MULTIVARIABLE INTEGRATION

(SPHERICAL POLAR COORDINATES)

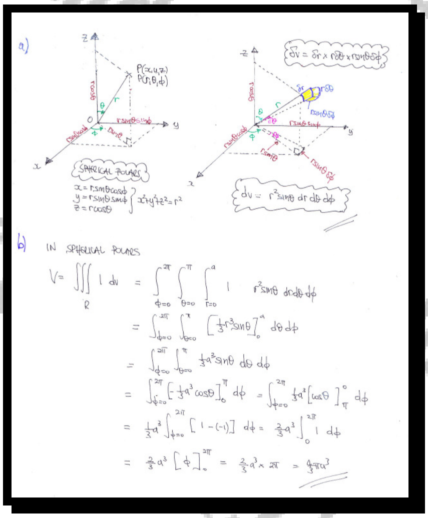
Question 1

a) Determine with the aid of a diagram an expression for the volume element in spherical polar coordinates, (r,θ,φ).

[*You may not use Jacobians in this part*]

b) Use spherical polar coordinates to obtain the standard formula for the volume of a sphere of radius a.

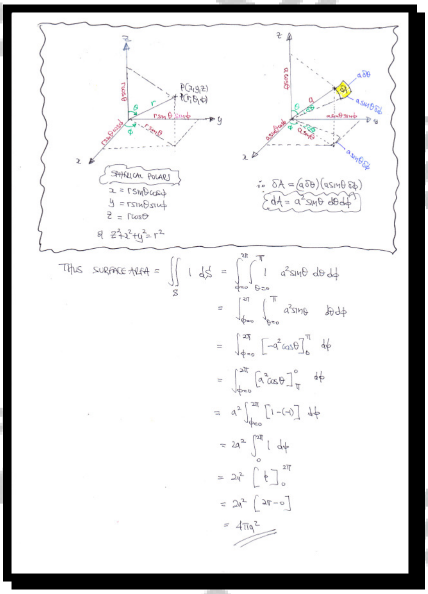
$$\boxed{dv = r^2 \sin\theta \, dr \, d\theta \, d\varphi}$$



Question 2

Use spherical polar coordinates, (r,θ,φ), to obtain the standard formula for the surface area of a sphere of radius a.

$$\boxed{dA = a^2 \sin\theta \; d\theta \; d\varphi \Rightarrow A = 4\pi a^2}$$



Question 3

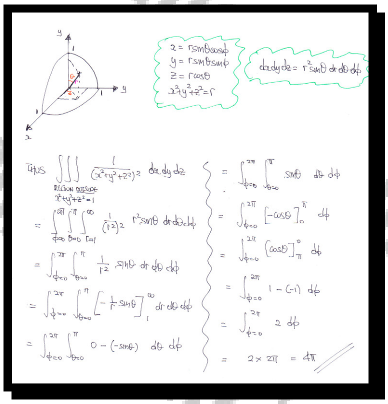
Determine an exact simplified value for

$$\int_R \frac{1}{\left(x^2+y^2+z^2\right)^2}\,dx\,dy\,dz\,,$$

where R is the region **outside** the sphere with equation

$$x^2+y^2+z^2=1.$$

$\boxed{4\pi}$



Question 4

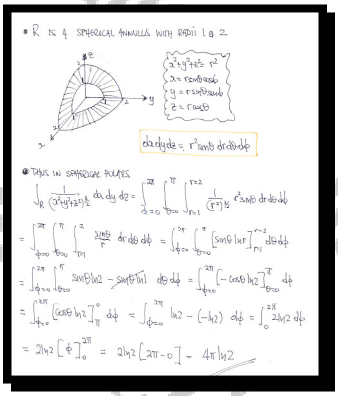
The finite R region is defined as

$$1 \le x^2 + y^2 + z^2 \le 4 .$$

Determine an exact simplified value for

$$\int_R \frac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} \, dx\,dy\,dz .$$

$\boxed{4\pi \ln 2}$



• R IS A SPHERICAL ANNULUS WITH RADII 1 & 2

$x^2+y^2+z^2=r^2$
$x=r\sin\theta\cos\phi$
$y=r\sin\theta\sin\phi$
$z=r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

• THUS IN SPHERICAL POLARS

$$\int_R \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dx\,dy\,dz = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=1}^{r=2} \frac{1}{(r^2)^{\frac{3}{2}}}\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=1}^{2} \frac{\sin\theta}{r}\,dr\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\sin\theta\ln r\right]_{r=1}^{r=2}\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta\ln 2 - \sin\theta\ln 1\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\left[-\cos\theta\ln 2\right]_{\theta=0}^{\pi}\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\left[\cos\theta\ln 2\right]_{\pi}^{0}\,d\phi = \int_{\phi=0}^{2\pi} \ln 2 - (-\ln 2)\,d\phi = \int_0^{2\pi} 2\ln 2\,d\phi$$

$$= 2\ln 2\left[\phi\right]_0^{2\pi} = 2\ln 2\left[2\pi - 0\right] = 4\pi\ln 2$$

Question 5

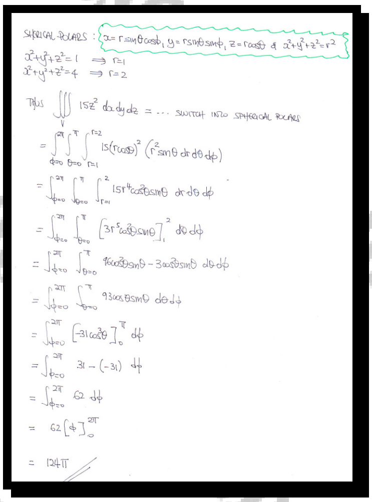
Determine an exact simplified value for

$$\int_R 15z^2 \ dx\,dy\,dz ,$$

where R is the region between the spheres with equations

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4 .$$

$\boxed{124\pi}$



SPHERICAL POLARS : $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ & $x^2+y^2+z^2=r^2$

$x^2+y^2+z^2=1 \implies r=1$

$x^2+y^2+z^2=4 \implies r=2$

THUS $\iiint_V 15z^2 \, dx\,dy\,dz = \ldots$ SWITCH INTO SPHERICAL POLARS

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=1}^{r=2} 15(r\cos\theta)^2 \left(r^2\sin\theta \, dr\,d\theta\,d\phi\right)$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=1}^{2} 15r^4\cos^2\theta\sin\theta \, dr\,d\theta\,d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[3r^5\cos^2\theta\sin\theta\right]_1^2 d\theta\,d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 96\cos^2\theta\sin\theta - 3\cos^2\theta\sin\theta \, d\theta\,d\phi$

$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 93\cos^2\theta\sin\theta \, d\theta\,d\phi$

$= \int_{\phi=0}^{2\pi} \left[-31\cos^3\theta\right]_0^{\pi} d\phi$

$= \int_{\phi=0}^{2\pi} 31 - (-31) \, d\phi$

$= \int_{\phi=0}^{2\pi} 62 \, d\phi$

$= 62\left[\phi\right]_0^{2\pi}$

$= 124\pi$

Question 6

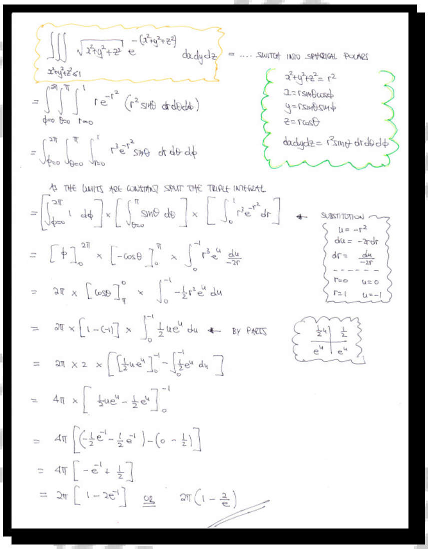
The finite R region is defined as the interior of the sphere with equation

$$x^2 + y^2 + z^2 = 1.$$

Determine an exact simplified value for

$$\int_R \sqrt{x^2 + y^2 + z^2}\, \mathrm{e}^{-\left(x^2+y^2+z^2\right)}\, dx\, dy\, dz.$$

$$\boxed{2\pi\left[1 - \frac{2}{\mathrm{e}}\right]}$$


$$\iiint_{x^2+y^2+z^2\le 1} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\,dy\,dz = \ldots \text{SWITCH INTO SPHERICAL POLARS}$$

$x^2+y^2+z^2 = r^2$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r e^{-r^2} \left(r^2 \sin\theta\, dr\, d\theta\, d\phi\right)$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^3 e^{-r^2} \sin\theta\, dr\, d\theta\, d\phi$$

AS THE LIMITS ARE CONSTANT, SPLIT THE TRIPLE INTEGRAL

$$= \left[\int_{\phi=0}^{2\pi} 1\, d\phi\right] \times \left[\int_{\theta=0}^{\pi} \sin\theta\, d\theta\right] \times \left[\int_0^1 r^3 e^{-r^2}\, dr\right] \leftarrow \text{SUBSTITUTION}$$

SUBSTITUTION: $u = -r^2$, $du = -2r\,dr$, $dr = \frac{du}{-2r}$; $r=0 \Rightarrow u=0$, $r=1 \Rightarrow u=-1$

$$= \left[\phi\right]_0^{2\pi} \times \left[-\cos\theta\right]_0^{\pi} \times \int_0^{-1} r^3 e^u \frac{du}{-2r}$$

$$= 2\pi \times \left[\cos\theta\right]_{\pi}^{0} \times \int_0^{-1} -\tfrac{1}{2} r^2 e^u\, du$$

$$= 2\pi \times \left[1-(-1)\right] \times \int_0^{-1} \tfrac{1}{2} u e^u\, du \leftarrow \text{BY PARTS}$$

$$= 2\pi \times 2 \times \left[\left[\tfrac{1}{2}ue^u\right]_0^{-1} - \int_0^{-1} \tfrac{1}{2}e^u\, du\right]$$

$$= 4\pi \times \left[\tfrac{1}{2}ue^u - \tfrac{1}{2}e^u\right]_0^{-1}$$

$$= 4\pi\left[\left(-\tfrac{1}{2}e^{-1} - \tfrac{1}{2}e^{-1}\right) - \left(0 - \tfrac{1}{2}\right)\right]$$

$$= 4\pi\left[-e^{-1} + \tfrac{1}{2}\right]$$

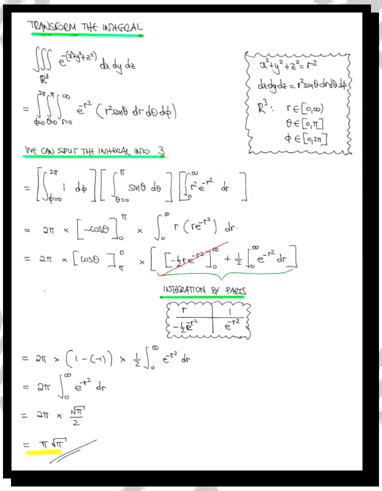
$$= 2\pi\left[1 - 2e^{-1}\right] \quad \text{OR} \quad 2\pi\left(1 - \tfrac{2}{e}\right)$$

Question 7

Use spherical polar coordinates to find an exact value for the following integral.

$$\iiint_{\mathbb{R}^3} e^{-\left(x^2+y^2+z^2\right)} \, dx \, dy \, dz \, .$$

MM2F, $\pi\sqrt{\pi}$



Question 8

A hemispherical solid piece of glass, of radius a m, has small air bubbles within its volume.

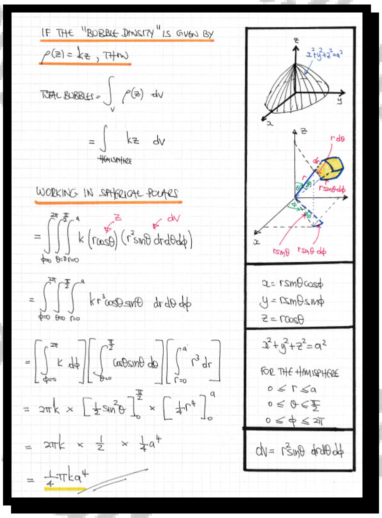
The air bubble density $\rho(z)$, in m^{-3}, is given by

$$\rho(z) = k\,z\,,$$

where k is a positive constant, and z is a standard cartesian coordinate, whose origin is at the centre of the flat face of the solid.

Given that the solid is contained in the part of space for which $z \geq 0$, determine the total number of air bubbles in the solid.

M23, $\boxed{\frac{1}{4}\pi k a^4}$



Question 9

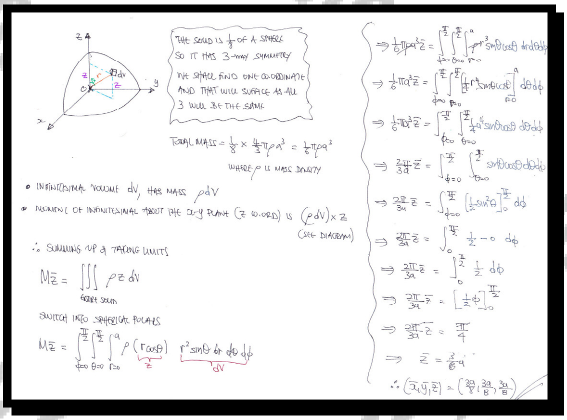
A uniform solid has equation

$$x^2 + y^2 + z^2 = a^2,$$

with $x > 0$, $y > 0$, $z > 0$, $a > 0$.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the solid.

$$\boxed{\left(\frac{3}{8}a, \frac{3}{8}a, \frac{3}{8}a\right)}$$



Question 10

A solid sphere has equation

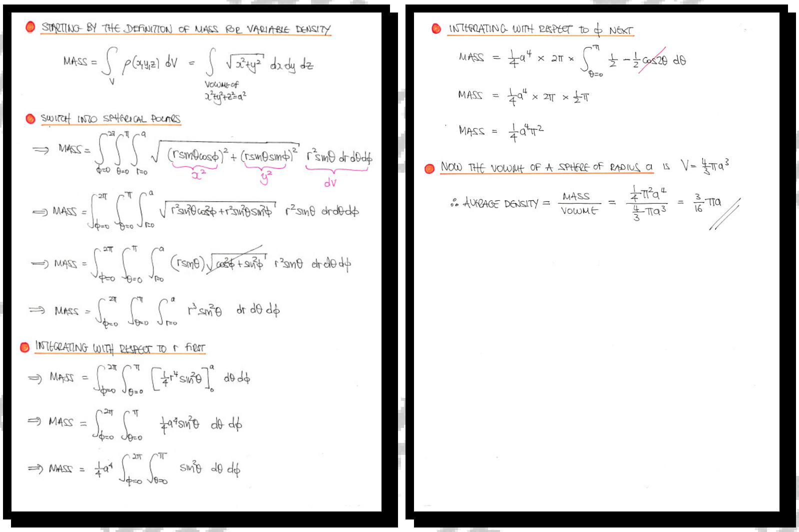
$$x^2 + y^2 + z^2 = a^2 .$$

The density, ρ, at the point of the sphere with coordinates (x_1, y_1, z_1) is given by

$$\rho = \sqrt{x_1^2 + y_1^2} .$$

Determine the **average** density of the sphere.

[illegible], $\boxed{\bar{\rho} = \frac{3}{16}\pi a}$



STARTING BY THE DEFINITION OF MASS FOR VARIABLE DENSITY

$$\text{MASS} = \int_V \rho(x,y,z)\, dV = \int_{\substack{\text{VOLUME OF} \\ x^2+y^2+z^2=a^2}} \sqrt{x^2+y^2}\, dx\, dy\, dz$$

SWITCH INTO SPHERICAL POLARS

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{\underbrace{(r\sin\theta\cos\phi)^2}_{x^2} + \underbrace{(r\sin\theta\sin\phi)^2}_{y^2}}\ \underbrace{r^2\sin\theta\, dr\, d\theta\, d\phi}_{dV}$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} (r\sin\theta)\sqrt{\cos^2\phi + \sin^2\phi}\ r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin^2\theta\, dr\, d\theta\, d\phi$$

INTEGRATING WITH RESPECT TO r FIRST

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \left[\tfrac{1}{4}r^4\sin^2\theta\right]_0^a d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \tfrac{1}{4}a^4\sin^2\theta\, d\theta\, d\phi$$

$$\Rightarrow \text{MASS} = \tfrac{1}{4}a^4\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin^2\theta\, d\theta\, d\phi$$

INTEGRATING WITH RESPECT TO ϕ NEXT

$$\text{MASS} = \tfrac{1}{4}a^4 \times 2\pi \times \int_{\theta=0}^{\pi} \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\, d\theta$$

$$\text{MASS} = \tfrac{1}{4}a^4 \times 2\pi \times \tfrac{1}{2}\pi$$

$$\text{MASS} = \tfrac{1}{4}a^4\pi^2$$

NOW THE VOLUME OF A SPHERE OF RADIUS a IS $V = \frac{4}{3}\pi a^3$

$$\therefore \text{AVERAGE DENSITY} = \frac{\text{MASS}}{\text{VOLUME}} = \frac{\frac{1}{4}\pi^2 a^4}{\frac{4}{3}\pi a^3} = \frac{3}{16}\pi a$$

Question 11

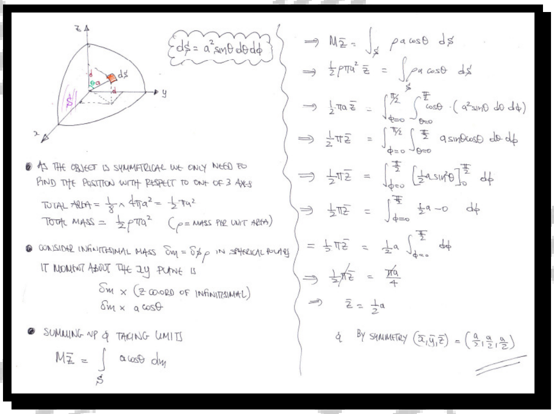
A thin uniform spherical shell with equation

$$x^2 + y^2 + z^2 = a^2, \ a > 0,$$

occupies the region in the first octant.

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the shell.

$$\boxed{\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)}$$



Question 12

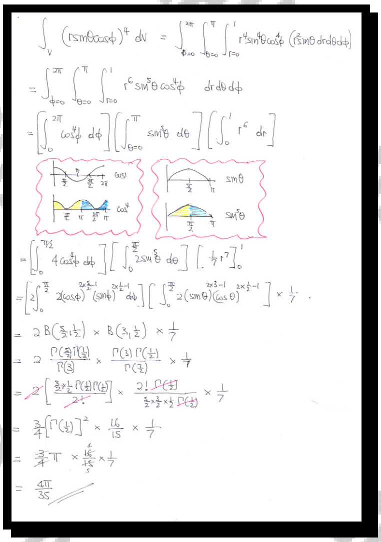
The finite region Ω is defined as

$$x^2 + y^2 + z^2 \le 1 .$$

Use Spherical Polar Coordinates (r, θ, φ), to evaluate the volume integral

$$\int_{\Omega} (r \sin\theta \cos\varphi)^4 \ dV .$$

$$\boxed{\frac{4\pi}{35}}$$



Question 13

A solid sphere has equation

$$x^2 + y^2 + z^2 = 1.$$

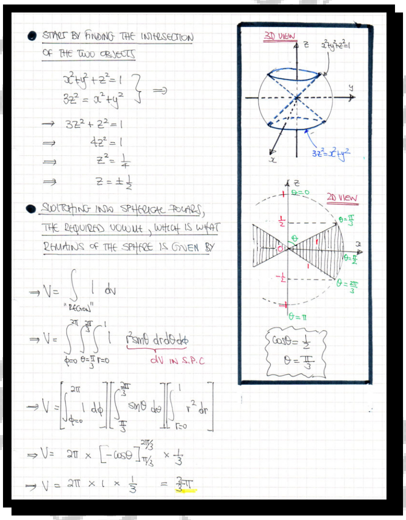
The region defined by the double cone with Cartesian equation

$$3z^2 \geq x^2 + y^2,$$

is bored out of the sphere.

Determine the volume of the remaining solid.

MM2C, $V = \frac{2}{3}\pi$



Question 14

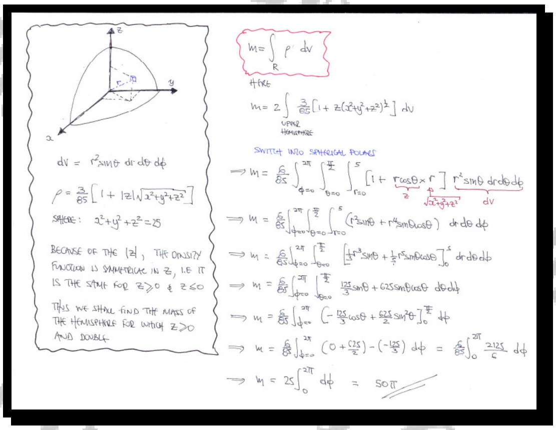
A solid sphere has radius 5 and is centred at the Cartesian origin O.

The density ρ at point $P(x_1, y_1, z_1)$ of the sphere satisfies

$$\rho = \frac{3}{85}\left[1 + |z_1|\sqrt{x_1^2 + y_1^2 + z_1^2}\right].$$

Use spherical polar coordinates, (r, θ, φ), to find the mass of the sphere.

$\boxed{m = 50\pi}$

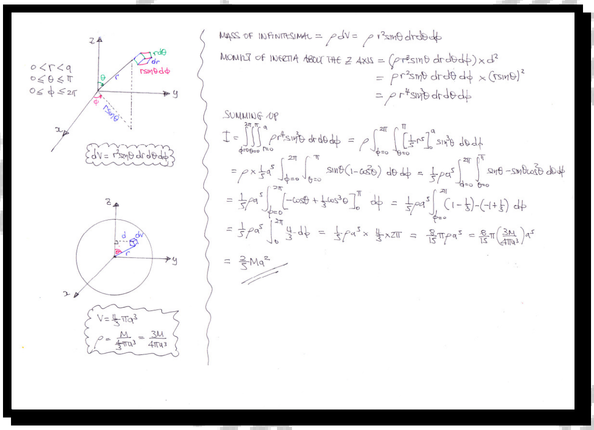


Question 15

A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), to show that the moment of inertial of this sphere about one of its diameters is $\frac{2}{5}Ma^2$.

proof

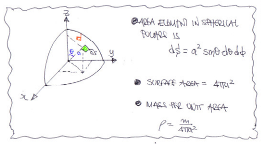


Question 16

A thin uniform spherical shell has mass m and radius a.

Use spherical polar coordinates, (r, θ, φ), to show that the moment of inertial of this spherical shell about one of its diameters is $\frac{2}{3}ma^2$.

proof



- AREA ELEMENT IN SPHERICAL POLARS IS $dS = a^2 \sin\theta \, d\theta \, d\phi$
- SURFACE AREA $= 4\pi a^2$
- MASS PER UNIT AREA $\rho = \frac{m}{4\pi a^2}$

WITHOUT LOSS OF GENERALITY TAKE THE z AXIS AS THE DIAMETER OF THE SHELL

- MASS OF INFINITESIMAL IS $\rho \, \delta S$
- MOMENT OF INERTIA ABOUT THE z-AXIS IS

$$(\rho \, \delta S) \times d^2 = \rho \, \delta S \times (a \sin\theta)^2 = \frac{m}{4\pi a^2} \times a^2 \sin^2\theta \, \delta S = \frac{m}{4\pi} \sin^2\theta \, \delta S = \frac{m}{4\pi} \sin^2\theta (a^2 \sin\theta \, \delta\theta \, \delta\phi) = \frac{ma^2}{4\pi} \sin^3\theta \, \delta\theta \, \delta\phi$$

SUMMING UP AND TAKING LIMITS

$$I = \iint_S \frac{ma^2}{4\pi} \sin^3\theta \, d\theta \, d\phi$$

$$I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{ma^2}{4\pi} \sin^3\theta \, d\theta \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \sin^2\theta \, d\theta \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta (1 - \cos^2\theta) \, d\theta \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{\pi} \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] \, d\phi$$

$$I = \frac{ma^2}{4\pi} \int_{\phi=0}^{2\pi} \frac{4}{3} \, d\phi$$

$$I = \frac{ma^2}{4\pi} \left[\frac{4}{3}\phi \right]_0^{2\pi}$$

$$I = \frac{ma^2}{\pi} \times \left[\frac{2\pi}{3} - 0 \right]$$

$$I = \frac{2}{3}ma^2$$

Question 17

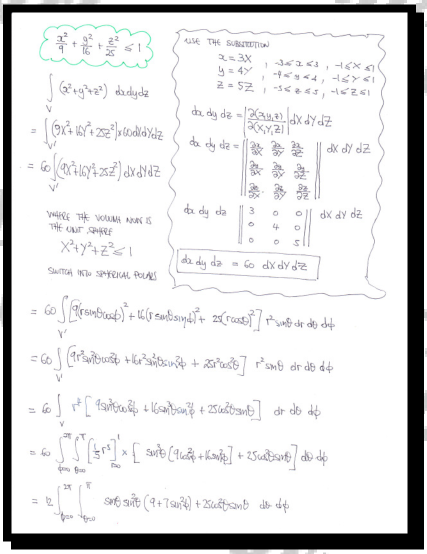
The finite region R is defined as the region enclosed by the ellipsoid with Cartesian equation

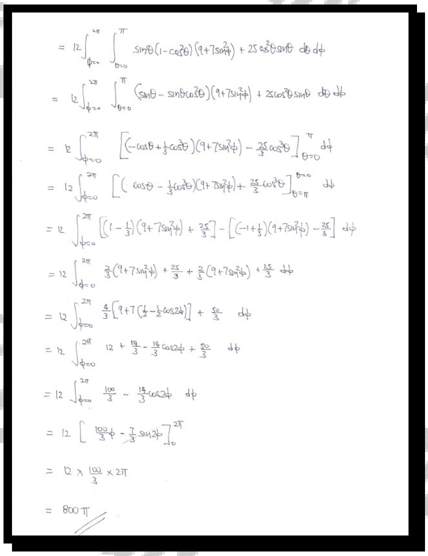
$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{25}=1.$$

By first transforming the Cartesian coordinates into a new Cartesian coordinate system, use spherical polar coordinates, (r,θ,φ), find the value of

$$\iiint_R \left(x^2+y^2+z^2\right)\ dx\,dy\,dz\,.$$

$\boxed{800\pi}$





Question 18

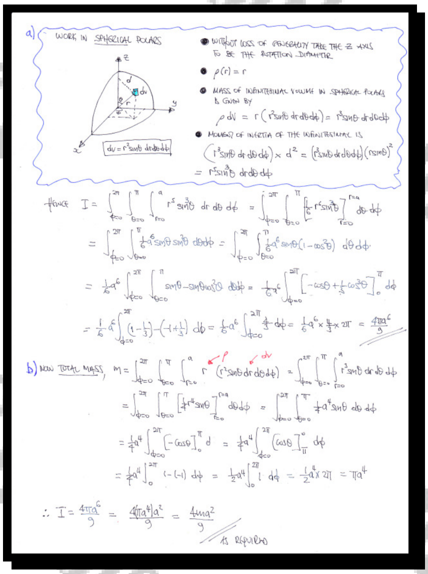
A solid uniform sphere of radius a, has variable density $\rho(r) = r$, where r is the radial distance of a given point from the centre of the sphere.

a) Use spherical polar coordinates, (r, θ, φ), to find the moment of inertia of this sphere I, about one of its diameters.

b) Given that the total mass of the sphere is m, show that

$$I = \tfrac{4}{9} m a^2 .$$

$$\boxed{I = \tfrac{4}{9}\pi a^6}$$



a) WORK IN SPHERICAL POLARS

$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

- WITHOUT LOSS OF GENERALITY TAKE THE z AXIS TO BE THE ROTATION DIAMETER
- $\rho(r) = r$
- MASS OF INFINITESIMAL VOLUME IN SPHERICAL POLARS IS GIVEN BY

$\rho \, dV = r(r^2 \sin\theta \, dr \, d\theta \, d\phi) = r^3 \sin\theta \, dr \, d\theta \, d\phi$

- MOMENT OF INERTIA OF THE INFINITESIMAL IS

$(r^3 \sin\theta \, dr \, d\theta \, d\phi) \times d^2 = (r^3 \sin\theta \, dr \, d\theta \, d\phi)(r\sin\theta)^2$

$= r^5 \sin^3\theta \, dr \, d\theta \, d\phi$

HENCE $I = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^5 \sin^3\theta \, dr \, d\theta \, d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{1}{6} r^6 \sin^3\theta\right]_{r=0}^{r=a} d\theta \, d\phi$

$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{6} a^6 \sin\theta \sin^2\theta \, d\theta \, d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{6} a^6 \sin\theta (1 - \cos^2\theta) \, d\theta \, d\phi$

$= \frac{1}{6} a^6 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta - \sin\theta\cos^2\theta \, d\theta \, d\phi = \frac{1}{6} a^6 \int_{\phi=0}^{2\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} d\phi$

$= \frac{1}{6} a^6 \int_{\phi=0}^{2\pi} \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) d\phi = \frac{1}{6} a^6 \int_{\phi=0}^{2\pi} \frac{4}{3} \, d\phi = \frac{1}{6} a^6 \times \frac{4}{3} \times 2\pi = \frac{4\pi a^6}{9}$

b) NOW TOTAL MASS, $m = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \, (r^2 \sin\theta \, dr \, d\theta \, d\phi) = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^3 \sin\theta \, dr \, d\theta \, d\phi$ (ρ, dV)

$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left[\frac{1}{4} r^4 \sin\theta\right]_{r=0}^{r=a} d\theta \, d\phi = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{1}{4} a^4 \sin\theta \, d\theta \, d\phi$

$= \frac{1}{4} a^4 \int_{\phi=0}^{2\pi} \left[-\cos\theta\right]_0^{\pi} d = \frac{1}{4} a^4 \int_{\phi=0}^{2\pi} \left[\cos\theta\right]_{\pi}^{0} d\phi$

$= \frac{1}{4} a^4 \int_0^{2\pi} 1 - (-1) \, d\phi = \frac{1}{2} a^4 \int_0^{2\pi} 1 \, d\phi = \frac{1}{2} a^4 \times 2\pi = \pi a^4$

$\therefore I = \frac{4\pi a^6}{9} = \frac{4(\pi a^4) a^2}{9} = \frac{4 m a^2}{9}$

AS REQUIRED

Question 19

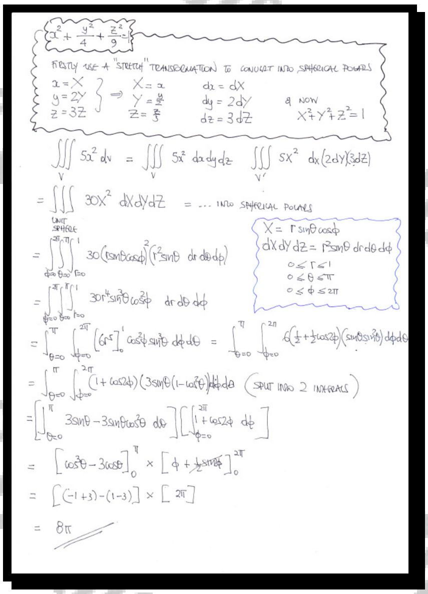
Evaluate the triple integral

$$\int_V 5x^2 \, dxdydz \,,$$

where V is the finite region contained within the closed surface with equation

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1 .$$

$\boxed{8\pi}$



Question 20

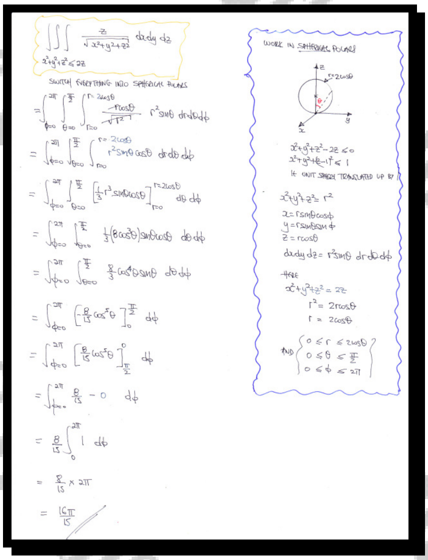
The finite R region is defined as

$$x^2 + y^2 + z^2 \le 2z .$$

Determine an exact simplified value for

$$\int_R \frac{z}{\sqrt{x^2+y^2+z^2}} \, dx\,dy\,dz .$$

$$\boxed{\frac{16\pi}{15}}$$



Question 21

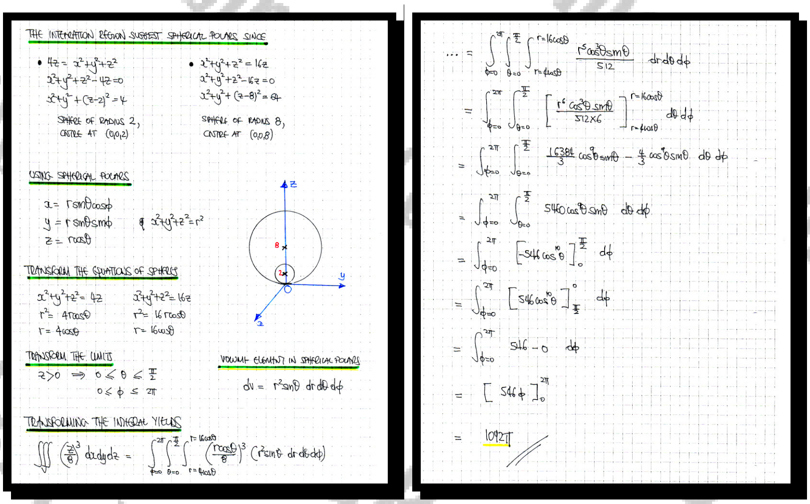
The finite R region is defined as

$$4z \le x^2 + y^2 + z^2 \le 16z \,.$$

Determine an exact simplified value for

$$\int_R \left(\frac{z}{8}\right)^3 \, dx\,dy\,dz \,.$$

$\boxed{}$, $\boxed{1092\pi}$



Question 22

A solid sphere has equation

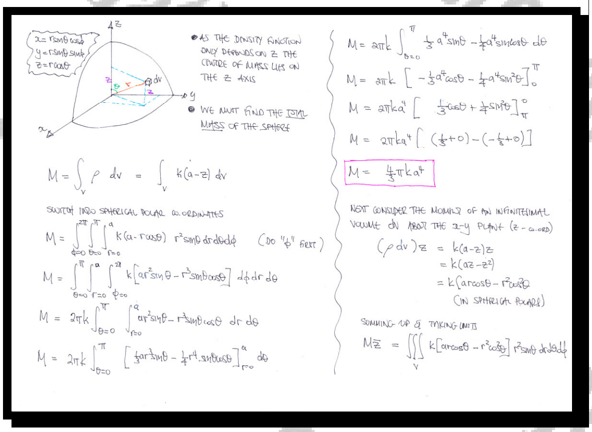
$$x^2 + y^2 + z^2 = a^2 , \ a > 0 .$$

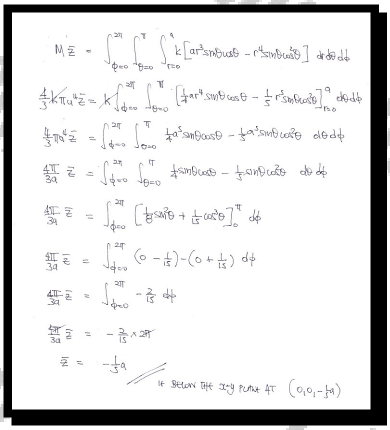
The sphere has variable density ρ, given by

$$\rho = k(a - z), \ k > 0 .$$

Use integration in spherical polar coordinates, (r, θ, φ), to find in Cartesian form the coordinates of the centre of mass of the sphere.

$$\boxed{\left(0, 0, -\tfrac{1}{5}a\right)}$$





Question 23

A solid is defined in a Cartesian system of coordinates by

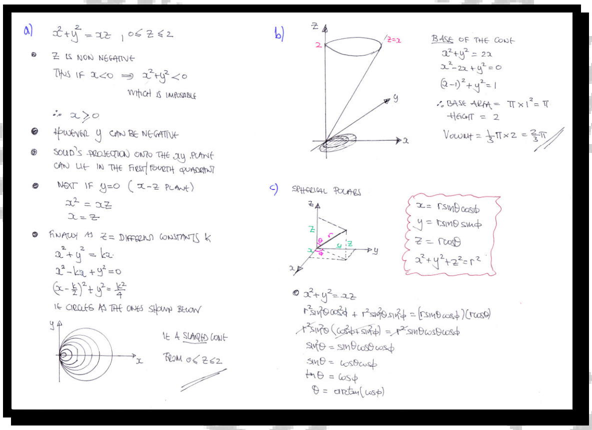
$$x^2 + y^2 = xz, \; 0 \le z \le 2.$$

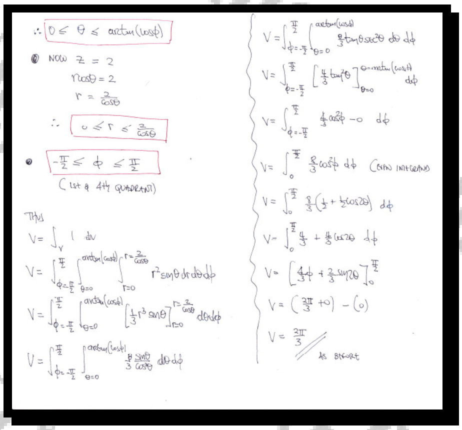
a) Describe the solid with the aid of a sketch.

b) Use standard elementary formulas to find the volume of the solid.

c) Use spherical polar coordinates to verify the answer to part (b)

$$\boxed{V = \frac{2}{3}\pi}$$





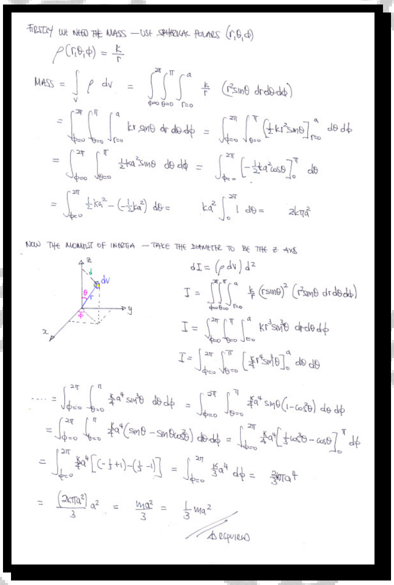
Question 24

A solid sphere has radius a and mass m.

The density ρ at any point in the sphere is inversely proportional to the distance of this point from the centre of the sphere

Show that the moment of inertia of this sphere about one of its diameters is $\frac{1}{3}ma^2$

proof



Question 25

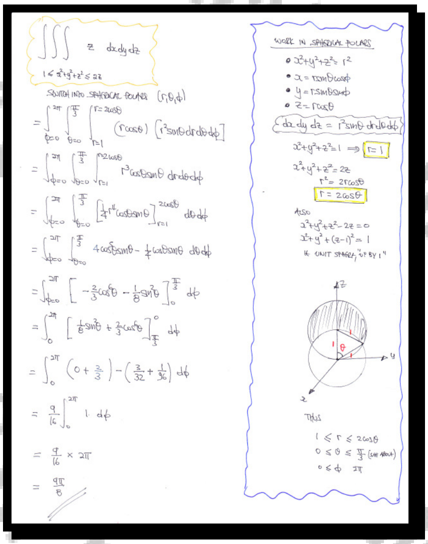
The finite R region is defined as

$$1 \le x^2 + y^2 + z^2 \le 2z\,.$$

Determine an exact simplified value for

$$\int_R z \quad dx\,dy\,dz\,.$$

$$\boxed{\frac{9\pi}{8}}$$



$$\iiint_{1 \le x^2+y^2+z^2 \le 2z} z \; dx\,dy\,dz$$

SWITCH INTO SPHERICAL POLARS (r,θ,ϕ)

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=1}^{r=2\cos\theta} (r\cos\theta)\left[r^2 \sin\theta \, dr\, d\theta\, d\phi\right]$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{3}} \int_{r=1}^{r=2\cos\theta} r^3 \cos\theta \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{3}} \left[\tfrac{1}{4} r^4 \cos\theta \sin\theta\right]_{r=1}^{2\cos\theta} d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{3}} 4\cos^5\theta \sin\theta - \tfrac{1}{4}\cos\theta \sin\theta \; d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} \left[-\tfrac{2}{3}\cos^6\theta - \tfrac{1}{8}\sin^2\theta\right]_0^{\frac{\pi}{3}} d\phi$$

$$= \int_0^{2\pi} \left[\tfrac{1}{8}\sin^2\theta + \tfrac{2}{3}\cos^6\theta\right]_{\frac{\pi}{3}}^{0} d\phi$$

$$= \int_0^{2\pi} \left(0 + \tfrac{2}{3}\right) - \left(\tfrac{3}{32} + \tfrac{1}{96}\right) d\phi$$

$$= \frac{9}{16}\int_0^{2\pi} 1 \; d\phi$$

$$= \frac{9}{16} \times 2\pi$$

$$= \frac{9\pi}{8}$$

WORK IN SPHERICAL POLARS

- $x^2+y^2+z^2 = r^2$
- $x = r\sin\theta\cos\phi$
- $y = r\sin\theta\sin\phi$
- $z = r\cos\theta$

$$dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$x^2+y^2+z^2 = 1 \implies \boxed{r=1}$

$x^2+y^2+z^2 = 2z$

$r^2 = 2r\cos\theta$

$\boxed{r = 2\cos\theta}$

ALSO

$x^2+y^2+z^2-2z = 0$

$x^2+y^2+(z-1)^2 = 1$

IE UNIT SPHERE, "UP BY 1"

THUS

$1 \le r \le 2\cos\theta$

$0 \le \theta \le \frac{\pi}{3}$ (SEE ABOVE)

$0 \le \phi \quad 2\pi$

Question 26

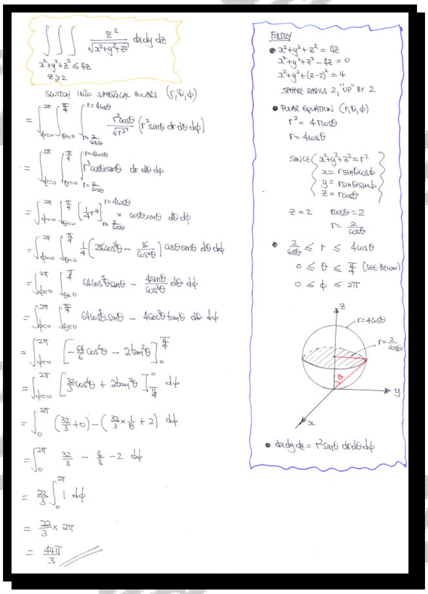
The finite R region is defined as

$$x^2 + y^2 + z^2 \le 4z \quad \text{and} \quad z \ge 2.$$

Determine an exact simplified value for

$$\int_R \frac{z^2}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz.$$

$$\boxed{\frac{44\pi}{3}}$$



Question 27

A **non right** circular cone has Cartesian equation

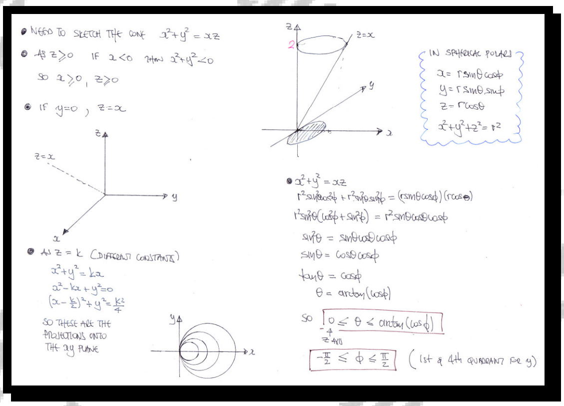
$$x^2 + y^2 = xz,\ 0 \le z \le 2.$$

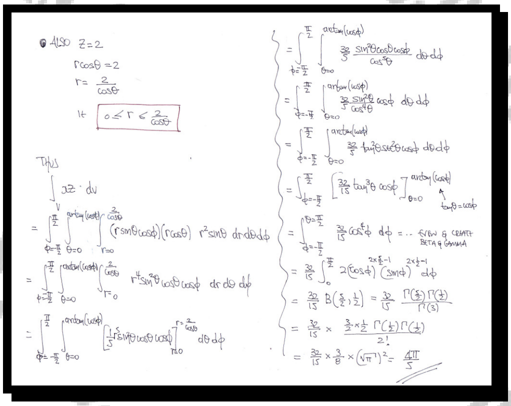
Use spherical polar coordinates to find an exact simplified value for

$$\int_R xz \ dV,$$

where R is the interior of the cone.

$$\boxed{\frac{4\pi}{5}}$$





Question 28

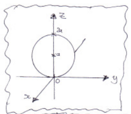
A solid uniform sphere has mass M and radius a.

Use spherical polar coordinates, (r,θ,φ), and direct calculus methods, to show that the moment of inertial of this sphere about one of its tangents is $\frac{7}{5}Ma^2$.

*You may **not** use any standard rules or standard results about moments of inertia in this question apart from the definition of moment of inertia.*

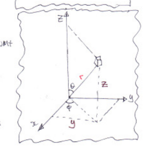
proof

- IN ORDER TO USE DIRECT INTEGRATION WE SHALL TRANSLATE THE SPHERE WITH EQUATION $x^2+y^2+z^2=a^2$ "UP" BY a SO THAT THE xy PLANE IS A TANGENT PLANE AND THE ROTATION AXIS IS THE x AXIS



$$x^2+y^2+(z-a)^2=a^2$$
$$x^2+y^2+z^2-2az+a^2=a^2$$
$$x^2+y^2+z^2=2az$$

- USE SPHERICAL POLARS (r,θ,ϕ)



  - LET $\rho = \frac{m}{\frac{4}{3}\pi a^3} = \frac{3m}{4\pi a^3}$ = MASS PER UNIT VOLUME
  - AN INFINITESIMAL VOLUME ELEMENT HAS MASS
$$\rho\, dV = \rho\, r^2\sin\theta\, dr\, d\theta\, d\phi$$
  - THE EQUATION OF THE SPHERE IN SPHERICAL POLARS IS
$$x^2+y^2+z^2=2az$$
$$r^2=2a(r\cos\theta)$$
$$r^2=2ar\cos\theta$$
$$\boxed{r=2a\cos\theta}$$
  - THE DISTANCE OF THE INFINITESIMAL FROM THE x AXIS IS $\sqrt{y^2+z^2}$, SO THE MOMENT OF INERTIA OF THE INFINITESIMAL ABOUT THE x AXIS IS

$$I = m\times d^2 = (\rho r^2\sin\theta\, dr\, d\theta\, d\phi)\times(y^2+z^2)$$

$x=r\sin\theta\cos\phi$
$y=r\sin\theta\sin\phi$
$z=r\cos\theta$

$\left[y^2+z^2 = r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta\right]$ SO $I=md^2$ BECOMES:

$$md^2 = (\rho r^2\sin\theta\, dr\, d\theta\, d\phi)(r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta)$$
$$= \rho r^4(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, dr\, d\theta\, d\phi$$

SUMMING UP: $\displaystyle\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{r=2a\cos\theta} \rho r^4(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, dr\, d\theta\, d\phi$

(NOTE THAT $0\le\theta\le\frac{\pi}{2}$ AS SPHERE IS TRANSLATED)

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}}\left[r^5\right]_{r=0}^{2a\cos\theta}(\sin^3\theta\sin^2\phi + \sin\theta\cos^2\theta)\, d\theta\, d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} 32a^5\cos^5\theta\sin^3\theta\sin^2\phi + 32a^5\cos^7\theta\sin\theta\ d\theta\, d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\left\{16a^5\sin^2\phi\int_{\theta=0}^{\frac{\pi}{2}} 2\cos^5\theta\sin^3\theta\, d\theta + \left[-4a^5\cos^8\theta\right]_0^{\frac{\pi}{2}}\right\} d\phi$$

$$B(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{2!\times 1!}{4!} = \frac{1}{12}$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\left\{\frac{4}{3}a^5\sin^2\phi + [0+4a^5]\right\} d\phi$$

$$I = \frac{1}{5}\rho\int_{\phi=0}^{2\pi}\frac{4}{3}a^5\sin^2\phi + 4a^5\, d\phi = \frac{1}{5}\rho a^5\int_{\phi=0}^{2\pi}\frac{4}{3}\sin^2\phi + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\int_0^{2\pi}\frac{4}{3}\left(\frac{1}{2}-\frac{1}{2}\cos 2\phi\right) + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\int_0^{2\pi}\frac{2}{3} - \frac{2}{3}\cos 2\phi + 4\ d\phi$$

$$I = \frac{1}{5}\rho a^5\times\int_0^{2\pi}\frac{14}{3}\, d\phi$$

$$I = \frac{14}{15}\rho a^5\int_0^{2\pi} 1\ d\phi$$

$$I = \frac{14}{15}\rho a^5\times 2\pi$$

$$I = \frac{28}{15}\rho a^5$$

$$I = \frac{28}{15}\left(\frac{3m}{4\pi a^3}\right)a^5$$

$$I = \frac{7}{5}ma^2$$

AS REQUIRED