{dx}\frac{d^2y}{dx^2}\cancel{-2y\frac{dy}{dx}}\cancel{-2x\left(\frac{dy}{dx}\right)^2}-2xy\frac{d^2y}{dx^2}\cancel{+2y\frac{dy}{dx}}=0$$

$$\Rightarrow -2(x^2-1)\frac{d^2y}{dx^2}\frac{dy}{dx}-2xy\frac{d^2y}{dx^2}=0$$

$$\Rightarrow -\frac{d^2y}{dx^2}\left[2(x^2-1)\frac{dy}{dx}+2xy\right]=0$$

- EITHER $\frac{d^2y}{dx^2}=0$
- OR $\cancel{2}(x^2-1)\frac{dy}{dx}+\cancel{2}xy=0$

$$\Rightarrow \frac{1}{y}\,dy=-\frac{x}{x^2-1}\,dx$$

$$\Rightarrow \ln y=-\tfrac{1}{2}\ln|x^2-1|+\ln A$$

$$\Rightarrow \ln y=\ln A-\ln(x^2-1)^{\frac{1}{2}}$$

$$\Rightarrow \ln y=\ln\frac{A}{\sqrt{x^2-1}}$$

$$\Rightarrow y=\frac{A}{\sqrt{x^2-1}}$$

$$\Rightarrow y\sqrt{x^2-1}-A=0 \quad \text{OR} \quad y\sqrt{x^2-1}+A=0$$

- SIMILARLY

$$\frac{d^2y}{dx^2}=0$$

$$\frac{dy}{dx}=B$$

$$y=Bx+C$$

$$y-Bx-C=0 \quad \text{OR} \quad y+Bx+C=0$$

- HENCE

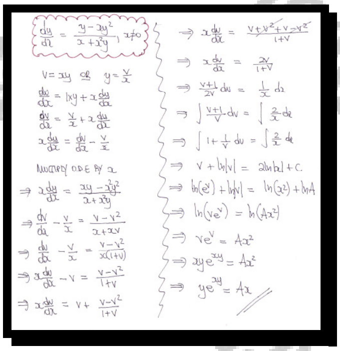
$$\left(y+Bx+C\right)\left(y\sqrt{x^2-1}+A\right)=0$$

Question 33 (*****)

Find a general solution for the differential equation

$$\frac{dy}{dx} = \frac{y - xy^2}{x + yx^2}, \ x \neq 0 .$$

$$\boxed{y\,e^{xy} = Cx}$$


$\dfrac{dy}{dx} = \dfrac{y - xy^2}{x + x^2y}, \ x \neq 0$

$v = xy$ or $y = \dfrac{v}{x}$

$\dfrac{dv}{dx} = 1 \times y + x\dfrac{dy}{dx}$

$\dfrac{dv}{dx} = \dfrac{v}{x} + x\dfrac{dy}{dx}$

$x\dfrac{dy}{dx} = \dfrac{dv}{dx} - \dfrac{v}{x}$

MULTIPLY O.D.E BY x

$\Rightarrow x\dfrac{dy}{dx} = \dfrac{xy - x^2y^2}{x + x^2y}$

$\Rightarrow \dfrac{dv}{dx} - \dfrac{v}{x} = \dfrac{v - v^2}{x + xv}$

$\Rightarrow \dfrac{dv}{dx} - \dfrac{v}{x} = \dfrac{v - v^2}{x(1+v)}$

$\Rightarrow x\dfrac{dv}{dx} - v = \dfrac{v - v^2}{1+v}$

$\Rightarrow x\dfrac{dv}{dx} = v + \dfrac{v - v^2}{1+v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{v + \cancel{v^2} + v - \cancel{v^2}}{1+v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{2v}{1+v}$

$\Rightarrow \dfrac{v+1}{2v}\,dv = \dfrac{1}{x}\,dx$

$\Rightarrow \displaystyle\int \frac{v+1}{v}\,dv = \int \frac{2}{x}\,dx$

$\Rightarrow \displaystyle\int 1 + \frac{1}{v}\,dv = \int \frac{2}{x}\,dx$

$\Rightarrow v + \ln|v| = 2\ln|x| + C$

$\Rightarrow \ln(e^v) + \ln|v| = \ln(x^2) + \ln A$

$\Rightarrow \ln(ve^v) = \ln(Ax^2)$

$\Rightarrow ve^v = Ax^2$

$\Rightarrow xye^{xy} = Ax^2$

$\Rightarrow ye^{xy} = Ax$

Question 34 (*****)

Solve the differential equation

$$\frac{dy}{dx} = -\frac{xy^2 + y}{x + yx^2 + x^3y^2}, \quad x \neq 0, \quad y > 0,$$

subject to the condition $y\left(\frac{1}{2}\right) = 1$.

$$\boxed{2x^2y^2 \ln y = 2xy + 1}$$

$$\boxed{\frac{dy}{dx} = -\frac{xy^2+y}{x+x^2y+x^3y^2} \quad \begin{matrix} x \neq 0 \\ y > 0 \end{matrix}}$$

- $V = xy$ OR $y = \frac{V}{x}$

$\frac{dv}{dx} = 1 \times y + x\frac{dy}{dx}$

$\frac{dv}{dx} = \frac{V}{x} + x\frac{dy}{dx}$

$x\frac{dy}{dx} = \frac{dv}{dx} - \frac{V}{x}$

- MULTIPLY O.D.E BY x

$\Rightarrow x\frac{dy}{dx} = \frac{-x^2y^2 - xy}{x + x^2y + x^3y^2}$

$\Rightarrow \frac{dv}{dx} - \frac{V}{x} = \frac{-V^2 - V}{x + xV + xV^2}$

$\Rightarrow \frac{dv}{dx} - \frac{V}{x} = \frac{-V^2 - V}{x(1 + V + V^2)}$

$\Rightarrow x\frac{dv}{dx} - V = \frac{-V^2 - V}{1 + V + V^2}$

$\Rightarrow x\frac{dv}{dx} = V - \frac{V^2 + V}{1 + V + V^2}$

$\Rightarrow x\frac{dv}{dx} = \frac{\cancel{V} + \cancel{V^2} + V^3 - \cancel{V^2} - \cancel{V}}{1 + V + V^2}$

$\Rightarrow \frac{1 + V + V^2}{V^3}\,dv = \frac{1}{x}\,dx$

$\Rightarrow \int \frac{1}{V^3} + \frac{1}{V^2} + \frac{1}{V}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow -\frac{1}{2V^2} - \frac{1}{V} + \ln|V| = \ln|x| + C$

$\Rightarrow \ln\left|\frac{V}{x}\right| = \frac{1}{2V^2} + \frac{1}{V} + C$

$\Rightarrow \ln|y| = \frac{1}{2x^2y^2} + \frac{1}{xy} + C$

$\Rightarrow 2x^2y^2\ln|y| = 1 + 2xy + C$

APPLY CONDITION $x = -\frac{1}{2}$, $y = 1$

$2\left(\frac{1}{4}\right)\cancel{\ln 1} = 1 + 2\left(-\frac{1}{2}\right) + C$

$C = 0$

$\therefore 2x^2y^2\ln y = 2xy + 1$

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Question 35 (*****)

The positive solution of the quadratic equation $x^2 - x - 1 = 0$ is denoted by ϕ, and is commonly known as the golden section or golden number.

a) Show, with a detailed method, that $F(x) = f(\phi)x^{g(\phi)}$ is a solution of the differential equation,

$$F'(x) = F^{-1}(x),$$

where f and g are constant expressions of ϕ, to be found in simplified form.

b) Verify the answer obtained in part **(a)** satisfies the differential equation, by differentiation and function inversion.

[You may assume that $F(x)$ is differentiable and invertible]

V, $\boxed{}$, $\boxed{F(x) = \left(\dfrac{1}{\phi}\right)^{\frac{1}{\phi}} x^{\phi} = \phi^{1-\phi}x^{\phi}}$

ASSUME A SOLUTION OF THE FORM $y = Ax^r$, WHERE A IS A CONSTANT & r IS ALSO A CONSTANT — FURTHER ASSUME THAT "FUNCTION" IS SMOOTH AND INVERTIBLE

- $y = Ax^r$

$\Rightarrow \dfrac{dy}{dx} = rAx^{r-1}$

$F'(x)$

- $y = Ax^r$

$\Rightarrow \dfrac{y}{A} = x^r$

$\Rightarrow \left(\dfrac{y}{A}\right)^{\frac{1}{r}} = (x^r)^{\frac{1}{r}}$

$\Rightarrow x = \left(\dfrac{1}{A}\right)^{\frac{1}{r}} y^{\frac{1}{r}}$

SWAP x & y

$\Rightarrow y = \left(\dfrac{1}{A}\right)^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow y = (A^{-1})^{\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow y = A^{-\frac{1}{r}} x^{\frac{1}{r}}$

$F^{-1}(x)$

SETTING EQUAL TO ONE ANOTHER, AS IN THE O.D.E

$\Rightarrow rAx^{r-1} = A^{-\frac{1}{r}} x^{\frac{1}{r}}$

$\Rightarrow \dfrac{x^{r-1}}{x^{\frac{1}{r}}} = \dfrac{A^{-\frac{1}{r}}}{rA}$

$\Rightarrow x^{r-1-\frac{1}{r}} = \dfrac{1}{r}A^{-1-\frac{1}{r}}$

NOW R.H.S IS A CONSTANT $\Rightarrow$ L.H.S MUST ALSO BE A CONSTANT

$\Rightarrow$ EXPONENT OF x MUST BE ZERO

$\Rightarrow r - 1 - \frac{1}{r} = 0$

BUT $r - 1 - \frac{1}{r} = 0$ CAN BE REARRANGED TO SEVERAL FORMS

LET $r = \phi$

$\phi = 1 + \frac{1}{\phi}$ $\quad$ $\phi^2 - \phi - 1 = 0$

$\phi^2 = \phi + 1$

ETC

BUT AS THE LHS IS A CONSTANT, THE ONLY CONSTANT IT CAN BE IS "ONE" AND THUS THE R.H.S MUST ALSO BE "ONE"

$\Rightarrow \frac{1}{r}A^{-1-\frac{1}{r}} = 1$

$\Rightarrow \frac{1}{\phi}A^{-1-\frac{1}{\phi}} = 1$

$\Rightarrow A^{-(1+\frac{1}{\phi})} = \phi$

$\Rightarrow A^{-\phi} = \phi$

$\Rightarrow A^{\phi} = \frac{1}{\phi}$

$\Rightarrow A = \left(\frac{1}{\phi}\right)^{\frac{1}{\phi}}$ OR EQUIVALENT $(\phi^{-1})^{\frac{1}{\phi}} = \phi^{-\frac{1}{\phi}}$

$= \phi^{1-\phi}$

$= \phi^{1-(\phi^2-1)}$

$= \phi^{2-\phi^2}$

ETC

$\therefore F(x) = \phi^{1-\phi}x^{\phi}$

b) DIFFERENTIATING $F(x) = \phi^{1-\phi}x^{\phi}$

$F'(x) = \phi\,\phi^{1-\phi}x^{\phi-1} = \phi^{2-\phi}x^{\phi-1}$

WE MAY HAVE TO MANIPULATE THIS FURTHER AT A LATER STAGE

INVERTING $F(x)$

$\Rightarrow y = \phi^{1-\phi}x^{\phi}$

$\Rightarrow \dfrac{y}{\phi^{1-\phi}} = x^{\phi}$

$\Rightarrow \dfrac{(y)^{\frac{1}{\phi}}}{(\phi)^{\frac{1-\phi}{\phi}}} = (x^{\phi})^{\frac{1}{\phi}}$

$\Rightarrow x = \phi^{-\frac{1-\phi}{\phi}} y^{\frac{1}{\phi}}$

$\Rightarrow F^{-1}(x) = \phi^{\frac{\phi-1}{\phi}} x^{\frac{1}{\phi}}$

LOOKING AT THE POWERS OF x, STARTING WITH $F^{-1}(x)$

$\frac{1}{\phi} = \phi - 1$ (SINCE $\phi = 1 + \frac{1}{\phi}$)

LOOKING AT THE CONSTANT, STARTING WITH THE EXPONENT AT $F^{-1}(x)$

$\frac{\phi-1}{\phi} = 1 - \frac{1}{\phi} = 1 - (\phi - 1) = 2 - \phi$

$\therefore \phi^{2-\phi}x^{\phi-1} = \phi^{\frac{\phi-1}{\phi}}x^{\frac{1}{\phi}}$

$\therefore F'(x) = F^{-1}(x)$

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