IYGB – SYNOPTIC PAPER R – QUESTION 1

CARRY OUT THE INTEGRATION FIRST

$$\int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}}\,dx = \int_k^{\frac{1}{2}} 6 \times e^{-(2-3x)}\,dx = \int_k^{\frac{1}{2}} 6e^{3x-2}\,dx$$

$$= \left[2e^{3x-2}\right]_k^{\frac{1}{2}} = 2e^{-\frac{1}{2}} - 2e^{3k-2}$$

NOW SETTING UP AN EQUATION

$$\Longrightarrow \int_k^{\frac{1}{2}} \frac{6}{e^{2-3x}}\,dx = 0.1998$$

$$\Longrightarrow 2\left(e^{-\frac{1}{2}} - e^{3k-2}\right) = 0.1998$$

$$\Longrightarrow \frac{1}{\sqrt{e}} - e^{3k-2} = 0.0999$$

$$\Longrightarrow \frac{1}{\sqrt{e}} - 0.0999 = e^{3k-2}$$

$$\Longrightarrow e^{3k-2} = 0.5066306597\ldots$$

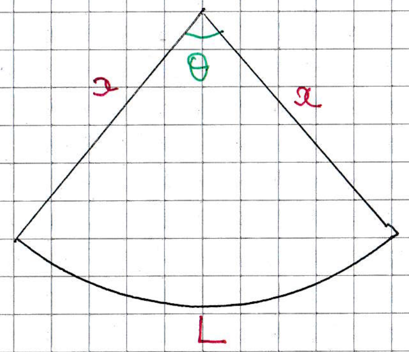
$$\Longrightarrow 3k-2 = \ln(0.5066\ldots)$$

$$\Longrightarrow k = 0.4400089924\ldots$$

$$\therefore k \simeq 0.44$$

IYGB - SYNOPTIC PAPER R - QUESTION 2

a)



CONSTRAINT ON AREA

$$A = 36$$

$$\text{"}\tfrac{1}{2}r^2\theta^c\text{"} = 36$$

$$\tfrac{1}{2}x^2\theta = 36$$

$$x^2\theta = 72$$

$$x(x\theta) = 72$$

$$x\theta = \frac{72}{x}$$

MAIN EQUATION

$$\Rightarrow P = L + 2x$$

$$\Rightarrow P = x\theta + 2x$$

$$\Rightarrow P = 2x + \frac{72}{x}$$

AS REQUIRED

b) DIFFERENTIATE & SOLVE FOR ZERO

$$P = 2x + 72x^{-1}$$

$$\frac{dP}{dx} = 2 - 72x^{-2}$$

$$\Rightarrow 0 = 2 - \frac{72}{x^2}$$

$$\Rightarrow \frac{72}{x^2} = 2$$

$$\Rightarrow 2x^2 = 72$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = +6$$

THIS VALUE OF x MINIMIZES OR MAXIMIZES P

TO CHECK THE "EFFECT" ON P USE $\frac{d^2P}{dx^2}$

$$\Rightarrow \frac{d^2P}{dx^2} = 144x^{-3} = \frac{144}{x^3}$$

$$\Rightarrow \left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{6^3} = \frac{2}{3} > 0$$

INDEED IT MINIMIZES

TO FIND THE MINIMUM VALUE OF P

$$P_{MIN} = 2\times 6 + \frac{72}{6} = 12 + 12 = 24\text{ cm}$$

c) USING THE CONSTRAINT EQUATION WITH $x = 6$

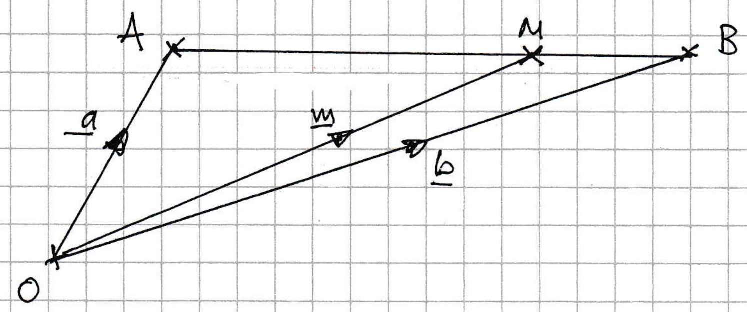
$$\Rightarrow x^2\theta = 72$$

$$\Rightarrow 36\theta = 72$$

$$\therefore \theta = 2^c$$

IYGB - SYNOPTIC PAPER R - QUESTION 3

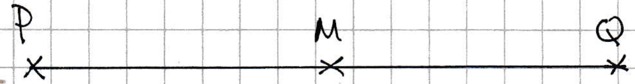
STARTING WITH A DIAGRAM



- $\underline{a} = \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix}$
- $\underline{b} = \begin{pmatrix} 11 \\ 6 \\ 20 \end{pmatrix}$

- $\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 11 \\ 6 \\ 20 \end{pmatrix} - \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 12 \end{pmatrix}$
- $\overrightarrow{AM} = \frac{3}{4}\overrightarrow{AB} = \frac{3}{4}\begin{pmatrix} 16 \\ 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 9 \end{pmatrix}$
- $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \begin{pmatrix} -5 \\ -2 \\ 8 \end{pmatrix} + \begin{pmatrix} 12 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 17 \end{pmatrix}$

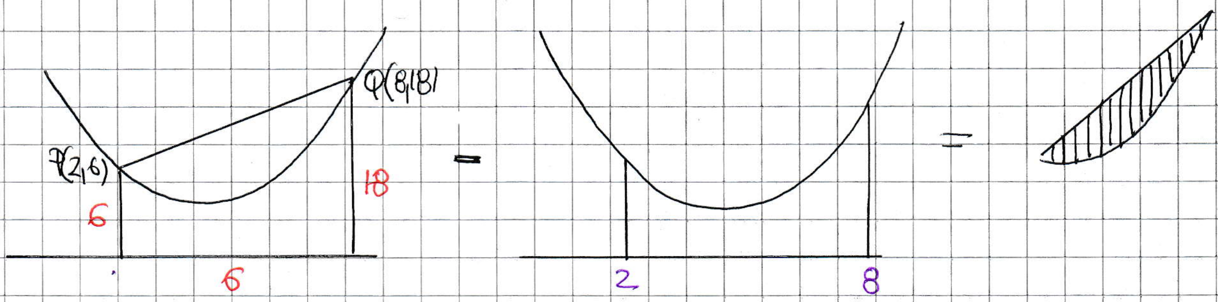
ANOTHER DIAGRAM NOW AND THE POSITION VECTOR Q CAN BE "READ OFF"



$$\begin{pmatrix} 10 \\ 8 \\ 19 \end{pmatrix} \xrightarrow{\begin{smallmatrix} -3 \\ -4 \\ -2 \end{smallmatrix}} \begin{pmatrix} 7 \\ 4 \\ 17 \end{pmatrix} \xrightarrow{\begin{smallmatrix} -3 \\ -4 \\ -2 \end{smallmatrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 15 \end{pmatrix}$$

IYGB - SYNOPTIC PAPER R - QUESTION 4

LOOKING AT THE DIAGRAM BELOW



AREA OF TRAPEZIUM

$$\frac{1}{2}(6+18)\times 6 = 72$$

AREA UNDER CURVE

$$\int_2^8 x^2-8x+18 \, dx$$

COMPLETING THE INTEGRATION

$$\int_2^8 x^2-8x+18 \, dx = \left[\frac{1}{3}x^3-4x^2+18x\right]_2^8$$

$$= \left(\frac{512}{3}-256+144\right)-\left(\frac{8}{3}-16+36\right)$$

$$= \frac{176}{3} - \frac{68}{3}$$

$$= 36$$

$\therefore$ REQUIRED AREA $= 72 - 36 = 36$

IYGB – SYNOPTIC PAPER R – QUESTION 5

USING THE POINT $(9,k)$ WITH THE LINE

$$y = 2x - 12$$
$$k = 2 \times 9 - 12$$
$$k = 6$$

THIS POINT $(9,6)$ ALSO LIES ON THE CIRCLE

$$\Rightarrow x^2 + y^2 - 8x + cy = 33$$
$$\Rightarrow 81 + 36 - 72 + 6c = 33$$
$$\Rightarrow 45 + 6c = 33$$
$$\Rightarrow 6c = -12$$
$$\Rightarrow c = -2$$

NEXT SOLVING SIMULTANEOUSLY – ALSO FIND THE CIRCLE PARTICULARS

$$\Rightarrow x^2 + y^2 - 8x - 2y = 33$$
$$\Rightarrow (x-4)^2 - 16 + (y-1)^2 - 1 = 33$$
$$\Rightarrow (x-4)^2 + (y-1)^2 = 50$$

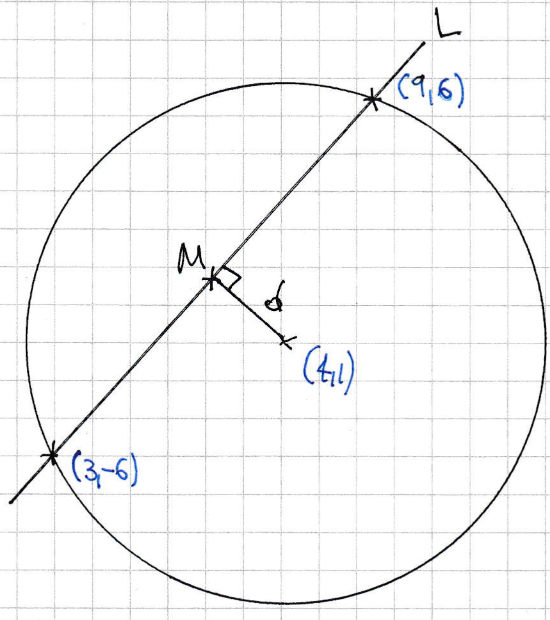
CENTRE AT $(4,1)$, $r = \sqrt{50}$

$$\Rightarrow (x-4)^2 + (2x-12-1)^2 = 50$$
$$\Rightarrow (x-4)^2 + (2x-13)^2 = 50$$
$$\Rightarrow \begin{matrix} x^2 - 8x + 16 \\ 4x^2 - 52x + 169 \end{matrix} = 50$$
$$\Rightarrow 5x^2 - 60x + 135 = 0$$
$$\Rightarrow x^2 - 12x + 27 = 0$$
$$\Rightarrow (x-3)(x-9) = 0$$

$$x = \begin{cases} 3 \\ 9 \end{cases} \leftarrow \text{ALREADY GIVEN} \qquad y = \begin{cases} -6 \\ 6 \end{cases}$$

IYGB - SYNOPTIC PAPER R - QUESTION 5

FINALLY WITH A DIAGRAM, NOTING THE POINTS $(9,6)$ & $(3,-6)$



$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$M\left(\frac{3+9}{2}, \frac{-6+6}{2}\right)$

$M(6,0)$

THE DISTANCE BETWEEN THE POINTS $(6,0)$ & $(4,1)$

$d = \sqrt{(6-4)^2 + (0-1)^2}$

$d = \sqrt{2^2 + (-1)^2}$

$d = \sqrt{5}$

IYGB - SYNOPTIC PAPER B - QUESTION 6

a) EXPAND UP & INCLUDING THE x^3 TERM

$$(1+ax)^k = 1 + \frac{k}{1}(ax)^1 + \frac{k(k-1)}{1\times2}(ax)^2 + \frac{k(k-1)(k-2)}{1\times2\times3}(ax)^3 + \dots$$

$$= 1 + \underbrace{ka}_{8}x + \underbrace{\tfrac{1}{2}k(k-1)a^2}_{30}x^2 + \underbrace{\tfrac{1}{6}k(k-1)(k-2)a^3}_{?}x^3 + \dots$$

SOLVING SIMULTANEOUS EQUATIONS

$$\left.\begin{array}{l} ka = 8 \\ \frac{1}{2}k(k-1)a^2 = 30 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} k^2a^2 = 64 \\ k^2(k-1)a^2 = 60k \end{array}\right\} \Longrightarrow$$

MULTIPLIED BY 2k

$\Longrightarrow \quad 64(k-1) = 60k$

$\Longrightarrow \quad 64k - 64 = 60k$

$\Longrightarrow \quad 4k = 64$

$\Longrightarrow \quad k = 16$

& SINCE $ka = 8$, $a = \frac{1}{2}$

b) TO FIND THE COEFFICIENT OF x^3

$$\frac{1}{6}k(k-1)(k-2)a^3 = \frac{1}{6}\times16\times15\times14\times\left(\frac{1}{2}\right)^3 = 70$$

IYGB – SYNOPTIC PAPER R – QUESTION 7

a) BY THE FACTOR THEOREM/REMAINDER THEOREM

$f(2)=0 \implies 2\times2^3+a\times2^2+b\times2+c=0$
$\implies 16+4a+2b+c=0$

$f(-1)=0 \implies 2(-1)^3+a(-1)^2+b(-1)+c=0$
$\implies -2+a-b+c=0$

$f(1)=14 \implies 2\times1^3+a\times1^2+b\times1+c=14$
$\implies 2+a+b+c=14$

SUBTRACT THE LAST TWO EQUATIONS

$4+2b=-14$
$2b=-18$
$b=-9$

THE EQUATIONS NOW BECOME

$\left.\begin{array}{l}4a+c=2\\ a+c=-7\end{array}\right\} \implies 3a=9$
$a=3$

$a+c=-7$

$c=-10$

$\therefore\ a=3 \quad b=-9 \quad c=-10$

b) FROM PART (a) $(x-2)$ & $(x+1)$ ARE FACTORS

BY INSPECTION

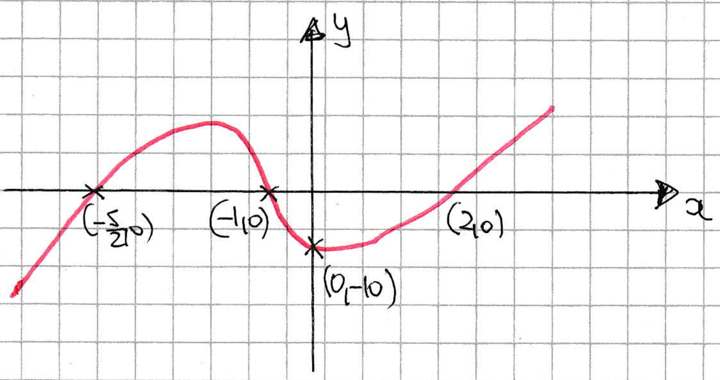
$f(x)=2x^3+3x^2-9x-10$

$f(x)=(x-2)(x+1)(2x+5)$

$+x^3\ldots \implies$ 

$x=0 \quad y=-10 \quad (0,-10)$

$y=0 \quad x=\begin{cases}-1 & (-1,0)\\ 2 & (2,0)\\ -\frac{5}{2} & (-\frac{5}{2},0)\end{cases}$



IYGB - SYNOPTIC PAPER R - QUESTION 8

a) MANIPULATE AS FOLLOWS

$$f(x) = \frac{4x-13}{x-3} = \frac{4(x-3)-1}{x-3} = \frac{4\cancel{(x-3)}}{\cancel{x-3}} - \frac{1}{x-3} = 4 - \frac{1}{x-3}$$

b) WORKING WITH TRANSFORMATIONS, STARTING WITH $y = 1/x$

$$\frac{1}{x} \longmapsto \frac{1}{(x-3)} \longmapsto -\left(\frac{1}{x-3}\right) \longmapsto \left(-\frac{1}{x-3}\right)+4$$

REPLACE x WITH $(x-3)$

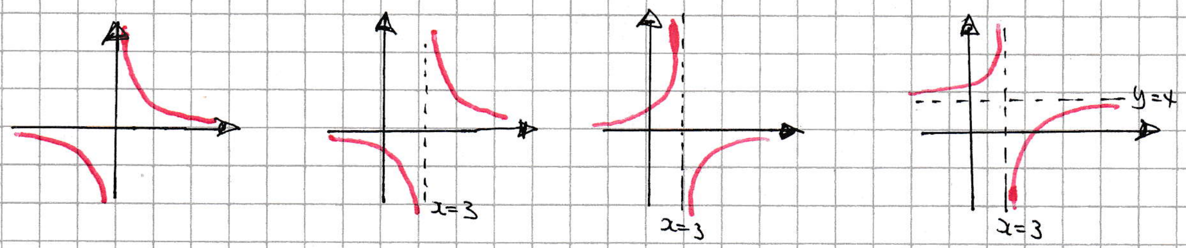
TRANSLATION BY 3 UNITS TO THE "RIGHT"

MULTIPLY THE ENTIRE "FUNCTION" BY -1

REFLECTION ABOUT THE x AXIS

ADD 4 TO THE "ENTIRE FUNCTION"

TRANSLATION BY 4 UNITS "UPWARDS"



OBTAIN THE x & y INTERCEPTS AND SKETCH

• $x=0$

$$y = \frac{4\times0-13}{0-3} = \frac{13}{3}$$

$$\left(0, \frac{13}{3}\right)$$

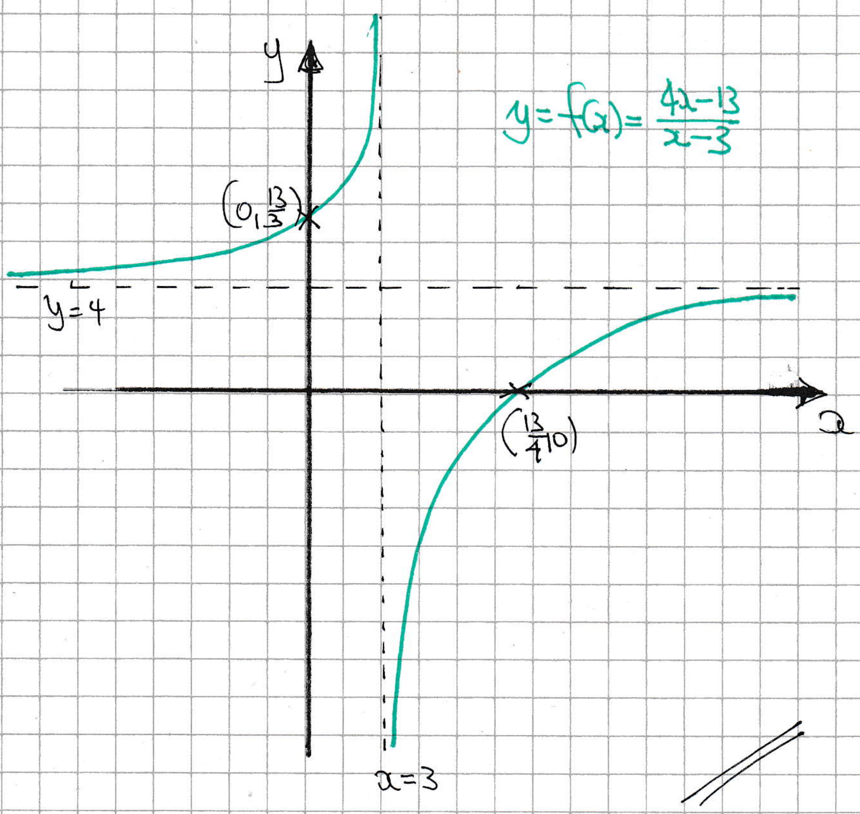
• $y=0$

$$0 = \frac{4x-13}{x-3}$$

$$4x-13=0$$

$$x = 13/4$$

$$\left(\frac{13}{4}, 0\right)$$



IYGB - SYNOPTIC PAPER R - QUESTION 8

c) USING THE ORIGINAL EXPRESSION FOR $f(x)$ TO SOLVE THE REQUIRED EQUATION

$$\Rightarrow \frac{4x-13}{x-3} = \frac{3}{x}$$

$$\Rightarrow 4x^2 - 13x = 3x - 9$$

$$\Rightarrow 4x^2 - 16x + 9 = 0$$

$$\Rightarrow x^2 - 4x + \frac{9}{4} = 0$$

COMPLETING THE SQUARE

$$\Rightarrow (x-2)^2 - 4 + \frac{9}{4} = 0$$

$$\Rightarrow (x-2)^2 = 4 - \frac{9}{4}$$

$$\Rightarrow (x-2)^2 = \frac{7}{4}$$

$$\Rightarrow x - 2 = \pm \frac{\sqrt{7}}{2}$$

$$\Rightarrow x = 2 \pm \frac{1}{2}\sqrt{7}$$

IYGB - SYNOPTIC PAPER R - QUESTION 9

IF $f(n)$ IS TO BE A SQUARE NUMBER, THEN IT MUST BE A PERFECT SQUARE IE REPEATED ROOTS

$$f(n) = n^2 - 2kn + k + 12$$

$$\Rightarrow b^2 - 4ac = 0$$

$$\Rightarrow (-2k)^2 - 4 \times 1 \times (k+12) = 0$$

$$\Rightarrow 4k^2 - 4(k+12) = 0$$

$$\Rightarrow k^2 - (k+12) = 0$$

$$\Rightarrow k^2 - k - 12 = 0$$

$$\Rightarrow (k-4)(k+3) = 0$$

$$\therefore k = \begin{cases} -3 \\ 4 \end{cases}$$

IYGB – SYNOPTIC PAPER R – QUESTION 10

a) START BY OBTAINING THE FIRST TWO DERIVATIVES OF $f(x)$

$$f(x) = e^{3x} - 4e^{-3x}$$
$$f'(x) = 3e^{3x} + 12e^{-3x}$$
$$f''(x) = 9e^{3x} - 36e^{-3x}$$
$$= 9\left[e^{3x} - 4e^{-3x}\right]$$
$$= 9f(x)$$

$$\therefore \; f''(x) > 0 \implies 9f(x) > 0 \implies f(x) > 0$$

INDEED THE SAME SOLUTION INTERVAL

b) SOLVING $f(x) > 0$

$$\implies e^{3x} - 4e^{-3x} > 0$$
$$\implies e^{-3x}\left(e^{6x} - 4\right) > 0$$
$$\implies e^{6x} - 4 > 0 \quad \left[e^{-3x} > 0\right]$$
$$\implies e^{6x} > 4$$
$$\implies 6x > \ln 4 \quad \left[= 2\ln 2\right]$$
$$\implies 3x > \ln 2$$
$$\implies x > \tfrac{1}{3}\ln 2$$

ALTERNATIVE

$$e^{3x} - 4e^{-3x} > 0$$
$$e^{3x} > 4e^{-3x}$$
$$e^{3x} > \frac{4}{e^{3x}}$$
$$e^{6x} > 4 \quad \left[e^{3x} > 0\right]$$
$$\vdots$$
$$x > \tfrac{1}{3}\ln 2$$

IYGB - SYNOPTIC PAPER R - QUESTION 11

DIFFERENTIATE IMPLICITLY WITH RESPECT TO x

$$\Rightarrow 8x^4 + 32xy^3 + 16y^4 = 1$$

$$\Rightarrow \frac{d}{dx}(8x^4) + \frac{d}{dx}(32xy^3) + \frac{d}{dx}(16y^4) = \frac{d}{dx}(1)$$

$$\Rightarrow 32x^3 + 32y^3 + 96xy^2\frac{dy}{dx} + 64y^3\frac{dy}{dx} = 0$$

$$\Rightarrow x^3 + y^3 + 3xy^2\frac{dy}{dx} + 2y^3\frac{dy}{dx} = 0$$

NOW GRADIENT OF $\frac{1}{2}$

$$\Rightarrow x^3 + \cancel{y^3} + \frac{1}{2}(3xy^2) + \cancel{y^3} = 0$$

$$\Rightarrow 2x^3 + 3xy^2 = 0$$

$$\Rightarrow x(2x^2 + 3y^2) = 0$$

THE TRIVIAL SOLUTION $x=y=0$, DOES NOT SATISFY THE CURVE,

$$\Rightarrow x=0 \qquad y \neq 0$$

RETURNING TO THE EQUATION WITH $x=0$

$$\Rightarrow 16y^4 - 1 = 0$$

$$\Rightarrow y^4 = \frac{1}{16}$$

$$\Rightarrow y = \pm\frac{1}{2}$$

$$\therefore \left(0,-\tfrac{1}{2}\right) \ \& \ \left(0,\tfrac{1}{2}\right)$$

IYGB - SYNOPTIC PAPER R - QUESTION 12

MANIPULATE AS FOLLOWS

$$\tan 2\theta - 3\cot\theta = 0$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = -\frac{3}{\tan\theta}$$

NOW NOTE THAT THE EQUATION WILL HOLD IF THE DENOMINATORS ARE $\pm\infty$, IE $\tan\theta = \pm\infty$ — EXCLUDE THIS SOLUTION

$$\Rightarrow 2\tan^2\theta = -3 + 3\tan^2\theta$$

$$\Rightarrow 3 = \tan^2\theta$$

$$\Rightarrow \tan\theta = \pm\sqrt{3}$$

COLLECTING ALL THE POSSIBILITIES, $\tan\theta = \pm\infty$, $\tan\theta = \sqrt{3}$, $\tan\theta = -\sqrt{3}$

$$\begin{cases} \theta = \frac{\pi}{3} \pm n\pi \\ \theta = -\frac{\pi}{3} \pm n\pi \\ \theta = \frac{\pi}{2} \pm n\pi \\ \theta = -\frac{\pi}{2} \pm n\pi \end{cases} \qquad n = 0, 1, 2, 3, 4$$

$$\therefore \theta = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$$

IYGB - SYNOPTIC PAPER R - QUESTION 13

START BY OBTAINING THE EQUATION OF THE NORMAL AT P

$\Rightarrow \quad y = x^3 - x^2 + 5$

$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 2x$

$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=1} = 3\times1^2 - 2\times1 = 1 \qquad \& \qquad y\big|_{x=1} = 1^3 - 1^2 + 5 = 5$

$\therefore$ NORMAL GRADIENT IS -1 & $P(1,5)$

$\Rightarrow \quad y - y_0 = m(x - x_0)$

$\Rightarrow \quad y - 5 = -1(x - 1)$

$\Rightarrow \quad y - 5 = -x + 1$

$\Rightarrow \quad y = -x + 6$

$\Rightarrow \quad y + x = 6$

SUPPOSE THIS NORMAL MEETS THE CURVE AGAIN

$$\left.\begin{array}{l} y = 6 - x \\ y = x^3 - x^2 + 5 \end{array}\right\} \Rightarrow x^3 - x^2 + 5 = 6 - x$$

$\Rightarrow \quad x^3 - x^2 + x - 1 = 0$

FACTORIZE BY FACTORIZATION IN PAIRS

$\Rightarrow \quad x^2(x-1) + (x-1) = 0$

$\Rightarrow \quad (x-1)(x^2+1) = 0$

ONLY SOLUTION $x = 1$, AS $x^2 + 1 \neq 0$

$\therefore$ ONLY INTERSECTION IS THE POINT OF NORMALITY $(1,5)$ & NO MORE !!

IYGB - SYNOPTIC PAPER R - QUESTION 14

a) NOTING THE REPEATED FACTOR IN THE DENOMINATOR

$$\frac{6x^2-21x+17}{(x-3)(x-1)^2} \equiv \frac{A}{x-3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$\boxed{6x^2-21x+17 \equiv A(x-1)^2 + B(x-3) + C(x-1)(x-3)}$$

- IF $x=1$

$6-21+17=-2B$

$2=-2B$

$B=-1$

- IF $x=3$

$54-63+17=4A$

$8=4A$

$A=2$

- IF $x=0$

$17=A-3B+3C$

$17=2+3+3C$

$12=3C$

$C=4$

$$\therefore f(x) = \frac{2}{x-3} - \frac{1}{(x-1)^2} + \frac{4}{x-1}$$

b) WORK AS FOLLOWS

$$\Rightarrow g(x) = \frac{x^2-11x+12}{(x-3)(x-1)^2} = \frac{(6x^2-5x^2)+(-21x+10x)+(17-5)}{(x-3)(x-1)^2}$$

$$\Rightarrow g(x) = \frac{6x^2-21x+17}{(x-3)(x-1)^2} + \frac{-5x^2+10x-5}{(x-3)(x-1)^2}$$

$$\Rightarrow g(x) = f(x) + \frac{-5\cancel{(x^2-2x+1)}}{(x-3)\cancel{(x-1)^2}}$$

$$\Rightarrow g(x) = f(x) - \frac{5}{x-3}$$

$$\Rightarrow g(x) = \frac{2}{x-3} - \frac{1}{(x-1)^2} + \frac{4}{x-1} - \frac{5}{x-3}$$

$$\Rightarrow g(x) = \frac{4}{x-1} - \frac{1}{(x-1)^2} - \frac{3}{x-3}$$

IYGB - SYNOPTIC PAPER R - QUESTION 15

TAKING NATURAL LOGARITHMS BOTH SIDES

$$e^{4x} = 16^{\frac{1}{\ln 2}}$$

$$\Longrightarrow 4x = \ln 16^{\frac{1}{\ln 2}}$$

$$\Longrightarrow 4x = \frac{1}{\ln 2} \times \ln 16$$

$$\Longrightarrow 4x = \frac{\ln 16}{\ln 2}$$

$$\Longrightarrow 4x = \frac{4\cancel{\ln 2}}{\cancel{\ln 2}}$$

$$\Longrightarrow x = 1$$

ALTERNATIVE FOR FUN

$\log_a b \equiv \dfrac{1}{\log_b a}$

$$e^{4x} = 16^{\frac{1}{\ln 2}} = (2^4)^{\frac{1}{\ln 2}} = \ldots \left[\frac{1}{\ln 2} = \frac{1}{\log_e 2} = \log_2 e\right]$$

$$\ldots = (2^4)^{\log_2 e}$$

$$= 2^{4\log_2 e}$$

$$= 2^{\log_2 e^4}$$

$$= e^4$$

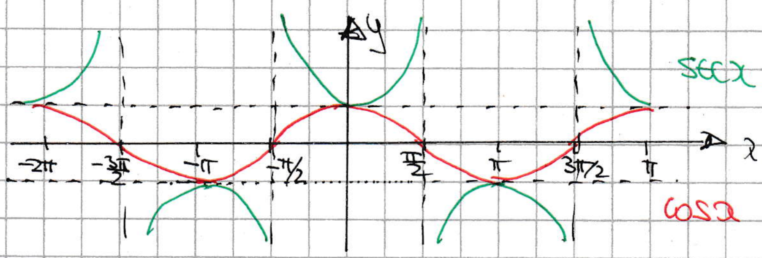
$a^{\log_a x} \equiv x$

$$\therefore e^{4x} = e^4$$

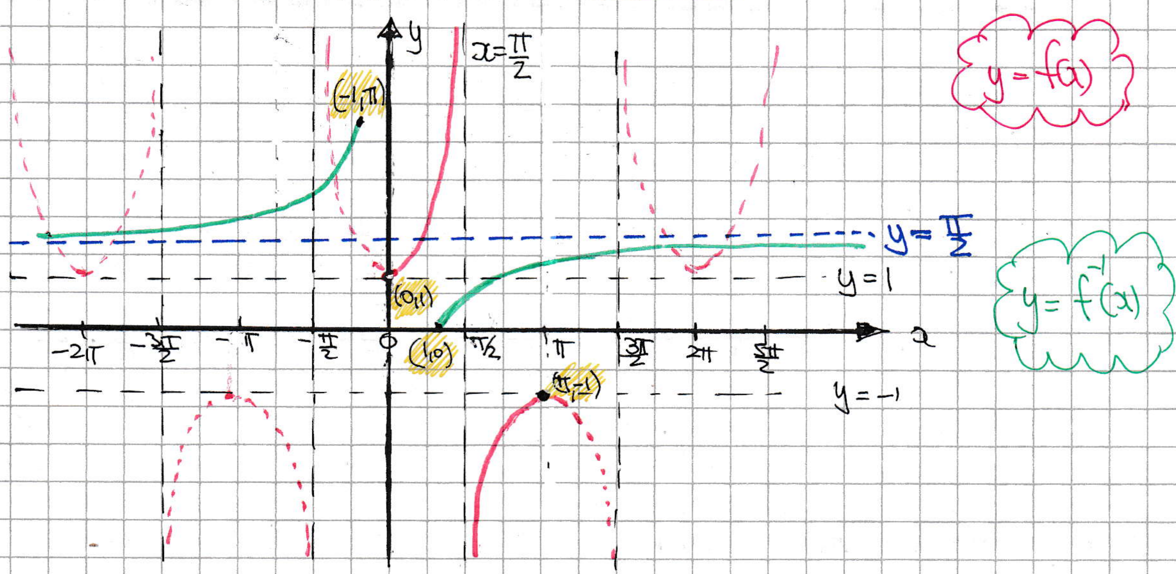
$$\Longrightarrow x = 1$$

IYGB - SYNOPTIC PAPER R - QUESTION 16

a) START BY DRAWING $\sec x$ FROM A COSINE GRAPH



HENCE DRAWING $f(x)$ AND ITS DEFINED INVERSE



b) LOOKING AT THE GRAPH ABOVE FOR $y = f^{-1}(x)$

DOMAIN: $x \leq -1 \cup x \geq 1$

RANGE: $0 \leq f^{-1}(x) \leq \pi, \quad f^{-1}(x) \neq \frac{\pi}{2}$

c) DIRECTLY FROM THE DEFINITION GIVEN

$\Rightarrow y = \text{arcsec}\, x$

$\Rightarrow \sec y = x$

$\Rightarrow \frac{1}{\cos y} = x$

IYGB – SYNOPTIC PAPER R – QUESTION 16

$$\Longrightarrow \cos y = \frac{1}{x}$$

$$\Longrightarrow y = \arccos\left(\frac{1}{x}\right)$$

$$\therefore \text{arcsec}\,x \equiv \arccos\left(\frac{1}{x}\right)$$

d) EITHER USE STANDARD RESULT & ANSWER OF PART (c)

$$\frac{d}{dx}(\text{arcsec}\,x) = \frac{d}{dx}\left(\arccos\left(\frac{1}{x}\right)\right) = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \left(-\frac{1}{x^2}\right) = +\frac{1}{\sqrt{\frac{x^2-1}{x^2}}} \times \frac{1}{\sqrt{x^4}}$$

$$= \frac{1}{\sqrt{\frac{x^2-1}{x^2} \times x^4}} = \frac{1}{\sqrt{(x^2-1)x^2}} = \frac{1}{\sqrt{x^4-x^2}}$$

AS REQUIRED

OR BY THE INVERSE RULE

$$\Longrightarrow y = \text{arcsec}\,x$$

$$\Longrightarrow \sec y = x$$

$$\Longrightarrow x = \sec y$$

$$\Longrightarrow \frac{dx}{dy} = \sec y \tan y$$

$$\Longrightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y \tan^2 y$$

$$\Longrightarrow \left(\frac{dx}{dy}\right)^2 = \sec^2 y(\sec^2 y - 1)$$

$$\Longrightarrow \left(\frac{dx}{dy}\right)^2 = \sec^4 y - \sec^2 y$$

$$\Longrightarrow \left(\frac{dx}{dy}\right)^2 = x^4 - x^2$$

$$\Longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4-x^2}$$

$$\Longrightarrow \frac{dy}{dx} = \pm\frac{1}{\sqrt{x^4-x^2}}$$

$$\Longrightarrow \frac{dy}{dx} = +\frac{1}{\sqrt{x^4-x^2}}$$

POSITIVE GRADIENT IN THE ENTIRE DOMAIN (GRAPH)

IYGB – SYNOPTIC PAPER R – QUESTION 17

USING THE SUBSTITUTION GIVEN

$$u = \frac{1}{x} + xe^{x}$$

$$\frac{du}{dx} = -\frac{1}{x^2} + e^{x} + xe^{x}$$

$$dx = \frac{du}{-\frac{1}{x^2} + e^{x} + xe^{x}}$$

MULTIPLY TOP BOTTOM BY x^2

$$dx = \frac{x^2}{x^3e^{x} + x^2e^{x} - 1}\,du$$

SUBSTITUTE INTO THE INTEGRAL

$$\int \frac{x^3 + x^2 - e^{-x}}{x^3 + xe^{-x}}\,dx = \int \frac{x^3+x^2-e^{-x}}{x^3 + e^{-x}} \times \frac{x^2}{x^3e^{x} + x^2e^{x} - 1}\,du$$

$$= \int \frac{x^3+x^2-e^{-x}}{x^3+xe^{-x}} \times \frac{x^2e^{-x}}{x^3e^{x}e^{-x} + x^2e^{x}e^{-x} - 1e^{-x}}\,du$$

$$= \int \frac{\cancel{x^3+x^2-e^{-x}}}{x^3+xe^{-x}} \times \frac{x^2e^{-x}}{\cancel{x^3+x^2-e^{-x}}}\,du$$

$$= \int \frac{x^2e^{-x}}{x^3+xe^{-x}}\,du$$

$$= \int \frac{e^{-x}}{x + \frac{1}{x}e^{-x}}\,du$$

$$= \int \frac{1}{xe^{x} + \frac{1}{x}}\,du$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln\left|\frac{1}{x} + xe^{x}\right| + C$$

IYGB – SYNOPTIC PAPER R – QUESTION 18

a) USING THE STANDARD FORMULA FOR THE SUM TO INFINITY

$$S_\infty = \frac{a}{1-r} \qquad -1 < r < 1$$

$$S_\infty = \frac{\sin\theta}{1-\cos\theta} = \ldots$$

$\sin 2A \equiv 2\sin A\cos A \Rightarrow \sin A \equiv 2\sin\frac{A}{2}\cos\frac{A}{2}$

$\cos 2A \equiv 1-2\sin^2 A \Rightarrow \cos A = 1-2\sin^2\frac{A}{2}$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-\left(1-2\sin^2\frac{\theta}{2}\right)} = \frac{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}{\cancel{2}\sin^{\cancel{2}}\theta/2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

AS REQUIRED

b) USING THE SAME FORMULA WITH $\sin\theta$ & $\cos\theta$ "REVERSED"

$$S_\infty = \frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cancel{\cos\theta}(1+\sin\theta)}{\cos^{\cancel{2}}\theta} = \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$

AS REQUIRED

IYGB - SYNOPTIC PAPER R - QUESTION 19

GET SIMPLIFIED EXPRESSIONS FOR $f(x+h)-f(x)$

$$f(x+h)-f(x) = \frac{(x+h)^2}{x+h-1} - \frac{x^2}{x-1} = \frac{(x-1)(x+h)^2 - x^2(x+h-1)}{(x-1)(x+h-1)}$$

$$= \frac{(x-1)(x^2+2xh+h^2) - x^3 - x^2h + x^2}{(x-1)(x+h-1)}$$

$$= \frac{\cancel{x^3} + 2x^2h + xh^2 - \cancel{x^2} - 2xh - h^2 - \cancel{x^3} - x^2h - \cancel{x^2}}{(x-1)(x+h-1)}$$

$$= \frac{x^2h - 2xh - h^2}{(x-1)(x+h-1)}$$

NOW THE LIMITING PROCESS

$$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\big(f(x+h)-f(x)\big)\right]$$

$$= \lim_{h \to 0}\left[\frac{1}{h}\left[\frac{x^2h - 2xh - h^2}{(x-1)(x+h-1)}\right]\right]$$

$$= \lim_{h \to 0}\left[\frac{x^2 - 2x - h}{(x-1)(x+h-1)}\right]$$

$$= \frac{x^2-2x}{(x-1)(x-1)}$$

$$= \frac{x(x-2)}{(x-1)^2}$$

IYGB - SYNOPTIC PAPER R - QUESTION 20

$$\boxed{\frac{dx}{dt} = k(4+x)(4-x)e^{-t} \qquad t=0 \qquad x=0}$$

SEPARATING VARIABLES

$$\Rightarrow \frac{1}{(4+x)(4-x)}\,dx = ke^{-t}\,dt$$

$$\Rightarrow \int_{x=0}^{x} \frac{1}{(4+x)(4-x)}\,dx = \int_{t=0}^{t} ke^{-t}\,dt$$

BY PARTIAL FRACTIONS IN THE LHS

$$\frac{1}{(4+x)(4-x)} \equiv \frac{A}{4+x} + \frac{B}{4-x}$$

$$1 \equiv A(4-x) + B(4+x)$$

- IF $x=4$ $\quad 1 = 8B$, $\quad B = \frac{1}{8}$
- IF $x=-4$ $\quad 1 = 8A$, $\quad A = \frac{1}{8}$

RETURNING TO THE O.D.E.

$$\Rightarrow \int_{x=0}^{x} \frac{\frac{1}{8}}{4+x} + \frac{\frac{1}{8}}{4-x}\,dx = \int_{t=0}^{t} ke^{-t}\,dt$$

$$\Rightarrow \int_{x=0}^{x} \frac{1}{4+x} + \frac{1}{4-x}\,dx = \int_{t=0}^{t} 8ke^{-t}$$

$$\Rightarrow \Big[\ln|4+x| - \ln|4-x|\Big]_0^x = \Big[-8ke^{-t}\Big]_0^t$$

$$\Rightarrow \Big[\ln\left|\frac{4+x}{4-x}\right|\Big]_0^x = \Big[8ke^{-t}\Big]_t^0$$

IYGB – SYNOPTIC PAPER R – QUESTION 20

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| - \cancel{\ln 1} = 8k - 8ke^{-t}$$

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| = 8k(1-e^{-t})$$

NOW AS $t \to +\infty$ $\quad x \to 2$

$$\Rightarrow \ln\left(\frac{4+2}{4-2}\right) = 8k(1-0)$$

$$\Rightarrow \ln 3 = 8k$$

$$\Rightarrow \left(k = \frac{1}{8}\ln 3\right)$$

FINALLY WE HAVE

$$\Rightarrow \ln\left|\frac{4+x}{4-x}\right| = (\ln 3)(1-e^{-t})$$

$$\Rightarrow \ln\left|\frac{4+1}{4-1}\right| = (\ln 3)(1-e^{-t})$$

$$\Rightarrow \ln\frac{5}{3} = \ln 3(1-e^{-t})$$

$$\Rightarrow \frac{\ln 5/3}{\ln 3} = 1 - e^{-t}$$

$$\Rightarrow e^{-t} = 1 - \frac{\ln 5/3}{\ln 3}$$

$$\Rightarrow e^{-t} = 0.535026\ldots$$

$$\Rightarrow -t = \ln(0.535026\ldots)$$

$$t \approx 0.625$$

IYGB - SYNOPTIC PAPER R - QUESTION 21

a) THE LINE WILL HAVE EQUATION $y = mx + c$ & PASSES THROUGH $(6,0)$

$$\Rightarrow y = mx + c$$
$$\Rightarrow 0 = 6m + c$$
$$\Rightarrow c = -6m$$

$$\therefore \ y = mx + c$$
$$y = mx - 6m$$
$$y = m(x-6)$$

// AS REQUIRED

b) SOLVING SIMULTANEOUSLY l_1 & l_2

- $y = 2x + 9$
- $y = m(x-6)$

$$\Rightarrow m(x-6) = 2x + 9$$
$$\Rightarrow mx - 6m = 2x + 9$$
$$\Rightarrow mx - 2x = 6m + 9$$
$$\Rightarrow x(m-2) = 6m + 9$$
$$\Rightarrow x = \frac{6m+9}{m-2}$$

- $y = 2x + 9$

$$\Rightarrow y = 2\left(\frac{6m+9}{m-2}\right) + 9$$
$$\Rightarrow y = \frac{12m+18}{m-2} + 9$$
$$\Rightarrow y = \frac{12m + \cancel{18} + 9m \cancel{-18}}{m-2}$$
$$\Rightarrow y = \frac{21m}{m-2}$$

$$\therefore \ A\left(\frac{6m+9}{m-2}, \frac{21m}{m-2}\right)$$

c) REPEAT WITH l_1 & l_3

- $y = 2x - 3$
- $y = m(x-6)$

$$\Rightarrow m(x-6) = 2x - 3$$
$$\Rightarrow mx - 6m = 2x - 3$$
$$\Rightarrow mx - 2x = 6m - 3$$
$$\Rightarrow (m-2)x = 6m - 3$$
$$\Rightarrow x = \frac{6m-3}{m-2}$$

- $y = 2x - 3$

$$\Rightarrow y = 2\left(\frac{6m-3}{m-2}\right) - 3$$
$$\Rightarrow y = \frac{12m-6}{m-2} - 3$$
$$\Rightarrow y = \frac{12m \cancel{-6} - 3m \cancel{+6}}{m-2}$$
$$\Rightarrow y = \frac{9m}{m-2}$$

IYGB - SYNOPTIC PAPER R - QUESTION 21

THUS $A\left(\frac{6m+9}{m-2}, \frac{21m}{m-2}\right)$ $B\left(\frac{6m-3}{m-2}, \frac{9m}{m-2}\right)$

$$\Rightarrow |AB| = \sqrt{\left[\frac{6m-3}{m-2} - \frac{6m+9}{m-2}\right]^2 + \left[\frac{21m}{m-2} - \frac{9m}{m-2}\right]^2}$$

$$\Rightarrow |AB| = \sqrt{\left(\frac{-12}{m-2}\right)^2 + \left(\frac{12m}{m-2}\right)^2}$$

$$\Rightarrow |AB| = \sqrt{\frac{144 + 144m^2}{(m-2)^2}}$$

$$\Rightarrow |AB| = \sqrt{\frac{144(1+m^2)}{m^2-4m+4}}$$

AS REQUIRED

d) FINALLY SOLVING $|AB| = 4\sqrt{2}$

$$\Rightarrow \sqrt{\frac{144(1+m^2)}{m^2-4m+4}} = 4\sqrt{2}$$

$$\Rightarrow \frac{144(1+m^2)}{m^2-4m+4} = 32$$

$$\Rightarrow 144 + 144m^2 = 32m^2 - \quad m + 128$$

$$\Rightarrow 112m^2 + 128m + 16 = 0$$

$\div 16$

$$\Rightarrow 7m^2 + 8m + 1 = 0$$

$$\Rightarrow (7m+1)(m+1) = 0$$

$m = -1, -\frac{1}{7}$ $\quad$ $c = 6, \frac{6}{7}$

$\therefore$

- $y = 6 - x$
- $y = -\frac{1}{7}x + \frac{6}{7}$

$7y = -x + 6$

$x + 7y = 6$

IYGB - SYNOPTIC PAPER R - QUESTION 22

USING THE SUMMATION FORMULA FOR AN A.P. WITH $a=-10$, $d=4$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\sum_{n=1}^{k} u_n = S_k = \frac{k}{2}\left[2(-10)+(k-1)\times 4\right] = \frac{k}{2}\left[-20+4k-4\right] = k(2k-12)$$

$$\sum_{n=1}^{2k} u_n = S_{2k} = \frac{2k}{2}\left[2(-10)+(2k-1)\times 4\right] = k\left[-20+8k-4\right] = k(8k-24)$$

THUS WE CAN WRITE

$$\sum_{n=1}^{2k} u_n - \sum_{n=1}^{k} u_n = 1728$$

$$k(8k-24) - k(2k-12) = 1728 \quad \div 2$$

$$k(4k-12) - k(k-6) = 864$$

$$k(4k-12-k+6) = 864$$

$$k(3k-6) = 864 \quad \div 3$$

$$k(k-2) = 288$$

BY INSPECTION AS WE ARE LOOKING FOR A POSITIVE INTEGER OR THE QUADRATIC FORMULA

$$k^2 - 2k - 288 = 0$$

$$(k+16)(k-18) = 0$$

$$k = 18$$