IYGB - SYNOPTIC PAPER U - QUESTION 1

a) AS THE TANGENT HAS GRADIENT 3, THE NORMAL MUST HAVE GRADIENT $-\frac{1}{3}$

$A(5,3)$

$$y - y_0 = m(x - x_0)$$
$$y - 3 = -\frac{1}{3}(x - 5)$$
$$3y - 9 = -x + 5$$
$$x + 3y = 14$$

b) THE NORMAL MUST PASS THROUGH THE CENTRE $C(8,k)$

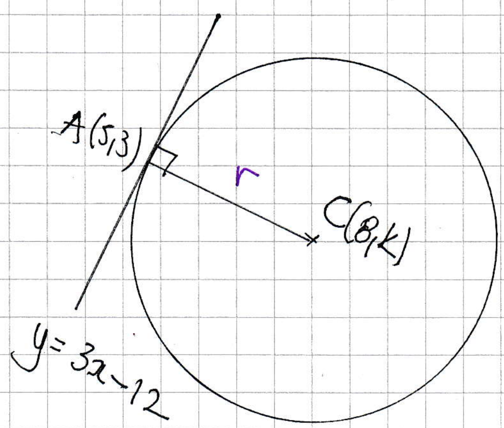
$$\Longrightarrow 8 + 3k = 14$$
$$\Longrightarrow 3k = 6$$
$$\Longrightarrow k = 2$$

$\therefore$ $A(5,3)$ & $C(8,2)$

THE EQUATION OF THE CIRCLE IS GIVEN BY

$$(x-8)^2 + (y-2)^2 = r^2$$
$$(x-8)^2 + (y-2)^2 = (\sqrt{10})^2$$
$$(x-8)^2 + (y-2)^2 = 10$$

$$|AC| = \sqrt{(2-3)^2 + (8-5)^2}$$
$$|AC| = \sqrt{1+9}$$
$$|AC| = \sqrt{10}$$



IYGB – SYNOPTIC PAPER U – QUESTION 2

THIS IS SUBSTITUTIONS AND INTEGRATION BY PARTS — IGNORE LIMITS TO START WITH

$$\int (x^3+x)e^{x^2}\,dx = \int x^3e^{x^2} + xe^{x^2}\,dx = \int x^2(xe^{x^2})\,dx + \int xe^{x^2}\,dx$$

↑PARTS "RECOGNISABLE"

x^2	$2x$
$\frac{1}{2}e^{x^2}$	xe^{x^2}

$$= \frac{1}{2}x^2e^{x^2} - \cancel{\int xe^{x^2}\,dx} + \cancel{\int xe^{x^2}\,dx}$$

BOTH CAN BE INTEGRATED, BUT THEY CANCEL OUT

THUS

$$\int_0^{\sqrt{2}} (x^3+x)e^{x^2}\,dx = \left[\frac{1}{2}x^2e^{x^2}\right]_0^{\sqrt{2}} = \frac{1}{2}\times 2\times e^2 - 0 = e^2$$

AS REQUIRED

ALTERNATIVE IS TO START WITH A SUBSTITUTION

$u = x^2$
$du = 2x\,dx$
$x=0 \mapsto u=0$
$x=\sqrt{2} \mapsto u=2$

$$\int_0^{\sqrt{2}} (x^3+x)e^{x^2}\,dx = \frac{1}{2}\int_0^{\sqrt{2}} 2x(x^2+1)e^{x^2}\,dx$$

$$= \frac{1}{2}\int_0^2 (u+1)e^u\,du = \int_0^2 \frac{1}{2}ue^u + \frac{1}{2}e^u\,du$$

$$= \int_0^2 \frac{1}{2}ue^u\,du + \int_0^2 \frac{1}{2}e^u\,du$$

INTEGRATION BY PARTS IN THE FIRST INTEGRAL

$\frac{1}{2}u$	$\frac{1}{2}$
e^u	e^u

$$= \left[\frac{1}{2}ue^u\right]_0^2 - \int_0^2 \cancel{\frac{1}{2}e^u\,du} + \int_0^2 \cancel{\frac{1}{2}e^u}\,du$$

$$= \left[\frac{1}{2}ue^u\right]_0^2$$

$$= \frac{1}{2}\times 2\times e^2 - 0$$

$$= e^2$$

AS BEFORE

IYGB ~ SYNOPTIC PAPER D - QUESTION 3

LOOKING AT THE EXPRESSION ON THE L.H.S OF THE EQUATION

$a^0 = 1 \implies x^2+2x-8=0$

$\implies (x-2)(x+4)=0$

$\implies x = \begin{cases} 2 \\ -4 \end{cases}$

THERE IS HOWEVER ANOTHER POSSIBILITY

$1^a = 1 \implies 2x^2-7x+4=1$

$\implies 2x^2-7x+3=0$

$\implies (2x-1)(x-3)=0$

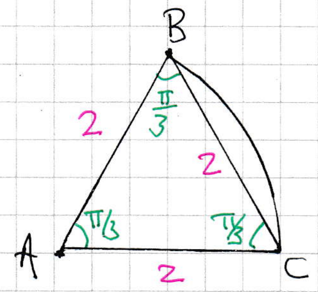
$\implies x = \begin{cases} \frac{1}{2} \\ 3 \end{cases}$

HENCE THERE ARE FOUR SOLUTIONS

$x = -4, \frac{1}{2}, 2, 3$ //

IYGB - SYNOPTIC PAPER U - QUESTION 4

LOOKING AT THE DIAGRAM BELOW



AREA OF THE TRIANGLE ABC

$$\frac{1}{2}|AB||AC|\sin(B\hat{A}C)$$
$$= \cancel{\tfrac{1}{2} \times 2} \times 2 \times \sin\frac{\pi}{3}$$
$$= 2 \times \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}$$

NEXT LOOKING AT THE SECTOR $\overset{\frown}{ABC}$

- AREA $\triangledown$ = "$\frac{1}{2}r^2\theta^c$" $= \frac{1}{2} \times 2^2 \times \frac{\pi}{3} = \frac{2\pi}{3}$
- AREA OF THE SEGMENT $= \frac{2\pi}{3} - \sqrt{3}$
- AREA OF "REULEUX TRIANGLE" = "3 SEGMENTS" + "TRIANGLE"

$$= 3\left(\frac{2\pi}{3} - \sqrt{3}\right) + \sqrt{3}$$
$$= 2\pi - 3\sqrt{3} + \sqrt{3}$$
$$= 2\pi - 2\sqrt{3}$$
$$= 2(\pi - \sqrt{3})$$

// AS REQUIRED

IYGB - SYNOPTIC PAPER U - QUESTION 5

a) T_1 : REFLECTION ABOUT THE x-AXIS

T_2 : TRANSLATION BY THE VECTOR $\begin{pmatrix}-1\\0\end{pmatrix}$, ("TO THE LEFT" BY 1 UNIT)

T_3 : TRANSLATION BY THE VECTOR $\begin{pmatrix}0\\2\end{pmatrix}$, ("UPWARDS" BY 2 UNITS)

b) WORK OUT INTERCEPTS WITH THE AXES & ASYMPTOTES

- $x=0$

$y = 2 - \frac{1}{0+1}$

$y = 1$

$(0,1)$

- $y=0$

$0 = 2 - \frac{1}{x+1}$

$\frac{1}{x+1} = 2$

$\frac{1}{2} = x+1$

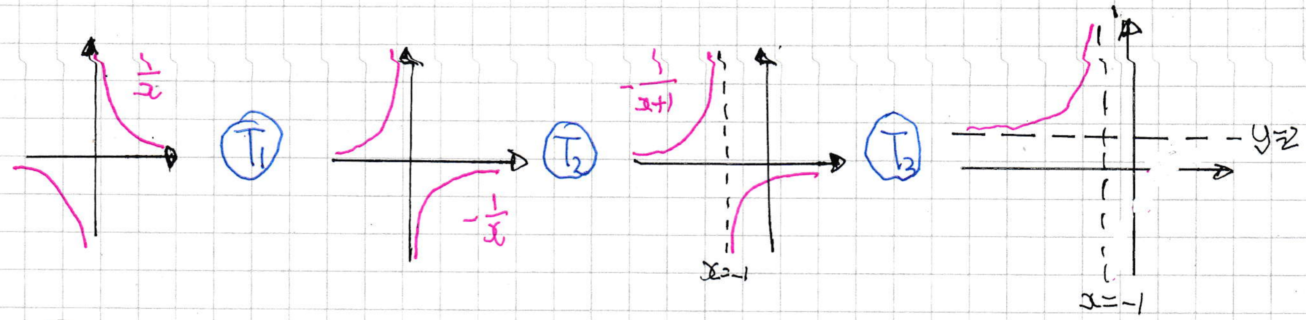
$x = -\frac{1}{2}$

$(-\frac{1}{2}, 0)$

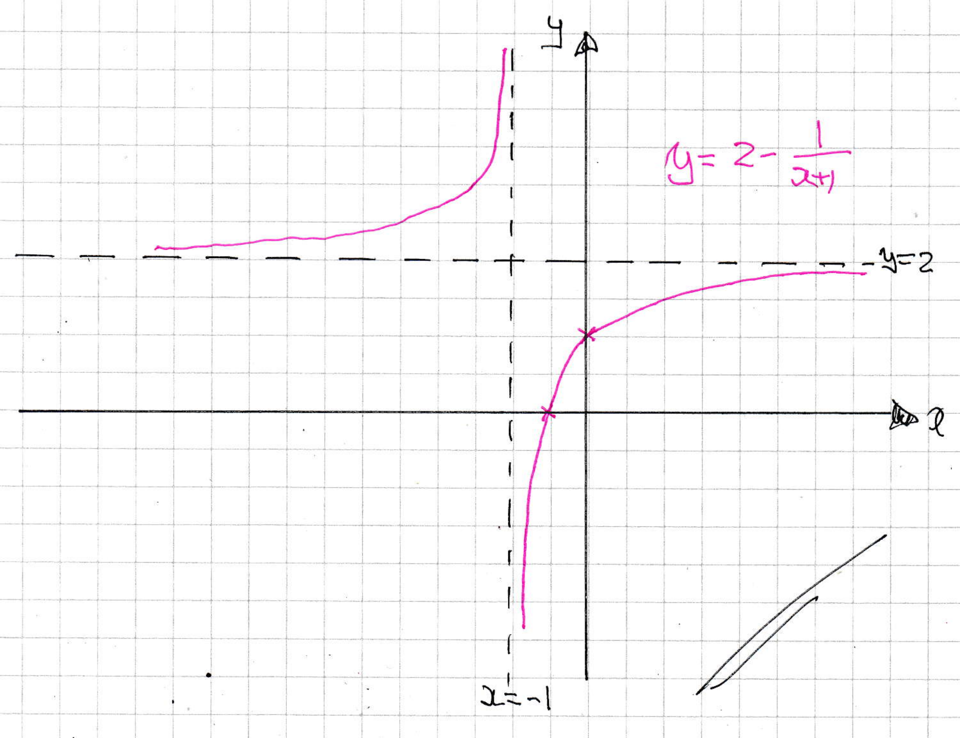
- $x=-1$ IS AN ASYMPTOTE (DIVIDING BY ZERO)

- $y=2$ IS AN ASYMPTOTE (AS $x \to \pm\infty$)

CAN ALSO BE SEEN BY FOLLOWING THE TRANSFORMATIONS



FINALLY A SKETCH



IYGB - SYNOPTIC PAPER U - QUESTION 5

c) FINALLY SOLVING THE FRACTIONAL EQUATION

$$2 - \frac{1}{x+1} = \frac{1}{x}$$

$\times x$

$$2x - \frac{x}{x+1} = 1$$

$\times (x+1)$

$$2x(x+1) - x = x+1$$

$$2x^2 + \cancel{2x} - \cancel{x} = \cancel{x} + 1$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$\therefore x = \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2}$$

IYGB - SYNOPTIC PAPER U - QUESTION 6

a) FORMING 2 EQUATIONS BY THE REMAINDER THEOREM

$f(2) = -7 \implies 2^5 + a \times 2^4 + b \times 2^3 - 2^2 + 4 \times 2 - 3 = -7$

$32 + 16a + 8b \cancel{-4} + 8 - \cancel{3} = \cancel{-7}$

$16a + 8b = -40$

$2a + b = -5$

$f(-1) = -16 \implies (-1)^5 + a \times (-1)^4 + b \times (-1)^3 - (-1)^2 + 4(-1) - 3 = -16$

$-1 + a - b - 1 - 4 - 3 = -16$

$a - b = -7$

ADDING THE EQUATIONS

$3a = -12$

$a = -4$

$a - b = -7$

$-4 - b = -7$

$b = 3$

b) QUICKEST WAY TO GET THIS PART IS AS FOLLOWS

$f(x) = (x+1)(x-2)\underbrace{g(x)}_{\text{CUBIC}} + \underbrace{Ax + B}_{\text{BECAUSE } (x+1)(x-2) \text{ IS A QUADRATIC}}$

$f(2) = -7 \implies 2A + B = -7$

$f(-1) = -16 \implies -A + B = -16$

SUBTRACT $3A = 9$

$A = 3, \quad B = -13$

$\therefore$ REMAINDER IS $3x - 13$

IYGB - SYNOPTIC PAPER U - QUESTION 2

STARTING BY FINDING THE VALUES OF m FOR REPEATED ROOTS

$\Rightarrow b^2 - 4ac = 0 \quad \leftarrow \quad [mx^2 - 4x + (m-3) = 0]$

$\Rightarrow (-4)^2 - 4 \times m \times (m-3) = 0$

$\Rightarrow 16 - 4m(m-3) = 0$

$\Rightarrow 16 - 4m^2 + 12m = 0$

$\Rightarrow 0 = 4m^2 - 12m - 16$

$\Rightarrow m^2 - 3m - 4 = 0$

$\Rightarrow (m+1)(m-4) = 0$

$\Rightarrow m = \begin{cases} -1 \\ 4 \end{cases}$

NOW SOLVE $mx^2 - 4x + m = 3$ FOR EACH OF THE TWO VALUES OF m

IF $m = -1$

$-x^2 - 4x - 1 = 3$

$-x^2 - 4x - 4 = 0$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$

IF $m = 4$

$4x^2 - 4x + 4 = 3$

$4x^2 - 4x + 1 = 0$

$(2x-1)^2 = 0$

$x = \frac{1}{2}$

IYGB - SYNOPTIC PAPER U - QUESTION 8

a) COLLECT AUXILIARIES FOR EACH OBJECT

- $y = 2|x^2 - 6x + 8|$

$y = 2|(x-2)(x-4)|$

$(2,0)\ ,\ (4,0)\ ,\ (0,16)$

- $y = 3x - 9$

$(3,0)\ ,\ (0,-9)$

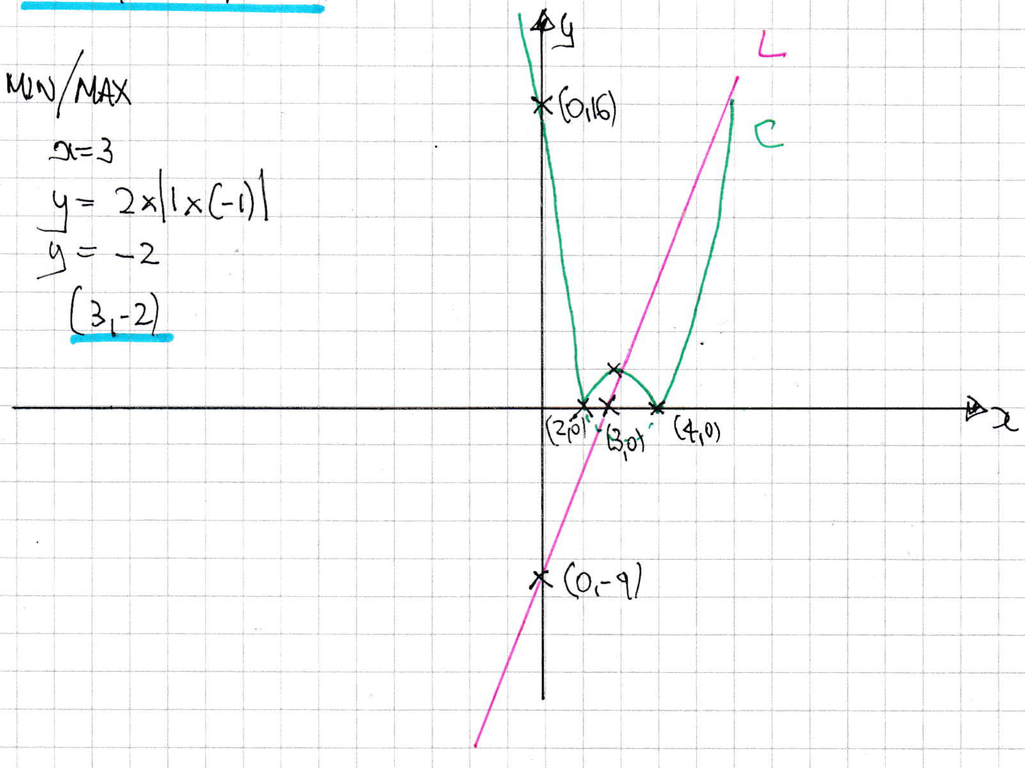
- MIN/MAX

$x = 3$

$y = 2 \times |1 \times (-1)|$

$y = -2$

$(3,-2)$



b) THERE ARE TWO INTERSECTIONS/SOLUTIONS

$2|x^2 - 6x + 8| = 3x - 9$

$\Rightarrow \left[\begin{array}{l} 2(x^2-6x+8) = 3x-9 \\ 2(x^2-6x+8) = 9-3x \end{array}\right.$

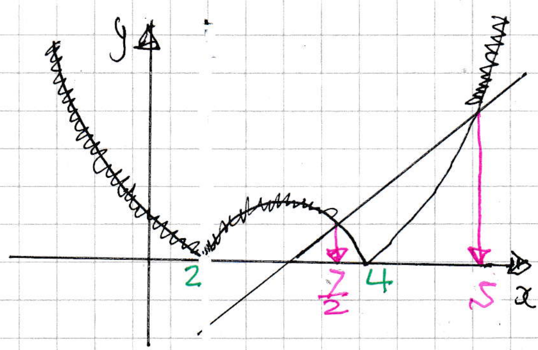
$\Rightarrow \left[\begin{array}{l} 2x^2 - 12x + 16 = 3x - 9 \\ 2x^2 - 12x + 16 = 9 - 3x \end{array}\right.$

$\Rightarrow \left[\begin{array}{l} 2x^2 - 15x + 25 = 0 \\ 2x^2 - 9x + 7 = 0 \end{array}\right.$

$\Rightarrow \left[\begin{array}{l} (2x-5)(x-5) = 0 \\ (2x-7)(x-1) = 0 \end{array}\right.$

$x = \begin{cases} \cancel{5/2} \\ 5 \\ 7/2 \\ \cancel{1} \end{cases}$

c) LOOKING AT A DIAGRAM WITH BETTER SCALE



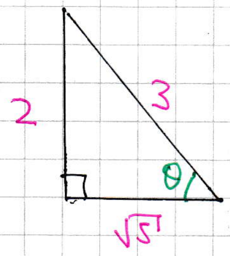
$\therefore\ x < \frac{7}{2}$ OR $x > 5$

IYGB - SYNOPTIC PAPER V - QUESTION 9

METHOD A — $2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$

LET $\theta = \arcsin\frac{2}{3}$, SO WE CAN GET RATIOS OFF A TRIANGLE

$\sin\theta = \frac{2}{3}$



THUS $2\theta = \psi$, FOR SOME ψ TO BE FOUND

$$\Rightarrow \cos 2\theta = \cos\psi$$
$$\Rightarrow 1 - 2\sin^2\theta = \cos\psi$$
$$\Rightarrow 1 - 2\times\left(\frac{2}{3}\right)^2 = \cos\psi$$
$$\Rightarrow 1 - \frac{8}{9} = \cos\psi$$
$$\Rightarrow \cos\psi = \frac{1}{9}$$
$$\Rightarrow \psi = \arccos\frac{1}{9}$$

$$\therefore \quad 2\theta = \psi$$
$$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$$

METHOD B — $2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$ (VARIANT)

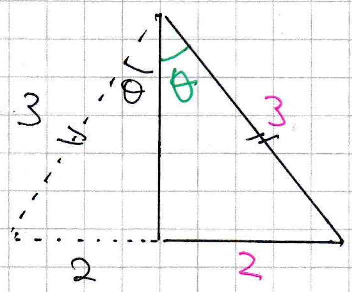
$$\sin\theta = \frac{2}{3} \qquad \left(\theta = \arcsin\frac{2}{3}\right)$$
$$\sin^2\theta = \frac{4}{9}$$
$$-\sin^2\theta = -\frac{4}{9}$$
$$-2\sin^2\theta = -\frac{8}{9}$$
$$1 - 2\sin^2\theta = 1 - \frac{8}{9}$$
$$\cos 2\theta = \frac{1}{9}$$
$$2\theta = \arccos\frac{1}{9}$$
$$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$$

IYGB

METHOD C — GEOMETRICAL

$$\arcsin\frac{2}{3} = \theta$$

$$\sin\theta = \frac{2}{3}$$



NOW BY THE COSINE RULE

$$4^2 = 3^2 + 3^2 - 2\times3\times3\times\cos2\theta$$

$$16 = 9 + 9 - 18\cos2\theta$$

$$18\cos2\theta = 2$$

$$\cos2\theta = \frac{1}{9}$$

$$2\theta = \arccos\frac{1}{9}$$

$$2\arcsin\frac{2}{3} = \arccos\frac{1}{9}$$

IYGB - SYNOPTIC PAPER U - QUESTION 10

MODEL AS FOLLOWS

1 SHEET CUTS 7% OF THE LIGHT $\Longrightarrow$ IT ALLOWS 93% = 0.93

2 SHEETS $\Longrightarrow$ ALLOW $0.93 \times 0.93 = 0.93^2$

3 SHEETS $\Longrightarrow$ ALLOW $0.93 \times 0.93^2 = 0.93^3$

ETC

WE NEED TO CUT OUT AT LEAST 95% OF THE LIGHT, IE ALLOW AT MOST 5%

$$u_n = ar^{n-1}$$

$$\Longrightarrow 0.93 \times 0.93^{n-1} \leqslant 0.05$$

$$\Longrightarrow 0.93^n \leqslant 0.05$$

$$\Longrightarrow \log(0.93^n) \leqslant \log(0.05)$$

$$\Longrightarrow n\log(0.93) \leqslant \log(0.05)$$

$$\Longrightarrow n \geqslant \frac{\log(0.05)}{\log(0.93)} \qquad \left[\log(0.93) < 0\right]$$

$$\Longrightarrow n \geqslant 41.2801\ldots$$

$$\therefore n = 42$$

IYGB – SYNOPTIC PAPER U – QUESTION 11

MANIPULATE AS FOLLOWS

$\Rightarrow y = 2\arcsin 3x$

$\Rightarrow \frac{1}{3}\pi = 2\arcsin 3x$

$\Rightarrow \frac{\pi}{6} = \arcsin 3x$

$\Rightarrow \sin\frac{\pi}{6} = \sin(\arcsin 3x)$

$\Rightarrow 3x = \frac{1}{2}$

$\Rightarrow x = \frac{1}{6}$

OBTAIN $\frac{dy}{dx}$ OR $\frac{dx}{dy}$

$\Rightarrow y = 2\arcsin 3x$

$\Rightarrow \frac{1}{2}y = \arcsin 3x$

$\Rightarrow \sin\left(\frac{1}{2}y\right) = 3x$

$\Rightarrow x = \frac{1}{3}\sin\left(\frac{1}{2}y\right)$

$\Rightarrow \frac{dx}{dy} = \frac{1}{6}\cos\left(\frac{1}{2}y\right)$

EVALUATE THE GRADIENT AT $\left(\frac{1}{6}, \frac{\pi}{3}\right)$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{6}\cos\left(\frac{1}{2}\times\frac{\pi}{3}\right) = \frac{1}{6}\cos\frac{\pi}{6} = \frac{1}{6}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{12}$$

$$\Rightarrow \frac{dy}{dx} = \frac{12}{\sqrt{3}}$$

$$\Rightarrow \frac{dy/dt}{dx/dt} = \frac{12}{\sqrt{3}}$$

$$\Rightarrow \frac{2}{dx/dt} = \frac{12}{\sqrt{3}}$$

$$\Rightarrow 12\frac{dx}{dt} = 2\sqrt{3}$$

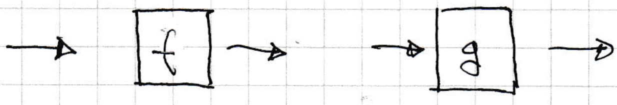
$$\Rightarrow \frac{dx}{dt} = \frac{1}{6}\sqrt{3}$$

IYGB - SYNOPTIC PAPER U - QUESTION 12

a) STANDARD METHODOLOGY

$$g(f(x)) = g(\sqrt{x+4}) = 2\left[\sqrt{x+4}\right]^2 - 3 = 2(x+4) - 3 = 2x+5$$

b) LOOKING AT A DIAGRAM



$x \geqslant -3$ $\qquad$ $x \leqslant 47$

$\Downarrow$

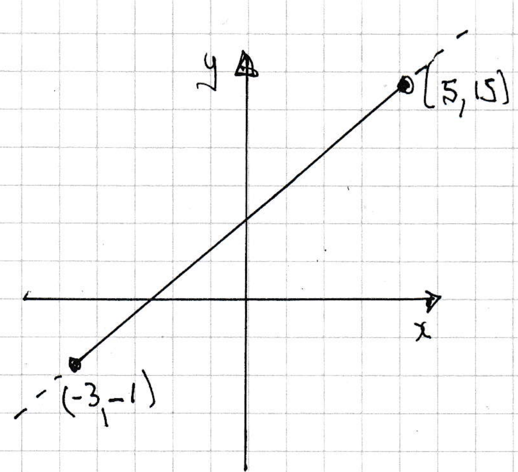
$f(x) \geqslant 1$ BEFORE IT GOES INTO g — COMBINING

$$g(x) \leqslant 47$$
$$2x^2 - 3 \leqslant 47$$
$$2x^2 \leqslant 50$$
$$x^2 \leqslant 25$$
$$-5 \leqslant x < 5$$

COMBINING THE INEQUALITIES $x \geqslant -3$ TO GO INTO INTO $f(x)$ AND THE OUTPUT OF f TO GO INTO $g(x)$ WE NEED $-5 \leqslant x \leqslant 5$

$\therefore$ DOMAIN $-3 \leqslant x \leqslant 5$

FOR THE RANGE OF $g[f(x)] = 2x+5$



RANGE $-1 \leqslant g(f(x)) \leqslant 15$

IYGB - SYNOPTIC PAPER U - QUESTION 12

c) FORMING $f(g(x))$ AS PART OF THE REQUIRED EQUATION

$$\Rightarrow f(g(x)) = 17$$

$$\Rightarrow f(2x^2-3) = 17$$

$$\Rightarrow \sqrt{(2x^2-3)+4} = 17$$

$$\Rightarrow \sqrt{2x^2+1} = 17$$

$$\Rightarrow 2x^2+1 = 17^2$$

$$\Rightarrow 2x^2 = 288$$

$$\Rightarrow x^2 = 144$$

$$\Rightarrow x = \begin{cases} \cancel{12} \\ -12 \end{cases}$$

CANNOT GO INTO $g(x)$

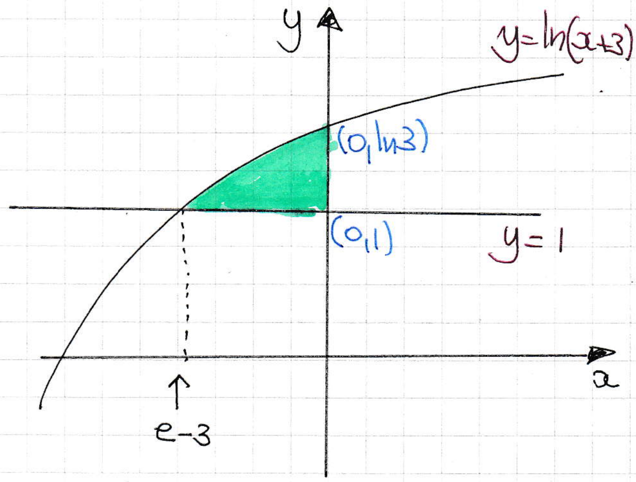
MYGB – SYNOPTIC PAPER U – QUESTION 13

LOOKING AT THE DIAGRAM

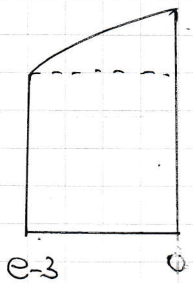
- $\ln(x+3) = 1$

  $x+3 = e$

  $x = e-3$



NEXT WE FIND THE AREA UNDER THE CURVE BETWEEN $e-3$ & 0



$$= \int_{e-3}^{0} \ln(x+3)\,dx = \dots \text{ BY PARTS}$$

$\ln(x+3)$	$\frac{1}{x+3}$
x	1

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} \frac{x}{x+3}\,dx$$

MANIPULATE, LONG DIVIDE OR ANOTHER SUBSTITUTION

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} \frac{(x+3)-3}{x+3}\,dx$$

$$= \Big[x\ln(x+3)\Big]_{e-3}^{0} - \int_{e-3}^{0} 1 - \frac{3}{x+3}\,dx$$

$$= \Big[x\ln(x+3) - x + 3\ln(x+3)\Big]_{e-3}^{0}$$

$$= (0 - 0 + 3\ln3) - \Big[(e-3)\ln e - (e-3) + 3\ln e\Big]$$

$$= 3\ln3 - \Big[\cancel{(e-3)} - \cancel{(e-3)} + 3\Big]$$

$$= -3 + 3\ln3$$

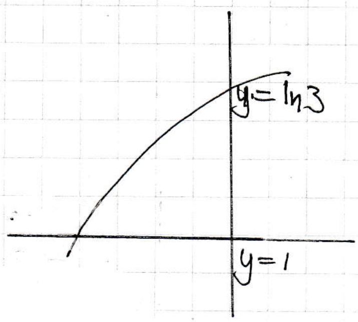
IYGB - SYNOPTIC PAPER U - QUESTION 13

FINALLY THE AREA OF THE RECTANGLE MEASURING $|e-3|$ BY 1

I.E $3-e$

$\therefore$ REQUIRED AREA IS $(-3+3\ln 3)-(3-e) = \underline{e+3\ln 3-6}$

ALTERNATIVE BY "y" INTEGRATION



- $y = \ln(x+3)$

$e^y = x+3$

$x = e^y - 3$

- $\text{AREA} = \int_{y_1}^{y_2} x(y)\,dy$

$$= \int_1^{\ln 3} (e^y - 3)\,dy$$

$$= \Big[e^y - 3y\Big]_1^{\ln 3}$$

$$= \left(e^{\ln 3} - 3\ln 3\right) - (e-3)$$

$$= 3 - 3\ln 3 - e + 3$$

$$= 6 - 3\ln 3 - e$$

BUT THIS IS "BELOW THE y AXIS" SO NEGATIVE

$\therefore$ AREA $= |6-3\ln 3 - e| = \underline{e+3\ln 3-6}$

AS BEFORE

IYGB - SYNOPTIC PAPER U - QUESTION 14

USING THE IDENTITY $\sin(A+B) \equiv \sin A \cos B + \cos A \sin B$ IN RADIANS

$$\Rightarrow \sin(1.01^c) = \underbrace{\sin 1^c}_{\text{GIVEN}} \underbrace{\cos(0.01^c)}_{\text{SMALL ANGLE}} + \underbrace{\cos 1^c}_{\text{GIVEN}} \underbrace{\sin(0.01)^c}_{\text{SMALL ANGLE}}$$

$$\simeq 0.8415 \times \left(1 - \frac{0.01^2}{2}\right) + 0.5403 \times 0.01$$

JUST USING 1 IS GOOD ENOUGH HERE

$$\simeq 0.841457\ldots + 0.005403\ldots$$

$$\simeq 0.84686\ldots$$

$$\therefore \sin(1.01^c) \simeq 0.847 \quad \text{3 d.p.}$$

IYGB - SYNOPTIC PAPER U - QUESTION 15

BY SUBSTITUTION

$$\left.\begin{array}{l}\sqrt{x} = 2y+3 \\ 2x + \sqrt{x}(1-2y) = 8\end{array}\right\} \Rightarrow 2(2y+3)^2 + (2y+3)(1-2y) = 8$$

$$\Rightarrow 2(4y^2+12y+9) + (2y-4y^2+3-6y) = 8$$

$$\Rightarrow \left(\begin{array}{l} 8y^2+24y+18 \\ -4y^2-4y+3\end{array}\right. = 8$$

$$\Rightarrow 4y^2 + 20y + 13 = 0$$

COMPLETING THE SQUARE (OR QUADRATIC FORMULA)

$$y = \frac{-20 \pm \sqrt{20^2 - 4\times4\times13}}{2\times4} = \frac{-20 \pm \sqrt{400-208}}{8} = \frac{-20 \pm \sqrt{192}}{8}$$

$$y = \frac{-20 \pm \sqrt{16\times12}}{8} = \frac{-20 \pm 4\sqrt{12}}{8} = \frac{-20 \pm 8\sqrt{3}}{8}$$

$$y = \left\langle \begin{array}{l} -\frac{5}{2} + \sqrt{3} \\ -\frac{5}{2} - \sqrt{3}\end{array}\right.$$

OBTAINING THE VALUE OF x

$$\Rightarrow \sqrt{x} = 2y+3$$

$$\Rightarrow \left(\begin{array}{l} \sqrt{x} = 2(-\frac{5}{2}+\sqrt{3})+3 = -5+2\sqrt{3}+3 = -2+2\sqrt{3} > 0 \\ \sqrt{x} = 2(-\frac{5}{2}-\sqrt{3})+3 = -5-2\sqrt{3}+3 = \cancel{-2-2\sqrt{3}} < 0\end{array}\right.$$

$$\Rightarrow x = (-2+2\sqrt{3})^2$$

$$\Rightarrow x = 4 - 8\sqrt{3} + 12$$

$$\Rightarrow x = 16 - 8\sqrt{3}$$

$$\therefore \left(16-8\sqrt{3}, -\frac{5}{2}-\sqrt{3}\right)$$

P.T.O

IYGB − SYNOPTIC PAPER U − QUESTION 15

ALTERNATIVE BY COMPLETING THE SQUARE − NO FORMULA

$$\Rightarrow 4y^2 + 20y + 13 = 0$$
$$\Rightarrow y^2 + 5y + \frac{13}{4} = 0$$
$$\Rightarrow \left(y + \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{13}{4} = 0$$
$$\Rightarrow \left(y + \frac{5}{2}\right)^2 = 3$$
$$\Rightarrow y + \frac{5}{2} = \pm\sqrt{3}$$

$\therefore y = -\frac{5}{2} \pm \sqrt{3}$

ETC ETC

TOTALLY DIFFERENT START

$$\left.\begin{aligned} 2y &= \sqrt{x} - 3 \\ 2x + \sqrt{x} - 2y\sqrt{x} &= 8 \end{aligned}\right\} \Rightarrow$$

$$2x + \sqrt{x} - (\sqrt{x} - 3)\sqrt{x} = 8$$
$$2x + \sqrt{x} - x + 3\sqrt{x} = 8$$
$$x + 4\sqrt{x} - 8 = 0$$
$$(\sqrt{x} + 2)^2 - 4 - 8 = 0$$
$$(\sqrt{x} + 2)^2 = 12$$
$$\sqrt{x} + 2 = \pm\sqrt{12}$$
$$\sqrt{x} = -2 \pm 2\sqrt{3}$$
$$+\sqrt{x} = -2 + 2\sqrt{3}$$
$$x = (-2 + 2\sqrt{3})^2$$
$$x = 4 - 8\sqrt{3} + 12$$
$$x = 6 - 8\sqrt{3}$$

AND
$$2y = \sqrt{x} - 3$$
$$2y = -2 + 2\sqrt{3} - 3$$
$$2y = -5 + 2\sqrt{3}$$
$$y = -\frac{5}{2} + \sqrt{3}$$

AS BEFORE

IYGB - SYNOPTIC PAPER U - QUESTION 16

REWRITE AS A QUADRATIC

$$\Rightarrow e^{\frac{3}{2}x} = e^{3x} - 2$$

$$\Rightarrow 0 = e^{3x} - e^{\frac{3}{2}x} - 2$$

$$\Rightarrow \underbrace{\left(e^{\frac{3}{2}x}\right)^2}_{e^{3x}} - e^{\frac{3}{2}x} + 2 = 0$$

LET $A = e^{\frac{3}{2}x}$ FOR SIMPLICITY

$$\Rightarrow A^2 - A + 2 = 0$$

$$\Rightarrow (A+1)(A-2) = 0$$

$$\Rightarrow A = \begin{cases} -1 \\ 2 \end{cases}$$

$$\Rightarrow e^{\frac{3}{2}x} = \begin{cases} \cancel{-1} \\ 2 \end{cases}$$

$$\Rightarrow \frac{3}{2}x = \ln 2$$

$$\Rightarrow x = \frac{2}{3}\ln 2$$

IYGB - SYNOPTIC PAPER U - QUESTION 17

USING THE SUBSTITUTION GIVEN

$$u = \sin x + x\tan x$$

$$\frac{du}{dx} = \cos x + \tan x + x\sec^2 x$$

$$dx = \frac{1}{\cos x + \tan x + x\sec^2 x}\,du$$

TRANSFORM THE GIVEN INTEGRAL

$$\int \frac{2x + \sin 2x + 2\cos^3 x}{(x+\cos x)\sin 2x}\,dx$$

$$= \int \frac{2x + 2\sin x\cos x + 2\cos^3 x}{(x+\cos x)(2\sin x\cos x)} \times \frac{1}{\cos x + \tan x + x\sec^2 x}\,du$$

$$= \int \frac{x + \sin x\cos x + \cos^3 x}{(x+\cos x)\sin x\cos x} \times \frac{1}{\cos x + \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}}\,du$$

MULTIPLY TOP & BOTTOM BY $\cos^2 x$

$$= \int \frac{\cancel{x + \sin x\cos x + \cos^3 x}}{(x+\cos x)\sin x\cos x} \times \frac{\cos^2 x}{\cancel{\cos^3 x + \sin x\cos x + x}}\,du$$

$$= \int \frac{\cos^2 x}{x\sin x\cos x + \sin x\cos^2 x}\,du$$

[DIVIDE TOP & BOTTOM BY $\cos^2 x$]

$$= \int \frac{1}{x\tan x + \sin x}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\sin x + x\tan x| + C$$

IYGB - SYNOPTIC PAPER U - QUESTION 18

a) REWRITE THE EXPRESSION IN SINE OR COSINE ONLY

$h(t) = 10 + \sqrt{3}\sin 30t + \cos 30t$

$h(t) = 10 + 2\left(\frac{\sqrt{3}}{2}\sin 30t + \frac{1}{2}\cos 30t\right)$

$h(t) = 10 + 2\left[\cos 30 \sin 30t + \sin 30 \cos 30t\right]$

$h(t) = 10 + 2\sin(30t + 30)$

$h(t) = 10 + \sqrt{3}\sin 30t + \cos 30t$

$h(t) = 10 + 2\left(\frac{\sqrt{3}}{2}\sin 30t + \frac{1}{2}\cos 30t\right)$

$h(t) = 10 + 2\left[\sin 60 \sin 30t + \cos 60 \cos 30t\right]$

$h(t) = 10 + 2\cos(30t - 60)$

$h(t)$ MAXIMUM $= 10 + 2 = 12$ (HIGH TIDE)

$h(t)$ MINIMUM $= 10 - 2 = 8$ (LOW TIDE)

- $\sin(30t + 30) = 1$
  $30t + 30 = 90$
  $30t = 60$
  $t = 2$

- $\sin(30t + 30) = -1$
  $30t + 30 = -90$
  $30t = -120$
  $t = -4 \quad \xrightarrow{+12}$
  $t = 8$

OR

- $\cos(30t - 60) = 1$
  $30t - 60 = 0$
  $30t = 60$
  $t = 2$

- $\cos(30t - 60) = -1$
  $30t - 60 = 180$
  $30t = 240$
  $t = 8$

$\therefore$ AT 02:00 HIGH TIDE OF 12m

AT 08:00 LOW TIDE OF 8m

b) SOLVING AN EQUATION $h(t) = 8.5$

$\Rightarrow 8.5 = 10 + 2\sin(30t + 30)$

$\Rightarrow 2\sin(30t + 30) = -1.5$

$\Rightarrow \sin(30t + 30) = -0.75$

$\arcsin(-0.75) = -48.59°$

IYGB - SYNOPTIC PAPER U - QUESTION 18

$$\begin{cases} 30t+30 = -48.59 \pm 360n \\ 30t+30 = 228.59 \pm 360n \end{cases}$$

$$\begin{cases} 30t = -78.59 \pm 360n \\ 30t = 198.59 \pm 360n \end{cases} \qquad n=0,1,2,3\ldots$$

$$\begin{cases} t = -2.62 \pm 12n \\ t = 6.62 \pm 12n \end{cases}$$

- $t = -2.62 \ (+12\ldots) = 9.38$
  $= 9 + 0.38 \times 60$
  $= 9 + 22.8$ — MINUTES

  i.e. 9:23

- $t = 6.62$

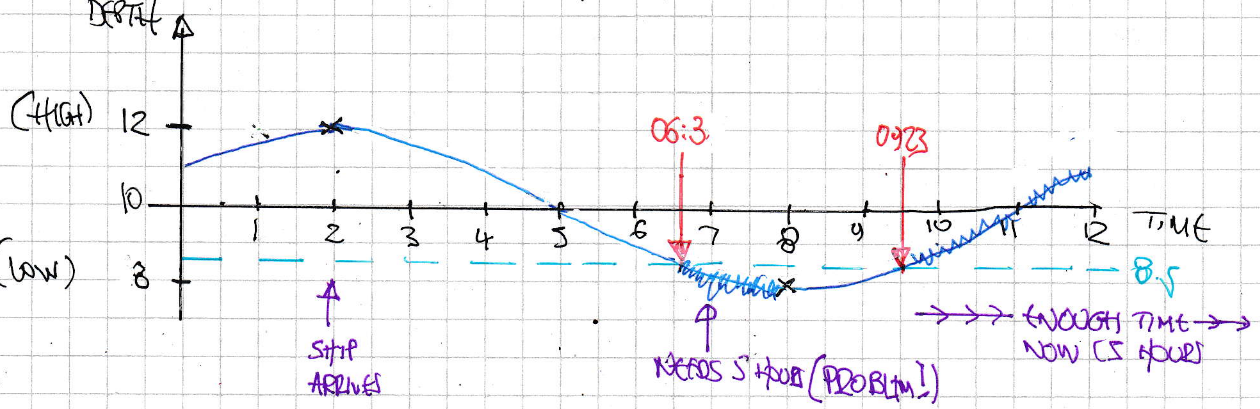
  i.e. 6 HOURS + 0.62×60
  6 HOURS + 37.2 MINUTES

  i.e. 06:37

INTERPRETING THE TIMES — LOOKING AT THE GRAPH

- ARRIVES AT HIGH TIDE (02:00)
- NEEDS 5 HOURS (07:00)
- BUT AT 06:37 THE DEPTH IS 8.5 SO IT WILL "RUN AGROUND"
- IF IT WAITS ON "ITS WAY" TO HIGH TIDE IT WILL BE O.K

$\therefore$ 09:23 // AS REQUIRED



IYGB - SYNOPTIC PAPER U - QUESTION 19

a) FORMING A DIFFERENTIAL EQUATION

$$\frac{dT}{dt} = -k(T-20)$$

RATE

COOLING

PROPORTIONAL

DIFFERENCE BETWEEN...

T = TEMPERATURE OF WATER (°C)

t = TIME (sec)

$t=0$, $T=40$

$\left.\frac{dT}{dt}\right|_{\substack{t=0\\T=40}} = -0.005$

APPLY THE CONDITION $\left.\frac{dT}{dt}\right|_{T=40} = -0.005$

$$\Rightarrow -0.005 = -k(40-20)$$

$$\Rightarrow -0.005 = -20k$$

$$\Rightarrow k = -\frac{1}{4000}$$

$$\therefore \frac{dT}{dt} = -\frac{1}{4000}(T-20)$$

// AS REQUIRED

b) SOLVE BY SEPARATION OF VARIABLES

$$\Rightarrow \frac{dT}{dt} = -\frac{1}{4000}(T-20)$$

$$\Rightarrow \frac{1}{T-20}\,dT = -\frac{1}{4000}\,dt$$

$$\Rightarrow \int \frac{1}{T-20}\,dT = \int -\frac{1}{4000}\,dt$$

$$\Rightarrow \ln|T-20| = -\frac{1}{4000}t + C$$

IYGB - SYNOPTIC PAPER U - QUESTION 19

$$\Longrightarrow \quad T - 20 = e^{-\frac{1}{4000}t + C}$$

$$\Longrightarrow \quad T - 20 = e^{-\frac{1}{4000}t} \times e^{C}$$

$$\Longrightarrow \quad T = 20 + Ae^{-\frac{1}{4000}t} \qquad (A = e^{C})$$

APPLY THE CONDITION $t=0$, $T=40$

$$\Longrightarrow \quad 40 = 20 + Ae^{0}$$

$$\Longrightarrow \quad A = 20$$

$$\Longrightarrow \quad T = 20 + 20e^{-\frac{1}{4000}t}$$

FINALLY WHEN $T=36$

$$\Longrightarrow \quad 36 = 20 + 20e^{-\frac{1}{4000}t}$$

$$\Longrightarrow \quad 16 = 20e^{-\frac{1}{4000}t}$$

$$\Longrightarrow \quad \frac{4}{5} = e^{-\frac{1}{4000}t}$$

$$\Longrightarrow \quad e^{\frac{1}{4000}t} = \frac{5}{4}$$

$$\Longrightarrow \quad \frac{1}{4000}t = \ln\left(\frac{5}{4}\right)$$

$$\Longrightarrow \quad t = 4000\ln(1.25)$$

$$\Longrightarrow \quad t \simeq 892.57\ldots \quad (\text{sec})$$

$$\Longrightarrow \quad t \simeq 14.876\ldots \quad (\text{min})$$

$\therefore$ APPROX 15 MIN

IYGB - SYNOPTIC PAPER U - QUESTION 20

a) STANDARD DIFFERENTIATION

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta + 4\cos2\theta}{-4\sin\theta - 4\sin2\theta} = -\frac{\cos\theta + \cos2\theta}{\sin\theta + \sin2\theta}$$

SOLVING FOR ZERO (NUMERATOR MUST EQUAL ZERO)

$$\Rightarrow \cos\theta + \cos2\theta = 0$$

$$\Rightarrow \cos\theta + 2\cos^2\theta - 1$$

$$\Rightarrow 2\cos^2\theta + \cos\theta - 1 = 0$$

$$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$$

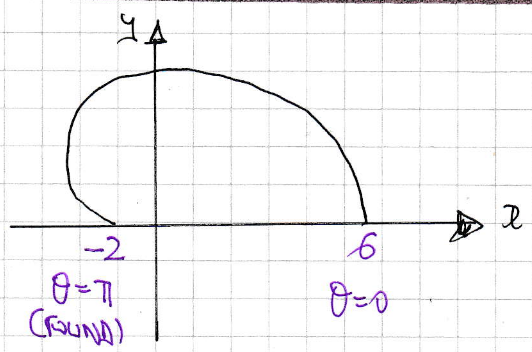
$$\Rightarrow \cos\theta = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$$

-1 ← CUSP AT A YIELDS $A(-2,0)$

$\frac{1}{2}$ YIELDS $\begin{cases} \pi/3 & \text{i.e. } P(1, 3\sqrt{3}) \\ 5\pi/3 & \text{i.e. } \cancel{(1, -3\sqrt{3})} \end{cases}$

↑ SYMMETRICALLY AT THE BOTTOM

b) LOOKING AT THE DIAGRAM (TOP HALF)



BY INSPECTION LOOKING AT

$0 \leq \theta < \pi$

$x_0 = 4\times1 + 2\times1 = 6$

$y_0 = 4\times0 + 2\times0 = 0$

i.e. $(6,0)$

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx$$

$$= \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta$$

$$= \int_{\pi}^{0} (4\sin\theta + 2\sin2\theta)(-4\sin\theta - 4\sin2\theta)\,d\theta$$

$$= \int_{\pi}^{0} -8(2\sin\theta + \sin2\theta)(\sin\theta + \sin2\theta)\,d\theta$$

$$= \int_{0}^{\pi} +8\left(2\sin^2\theta + 3\sin\theta\sin2\theta + \sin^2 2\theta\right)d\theta$$

IYGB - SYNOPTIC PAPER U - QUESTION 20

FINALLY WE HAVE THE REQUIRED ANSWER

$$\text{AREA OF "TOP HALF"} = \int_0^{\pi} 16\sin^2\theta + 24\sin\theta\sin 2\theta + 8\sin^2 2\theta \, d\theta$$

$$= \int_0^{\pi} 16\sin^2\theta + 24\sin\theta(2\sin\theta\cos\theta) + 8\sin^2 2\theta \, d\theta$$

$$= \int_0^{\pi} 16\sin^2\theta + 8\sin^2 2\theta + 48\sin^2\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi} 16\left[\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right] + 8\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4\theta\right) + 48\sin^2\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi} 12 - 8\cos 2\theta - 4\cos 4\theta + 48\sin^2\theta\cos\theta \, d\theta$$

$$= \left[12\theta - 4\sin 2\theta - \sin 4\theta + 16\sin^3\theta\right]_0^{\pi}$$

$$= (12\pi + 0 + 0 + 0) - (0 + 0 + 0 + 0)$$

$$= 12\pi$$

$\therefore$ TOTAL AREA $= 24\pi$

IYGB - SYNOPTIC PAPER U - QUESTION 21

a) BY COVER UP OR STANDARD TECHNIQUES

$$\frac{ax+b}{(1-x)(1-2x)} \equiv \frac{P}{1-x} + \frac{Q}{1+2x}$$

$$ax+b \equiv P(1+2x) + Q(1-x)$$

• IF $x=1$

$a+b=3P$

$P=\dfrac{a+b}{3}$

• IF $x=-\frac{1}{2}$

$-\frac{1}{2}a+b = \frac{3}{2}Q$

$-a+2b = 3Q$

$Q=\dfrac{2b-a}{3}$

b) COMPARING $f(x)$ IN TERMS OF a & b IN THE EXPANSION GIVEN

$$f(x) = 1 + 13x + Ax^2 + Bx^3 + \dots$$

$$\Rightarrow \frac{\frac{a+b}{3}}{1-x} + \frac{\frac{2b-a}{3}}{1+2x} \equiv 1 + 13x + Ax^2 + Bx^3 + \dots \quad \times 3$$

$$\Rightarrow (a+b)(1-x)^{-1} + (2b-a)(1+2x)^{-1} \equiv 3 + 39x + 3Ax^2 + 3Bx^3 + \dots$$

$$\Rightarrow (a+b)(1+x+x^2+x^3) + (2b-a)(1-2x+4x^2-8x^3) \equiv 3 + 39x + 3Ax^2 + 3Bx^3$$

HERE WE USED STANDARD EXPANSIONS

$$\frac{1}{1-x} = 1+x+x^2+x^3+\dots$$

$$\frac{1}{1+x} = 1-x+x^2-x^3+\dots$$

$$\frac{1}{1+2x} = 1-(2x)+(2x)^2-(2x)^3+\dots = 1-2x+4x^2-8x^3$$

COMPARING CONSTANTS

$(a+b)+(2b-a)=3$

$3b=3$

$b=1$

COMPARING x^1

$(a+b)-2(2b-a)=39$

$3a-3b=39$

$a-b=13$

$\therefore a=14$

IYGB - SYNOPTIC PAPER U - QUESTION 21

FINALLY COMPARING $[x^2]$ & $[x^3]$

$(9+b)+4(2b-a)=3A$

$9b-3a=3A$

$A=3b-a$

$A=3-14$

$A=-11$

$(a+b)-8(2b-a)=3B$

$9a-15b=3B$

$3a-5b=B$

$B=3\times14-5\times1$

$B=42-5$

$B=37$

IYGB - SYNOPTIC PAPER U - QUESTION 22

THE QUOTIENT RULE COULD BE USED HERE, BUT A QUICK SUBSTITUTION WHICH ALLOWS TO SPLIT THE FRACTION WOULD BE QUICKER HERE

$$\Rightarrow y = \frac{2x+3}{\sqrt{2x-1}}$$

- LET $t = 2x-1$ — THIS TRANSLATES THE GRAPH 1 UNIT TO THE "RIGHT", THEN IT HALVES THE x CO-ORDS

$$\Rightarrow y = \frac{(t+1)+3}{\sqrt{t}} = \frac{t+4}{t^{1/2}} = t^{1/2} + 4t^{-\frac{1}{2}}$$

- DIFFERENTIATE W.R.T t

$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

- SOLVING FOR ZERO YIELDS

$$\Rightarrow 0 = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

$$\Rightarrow 2t^{-\frac{3}{2}} = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\Rightarrow \frac{2}{t^{3/2}} = \frac{1}{2t^{1/2}}$$

$$\Rightarrow 4t^{\frac{1}{2}} = t^{3/2} \qquad (t \neq 0)$$

$$\Rightarrow \frac{t^{3/2}}{t^{1/2}} = 4$$

$$\Rightarrow t = 4$$

IYGB - SYNOPTIC PAPER V - QUESTION 22

- DIFFERENTIATE AGAIN TO CHECK THE NATURE, WHICH IS NOT AFFECTED BY THESE TRANSFORMATIONS

$$\frac{dy}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - 2t^{-\frac{3}{2}}$$

$$\frac{d^2y}{dt^2} = -\frac{1}{4}t^{-\frac{3}{2}} + 3t^{-\frac{5}{2}} = \frac{1}{4}t^{-\frac{5}{2}}\left[12 - t\right]$$

$$\left.\frac{dy}{dt^2}\right|_{t=4} = \frac{1}{4} \times 4^{-\frac{5}{2}} \times (12-4) = 2 \times \frac{1}{32} = \frac{1}{16} > 0$$

- USING $t=4$ TO FIND THE VALUE OF y (NOT AFFECTED)

$$y = \frac{t+4}{\sqrt{t}}$$

$$\left.y\right|_{t=4} = \frac{4+4}{\sqrt{4}} = 4$$

- REVERSING THE TRANSFORMATION IN x

$$t = 2x-1$$

$$4 = 2x-1$$

$$5 = 2x$$

$$x = \frac{5}{2}$$

HENCE THERE IS A MINIMUM AT $\left(\frac{5}{2}, 4\right)$ //